Dispersive analysis of pion-kaon scattering

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Motivation to study $\pi K$ scattering

- $\pi,K$ appear as final products of almost all hadronic strange processes: Examples: B,D, decays, CP violation studies, etc…

- $\pi,K$ are Goldstone Bosons of QCD → Test Chiral Symmetry Breaking

- Many light resonances appear → Strange SPECTROSCOPY

Particularly interesting:
- $\kappa/K_0^*(800)$ light scalar meson. “needs confirmation”. Light scalar mesons longstanding candidates for non-ordinary mesons. Settle multiplet classification?

- $K_0^*(1430)$ smaller discussion on parameters and nature
Most reliable sets:
Estabrooks et al. 78 (SLAC)
Aston et al. 88 (SLAC-LASS)

I=1/2 and 3/2 combination

No clear “peak” or phase movement of $\kappa/K_0^*(800)$ resonance

Definitely NO BREIT-WIGNER shape

Mathematically correct to use POLES
Resonances as poles

The Breit-Wigner shape is just an approximation for narrow and isolated resonances

The universal features of resonances are their pole positions and residues *

$$\sqrt{s_{pole}} \approx M - i \Gamma / 2$$

*in the Riemann sheet obtained from an analytic continuation through the physical cut
Why use dispersion relations?

CAUSALITY: Amplitudes $T(s,t)$ are ANALYTIC in complex $s$ plane but for cuts for thresholds. Crossing implies left cut from u-channel threshold.

Cauchy Theorem determines $T(s,t)$ at ANY $s$, from an INTEGRAL on the contour.

If $T->0$ fast enough at high $s$, curved part vanishes.

$$T(s, t) = \int_{th}^{\infty} \frac{Im T(s', t)}{s - s'} ds' + LC$$

Otherwise, determined up to polynomial (subtractions)

Left cut usually a problem

Good for:
1) Calculating $T(s,t)$ where there is not data
2) Constraining data analysis
3) ONLY MODEL INDEPENDENT extrapolation to complex $s$-plane without extra assumptions
Why so much worries about low energy and CORRECT ANALYTIC STRUCTURE?

Analyticity is expressed in the $s$-variable, not in $\sqrt{s}$.
Why so much worries about low energy and CORRECT ANALYTIC STRUCTURE?

Analyticity is expressed in the $s$-variable, not in $\sqrt{s}$

Important for the $\kappa/K_0^*(800)$
- Threshold behavior (chiral symmetry)
- Subthreshold behavior (chiral symmetry → Adler zeros)
- Other cuts (Left & circular)
- Avoid spurious singularities

Problem shared by lattice!

Less important for other resonances…
So, we need to get rid of ONE VARIABLE to write CAUCHY THEOREM in terms of the other one.

TWO MAIN APPROACHES

1) Integrate one variable and keep the other
   (partial wave dispersion relations)
Due to elastic unitarity:

\[ S^{II}(s) = \frac{1}{S^I(s)} \]

Recalling \( s(s) = 1 + 2i\sigma t(s), \quad \sigma(s) = \frac{k}{2\sqrt{s}} \)

The second sheet is then:

\[ t^{II}(s) = \frac{t^I(s)}{1 + 2i\sigma t^I(s)} \]

Looking for resonance poles is nothing but looking for a zero in that denominator on the first Riemann sheet accessible with the pw DR

The problem is the left (and circular) cut
**Unitarized ChPT**

90’s Truong, Dobado, Herrero, JRP, Oset, Oller, Ruiz Arriola, Nieves, Meissner,…

Uses Chiral Perturbation Theory amplitudes inside dispersion relation.

Relatively simple, although different levels of rigour. Generates all scalars

LEFT CUT APPROXIMATED, not so good for precision: \((753 \pm 52)-i(235 \pm 33)\)MeV

But good for connecting with QCD. Strong hints of non-ordinary nature:

![Graphs showing Nc behavior and m_q dependence](graph.png)

Correct behavior obtained for vectors

Virtual state recently found on lattice

Both suggest important “molecular” component
**Roy-like equations.** 70's Roy, Basdevant, Pennington, Petersen…  
00's Ananthanarayan, Caprini, Colangelo, Gasser, Leutwyler, Moussallam, Decotes Genon, Lesniak, Kaminski, JRP, Ruiz de Elvira, Yndurain…

**LEFT CUT WITH PRECISION.**

**PRICE:** Infinite set of coupled integral equations. VALIDITY LIMITED at ~1.1 GeV

Use data on all waves + high energy. Optional: ChPT predictions for subtraction constants

The most precise and model independent pole determinations

\[
f_0(500) \text{ and } K_0^*(800) \text{ existence, mass and width firmly established with precision}
\]

\[
(658\pm13)-i(278.5\pm12) \text{ MeV} \\
\text{Descotes-Genon, B. Moussallam}
\]

Listed @PDG, but not enough for PDG

This approach already summarized yesterday by J. Ruiz de Elvira in his talk on applications to threshold region
SOLVE equations: (Ananthanarayan, Colangelo, Gasser, Leutwyler, Caprini, Moussallam, Stern…)

S and P wave solution for Roy-like equations unique at low energy if high-energy, higher waves and scattering lengths known. (in isospin limit)

NO scattering DATA used at low energies ($\sqrt{s} \leq 1 \text{ GeV}$)

Good if interested in low energy scattering and do not trust data.

Uses ChPT/other input for threshold parameters

(see B. Moussallam’s talk)

Impose Dispersion Relations on fits to data. (García-Martín, Kaminski,JRP, Ruiz de Elvira, Ynduráin)

Use any functional form and fit to DATA imposing DR within uncertainties.

Also needs input on other waves and high energy.

(But you can use physical inspiration for clever choices of parameterizations)
So, we need to get rid of ONE VARIABLE to write CAUCHY THEOREM in terms of the other one.

TWO MAIN APPROACHES

1) Integrate one variable and keep the other (partial wave dispersion relations)

2) Fix one variable in terms of the other (fixed-t, hyperbolic relations…)
Fixed-t Dispersion Relations (DR)

Simple analytic structure in s-plane, simple derivation and use
Left cut: With crossing can be rewritten in terms of physical region

Most popular: $t_0=0$, FORWARD DISPERSION RELATIONS (FDRs).
(Kaminski, Pelaez, Yndurain, Garcia Martin, Ruiz de Elvira, Rodas)

One equation per amplitude.
High Energy part known since Forward Amplitude~ Total cross section

Calculated up $1.7 \text{ GeV for } \pi K$ (and $1400 \text{ MeV for } \pi \pi$)

Not directly usable for unphysical sheets but very useful to constraint physical amplitudes up to relatively high energies
Since interested in the resonance region, we use minimal number of subtractions

Defining the $s\leftrightarrow u$ symmetric and anti-symmetric amplitudes at $t=0$

\[
\begin{align*}
T^+(s) &= \frac{T^{1/2}(s) + 2T^{3/2}(s)}{3} = \frac{T^{I_1-0}(s)}{\sqrt{6}}, \\
T^-(s) &= \frac{T^{1/2}(s) - T^{3/2}(s)}{3} = \frac{T^{I_1-1}(s)}{2}.
\end{align*}
\]

We need one subtraction for the symmetric amplitude

\[
\text{Re}T^+(s) = T^+(s_{th}) + \frac{(s - s_{th})}{\pi} P \int_{s_{th}}^{\infty} ds' \left[ \frac{\text{Im}T^+(s')}{(s' - s)(s' - s_{th})} - \frac{\text{Im}T^+(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],
\]

And none for the antisymmetric

\[
\text{Re}T^-(s) = \frac{(2s - 2\Sigma_{\pi K})}{\pi} P \int_{s_{th}}^{\infty} ds' \frac{\text{Im}T^-(s')}{(s' - s)(s' + s - 2\Sigma_{\pi K})}.
\]

where $\Sigma_{\pi K} = m_{\pi}^2 + m_{K}^2$
Dispersive analysis of $\pi K$ scattering DATA up to 1.6 GeV

(not a solution of dispersión relations, but a constrained fit)
A. Rodas & JRP, PRD93,074025 (2016)

First observation:
Forward Dispersion relations
Not well satisfied by data
Particularly at high energies

So we use
Forward Dispersion Relations
as CONSTRAINTS on fits
How well Dispersion Relations are satisfied by unconstrained fits

Every 22 MeV calculate the difference between both sides of the DR /uncertainty

Define an averaged $\chi^2$ over these points, that we call $d^2$

$d^2$ close to 1 means that the relation is well satisfied

d$^2 \gg 1$ means the data set is inconsistent with the relation.

This can be used to check DR

To obtain CONSTRAINED FITS TO DATA (CFD) we minimize:

$$\chi^2 = W \left\{ \overline{d_{T+}}^2 + \overline{d_{T-}}^2 \right\} + \overline{d_{1/2}}^2 + \overline{d_{3/2}}^2 + \sum_k \frac{(p_k - p_k^{\text{exp}})^2}{\delta p_k^2}$$

2 FDR’s

Sum Rules threshold

Parameters of the unconstrained data fits

W roughly counts the number of effective degrees of freedom
(sometimes we add weight on certain energy regions)
S-waves. The most interesting for the $K_0^*$ resonances

Largest changes from UFD to CFD
at higher energies
From Unconstrained (UFD) to Constrained Fits to data (CFD)

P-waves: Small changes

Our fits describe data well
SOLUTION from previous Roy-Steiner approach
From Unconstrained (UFD) to Constrained Fits to data (CFD)

D-waves: Largest changes of all, but at very high energies

F-waves: Imperceptible changes

Regge parameterizations allowed to vary: Only \( \pi K - \rho \) residue changes by 1.4 deviations
Consistency up to 1.6 GeV!!

$\text{Re}T^+(s)$
- Dispersive UFD
- Input UFD

$d^2 = 4.1$

Consistency up to 1.74 GeV!!

$\text{Re}T^+(s)$
- Dispersive CFD
- Input CFD

$d^2 = 1.3$

$d^2 = 3.4$

$d^2 = 0.37$
We have used FORWARD DISPERSION RELATIONS to constraint $\pi K$ scattering amplitudes up to 1.6 GeV:

- Simple parameterizations. Easy to use
- Still describe data
- Consistent with unitarity, ANALYTICITY and crossing

In progress:

We are about to finish the $\pi\pi \rightarrow KK$ Roy-Steiner analysis up to 1.5 GeV
Working on the Roy-Steiner analysis for $\pi K \rightarrow \pi K$
Strange scalar resonances from dispersive analysis and analyticity

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Model independent analysis

- DR provide 1st Riemann Sheet
- For partial waves, IF in elastic regime, poles of S in 2nd sheet are zeros on 1st

- ONLY ONE MODEL INDEPENDENT ANALYSIS from a Roy-Steiner dispersive formalism Decotes Genon et al 2006

AT low energies is a SOLUTION it does NOT use data. Call it prediction?

\[(658\pm13)-i(278.5\pm12)\text{ MeV}\]

Listed @PDG, but not enough for PDG
They ask for more dispersive determinations
Possibly with different approaches
We have amplitudes that describe data and satisfy dispersion relations up to 1.6 GeV.

There is also a $\kappa$ Pole in our CFD parameterizations.
We use the unitary functional form for the partial waves

$$t^l(s) = \frac{1}{\sigma(s)} \frac{1}{\cot \delta^l(s) - i}$$  \hspace{1cm} (5)

Where

$$\cot \delta^l(s) = \frac{\sqrt{s}}{2q^{2l+1}} \sum B_n \omega(s)^n$$  \hspace{1cm} (6)

with \( \omega(s) = \frac{\sqrt{y(s) - \alpha \sqrt{y(s^0) - y(s)}}}{\sqrt{y(s) + \alpha \sqrt{y(s^0) - y(s)}}} \) as our new variable (conformal mapping).

Here \( y(s) = (\frac{s-su}{s+su})^2 \) defines the circular cut on the next figure. \( \omega \) used to maximize the analyticity domain.
Figure: Structure of the PW.

- $\alpha$ is used to center the point of energy $s_c$ for the expansion.
Kappa pole from CFD

We have amplitudes that describe data and satisfy dispersion relations up to 1.6 GeV

There is also a $\kappa$ POLE in our CFD parameterizations

- Extracted from our conformal CFD parameterization A.Rodas & JRP, PRD93,074025 (2016)
  Fantastic analyticity properties, but still not completely model independent

  $(680\pm15)-i(334\pm7.5)$ MeV
The method is suitable for the calculation of both elastic and inelastic resonances.

The Padé sequence gives us the continuation to the continuous Riemann Sheet.

We take care of the calculation of the errors. Apart from the experimental and systematic errors of each parameterization we also include different fits.
Kappa pole from CFD

We have amplitudes that describe data and satisfy dispersion relations up to 1.6 GeV

| There is also a $\kappa$ POLE in our CFD parameterizations |

- Extracted from our conformal CFD parameterization A.Rodas & JRP, PRD93,074025 (2016)
  Fantastic analyticity properties, but still not completely model independent
  $(680\pm 15)-i(334\pm 7.5)$ MeV

  Almost model independent: Do not assume any functional form
  (but local determination)
  $(680\pm 13)-i(325\pm 7)$ MeV

Compare to PDG: $(682\pm 29)-i(273\pm 12)$ MeV
Summary

- Dispersion relations have been useful for establishing the existence of resonances and for rigorous determinations of their parameters.

- For light scalars, they have settled the longstanding $\sigma$-meson controversy and are on the way to settle that of the $\kappa$-meson.

Still in progress:

A second dispersive determination with Roy-Steiner and FDRs will finally settle the $\kappa/K_0^*(800)$ issue at the PDG. Our group has been asked to do it.

We are about to finish the $\pi\pi\rightarrow KK$ analysis needed as input for $\pi K\rightarrow\pi K$. 
SPARE SLIDES