# Energy Correlations without scattering amplitudes 

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## $\mathrm{e}^{+} \mathrm{e}^{-}$annihilation into hadrons


$\checkmark$ The lepton pair annihilates into a virtual photon (Z-boson) that in turn decays into quark and gluons that undergo a hadronization process into mesons and baryons
$\checkmark$ The total (inclusive) cross section

$$
\sigma_{\text {tot }}(q)=\sum_{X}(2 \pi)^{4} \delta^{(4)}\left(q-k_{X}\right)\left|\mathcal{M}_{\gamma^{*}(q) \rightarrow X}\right|^{2}
$$

requires a calculation of parton-level amplitudes (and understanding of fragmentation?)

$$
\mathcal{M}_{\gamma^{*}(q) \rightarrow X}=\langle X| \epsilon^{\mu}(q) j_{\mu}(0)|0\rangle
$$

$\checkmark$ Completeness condition for hadronic states saves the day
$\sigma_{\text {tot }}(q)=\epsilon_{\mu}^{*}(q) \epsilon_{\mu}(q) \int d^{4} x \mathrm{e}^{i q \cdot x}\langle 0| j^{\nu}(x) j^{\mu}(0)|0\rangle$
$\checkmark$ Whightman function (non time-ordered) can be related to Green function via optical theorem (use OPE to analyze!)

## Weighted cross sections

$\checkmark$ If the cross section is not totally inclusive w.r.t. final state, i.e., it measures the properties of the latter, then its value computed in QCD perturbation theory will deviate from the experimental one. This difference is due to hadronization.
$\checkmark$ However, we do not measure the fate of individual particles but rather only the energy flow into final states.

The final states can be described by a class of infrared safe observables known as event shapes.
$\frac{d \sigma}{d e}=\sum_{X}(2 \pi)^{4} \delta^{(4)}\left(q-k_{X}\right)\left|\mathcal{M}_{\gamma^{*}(q) \rightarrow X}\right|^{2} \delta\left(e-e\left(k_{X}\right)\right)$
Here $e(X)$ is a function of the momenta of the particles populating the final state
$\checkmark$ Infrared safety implies linear suppression of contributions from soft-gluon radiation in the weight factor!

$\checkmark$ Independent of the jet definition/algorithm

## Event shapes

Event shape variables are given by the following weight functions:
$x$ Thrust:

$$
T=\max _{\vec{n}} \sum_{n}\left|\vec{n} \cdot \vec{k}_{n}\right| / \sum_{n}\left|\vec{k}_{n}\right|
$$

$x$ Heavy-jet mass

$$
\rho_{H}=\max \left(\sum_{n \in L, R} k_{n}^{\mu}\right)^{2} / q^{2}
$$

$x$ Broadening

$$
B=\frac{1}{2} \sum_{n}\left|\vec{k}_{n}^{\perp}\right| / \sum_{n}\left|\vec{k}_{n}\right|
$$

Allow to measure different properties of the event.
Event shape variables vary depending on the geometry of the underlying event.
$\checkmark$ Extract precise information about QCD strong coupling.
$\checkmark$ Hadronization corrections show up power suppressed by the c -of-m energy!

| event shape | $1-T$ | $\rho_{H}$ | $B$ |
| :--- | :---: | :---: | :---: |
| pencil-like | 0 | 0 | 0 |
| spherical | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{\pi}{8}$ |

## Energy flow observables

All event shape observables are related to energy flow into the final state, as can be seen from their moments
$\int e^{N} d \sigma=\sum_{X}(2 \pi)^{4} \delta^{(4)}\left(q-k_{X}\right) e^{N}\left(k_{X}\right)\left|\mathcal{M}_{\gamma^{*}(q) \rightarrow X}\right|^{2}$

Direct access to energy distribution and correlation of final state particles is achieved through energy correlations as measured by detectors located at "spatial infinity" in the direction of the vectors $\vec{n}$.
$x$ Single detector weight:

$$
w_{\mathcal{E}}(\vec{n})=\sum_{X} E_{X} \delta^{(2)}\left(\Omega_{\vec{k}_{X}}-\Omega_{\vec{n}}\right)
$$


$x$ Double detector weight (energy-energy correlations):

$$
\begin{aligned}
w_{\mathrm{EEC}}(\chi) & =\int d \Omega_{\vec{n}_{1}} d \Omega_{\vec{n}_{2}} \delta\left(\vec{n}_{1} \cdot \vec{n}_{2}-\cos \chi\right) w_{\mathcal{E}}\left(\vec{n}_{1}\right) w_{\mathcal{E}}\left(\vec{n}_{2}\right) \\
& =\sum_{X, X^{\prime}} E_{X} E_{X^{\prime}} \delta\left(\cos \theta_{X X^{\prime}}-\cos \chi\right)
\end{aligned}
$$

$x$ etc.

## Energy-energy correlations

$\checkmark$ Function of the angle $0 \leq \chi \leq \pi$ between detected particles [Basham,Brown,Ellis,Love]
$\operatorname{EEC}(\chi)=\left\langle\frac{1}{\Delta \chi} \sum_{a, b} \frac{E_{a} E_{b}}{Q^{2}} \theta\left(\Delta \chi-\left|\cos \theta_{a b}-\cos \chi\right|\right)\right\rangle$
events
Total energy $\sum_{a} E_{a}=Q$
$\checkmark$ Conventional ('amplitude') approach


$$
\operatorname{EEC}(\chi)=\frac{1}{\sigma_{\text {tot }}} \sum_{a, b} \int d \sigma_{a+b+X} \frac{E_{a} E_{b}}{Q^{2}} \delta\left(\cos \theta_{a b}-\cos \chi\right)
$$

$\sigma_{\text {tot }}$ total cross section $e^{+} e^{-} \rightarrow$ hadrons
$\checkmark$ Weak coupling expansion in QCD

$$
\operatorname{EEC}(\chi)=a_{\mathrm{S}} A(\chi)+a_{\mathrm{S}}^{2} B(\chi)+O\left(a_{\mathrm{S}}^{3}\right)
$$

$\checkmark$ Current status (1978 - today):
$x$ Very precise experimental data
$\times$ Poor analytical control, $B(\chi)$ is known numerically


## Amplitudes vs. correlators

$\checkmark$ 'Amplitude approach' has the following disadvantages:
x presence of intrinsic infrared divergences inside transition amplitudes $\mathcal{M}_{\gamma^{*}(q) \rightarrow X}$
$x$ integration over the phase space of the final states and subsequent intricate IR cancellations
$x$ necessity for summation over all final states

$$
\left.\sigma_{w}(q)=\sum_{X}(2 \pi)^{2} \delta^{(4)}\left(q-k_{X}\right) w(X)|\langle X| O(0)| 0\right\rangle\left.\right|^{2}
$$






$\checkmark$ New approach: event shapes (energy correlations) from Wightman correlation functions

$$
\sigma_{w}(q)=\int d^{4} x \mathrm{e}^{i q x}\langle 0| O(x) \mathcal{E}[w] O(0)|0\rangle
$$

$x$ no IR divergences are present in the correlation functions
$x$ no summation over all final states is needed
$x$ no integration over the phase space is required

## Energy-flow operator

To be able to recover an "optical theorem" for weighted cross sections, we have to find find the operator that produces the weight when acting on the final state

$$
w_{\mathcal{E}}(\vec{n})=\sum_{X} E_{X} \delta^{(2)}\left(\Omega_{\vec{k}_{X}}-\Omega_{\vec{n}}\right)
$$

The energy-flow operator obeys

$$
\mathcal{E}(\vec{n})|X\rangle=w_{\mathcal{E}}(X)|X\rangle
$$

It is expressed in terms of stress-tensor

$$
\mathcal{E}(\vec{n})=\int_{0}^{\infty} d t \lim _{r \rightarrow \infty} r^{2} \vec{n}^{i} T_{0 i}(t, r \vec{n})
$$

Since the operator acts at spatial infinity, the fields are noninteracting and one can represent the energy-flow operator in terms of free on-shell states

$$
\mathcal{E}(\vec{n})=\int \frac{d^{4} k}{(2 \pi)^{4}} 2 \pi \delta_{+}\left(k^{2}\right) k^{0} \delta\left(\Omega_{\vec{k}}-\Omega_{\vec{n}}\right) \sum_{f} a_{f}^{\dagger}(k) a_{f}(k)
$$



## Energiy correlations

$\checkmark$ Single correlator

$$
\sum_{X}\langle 0| O(x)|X\rangle w_{\mathcal{E}}(X)\langle X| O(0)|0\rangle=\sum_{X}\langle 0| O(x) \mathcal{E}(\vec{n})|X\rangle\langle X| O(0)|0\rangle=\langle 0| O(x) \mathcal{E}(\vec{n}) O(0)|0\rangle
$$

Wightman correlation function (no time ordering!) due to real-time evolution
The weighted cross section:

$$
\sigma_{w}(q)=\int d^{4} x \mathrm{e}^{i q \cdot x}\langle 0| O(x) \mathcal{E}(\vec{n}) O(0)|0\rangle
$$

Single-energy correlation:

$$
\langle\mathcal{E}(\vec{n})\rangle=\sigma_{w}(q) / \sigma_{\mathrm{tot}}(q)
$$

$\checkmark$ Multi-energy correlations:

$$
\begin{aligned}
& \left\langle\mathcal{E}\left(\vec{n}_{1}\right) \ldots \mathcal{E}\left(\vec{n}_{\ell}\right)\right\rangle \\
& \quad=\sigma_{\text {tot }}^{-1} \int d^{4} x \mathrm{e}^{i q x}\langle 0| O(x) \mathcal{E}\left(\vec{n}_{1}\right) \ldots \mathcal{E}\left(\vec{n}_{\ell}\right) O(0)|0\rangle
\end{aligned}
$$

Energy flow in the direction of $\vec{n}_{1}, \ldots, \vec{n}_{\ell}$
Depends on the relative angles $\cos \theta_{i j}=\left(\vec{n}_{i} \cdot \vec{n}_{j}\right)$

$\checkmark$ Everything is boils down to the calculation of Wightman correlation functions.
$\checkmark$ A lot of recent progress in calculation of Euclidean correlation functions in $\mathbb{N}=4$ SYM. Can we use it?

## Initial and final states in $\mathcal{N}=4$ SYM

$\checkmark$ Use the protected half-BPS operator $\mathrm{O}_{20}$ as an analogue of the QCD electromagnetic current

$$
O(x)=Y^{I} Y^{J} O_{20^{\prime}}^{I J}=Y^{I} Y^{J} \operatorname{tr}\left[\Phi^{I}(x) \Phi^{J}(x)\right]
$$

The null vector $Y^{I}$ defines the orientation of the projected operator in the isotopic $\mathrm{SO}(6)$ space
$\checkmark$ To lowest order in the coupling, $O(x)$ produces a pair of scalars out of the vacuum
$\checkmark$ For arbitrary coupling, the state $O(x)|0\rangle$ can be decomposed into an infinite sum over on-shell states with an arbitrary number of scalars ( $s$ ), gauginos $(\lambda)$ and gauge fields $(g)$

$$
\int d^{4} x \mathrm{e}^{i q x} O(x)|0\rangle=|s s\rangle+|s s g\rangle+|s \lambda \lambda\rangle+\ldots
$$

$\checkmark$ The amplitude of creation of a particular final state $|X\rangle$ out of the vacuum

$$
\langle X| \int d^{4} x \mathrm{e}^{i q x} O(x)|0\rangle=(2 \pi)^{4} \delta^{(4)}\left(q-p_{X}\right) \mathcal{M}_{O_{20^{\prime}} \rightarrow X}
$$

$p_{X}$ is the total momentum of the state $|X\rangle$
$\checkmark$ The amplitude $\mathcal{M}_{O \rightarrow X}$ has the meaning of a (IR divergent) form-factor

$$
\mathcal{M}_{O_{20^{\prime}} \rightarrow X}=\langle X| O(0)|0\rangle
$$



## $N=4$ total "cross section"

$\checkmark$ The analogue of $e^{+} e$ annihilation into everything

$$
\sigma_{\text {tot }}(q)=\sum_{X}(2 \pi)^{4} \delta^{(4)}\left(q-p_{X}\right)\left|\mathcal{M}_{O_{20^{\prime}} \rightarrow X}\right|^{2}
$$

$\checkmark$ To lowest order in the coupling, the production of a pair of scalars

$$
\sigma_{\mathrm{tot}}(q)=\frac{1}{2}\left(N^{2}-1\right) \int \frac{d^{4} k}{(2 \pi)^{4}}(2 \pi)^{2} \delta_{+}\left(k^{2}\right) \delta_{+}\left((q-k)^{2}\right)+\ldots
$$

$\checkmark$ To higher order in the coupling, each term in the sum $\sum_{X}$ has IR / collinear divergences
$\checkmark$ How to avoid divergences? Use the completeness condition $\sum_{X}|X\rangle\langle X|=1$

$$
\begin{aligned}
\sigma_{\text {tot }}(q) & =\int d^{4} x \mathrm{e}^{i q x} \sum_{X}\langle 0| O(0)|X\rangle \mathrm{e}^{-i x p_{X}}\langle X| O(0)|0\rangle \\
& =\int d^{4} x \mathrm{e}^{i q x}\langle 0| O(x) O(0)|0\rangle \quad \text { The operators are not time ordered! }
\end{aligned}
$$

Wightman correlation function (protected for half-BPS operators)
$\checkmark$ All-loop result in $\mathcal{N}=4$ SYM

$$
\sigma_{\text {tot }}(q)=\frac{1}{16 \pi}\left(N^{2}-1\right) \theta\left(q^{0}\right) \theta\left(q^{2}\right)
$$

Perturbative corrections cancel order by order

## $N=4$ correlations from amplitudes

$\checkmark$ Transition amplitude at one loop

$\checkmark$ Energy correlations
$\sigma_{\mathcal{E}}(q)=\int \mathrm{dPS}_{2} w_{\mathcal{E}}(1,2)\left|\mathcal{M}_{O_{20^{\prime}} \rightarrow s s}\right|^{2}+\int \mathrm{dPS}_{3} w_{\mathcal{E}}(1,2,3)\left(\left|\mathcal{M}_{O_{20^{\prime}} \rightarrow s s g}\right|^{2}+\left|\mathcal{M}_{O_{20^{\prime}} \rightarrow s \lambda \lambda}\right|^{2}\right)+\ldots$
$x$ Single detector correlation (protected from loop corrections)

$$
\langle\mathcal{E}(\vec{n})\rangle=\frac{q_{0}}{4 \pi}
$$

$x$ Two detectors oriented along $\vec{n}_{i}$ (unprotected quantity)

$$
\left\langle\mathcal{E}\left(\vec{n}_{1}\right) \mathcal{E}\left(\vec{n}_{2}\right)\right\rangle=-\frac{q_{0}^{2}}{(4 \pi)^{4}}\left[-a \frac{\ln (1-z)}{2 z^{2}(1-z)}+O\left(a^{2}\right)\right]
$$

The scaling variable in the rest frame of the source $z=\left(1-\cos \theta_{12}\right) / 2$
$x$ Two-loop corrections to $\left\langle\mathcal{E}\left(\vec{n}_{1}\right) \mathcal{E}\left(\vec{n}_{2}\right)\right\rangle$ are hard to compute (in components, not easier than in QCD)

## $N=4$ correlations from correlators

$\checkmark$ Energy flow operator

$$
\begin{aligned}
\left\langle\mathcal{E}\left(\vec{n}_{1}\right)\right\rangle & \sim \int d^{4} x \mathrm{e}^{i q x}\langle 0| O(x) \mathcal{E}\left(\vec{n}_{1}\right) O(0)|0\rangle \\
& =\left.\underbrace{\int d^{4} x \mathrm{e}^{i q x}}_{\text {Fourier }} \underbrace{\int_{0}^{\infty} d t \lim _{r \rightarrow \infty} r^{2}}_{\text {Detector limit }} \underbrace{\langle 0| O(x) T_{0 \vec{n}_{1}}\left(x_{1}\right) O(0)|0\rangle}_{\text {Wightman corr. function }}\right|_{x_{1}}=\left(t, r \vec{n}_{1}\right)
\end{aligned}
$$

$\checkmark$ Generalization for $\ell$ detectors

$$
\left\langle\mathcal{E}\left(\vec{n}_{1}\right) \ldots \mathcal{E}\left(\vec{n}_{\ell}\right)\right\rangle=\text { Fourier } \times \operatorname{Limit}\left[\left.\langle 0| O(x) T_{0 \vec{n}_{1}}\left(x_{1}\right) \ldots T_{0 \vec{n}_{\ell}}\left(x_{\ell}\right) O(0)|0\rangle\right|_{x_{i}=\left(t_{i}, r_{i} \vec{n}_{i}\right)}\right]
$$

$\checkmark$ How to compute energy flow correlators:
$\times$ Start with corr.function $\left\langle O(x) T\left(x_{1}\right) \ldots T\left(x_{\ell}\right) O(0)\right\rangle$ in Euclid
$x$ Continue to Minkowski with Wightman prescription
$x$ Take detector limit + perform Fourier
$\checkmark$ Correlation functions in $\mathcal{N}=4 \mathrm{SYM}$ have a lot of symmetry :
$x\left\langle O(x) T\left(x_{1}\right) O(0)\right\rangle$ is fixed by conformal symmetry $\rightarrow$ exact result for $\left\langle\mathcal{E}\left(\vec{n}_{1}\right)\right\rangle$
$x\left\langle O(x) T\left(x_{1}\right) T\left(x_{2}\right) O(0)\right\rangle$ is not fixed by conformal symmetry

## Step 1: Correlator in Euclid

$\checkmark$ Perturbative expansion can be cast in terms of conformal integrals:
$\Phi(u, v ; a)=a \Phi^{(1)}(u, v)$

$$
\begin{aligned}
& +a^{2}\left\{\frac{1}{2}(1+u+v)\left[\Phi^{(1)}(u, v)\right]^{2}\right. \\
& \left.\quad+2\left[\Phi^{(2)}(u, v)+\frac{1}{u} \Phi^{(2)}(v / u, 1 / u)+\frac{1}{v} \Phi^{(2)}(1 / v, u / v)\right]\right\}+O\left(a^{3}\right)
\end{aligned}
$$


$\checkmark$ Available AdS/CFT prediction for the function at strong coupling


## Step 2: From Euclid to Minkowski

$\checkmark$ Brute force method: compute anew using Schwinger-Keldysh technique (too hard)
$\checkmark$ Better method: analytically continue correlation functions from Euclid to Minkowski+Wightman
$\checkmark$ Warm-up example: free scalar propagator $D_{\text {Euclid }}(x)=\langle\phi(x) \phi(0)\rangle \sim 1 / x^{2}$

$$
\begin{aligned}
\langle 0| \phi(x) \phi(0)|0\rangle & =\sum_{n}\langle 0| \phi(x)|n\rangle\langle n| \phi(0)|0\rangle \\
& =\sum_{E_{n}>0} \mathrm{e}^{-i E_{n}\left(x^{0}-i 0\right)+i \vec{p} \vec{x}}\langle 0| \phi(0)|n\rangle\langle n| \phi(0)|0\rangle \sim \frac{1}{\left(x^{0}-i 0\right)^{2}-\vec{x}^{2}}
\end{aligned}
$$

$\checkmark$ How to get Wightman correlation functions ('magic' recipe):
$x$ Go to Mellin space:

$$
\Phi_{\mathrm{Euclid}}=\int_{-\delta-i \infty}^{-\delta+i \infty} \frac{d j_{1} d j_{2}}{(2 \pi i)^{2}} M\left(j_{1}, j_{2} ; a\right) u^{j_{1}} v^{j_{2}},
$$

$x$ Nontrivial Wick rotation

$$
\Phi_{\text {Wightman }}=\Phi_{\text {Euclid }}\left(x_{i j}^{2} \rightarrow x_{i j,+}^{2}=x_{i j}^{2}-i 0 \cdot x_{i j}^{0}\right)
$$

$\checkmark M\left(j_{1}, j_{2} ; a\right)$ is known both at weak and strong coupling in planar $\mathcal{N}=4$ SYM

## Step 3: Detector limit

$\checkmark$ The detector limit yields the scalar correlations, related to energy flow by susy (as shown later):

$$
\left\langle\mathcal{O}\left(n_{1}\right) \mathcal{O}\left(n_{2}\right)\right\rangle=\frac{1}{4 \pi^{2}} \frac{\mathcal{F}(z ; a)}{q^{2}\left(n_{1} \cdot n_{2}\right)}
$$

$\checkmark$ The event shape function $\mathcal{F}(z ; a)$ at any coupling:

$$
\mathcal{F}(z ; a)=\int_{-\delta-i \infty}^{-\delta+i \infty} \frac{d j_{1} d j_{2}}{(2 \pi i)^{2}} \underbrace{M\left(j_{1}, j_{2} ; a\right)}_{\text {corr. function }} \underbrace{K\left(j_{1}, j_{2} ; z\right)}_{\text {detector }}
$$

$x$ The detector function is coupling independent:

$$
K\left(j_{1}, j_{2}, z\right)=\frac{\Gamma\left(1-j_{1}-j_{2}\right)}{\Gamma\left(j_{1}+j_{2}\right)\left[\Gamma\left(1-j_{1}\right) \Gamma\left(1-j_{2}\right)\right]^{2}}\left(\frac{z}{1-z}\right)^{1-j_{1}-j_{2}}
$$

$x$ The Mellin transform of Euclidean correlator is known

$$
M\left(j_{1}, j_{2} ; a\right)=\underbrace{a M^{(1)}\left(j_{1}, j_{2}\right)+a^{2} M^{(2)}\left(j_{1}, j_{2}\right)}_{\text {are known }}+\ldots
$$

$\checkmark$ Weak and strong coupling:

$$
\mathcal{F}(z ; a \ll 1)=-\frac{a}{8} \frac{z \ln (1-z)}{1-z}+O\left(a^{2}\right) \quad \mathcal{F}(z ; a \rightarrow \infty)=-\frac{z^{3}}{2}+O(1 / a)
$$

## EEC@2-loops

Four-point correlator at weak coupling

$$
\begin{aligned}
\Phi(u, v ; a) & =a \Phi^{(1)}(u, v)+a^{2}\left\{\frac{1}{2}(1+u+v)\left[\Phi^{(1)}(u, v)\right]^{2}\right. \\
& \left.+2\left[\Phi^{(2)}(u, v)+\frac{1}{u} \Phi^{(2)}(v / u, 1 / u)+\frac{1}{v} \Phi^{(2)}(1 / v, u / v)\right]\right\}
\end{aligned}
$$

Euclidean 'scalar box' integrals $\Phi^{(1)}$ and $\Phi^{(2)}$
Mellin amplitude to two loops:

$$
\begin{aligned}
M\left(j_{1}, j_{2}\right)= & a M^{(1)}\left(j_{1}, j_{2}\right)+a^{2}\left[\frac{1}{2} \widetilde{M}^{(2)}\left(j_{1}, j_{2}\right)+\widetilde{M}^{(2)}\left(j_{1}, j_{2}-1\right)\right. \\
& \left.+2 M^{(2)}\left(j_{1}, j_{2}\right)+4 M^{(2)}\left(j_{1},-1-j_{1}-j_{2}\right)\right] \\
M^{(1)}\left(j_{1}, j_{2}\right)= & -\frac{1}{4}\left[\Gamma\left(-j_{1}\right) \Gamma\left(-j_{2}\right) \Gamma\left(1+j_{1}+j_{2}\right)\right]^{2} \\
M^{(2)}\left(j_{1}, j_{2}\right)= & -\frac{1}{4} \Gamma\left(-j_{1}\right) \Gamma\left(-j_{2}\right) \Gamma\left(1+j_{1}+j_{2}\right) \\
& \times \int \frac{d j_{1}^{\prime} d j_{2}^{\prime}}{(2 \pi i)^{2}} M^{(1)}\left(j_{1}^{\prime}, j_{2}^{\prime}\right) \frac{\Gamma\left(j_{1}^{\prime}-j_{1}\right) \Gamma\left(j_{2}^{\prime}-j_{2}\right) \Gamma\left(1+j_{1}+j_{2}-j_{1}^{\prime}-j_{2}^{\prime}\right)}{\Gamma\left(1-j_{1}^{\prime}\right) \Gamma\left(1-j_{2}^{\prime}\right) \Gamma\left(1+j_{1}^{\prime}+j_{2}^{\prime}\right)} \\
\widetilde{M}^{(2)}\left(j_{1}, j_{2}\right) & =\int \frac{d j_{1}^{\prime} d j_{2}^{\prime}}{(2 \pi i)^{2}} M^{(1)}\left(j_{1}-j_{1}^{\prime}, j_{2}-j_{2}^{\prime}\right) M^{(1)}\left(j_{1}^{\prime}, j_{2}^{\prime}\right)
\end{aligned}
$$

## Warm-up

$\checkmark$ Master formula at one loop

$$
\operatorname{EEC}^{(1-\text { loop })}=\frac{a}{4 z^{2}(1-z)} \int_{-\delta-i \infty}^{-\delta+i \infty} \frac{d j_{1} d j_{2}}{(2 \pi i)^{2}} M^{(1)}\left(j_{1}, j_{2} ; a\right) K\left(j_{1}, j_{2}\right)\left(\frac{1-z}{z}\right)^{j_{1}+j_{2}}
$$

Mellin amplitude

$$
\begin{aligned}
& M^{(1)}\left(j_{1}, j_{2}\right)=-\frac{1}{4}\left[\Gamma\left(-j_{1}\right) \Gamma\left(-j_{2}\right) \Gamma\left(1+j_{1}+j_{2}\right)\right]^{2} \\
& K\left(j_{1}, j_{2}\right)=\frac{2 \Gamma\left(1-j_{1}-j_{2}\right)}{\Gamma\left(j_{1}+j_{2}\right)\left[\Gamma\left(1-j_{1}\right) \Gamma\left(1-j_{2}\right)\right]^{2}}
\end{aligned}
$$

$\checkmark$ Change integration variable $j_{1}+j_{2} \rightarrow j_{1}$

$$
\begin{aligned}
\mathrm{EEC}^{(1-\mathrm{loop})} & =-\frac{a}{4 z^{2}(1-z)} \int \frac{d j_{1} d j_{2}}{(2 \pi i)^{2}} \frac{j_{1}^{2}}{2\left(j_{1}-j_{2}\right)^{2} j_{2}^{2}} \frac{\pi}{\sin \left(\pi j_{1}\right)}\left(\frac{1-z}{z}\right)^{j_{1}} \\
& =\frac{a}{4 z^{2}(1-z)} \int \frac{d j_{1}}{2 \pi i} \frac{\pi}{j_{1} \sin \left(\pi j_{1}\right)}\left(\frac{1-z}{z}\right)^{j_{1}} \\
& =\frac{a}{4 z^{2}(1-z)} \sum_{k=-1}^{-\infty} \frac{(-1)^{k}}{k}\left(\frac{1-z}{z}\right)^{k} \\
& =\frac{a}{4 z^{2}(1-z)} \ln \frac{1}{1-z}
\end{aligned}
$$

## EEC@R-loops (II)

Final result for EEC

$$
\operatorname{EEC}_{\mathcal{N}=4}=\frac{1}{4 z^{2}(1-z)}\left\{a F_{1}(z)+a^{2}\left[(1-z) F_{2}(z)+\frac{1}{4} F_{3}(z)\right]\right\}, \quad z=\frac{1}{2}(1-\cos \chi)
$$

$F_{w}(z)$ are linear combinations of functions of homogenous weight $w=1,2,3$

$$
\begin{aligned}
& F_{1}(z)=-\ln (1-z) \\
& \begin{aligned}
F_{2}(z) & =4 \sqrt{z}\left[\operatorname{Li}_{2}(-\sqrt{z})-\operatorname{Li}_{2}(\sqrt{z})+\frac{1}{2} \ln z \ln \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right] \\
& +(1+z)\left[2 \operatorname{Li}_{2}(z)+\ln ^{2}(1-z)\right]+2 \ln (1-z) \ln \left(\frac{z}{1-z}\right)+z \frac{\pi^{2}}{3}, \\
F_{3}(z) & =(1-z)(1+2 z)\left[\ln ^{2}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) \ln \left(\frac{1-z}{z}\right)-8 \operatorname{Li}_{3}\left(\frac{\sqrt{z}}{\sqrt{z}-1}\right)-8 \operatorname{Li}_{3}\left(\frac{\sqrt{z}}{\sqrt{z}+1}\right)\right] \\
& -4(z-4) \operatorname{Li}_{3}(z)+6\left(3+3 z-4 z^{2}\right) \operatorname{Li}_{3}\left(\frac{z}{z-1}\right)-2 z(1+4 z) \zeta_{3}+2[(3-4 z) z \ln z \\
& \left.+2\left(2 z^{2}-z-2\right) \ln (1-z)\right] \operatorname{Li}_{2}(z)+\frac{1}{3} \ln ^{2}(1-z)\left[4\left(3 z^{2}-2 z-1\right) \ln (1-z)\right. \\
& +3(3-4 z) z \ln z]+\frac{\pi^{2}}{3}\left[2 z^{2} \ln z-\left(2 z^{2}+z-2\right) \ln (1-z)\right]
\end{aligned}
\end{aligned}
$$

## EEC@2-loops (III)

$\checkmark$ Sum of basis functions with nontrivial arguments and rational prefactors $R_{2}$ and $R_{3}$

$$
\mathrm{EEC}^{(2 \mathrm{loops})}=\sum R_{2}(z, \sqrt{z}) W_{2}(z, \sqrt{z})+\sum R_{3}(z, \sqrt{z}) W_{3}(z, \sqrt{z})
$$

Weight two $W_{2}=\left\{\mathrm{Li}_{2}, \ln \ln , \pi^{2}\right\}$
Weight three $W_{3}=\left\{\mathrm{Li}_{3}, \mathrm{Li}_{2} \ln , \ln \ln \ln , \pi^{2} \ln , \zeta_{3}\right\}$
$\checkmark W_{2}$ and $W_{3}$ depend on $\sqrt{z}=|\sin (\chi / 2)|$, but EEC is manifestly invariant under $\sqrt{z} \rightarrow-\sqrt{z}$
$\checkmark$ Scattering amplitudes have homogenous weight in planar $\mathcal{N}=4 \mathrm{SYM}$ at weak coupling

$$
A_{a+b+X} \sim \exp \left(-\operatorname{Div}(1 / \epsilon)+\sum_{\ell} a^{\ell} W_{2 \ell}+O(\epsilon)\right)
$$

This property is 'minimally' violated for EEC after the phase space integration

$$
\operatorname{EEC}(\chi)=\frac{1}{\sigma_{\mathrm{tot}}} \sum_{a, b} \int \mathrm{dLIPS}\left|A_{a+b+X}\right|^{2} \frac{E_{a} E_{b}}{Q^{2}} \delta\left(\cos \chi-\cos \theta_{a b}\right)
$$

But it is restored in the back-to-back kinematics $\chi \rightarrow \pi$, or $z \rightarrow 1$ (see below)
$\checkmark \mathrm{EEC}^{(2 \mathrm{loops})}$ involves some of the transcendental functions that also appear in the two-loop result for the quarks ( $=n_{f}$ dependent) contribution to EEC in QCD

## Sudakov scaling

EEC in the back-to-back kinematics $\chi \rightarrow \pi$ (or $\left.y \equiv 1-z \sim(\pi-\chi)^{2} \rightarrow 0\right)$

$$
\mathrm{EEC} \stackrel{z \rightarrow^{1}}{\sim} \frac{1}{4 y}\left\{a \ln (1 / y)-\frac{a^{2}}{2}\left[\ln ^{3}(1 / y)+\frac{\pi^{2}}{2} \ln (1 / y)\right]\right\}
$$


$\checkmark$ Large (Sudakov) corrections $a^{k} \ln ^{n} y$ come from the emission of soft and collinear particles
$\checkmark$ All order resummation

$$
\mathrm{EEC} \sim \frac{1}{8 y} H(a) \int_{0}^{\infty} d b b J_{0}(b) S\left(b^{2} / y ; a\right)
$$

$J_{0}(b)$ Bessel function; $S\left(b^{2} / y ; a\right)$ the Sudakov form factor (with $b_{0}=2 \mathrm{e}^{-\gamma_{\mathrm{E}}}$ )

$$
S=\exp \left[-\frac{1}{2} \Gamma_{\text {cusp }}(a) \ln ^{2}\left(\frac{b^{2}}{y b_{0}^{2}}\right)-\Gamma(a) \ln \left(\frac{b^{2}}{y b_{0}^{2}}\right)\right]
$$

Dependence on the coupling constant is encoded in three functions

$$
\Gamma_{\text {cusp }}(a)=a-\frac{1}{2} \zeta_{2} a^{2}, \quad \Gamma(a)=-\frac{3}{2} \zeta_{3} a^{2}, \quad H(a)=1-\zeta_{2} a
$$

$\checkmark$ Perturbative corrections to $\operatorname{EEC}(z \rightarrow 1)$ have homogeneous transcedentality

$$
\left[\mathrm{EEC}_{\mathrm{QCD}}(z \rightarrow 1)\right]_{\text {maximal transcedentality }}=\mathrm{EEC}_{\mathcal{N}=4}(z \rightarrow 1)
$$

## Collinear colarimeters

Small angle correlations $\chi \rightarrow 0$ (or $z \sim \chi^{2} \rightarrow 0$ ): calorimeters measure nearly collinear particles

$$
\mathrm{EEC}^{z \rightarrow 0} \stackrel{a}{4 z}\left[1+a\left(\ln z-\frac{1}{2} \zeta_{3}+\zeta_{2}-3\right)\right]
$$


$\checkmark$ Corrections are enhanced by $\ln z$, no homogenous transcedentality
$\checkmark$ EEC ${ }^{\text {(2loops) }}$ involves $\sqrt{z} \sim|\chi|$ but expansion runs in integer positive powers of $z \sim \chi^{2}$
$\checkmark$ Resummation of leading log's $a(a \ln z)^{k}$ using the "jet calculus"

$$
\begin{aligned}
\mathrm{EEC} \stackrel{z \rightarrow 0}{\sim} & \frac{a}{4 z} \int_{0}^{1} d x x^{2} D\left(x, Q^{2} / S_{a b}\right) \\
& =\frac{a}{4 z}\left(Q^{2} / S_{a b}\right)^{-\gamma_{T}(3)}=\frac{a}{4} z^{-1+a+O\left(a^{2}\right)}
\end{aligned}
$$

$D\left(x, Q^{2} / S_{a b}\right)$ probability to fragment into a pair of partons with $S_{a b}=2 E_{a} E_{b}(1-\cos \chi) \sim Q^{2} z$ $\gamma_{T}(S)=a \sum_{k=1}^{S-2} 1 / k+O\left(a^{2}\right)$ the twist-two time-like anomalous dimension of spin $S$
$\checkmark$ Resummation weakens singularity of EEC for $\chi \rightarrow 0$, jets at weak coupling

$$
\int_{0}^{\chi_{0}} d \cos \chi \mathrm{EEC} \sim 1, \quad\left(\chi_{0} \ll 1\right)
$$

## From weak to strong coupling



$\checkmark$ At weak coupling $\mathrm{EEC}_{\mathcal{N}=4}$ has a shape which is remarkably similar to the one in QCD
$\checkmark$ Going from one to two loops, EEC flattens
$\checkmark$ This agrees with strong coupling prediction for EEC in planar $\mathcal{N}=4$ SYM

$$
\mathrm{EEC}_{\mathcal{N}=4} \stackrel{a \rightarrow \infty}{\sim} \frac{1}{2}\left[1+a^{-1}(1-6 z(1-z))+O\left(a^{-3 / 2}\right)\right]
$$

No jets at strong coupling

## Conclusions and questions

$\checkmark$ Observables of scattering experiments calculated bypassing amplitudes
$\checkmark$ Symmetries of the theory preserved at every step of the calculation
$\checkmark$ Event shapes can be used to constrain correlation functions
$\checkmark$ Does the $\mathrm{N}=4$ result provides the most complicated part of QCD expressions?
$\checkmark$ Can one devise an interpolation between weak and strong coupling?
$\checkmark$ What is the manifestation of integrability in event shapes?

## Correlators with stress tensor

$\checkmark$ Single stress-tensor component $\left(\left(\theta_{1} \bar{\theta}_{1}\right)^{2}\right.$ term minus two-level descendants)

$$
\left\langle T_{\alpha \beta, \dot{\alpha} \dot{\beta}}(1) \mathcal{O O O}\right\rangle=\frac{y_{23}^{2} y_{34}^{2} y_{42}^{2}}{x_{23}^{2} x_{34}^{2} x_{42}^{2}}\left(\partial_{x_{1}}\right)_{\dot{\alpha}}^{\rho}\left(\partial_{x_{1}}\right)_{\dot{\beta}}^{\gamma}\left[\mathcal{M}_{\alpha \beta \gamma \delta} \Phi(u, v) \frac{x_{12}^{2} x_{14}^{2}}{x_{24}^{2}}\right]
$$

$x$ Conserved due to the totally symmetric matrix

$$
\mathcal{M}_{\alpha \beta \gamma \delta}=\left[X_{134}, X_{124}\right]_{(\alpha \beta}\left[X_{134}, X_{124}\right]_{\gamma \delta)}, \quad X_{a b c}=x_{a b}^{-1}-x_{a c}^{-1}
$$

$x$ No dependence on $y_{1}(T$ is an $S U(4)$ singlet)
$\checkmark$ Two stress-tensors:

$$
\begin{aligned}
& \left\langle T_{\alpha_{1} \beta_{1}, \dot{\alpha}_{1} \dot{\beta}_{1}}(1) T_{\alpha_{2} \beta_{2}, \dot{\alpha}_{2} \dot{\beta}_{2}}(2) \mathcal{O O}\right\rangle \\
& \quad=\left(\partial_{x_{1}}\right)_{\dot{\alpha}_{1} \delta_{1}}\left(\partial_{x_{1}}\right)_{\dot{\beta}_{1} \gamma_{1}}\left(\partial_{x_{2}}\right)_{\dot{\alpha}_{2} \delta_{2}}\left(\partial_{x_{2}}\right)_{\dot{\beta}_{2} \gamma_{2}}\left[\mathcal{M}_{\left(\alpha_{1} \beta_{1}\right) ;\left(\alpha_{2} \beta_{2}\right)}^{\left(\delta_{1} \gamma_{1}\right) ;\left(\delta_{2} \gamma_{2}\right)} \frac{\Phi\left(x_{1}, x_{2}\right)}{x_{1}^{4} x_{12}^{2} x_{2}^{2}}\right]
\end{aligned}
$$

$x$ The expression becomes extremely complicated if we distribute the 4 space-time derivatives.
$x$ Miracle happens when we plug it into the energy-energy correlation $\left\langle\mathcal{E}\left(n_{1}\right) \mathcal{E}\left(n_{2}\right)\right\rangle$, the result is very simple!

$$
\left\langle\mathcal{E}\left(n_{1}\right) \mathcal{E}\left(n_{2}\right)\right\rangle=\frac{4\left(q^{2}\right)^{2}}{\left(n_{1} \cdot n_{2}\right)^{2}}\left\langle\mathcal{O}\left(n_{1}\right) \mathcal{O}\left(n_{2}\right)\right\rangle_{\mathbf{1 0 5}}
$$

