Rapidity evolution of Wilson lines

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JLAB & ODU

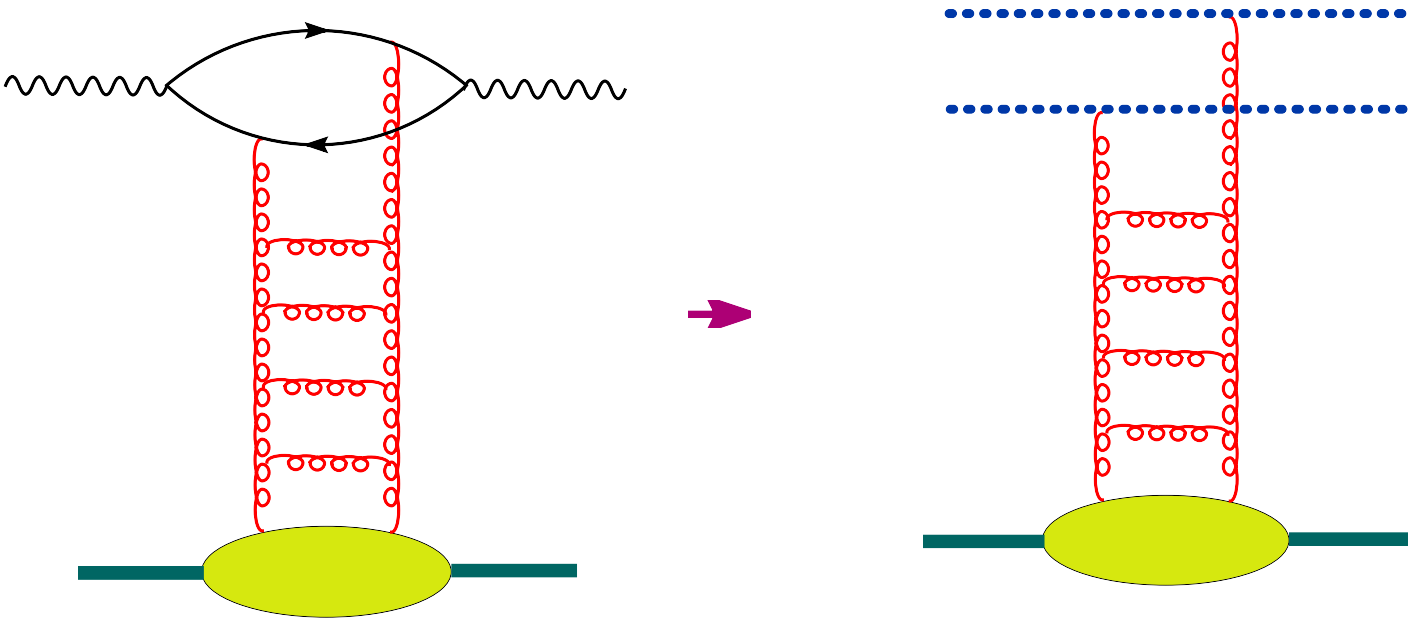
QCD evolution 2014 13 May 2014

Outline

1. High-energy scattering and Wilson lines
   - High-energy scattering and Wilson lines.
   - Evolution equation for color dipoles.
   - Leading order: BK equation.
2. NLO high-energy amplitudes
   - Conformal composite dipoles
   - NLO BK kernel in QCD.
   - NLO hierarchy of Wilson-lines evolution.
   - Evolution of 3-Wilson-line baryon operator
   - Conclusions

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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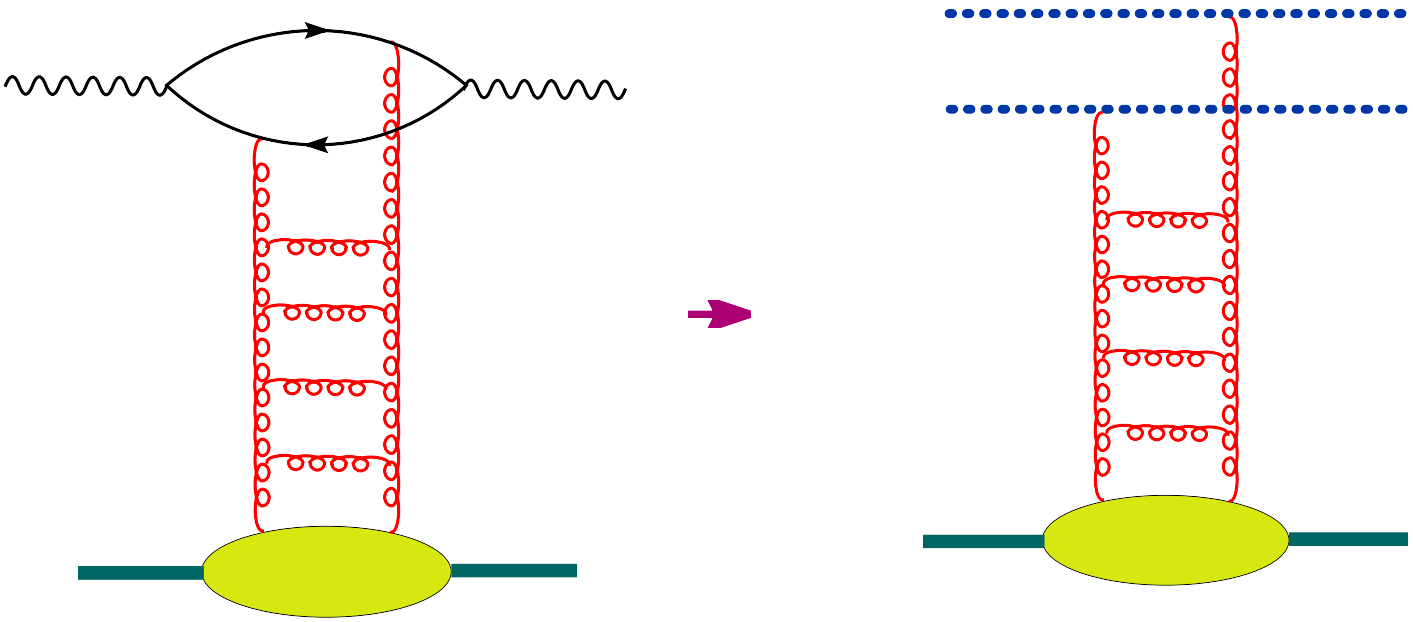

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle$$

Formally, $\rightarrow$ means the operator expansion in Wilson lines

Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

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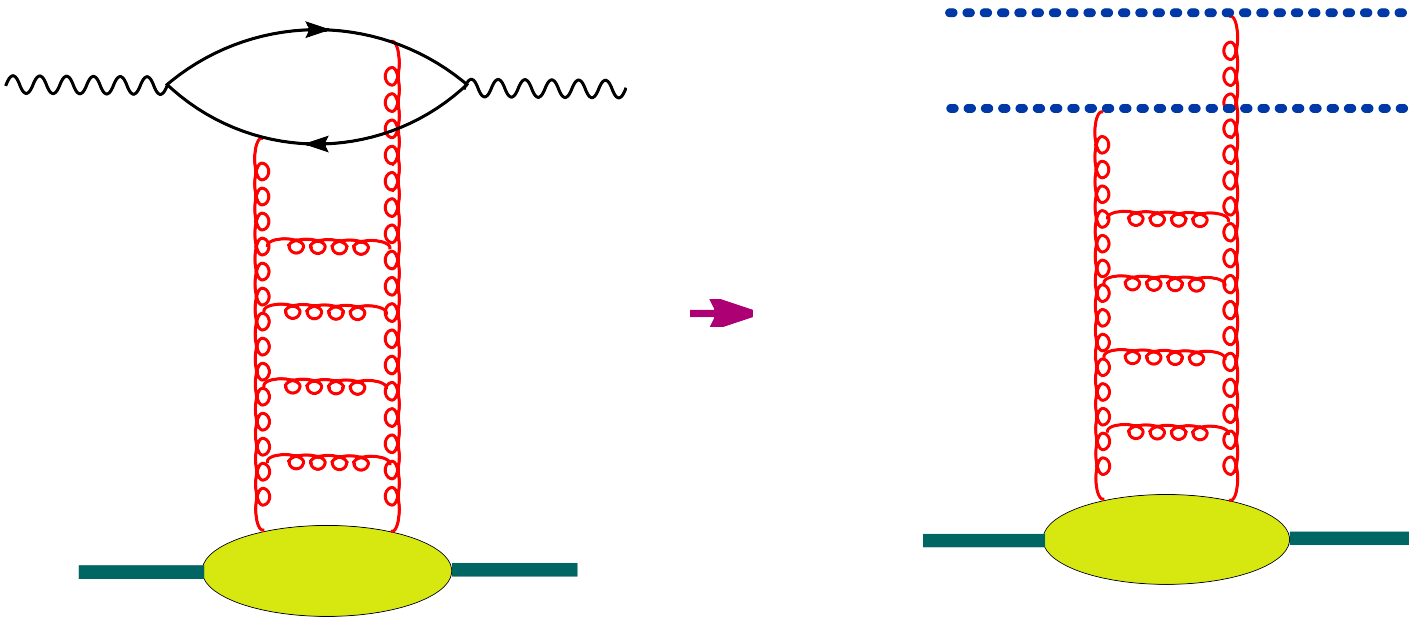

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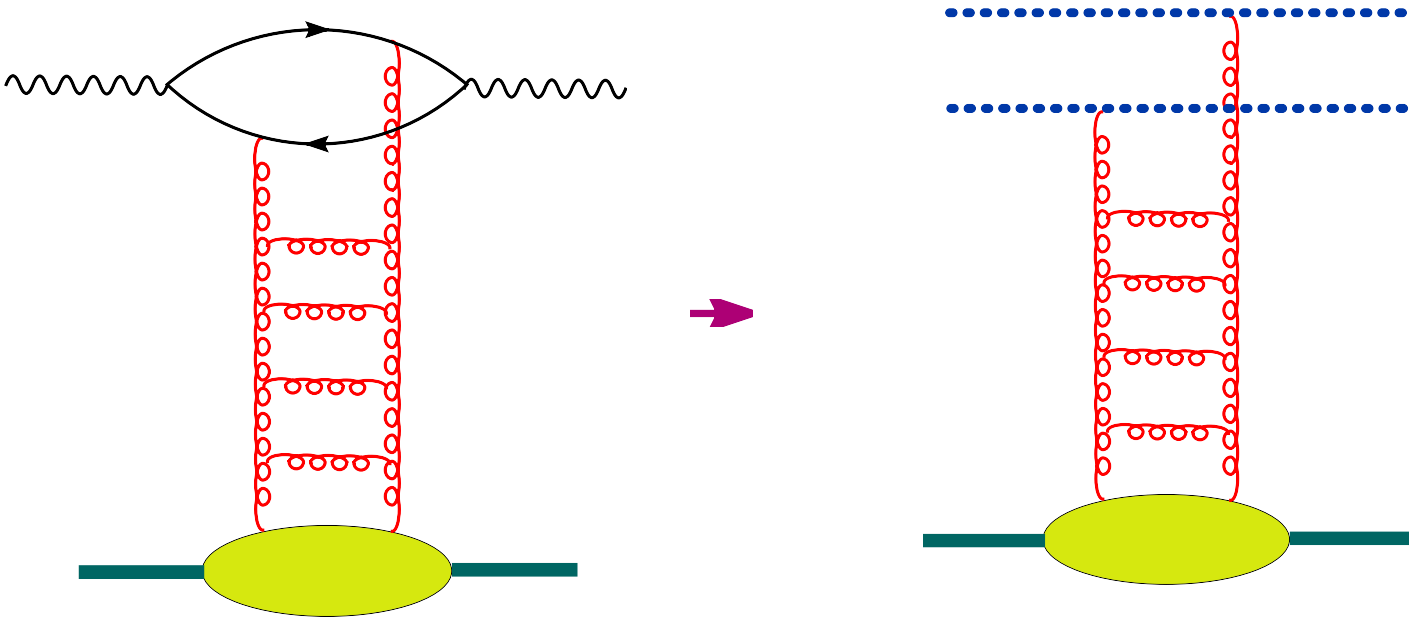

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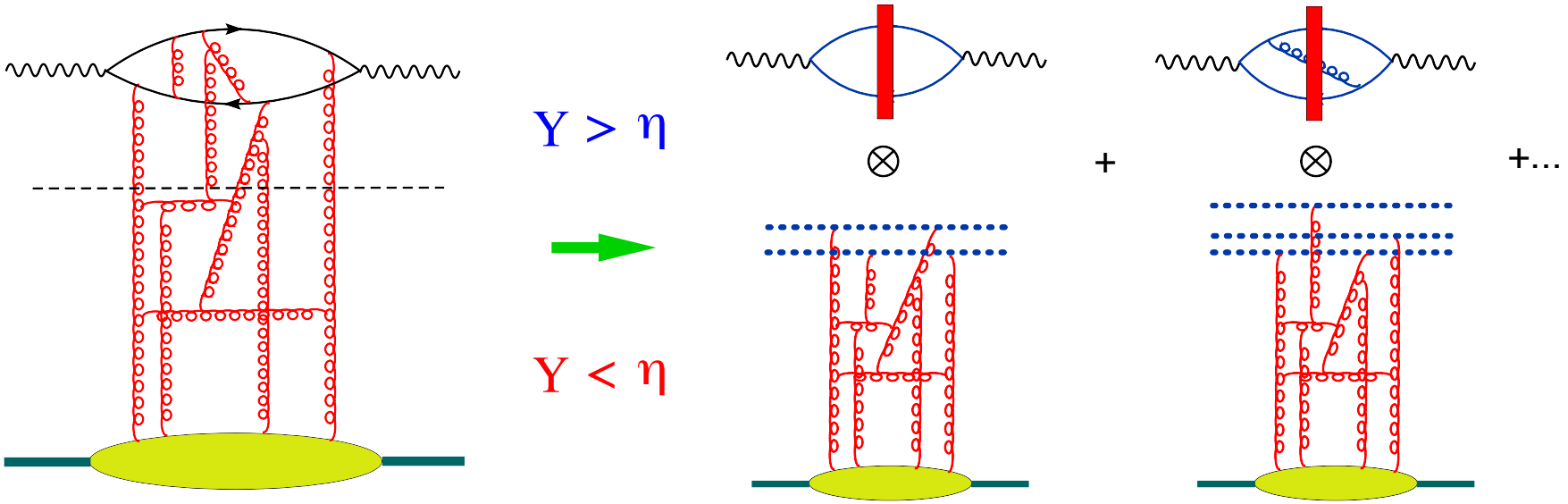
Rapidity factorization



η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")

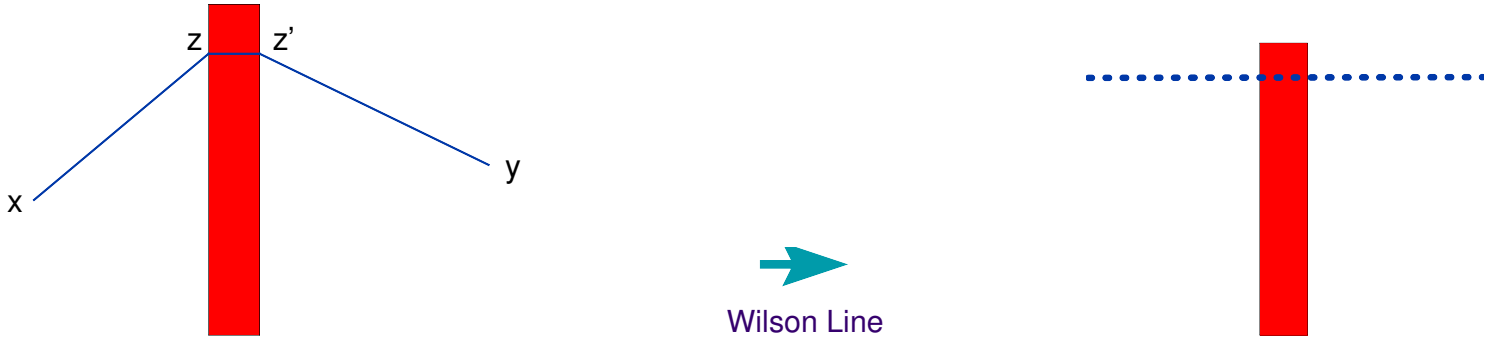
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\left[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

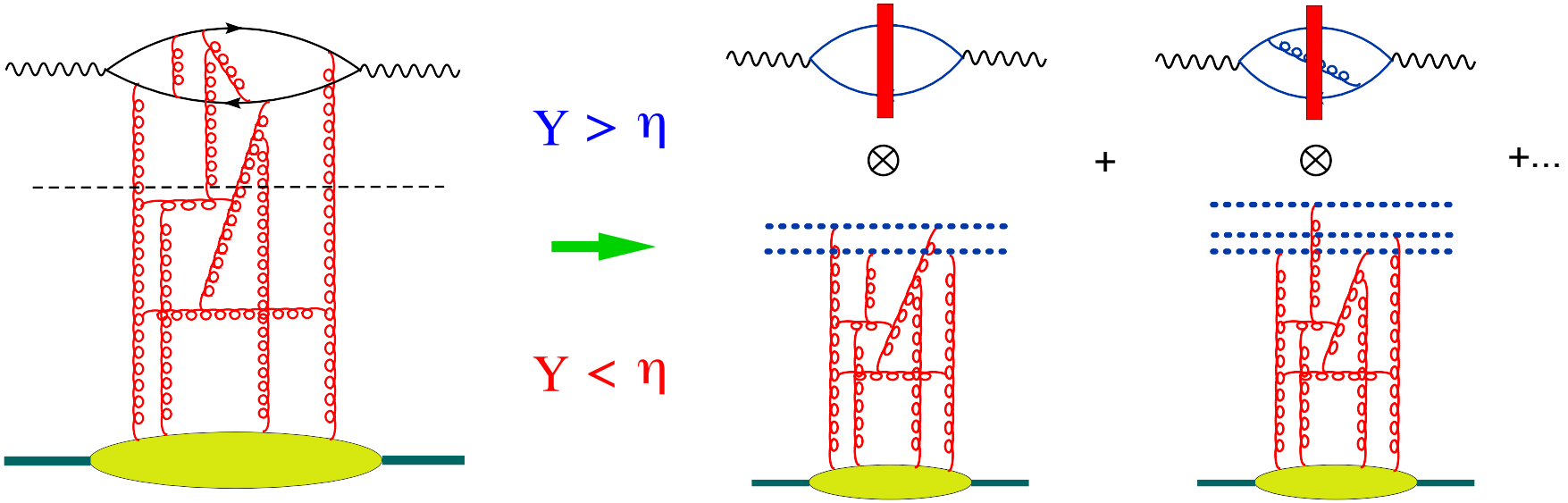
Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \to z$: free propagation]$\times$
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]$\times$
[$z \to y$: free propagation]

High-energy expansion in color dipoles

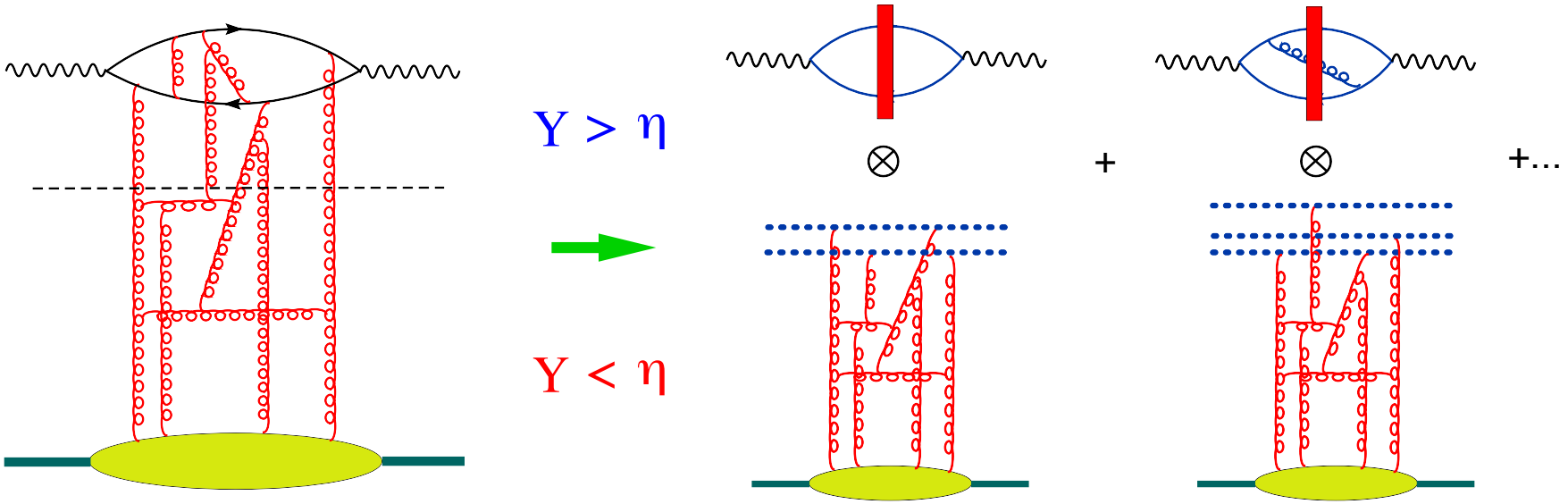


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$+$ NLO contribution

High-energy expansion in color dipoles



η - rapidity factorization scale

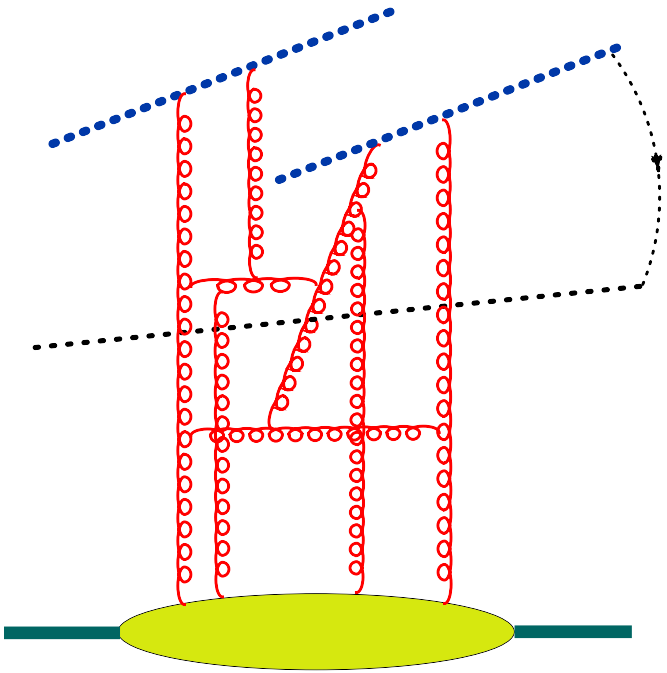
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} - N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

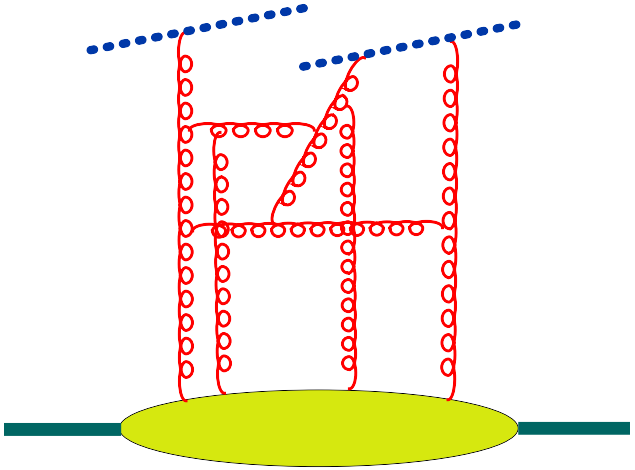
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



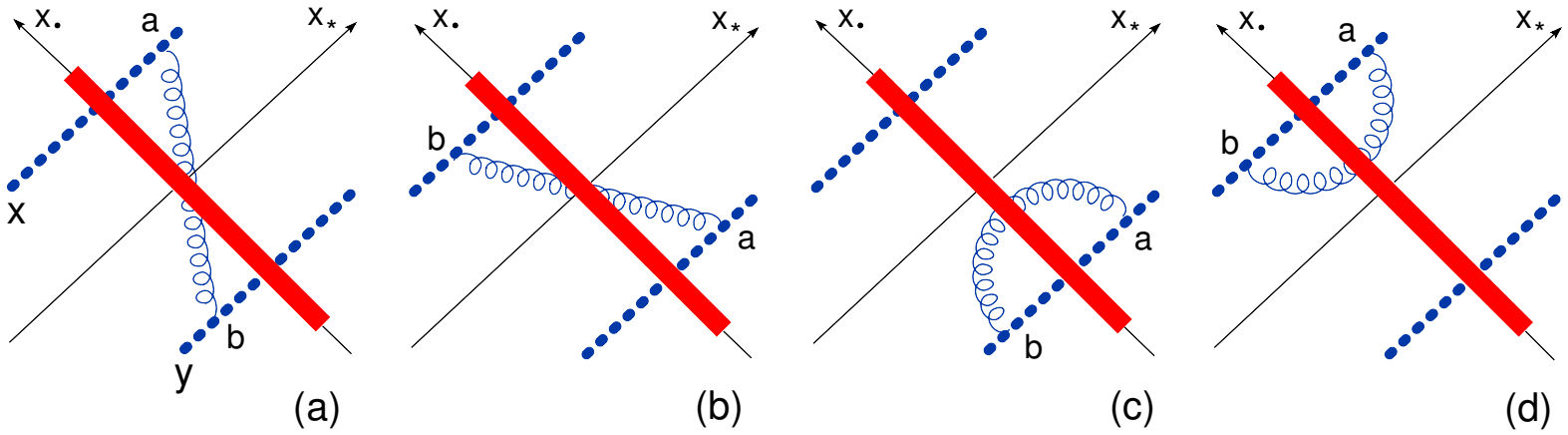
$$\alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \ \otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

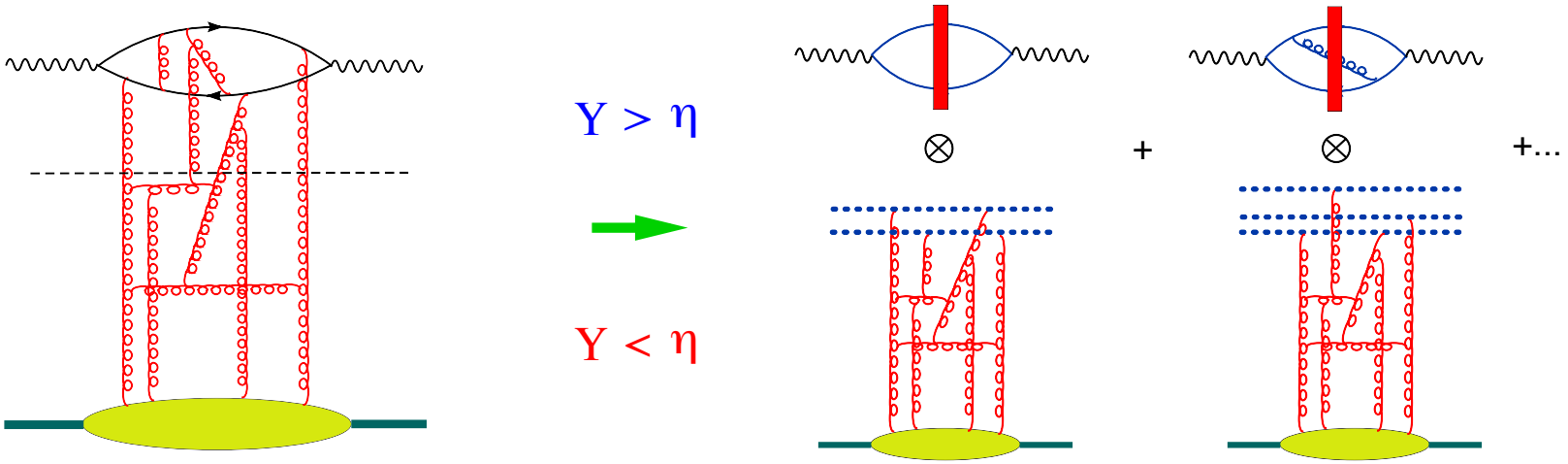
LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To check conformal invariance of the NLO BK equation(in $\mathcal{N}$=4 SYM)
- To determine the argument of the coupling constant of the BK equation(in QCD).
- To get the region of application of the leading order evolution equation.

Hign-energy OPE In the next-to-leading order



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

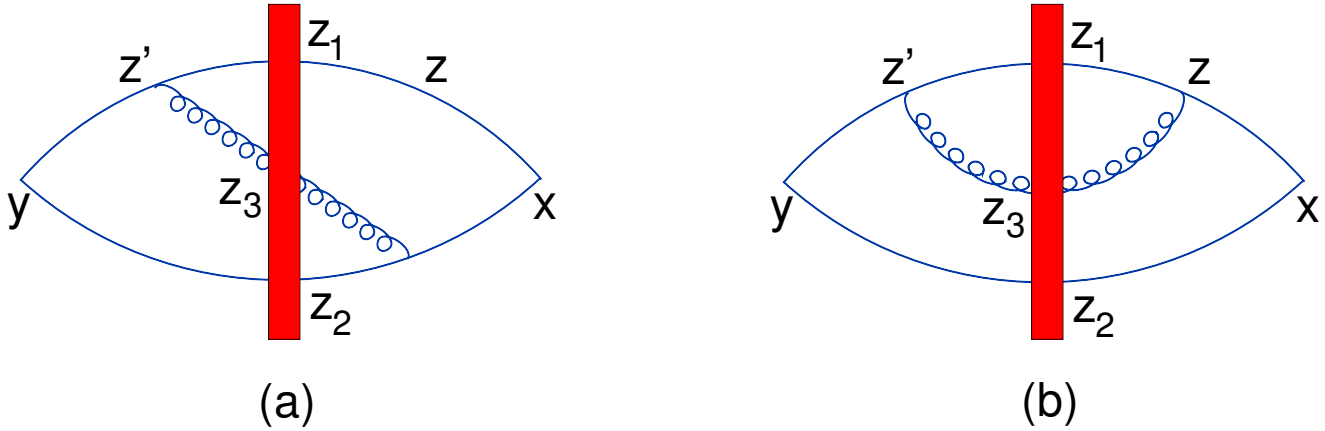
$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\kappa^4}{\pi^6(\kappa\cdot\zeta_1)^3(\kappa\cdot\zeta_2)^3}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\left[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\right].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\left(\frac{p_1}{s}-x^2p_2+x_\perp\right)-x\leftrightarrow y, \quad \zeta_i \equiv \left(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\right)$$

NLO impact factor is calculated recently (G.A. Chirilli and I.B, 2013)

NLO impact factor and conformal Möbius invariance



$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{e^{\eta}s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant $\Leftarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\big[{\rm Tr}\{\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf} \;=\; {\rm Tr}\{\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\;\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\big[{\rm Tr}\{T^n\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^{\eta}_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c{\rm Tr}\{\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]\ln\frac{az_{12}^2}{z_{13}^2 z_{23}^2}\;+\;O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\left[\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\right]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ $\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant (in $\mathcal{N}=4$ SYM)

We think that one can construct the composite conformal (in $\mathcal{N}=4$ SYM) dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

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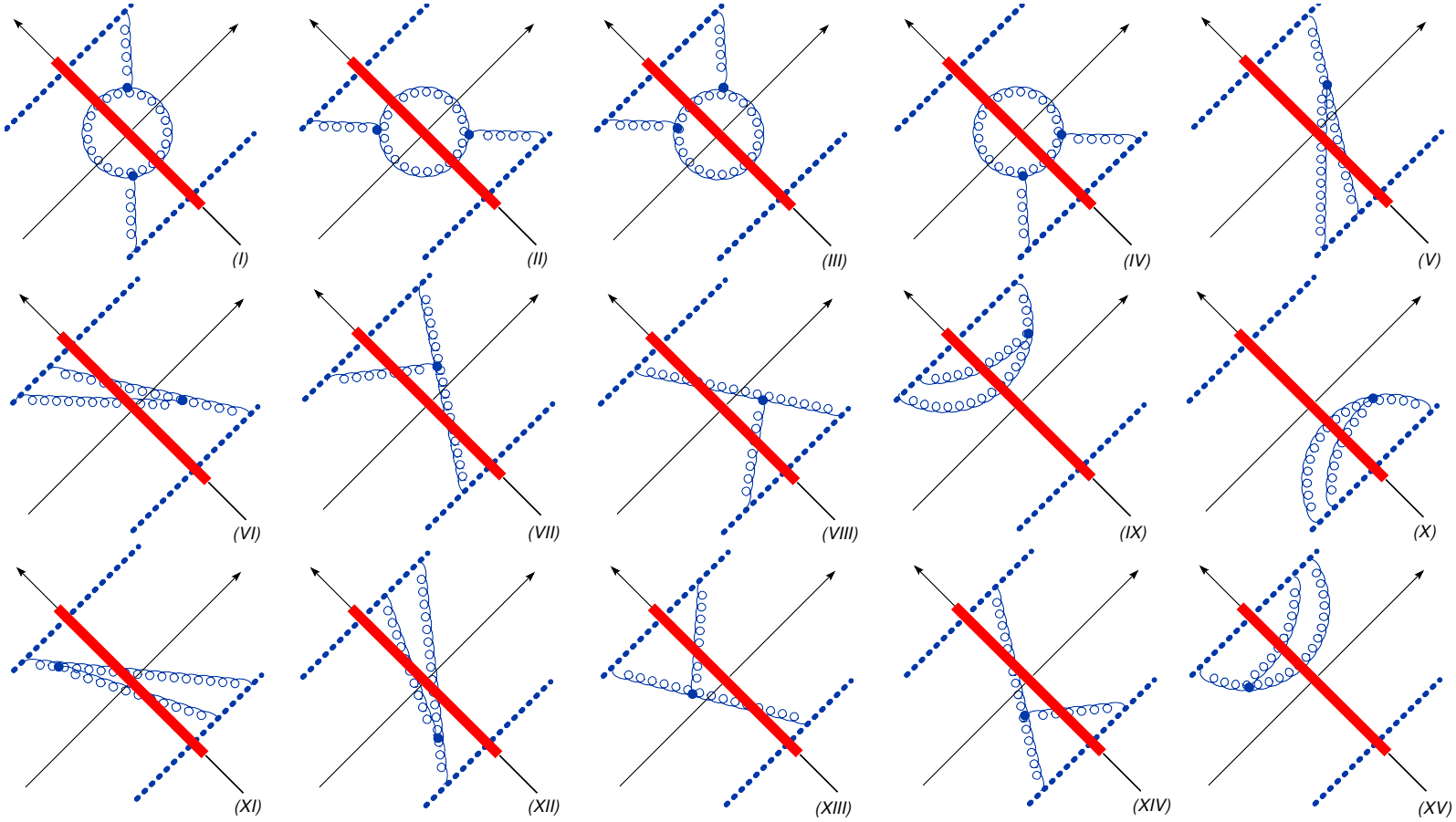
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow \quad \left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

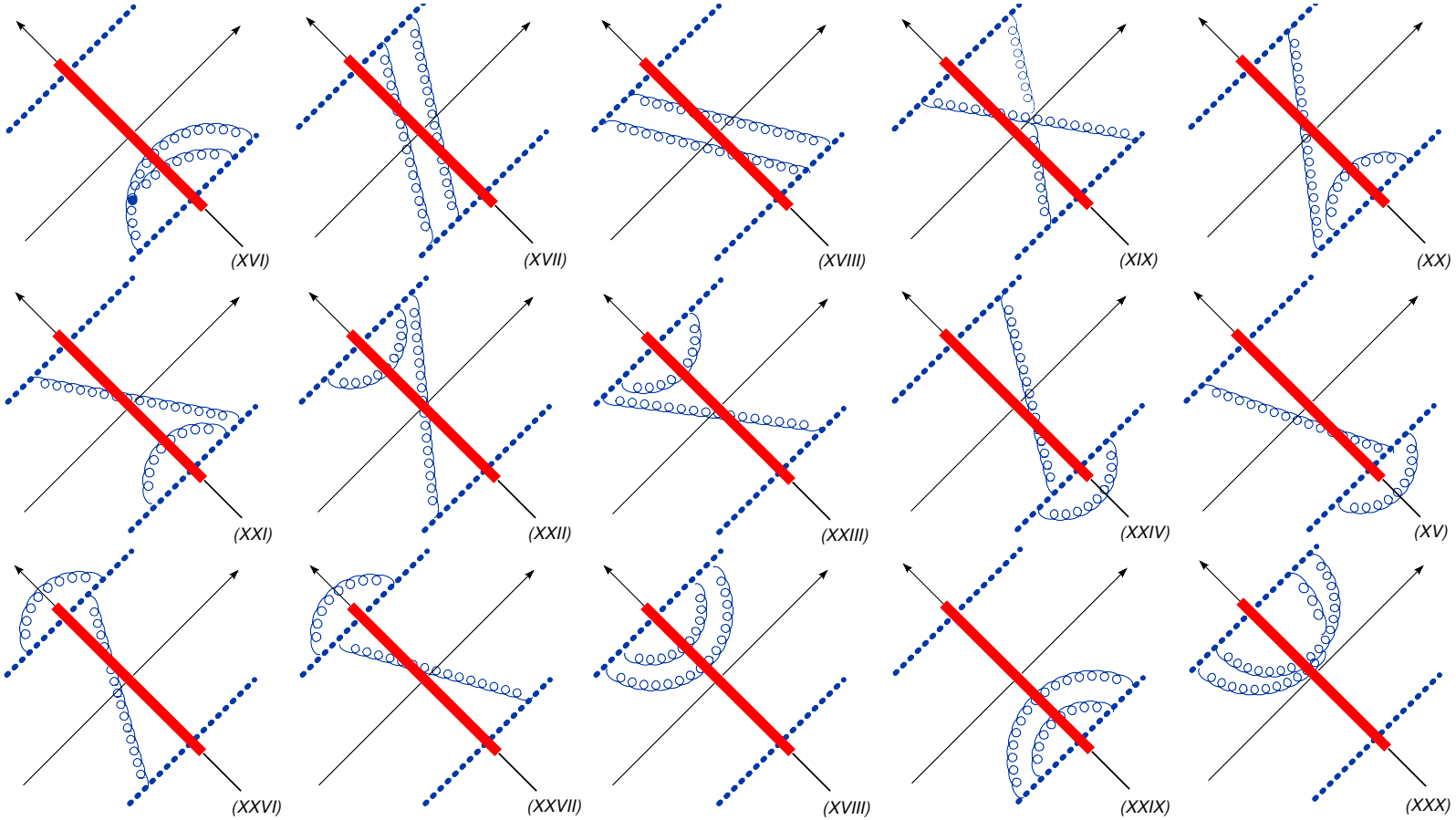
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Gluon part of the NLO BK kernel: diagrams



Diagrams for 1→3 dipoles transition



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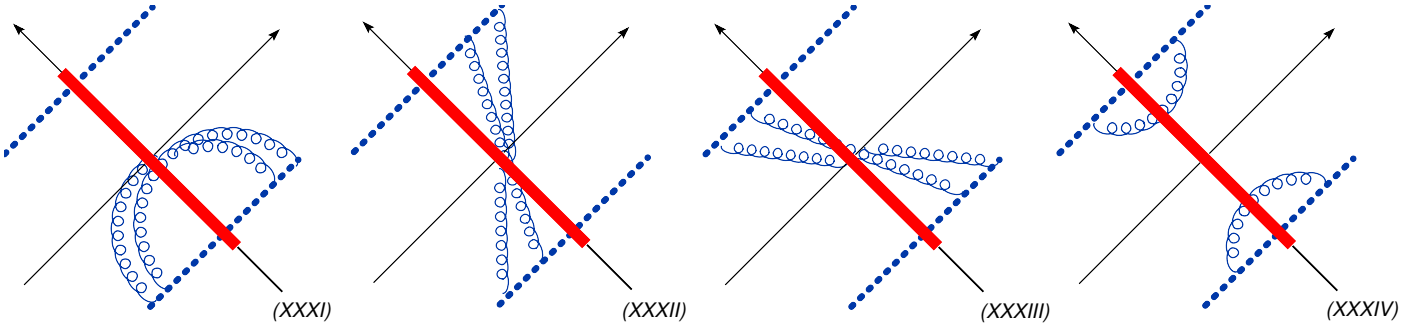
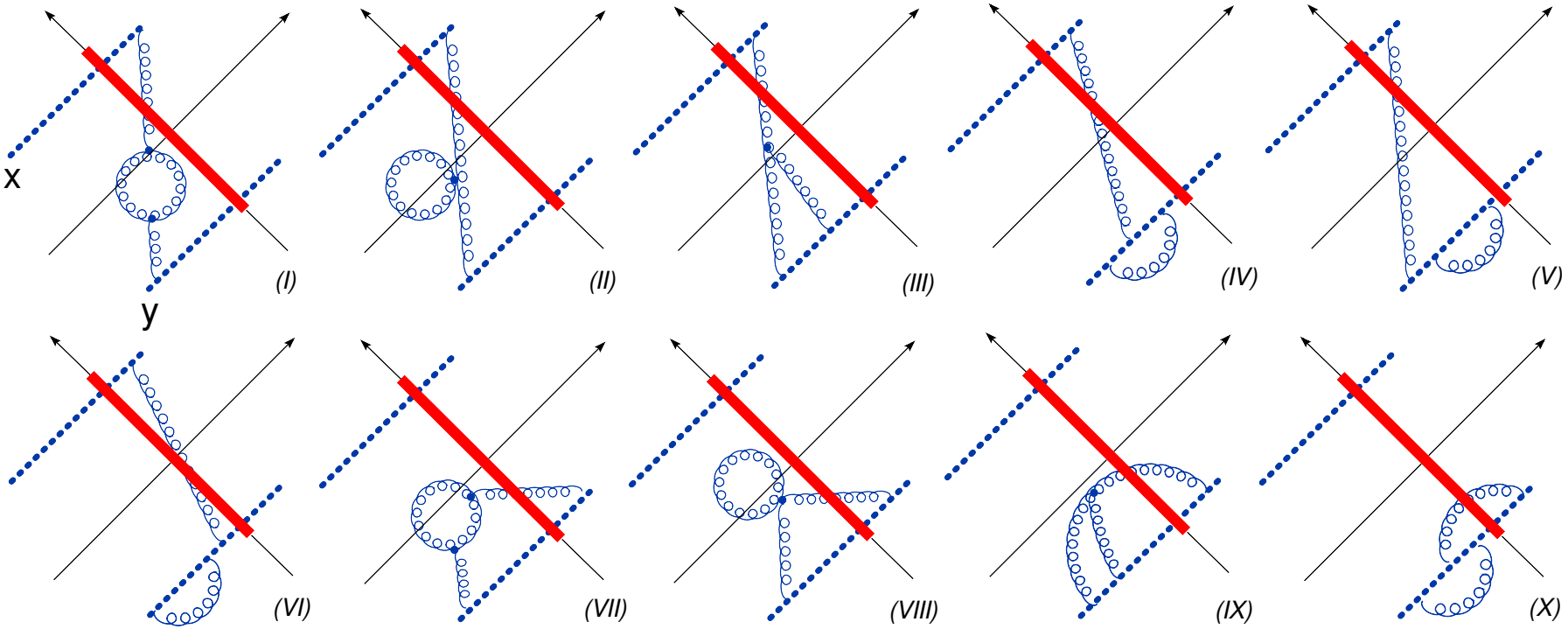
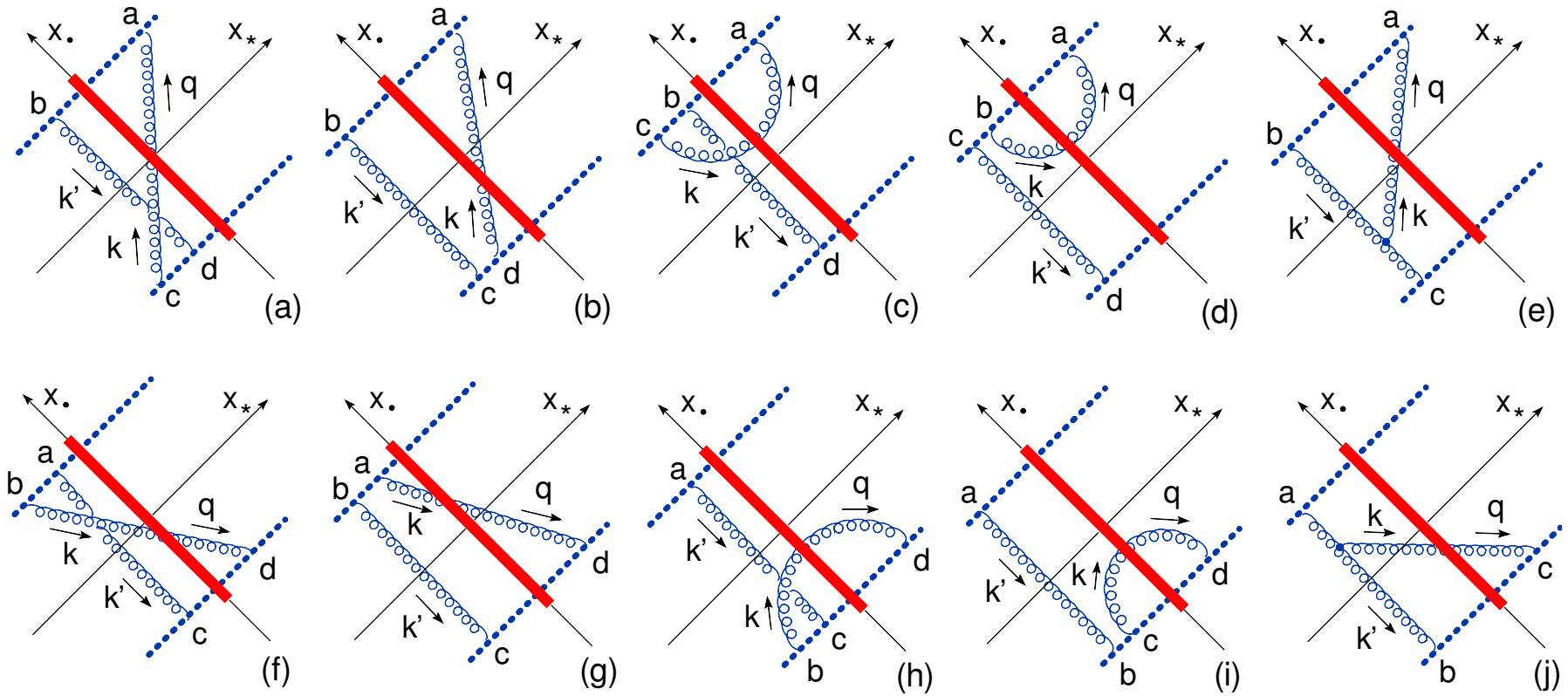


"Running coupling" diagrams



$1 \to 2$ dipole transition diagrams



NLO evolution of composite "conformal" dipoles in QCD

I. B. and G. Chirilli

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{23}}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{23}}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{{z_{23}}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{{z_{23}}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-{z_{23}}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{{z_{23}}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ (for composite dipoles) reproduces the result for the forward NLO BFKL kernel for Green function of two reggeized gluons.

NLO hierarchy of evolution of Wilson lines (G.A.C. and I.B., 2013)

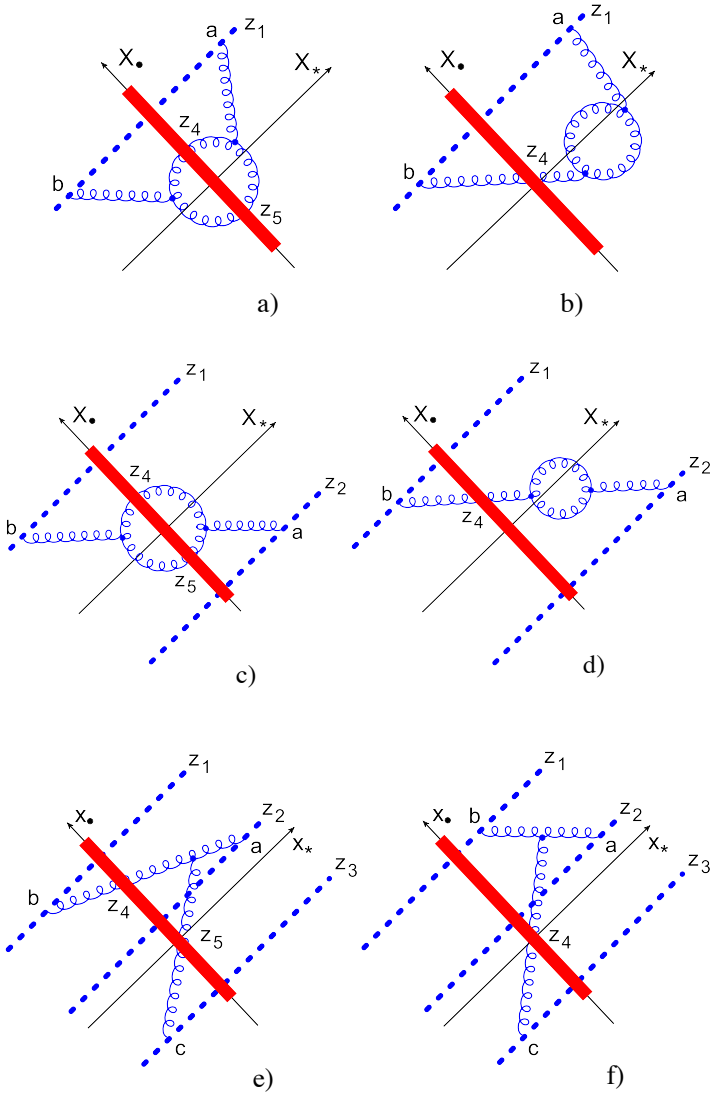
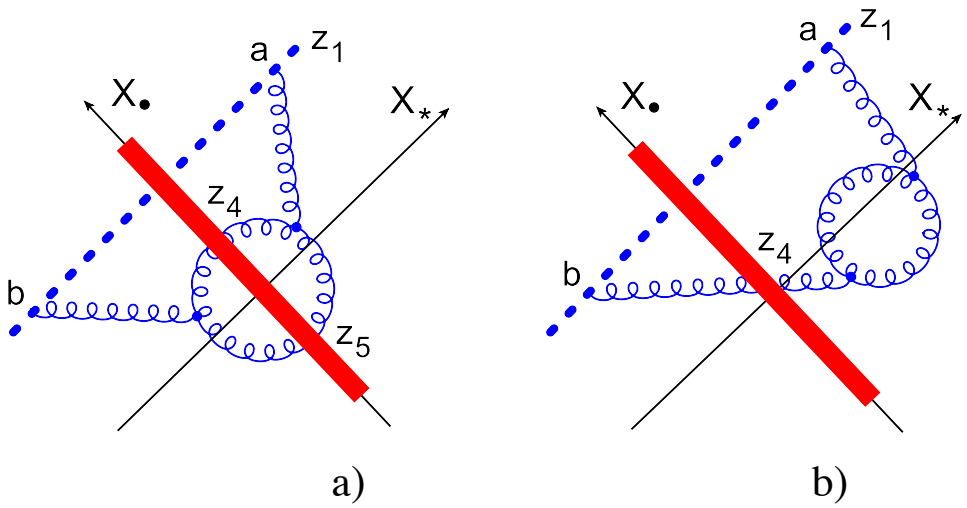


Figure : Typical NLO diagrams: self-interaction (a,b), pairwise interactions (c,d), and triple interaction (e,f)

Self-interaction (gluon reggeization)



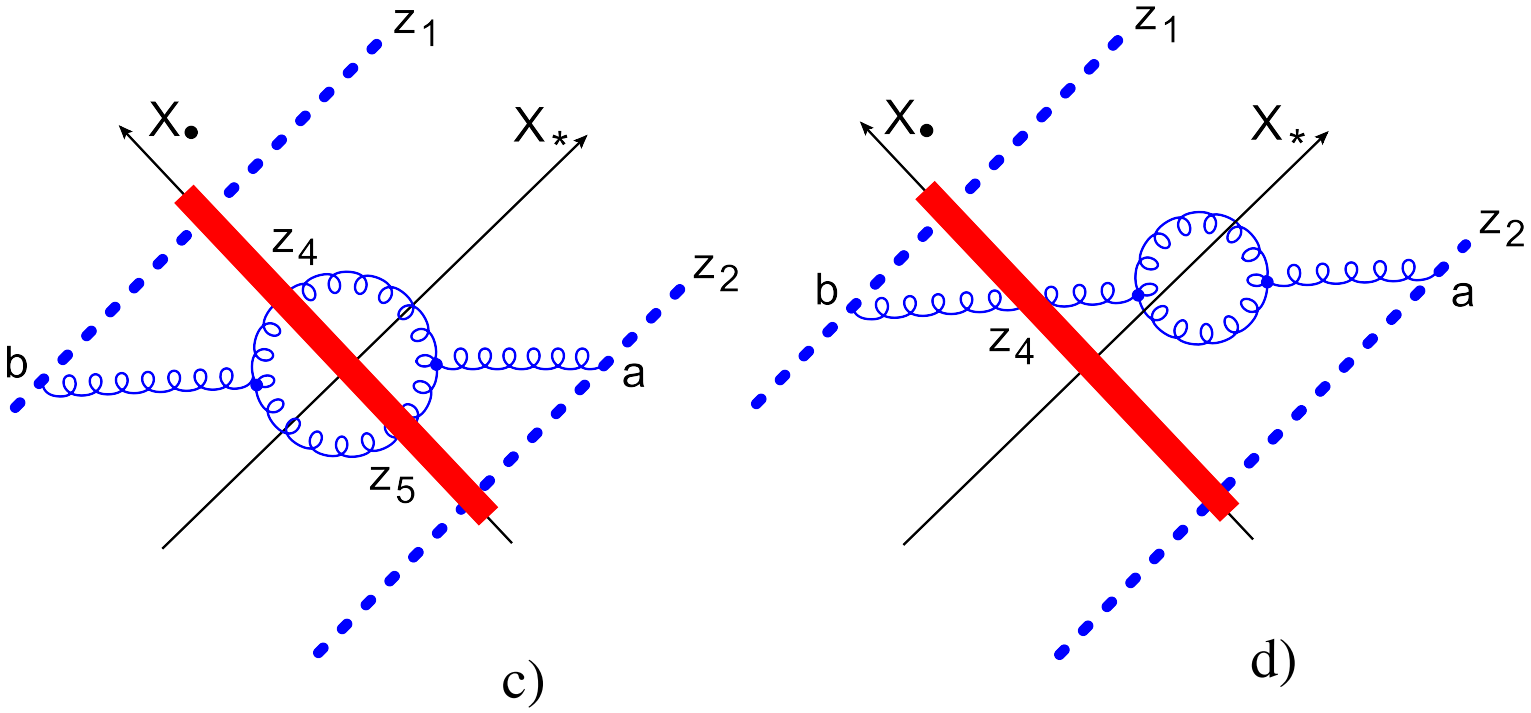
$$\frac{d}{d\eta}(U_1)_{ij} = \frac{\alpha_s^2}{8\pi^4}\int \frac{d^2z_4 d^2z_5}{z_{45}^2}\Big\{U_4^{dd'}(U_5^{ee'} - U_4^{ee'})$$

$$\times\Big(\Big[2I_1 - \frac{4}{z_{45}^2}\Big]f^{ade}f^{bd'e'}(t^aU_1t^b)_{ij} + \frac{(z_{14},z_{15})}{z_{14}^2 z_{15}^2}\ln\frac{z_{14}^2}{z_{15}^2}\big[if^{ad'e'}(\{t^d,t^e\}U_1t^a)_{ij} - if^{ade}(t^aU_1\{t^{d'},t^{e'}\})_{ij}\big]\Big)\Big\}$$

$$+\frac{\alpha_s^2 N_c}{4\pi^3}\int\frac{d^2z_4}{z_{14}^2}(U_4^{ab} - U_1^{ab})(t^aU_1t^b)_{ij}\Big\{\Big[\frac{11}{3}\ln z_{14}^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big]$$

$$I_1 \equiv I(z_1,z_4,z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2}\Big[\frac{z_{14}^2+z_{15}^2}{z_{45}^2} - \frac{(z_{14},z_{15})}{z_{14}^2} - \frac{(z_{14},z_{15})}{z_{15}^2} - 2\Big]$$

Pairwise interaction



$$\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl} = \frac{\alpha_s^2}{8\pi^4}\int d^2z_4 d^2z_5(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3) + \frac{\alpha_s^2}{8\pi^3}\int d^2z_4(\mathcal{B}_1+N_c\mathcal{B}_2)$$

Pairwise interaction

$$\mathcal{A}_1 = \left[(t^a U_1)_{ij}(U_2 t^b)_{kl} + (U_1 t^b)_{ij}(t^a U_2)_{kl}\right]$$
$$\times \left[f^{ade} f^{bd'e'} U_4^{dd'}(U_5^{ee'} - U_4^{ee'})\Big(-K - \frac{4}{z_{45}^4} + \frac{I_1}{z_{45}^2} + \frac{I_2}{z_{45}^2}\Big)\right]$$

K = NLO BK kernel for $\mathcal{N} = 4$ SYM

$$\mathcal{A}_2 = 4(U_4 - U_1)^{dd'}(U_5 - U_2)^{ee'}$$
$$\Big\{ i\big[f^{ad'e'}(t^d U_1 t^a)_{ij}(t^e U_2)_{kl} - f^{ade}(t^a U_1 t^{d'})_{ij}(U_2 t^{e'})_{kl}\big] J_{1245} \ln \frac{z_{14}^2}{z_{15}^2}$$
$$+ i\big[f^{ad'e'}(t^d U_1)_{ij}(t^e U_2 t^a)_{kl} - f^{ade}(U_1 t^{d'})_{ij}(t^a U_2 t^{e'})_{kl}\big] J_{2154} \ln \frac{z_{24}^2}{z_{25}^2} \Big\}$$

$$J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} - 2\frac{(z_{15}, z_{45})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{25}^2 z_{45}^2} + 2\frac{(z_{25}, z_{45})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

Pairwise interaction

$$
\begin{aligned}
\mathcal{A}_3 \;=\; & 2U_4^{dd'}\Big\{ i\big[f^{ad'e'}(U_1t^a)_{ij}(t^dt^eU_2)_{kl} - f^{ade}(t^aU_1)_{ij}(U_2t^{e'}t^{d'})_{kl}\big] \\
& \times \left[\mathcal{J}_{1245}\ln\frac{z_{14}^2}{z_{15}^2} + (J_{2145}-J_{2154})\ln\frac{z_{24}^2}{z_{25}^2}\right](U_5-U_2)^{ee'} \\
& + i\big[f^{ad'e'}(t^dt^eU_1)_{ij}(U_2t^a)_{kl} - f^{ade}(U_1t^{e'}t^{d'})_{ij}(t^aU_2)_{kl}\big] \\
& \times \left[\mathcal{J}_{2145}\ln\frac{z_{24}^2}{z_{25}^2} + (J_{1245}-J_{1254})\ln\frac{z_{14}^2}{z_{15}^2}\right](U_5-U_1)^{ee'}\Big\}
\end{aligned}
$$

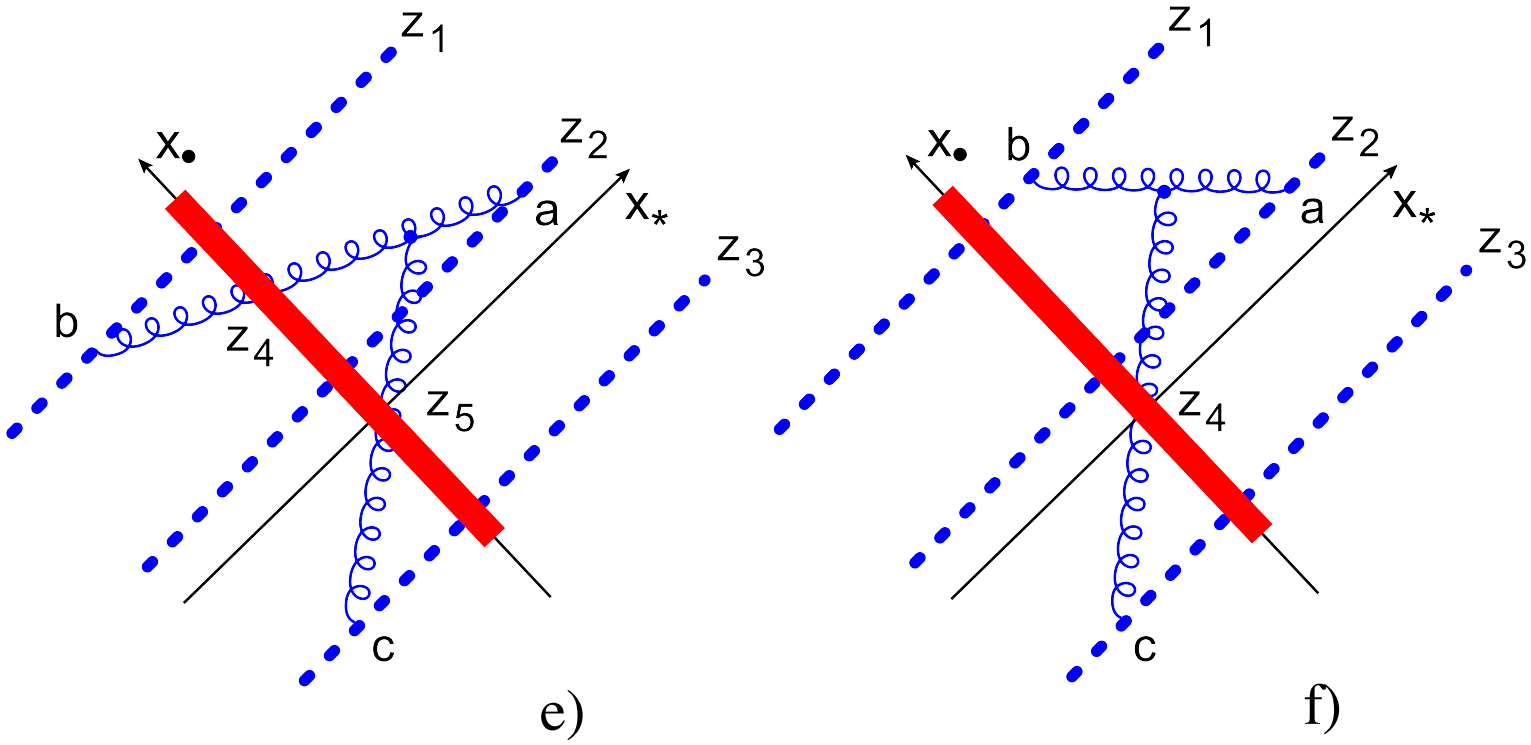
$$
\begin{aligned}
\mathcal{J}_{1245} \;&\equiv\; \mathcal{J}(z_1,z_2,z_4,z_5) \\
&= \frac{(z_{24},z_{25})}{z_{24}^2z_{25}^2z_{45}^2} - \frac{2(z_{24},z_{45})(z_{15},z_{25})}{z_{24}^2z_{25}^2z_{15}^2z_{45}^2} + \frac{2(z_{25},z_{45})(z_{14},z_{24})}{z_{14}^2z_{24}^2z_{25}^2z_{45}^2} - 2\frac{(z_{14},z_{24})(z_{15},z_{25})}{z_{14}^2z_{15}^2z_{24}^2z_{25}^2}
\end{aligned}
$$

Pairwise interaction

$$\mathcal{B}_1 \;=\; 2\ln\frac{z_{14}^2}{z_{12}^2}\ln\frac{z_{24}^2}{z_{12}^2}$$
$$\times\Big\{(U_4-U_1)^{ab}i\big[f^{bde}(t^aU_1t^d)_{ij}(U_2t^e)_{kl}+f^{ade}(t^eU_1t^b)_{ij}(t^dU_2)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{14}^2}\Big]$$
$$+\,(U_4-U_2)^{ab}i\big[f^{bde}(U_1t^e)_{ij}(t^aU_2t^d)_{kl}+\,f^{ade}(t^dU_1)_{ij}(t^eU_2t^b)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{24}^2}\Big]\Big\}$$

$$\mathcal{B}_2 \;=\; \big[2U_4^{ab}-U_1^{ab}-U_2^{ab}\big]\big[(t^aU_1)_{ij}(U_2t^b)_{kl}+(U_1t^b)_{ij}(t^aU_2)_{kl}\big]$$
$$\times\Big\{\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}\big[\frac{11}{3}\ln z_{12}^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big]+\;\frac{11}{3}\big[\frac{1}{2z_{14}^2}\ln\frac{z_{24}^2}{z_{12}^2}+\frac{1}{2z_{24}^2}\ln\frac{z_{14}^2}{z_{12}^2}\big]\Big\}$$

Triple interaction



$$\mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) = -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2}$$
$$- \frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

Triple interaction

$$
\begin{aligned}
&\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl}(U_3)_{mn} \\
&= i\frac{\alpha_s^2}{2\pi^4}\int d^2z_4 d^2z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\
&\times f^{cde}\left[(t^aU_1)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{ad}(U_5-U_2)^{be}\right. \\
&\left. - (U_1t^a)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{da}(U_5-U_2)^{eb}\right] \\
&+ \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\
&\times f^{ade}\left[(U_1t^a)_{ij}(t^bU_2)_{kl}(t^cU_3)_{mn}(U_4-U_3)^{cd}(U_5-U_2)^{be}\right. \\
&\left. - (t^aU_1)_{ij}\otimes(U_2t^b)_{kl}(U_3t^c)_{mn}(U_4^{dc}-U_3^{dc})(U_5^{eb}-U_2^{eb})\right] \\
&+ \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\
&\times f^{bde}\left[(t^aU_1)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{ad}(U_5-U_3)^{ce}\right. \\
&\left. - (U_1t^a)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{da}(U_5-U_3)^{ec}\right]
\end{aligned}
\tag{1}
$$

Rapidity evolution of the baryon "tripole"

Baryon operator

$$B_{123} = \varepsilon^{i'j'h'} \varepsilon_{ijh} U^i_{i'}(r_{1\perp}) U^j_{j'}(r_{1\perp}) U^h_{h'}(r_{3\perp}) \equiv U_1 \cdot U_2 \cdot U_3,$$

Evolution equation in the LO (A. Grabovsky, 2013)

$$\frac{d}{d\eta} B_{123} = \frac{\alpha_s 3}{8\pi^2} \int d\vec{r}_4 \left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2} \vec{r}_{42}^{\,2}} \ln \left(\frac{a \vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2} \vec{r}_{42}^{\,2}} \right) \right.$$
$$\left. \times (-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right].$$

Composite "conformal" baryon operator in the NLO

General prescription:

$$O^{conf} = O + \frac{1}{2}\frac{\partial O}{\partial \eta}\Bigg|_{\frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}} \to \frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}} \ln\left(\frac{\vec{r}_{mn}^{\,2} a}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}}\right)}$$

(cf. "Conformal JIMWLK" by Kovner and Lublinsky, 2014)

Composite "conformal" baryon operator

$$B_{123}^{conf} = B_{123} + \frac{\alpha_s 3}{8\pi^2}\int d^2\vec{r}_4 \left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}} \ln\left(\frac{a\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\right)\right.$$
$$\left.\times\left(-B_{123} + \frac{1}{6}\left(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}\right)\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right].$$

NLO evolution of baryon operator (A. Grabovsky and I.B., 2014)

$$
\begin{aligned}
\frac{dB_{123}^{conf}}{d\eta} &= \mathrm{LO} - \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 \; \Big(\Big\{\tilde{L}_{12}^C\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 \\
&+L_{12}^C\Big[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 + tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4 \\
&\qquad -\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\Big] \\
&+M_{12}^C\Big[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\Big] \\
&\qquad +Z_{12}B_{355}B_{125}+\ (\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\Big\}+(0\leftrightarrow 4)\Big) \\
&-\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_5\left(\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{15}^{\,2}}{\vec{r}_{25}^{\,2}}\right)\left(\frac{1}{\vec{r}_{25}^{\,2}}-\frac{1}{\vec{r}_{15}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{25}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right. \\
&\times\left(\frac{3}{2}(B_{155}B_{235}+B_{255}B_{135}-B_{355}B_{125})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Big).
\end{aligned}
$$

NLO evolution kernels

Here

$$L_{12}^{C} = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}{}^{2}}{4\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{14}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}}\right) + \frac{\vec{r}_{12}{}^{2}}{4\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{14}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{2}\vec{r}_{24}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}}\right),$$

$$\tilde{L}_{12}^{C} = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}{}^{2}}{4\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{14}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}}\right) - \frac{\vec{r}_{12}{}^{2}}{4\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{14}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{2}\vec{r}_{24}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}}\right),$$

$$Z_{12} = \frac{\vec{r}_{12}{}^{2}}{8\vec{r}_{15}{}^{2}\vec{r}_{25}{}^{2}} \left[\left(\frac{\vec{r}_{35}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{34}{}^{2}} - \frac{\vec{r}_{25}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}}\right) \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{14}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}}\right)\right.$$

$$\left. + \frac{\vec{r}_{15}{}^{2}}{\vec{r}_{45}{}^{2}\vec{r}_{14}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{34}{}^{2}}{\vec{r}_{35}{}^{2}\vec{r}_{24}{}^{2}}\right) + \frac{\vec{r}_{13}{}^{2}}{\vec{r}_{14}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{35}{}^{2}\vec{r}_{12}{}^{2}}{\vec{r}_{25}{}^{2}\vec{r}_{13}{}^{2}}\right)\right] - (1 \leftrightarrow 3),$$

NLO evolution kernels

$$M_{12}^{C} = \frac{\vec{r}_{12}{}^{2}}{16\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{14}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{2}\vec{r}_{25}{}^{2}\vec{r}_{34}{}^{4}}{\vec{r}_{35}{}^{4}\vec{r}_{14}{}^{2}\vec{r}_{24}{}^{2}}\right) + \frac{\vec{r}_{12}{}^{2}}{16\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}} \ln\left(\frac{\vec{r}_{35}{}^{4}\vec{r}_{45}{}^{4}\vec{r}_{12}{}^{4}\vec{r}_{24}{}^{2}}{\vec{r}_{15}{}^{2}\vec{r}_{25}{}^{6}\vec{r}_{14}{}^{2}\vec{r}_{34}{}^{4}}\right)$$

$$+ \frac{\vec{r}_{23}{}^{2}}{16\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{4}\vec{r}_{35}{}^{2}\vec{r}_{24}{}^{6}\vec{r}_{34}{}^{2}}{\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{4}\vec{r}_{14}{}^{4}\vec{r}_{23}{}^{4}}\right) + \frac{\vec{r}_{23}{}^{2}}{16\vec{r}_{35}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{35}{}^{2}\vec{r}_{14}{}^{4}}{\vec{r}_{15}{}^{4}\vec{r}_{24}{}^{2}\vec{r}_{34}{}^{2}}\right)$$

$$+ \frac{\vec{r}_{13}{}^{2}}{16\vec{r}_{35}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{14}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{4}\vec{r}_{14}{}^{2}\vec{r}_{34}{}^{2}}{\vec{r}_{15}{}^{2}\vec{r}_{35}{}^{2}\vec{r}_{24}{}^{4}}\right) + \frac{\vec{r}_{13}{}^{2}}{16\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{4}\vec{r}_{14}{}^{2}\vec{r}_{34}{}^{2}}{\vec{r}_{15}{}^{2}\vec{r}_{35}{}^{2}\vec{r}_{24}{}^{4}}\right)$$

$$+ \frac{\vec{r}_{35}{}^{2}\vec{r}_{12}{}^{2}}{8\vec{r}_{15}{}^{2}\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{2}\vec{r}_{35}{}^{2}\vec{r}_{24}{}^{4}}{\vec{r}_{25}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{12}{}^{2}\vec{r}_{34}{}^{2}}\right) + \frac{\vec{r}_{23}{}^{2}\vec{r}_{12}{}^{2}}{8\vec{r}_{15}{}^{2}\vec{r}_{25}{}^{2}\vec{r}_{24}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{25}{}^{2}\vec{r}_{12}{}^{2}\vec{r}_{34}{}^{2}}{\vec{r}_{15}{}^{2}\vec{r}_{23}{}^{2}\vec{r}_{24}{}^{2}}\right)$$

$$+ \frac{\vec{r}_{14}{}^{2}\vec{r}_{23}{}^{2}}{8\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{24}{}^{2}\vec{r}_{34}{}^{2}} \ln\left(\frac{\vec{r}_{15}{}^{2}\vec{r}_{45}{}^{2}\vec{r}_{23}{}^{2}\vec{r}_{24}{}^{2}}{\vec{r}_{25}{}^{4}\vec{r}_{14}{}^{2}\vec{r}_{34}{}^{2}}\right)$$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- It gives the anomalous dimensions of gluon light-ray operators at "the BFKL point" $j \to 1$
- NLO photon impact factor is calculated.
- NLO hierarchy of Wilson-line evolution is derived.
- NLO evolution of baryon operator ($\ni$ odderon contribution) is obtained.