Large N_c, Constituent Quarks, and N, Δ Charge Radii

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Abstract

We show how one may define baryon constituent quarks in a rigorous manner, given physical assumptions that hold in the large-N_c limit of QCD. This constituent picture gives rise to an operator expansion that has been used to study large-N_c baryon observables; here we apply it to the case of charge radii of the N and Δ states. For example, one finds the relation $r_p^2 - r_{\Delta^+}^2 = r_n^2 - r_{\Delta^0}^2$ to be broken only by three-body, $O(1/N_c^2)$ effects for any N_c.

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I. INTRODUCTION

The only known path to rendering QCD-like theories perturbative at all energy scales is to increase the number N_c of color charges [1], so that $1/N_c$ itself becomes the small expansion parameter. While mesons in large N_c continue to exhibit the quantum numbers of a single quark-antiquark pair, the large-N_c baryon requires N_c valence quarks, since SU(N_c) group theory requires a minimum of N_c fundamental representation indices to form a color singlet.[1] However, physical baryons consist also of myriad gluons and sea quark-antiquark pairs; does this then imply that large N_c baryons have a meaning only within the context of the valence quark model? In this paper we claim that this is not the case, and indeed argue that it is possible to use the very existence of baryons boasting well-defined quantum numbers and large-N_c arguments to derive a *rigorous* constituent quark picture. These assumptions are clearly independent of the momentum transfer scale, and therefore this constituent picture holds from the low-energy to deep-inelastic scattering regimes.[2]

This is actually the same picture, in a somewhat different language, used to derive an effective Hamiltonian $1/N_c$ operator expansion for baryon observables. The operator expansion has been used to analyze phenomenologically the baryon mass spectrum of the ground-state [3], orbitally-excited [4], and heavy-quark [5] baryons, as well as magnetic moments [6,7], axial-vector couplings [7,8], and photoproduction [9] and pionic [10] transitions of N^*s in large N_c.

We then apply this knowledge to a study of the charge radii of the nonstrange baryons N and Δ. We first present the generic expansion demanded by $1/N_c$ when no other physical input is included, and then specialize to include physical restrictions, such as the statement that the operators representing the charge radii must be proportional to the constituent quark charges. We find that there are actually two independent contributions at the leading order, $O(N_c^0)$, and one at $O(1/N_c)$. Since there are six baryons in the N, Δ multiplets, this implies a number of relations between the charge radii that are expected to be satisfied particularly well, as we explore below. For example, we show that a relation found previously in an $N_c = 3$ quark model with two-body currents [11] holds for arbitrary N_c with $O(1/N_c^2)$ corrections.

The paper is organized as follows. In Sec. II, we elucidate the promised relation between constituent quarks and baryon symmetry properties. In Sec. III we restrict to the two-flavor case and exhibit the complete $1/N_c$ operator expansion for scalar observables such as N, Δ charge form factors. We then consider this expansion in the "general parametrization method" [12] generalized to large N_c, which places additional restrictions on the allowed operators based on the observable at hand. We present and discuss results in Sec. IV and conclude in Sec. V.

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[1]See Ref. [2] for a pedagogical introduction to large N_c.

[2]Of course, for any finite N_c, the individual coefficients of the terms in the $1/N_c$ expansion might grow large for high momentum transfers, spoiling the utility of the expansion. It is not known where this transition occurs.

II. LARGE N_C AND CONSTITUENT QUARKS

We begin with the quantum numbers of the current quarks themselves. To obtain the electric charge and hypercharge of the quarks for arbitrary N_c, we require only that (u, d), (c, s), and (t, b) remain weak isospin doublets with $I_3 = +1/2$ and $-1/2$, respectively, that under strong isospin the up quark and down quark still form a doublet with $I_3 = +1/2$ and $-1/2$, respectively, while the strange and all other quarks are isosinglets, and that all quarks in the electroweak interaction and u, d, s quarks in the strong interaction satisfy the Gell-Mann–Nishijima condition

$$Q = I_3 + Y/2. \tag{2.1}$$

Then the cancellation of the SU(N_c)×SU(2)×U(1) standard model chiral anomalies imposes

$$Q_{u,c,t} = (N_c + 1)/2N_c, \quad Q_{d,s,b} = (-N_c + 1)/2N_c, \tag{2.2}$$

while under strong hypercharge one finds

$$Y_u = Y_d = 1/N_c, \quad Y_s = (-N_c + 1)/N_c. \tag{2.3}$$

It is interesting to note that these results maintain for arbitrary N_c the usual electric charge and hypercharge assignments familiar in $N_c = 3$, such as the proton quantum numbers $Q_p = Y_p = +1$.

Baryons in large N_c have masses of $O(N_c)$, owing to both the intrinsic $O(1)$ masses of the quarks and interaction terms which also scale as N_c [13]. The emergence of an exact spin-flavor symmetry in the large-N_c limit for any number of flavors was first demonstrated in Ref. [14], so that it is meaningful to classify baryons into spin-flavor representations at leading order in $1/N_c$.

The ground-state multiplet of baryons for arbitrary N_c fills, by assumption, a spin-flavor multiplet described by a tensor completely symmetric on N_c indices (Fig. 1). For three flavors (u, d, s), this is an SU(6) multiplet that for $N_c = 3$ reduces to the familiar positive-parity 56-plet containing the spin-1/2 SU(3) octet and spin-3/2 decuplet. When $N_c > 3$, these multiplets are much larger.[3] Then each multiplet possesses, in general, a number of states whose quantum numbers reduce to those of the familiar baryons in $N_c = 3$. For example, the spin-flavor multiplet of Fig. 1 decomposes into N_c distinct flavor multiplets with spins $1/2, 3/2, \ldots, N_c/2$: Is the Δ to be identified as a spin-3/2 or spin-$N_c/2$ state? In this case, one finds that [16] $(M_\Delta - M_N) \propto J(J+1)/N_c$, compared to $M_{\Delta,N} = O(N_c)$. The observed relatively small Δ-N mass splitting suggests that one should take $J = 3/2$ rather than $J = N_c/2$.

Similar considerations [2] lead one to take the large-N_c analogues of the familiar baryons to have the usual spins, isospins, and hypercharges values of $O(1)$ rather than $O(N_c)$. In particular, this identifies the proton as a state with $I = I_3 = 1/2$, $J = 1/2$, and valence quark content consisting of the usual triple of uud in an $I = J = 1/2$ combination, augmented by

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[3]The multiplets are exhibited in Refs. [2,3,15]

$(N_c - 3)/2$ ud pairs, each in a spin-singlet, isosinglet combination. Then $N_u = (N_c + 1)/2$ and $N_d = (N_c - 1)/2$, and one may verify the previous claim that $Q_p = Y_p = +1$.

Obtaining a rigorous constituent picture for baryons requires that each baryon truly resides in a unique spin-flavor multiplet. In the case of the familiar SU(3) octet and decuplet baryons, this is the completely symmetric 56-plet of SU(6). Such an assumption is subject to two conditions:

1. The baryons are stable under strong interactions, so that they are true narrow-width eigenstates of the strong Hamiltonian. This is true in large N_c, since the production of each meson costs one power of $1/\sqrt{N_c}$ in the amplitude. It is also true for physical nucleons, where only weak decays are permitted, and to a lesser extent for the other ground-state baryons, where phase space suppresses such decays.

2. Configuration mixing between the dominant ground-state multiplet and higher multiplets is suppressed.[4] This is also true in large N_c, where such mixing requires the exchange of gluons to excite the ground state into an overlap with the higher state. These gluon couplings introduce additional $1/N_c$ suppressions. For example, consider flipping the spin of one of the N_c quarks in a proton to form a Δ^+. Dynamics tells us that the spatial wavefunction of the baryon should adjust itself to the new spin configuration; however, since only one of the quarks in N_c has changed, one expects this effect to be suppressed by some power of $1/N_c$.

Once these conditions are satisfied, it becomes a matter of mathematics alone to identify individual "constituent" quarks within the baryon. This is seen from the spin-flavor Young tableau for the ground state (Fig. 1); the spin-flavor wavefunction is a completely symmetric tensor with N_c indices, represented by N_c boxes in the tableau. Each index corresponds to a fundamental representation of the spin-flavor group, and carries precisely the same quantum numbers as one of the current quarks within the baryon. One may use spin-flavor projection operators to isolate these representation "quarks" (which we call r-quarks), the collective action of which is to reproduce the *entire* baryon spin-flavor wavefunction.[5] In terms of field theory, the r-quarks are interpolating fields carrying the spin-flavor quantum numbers of current quarks, such that an appropriately symmetrized set of N_c boast complete overlap with the baryon wavefunction (Fig. 2). The r-quarks are then true "constituent" quarks, in that the baryon is constituted entirely of them and nothing else. It turns out that the "Naive quark model for an arbitrary number of colors" presented in Ref. [17], based on the constituent quark model, is not so naive after all.

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[4] "Configuration mixing" has two meanings here: One, such as that used in the text, indicates the change of a baryon wavefunction when spins or flavors of individual quarks are altered. There is also a narrower meaning of mixing between two spin-flavor eigenstates with the same global quantum numbers, such as between nucleon and Roper states. In both cases, the mixing between pure spin-flavor eigenstates requires gluon exchanges and thus is suppressed in $1/N_c$.

[5] The spatial wavefunction of each r-quark then has the same functional behavior as the spatial wavefunction of the whole baryon, restating the assumption that configuration mixing is neglected.

Obviously, such a trick cannot possibly tell us everything about the baryon structure. As an explicit example, consider the strangeness content of the proton. We have argued that the flavor structure of the proton for arbitrary N_c consists of the usual valence uud triple and $(N_c - 3)/2$ ud pairs each in a spin-singlet, isosinglet combination. But if these are all of the r-quarks, how can the proton have strange content? The answer is that $s\bar{s}$ pairs are present, as are other sea quarks and gluons, but all of these have been incorporated into the r-quarks. In terms of field theory, these other components have been integrated out in favor of the r-quark fields.[6] Thus, the proton may have strange content even if it has no strange r-quarks.

The r-quark decomposition clearly does not indicate in detail how constituent quarks are formed from the fundamental degrees of freedom in the baryon. But it does give, by construction, values for observable matrix elements that an arbitrarily good constituent quark model, i.e., one that gives all of the correct baryon observables, must satisfy. In this way it serves as a means to improve explicit quark model calculations. As an example given in the next section, one can extract the r-quark masses and interaction energy terms from the N, Δ spectrum.

Finally, it should be pointed out that this decomposition has nothing to do with large N_c *per se*, except that identifying physical baryons with distinct irreducible spin-flavor representations for larger N_c is on somewhat more solid theoretical ground because of the conditions listed above. If one declares that the proton lies entirely in an SU(6) 56-plet in the physical case of $N_c = 3$, there is no problem in defining the three r-quarks.

III. OPERATOR ANALYSES

The analysis of any observable with given spin-flavor quantum numbers in the $1/N_c$ expansion may be carried out in essentially the same way: One simply writes down all operators with the same spin-flavor transformation properties as the observable, weighted with the appropriate suppression power of $1/N_c$. The number of such operators is finite since the number of spin-flavor structures connecting the initial- and final-state baryons is finite. As a trivial example, consider the problem of mass operators of the $I = 1/2$ nucleon states. The Wigner-Eckart theorem tells us that only operators with isospins $I = 0$ or 1 can connect the states. Indeed, the most general decomposition, as was done for the ground-state baryon masses [3], or the orbitally-excited baryons [4], or here for the charge radii, may be considered the application of the Wigner-Eckart theorem in spin-flavor space.

We also see from this example that there are precisely as many operators (2) as independent mass observables, which permits arbitrary masses for the p and n states. In the given example, the $I = 0$ and $I = 1$ operators contribute to $(m_n + m_p)/2$ and $(m_n - m_p)$, respectively. Unless some of the operators in a given expansion may be eliminated or suppressed, the operators merely provide a reparametrization of the data, i.e., a different basis for the

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[6] Indeed, $s\bar{s}$ pairs in the proton appear only in vacuum loops, which introduce $1/N_c$ suppressions compared to pure glue interactions. The same is not necessarily true for $u\bar{u}$ or $d\bar{d}$ pairs in the proton, which can appear in Z-graphs.

same vector space of observables.

However, we have not yet taken into account suppressions of operators by powers of $1/N_c$. In order to identify these suppressions for baryons, it is most convenient to work with r-quarks. Let us define an *n-body operator* as one that requires the participation of n r-quarks; that is, the Feynman diagram has a piece that is n-particle irreducible. Since r-quarks each carry a fundamental color index, they exchange gluons just like current quarks and hence obey the same large-N_c counting rules. Indeed, it is not difficult to see that an n-body operator requires the exchange of a minimum of $n-1$ gluons and hence a suppression of $1/N_c^{n-1}$, since [1] $\alpha_s \propto 1/N_c$.

The most general possible n-body operators can be built from nth-degree polynomials in 1-body operators, whose members fill the adjoint representation of the spin-flavor group. We denote these

$$ J^i \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes \mathbb{1} \right) q^\alpha, $$

$$ T^a \equiv q_\alpha^\dagger \left(\mathbb{1} \otimes \frac{\lambda^a}{2} \right) q^\alpha, $$

$$ G^{ia} \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes \frac{\lambda^a}{2} \right) q^\alpha, \qquad (3.1) $$

where σ^i are the usual Pauli spin matrices, λ^a denote Gell-Mann flavor matrices, and the index α sums over all N_c quark lines in the baryons. In the two-flavor case considered here, T^a is replaced with the isospin operator I^a. One then builds polynomials in J, I, and G with the same spin-flavor quantum numbers as the observable in question.

However, there are still three important points to take into account before the analysis is complete. First, the operators in Eq. (3.1) sum over all the r-quarks in the baryon and may add coherently to give combinatoric powers of N_c that compensate some of the $1/N_c$ suppressions. Generally, this occurs for G and not I or J, since we have chosen baryons to have spins and isospins of $O(1)$ rather than $O(N_c)$.

Second, there exist relations, called operator reduction rules [4,15], between some combinations of operators due to the spin-flavor symmetry of the baryon representation. For example, one particular combination of J^2, I^2, and G^2 is the quadratic Casimir of the spin-flavor algebra, and just gives the same number when applied to all baryons in the same representation. In the two-flavor case with scalar operators, the operator reduction rules of Ref. [15] tell us that the G^{ia} never need appear, since every possible contraction of its spin index leads to a reducible combination. Likewise, $I^2 = J^2$ in the two-flavor case.

Third, the most complicated operator necessary to describe a baryon with N_c r-quarks is an N_c-body operator. However, ultimately we are interested in the subset of these baryons that persist when $N_c = 3$, and by the same logic, these are completely described by expanding only out to 3-body operators. The 4-, 5-, ..., N_c-body operators would be linearly independent when acting upon the *full* baryon representation, but must be linearly dependent on the 0-, 1-, 2-, and 3-body operators when acting upon the baryons that persist for $N_c = 3$. Since we are not taking the strict $N_c \to \infty$ limit but rather N_c large and finite, the question of losing information due to noncommutativity with the chiral limit [18] does not arise.

Using these rules, it is straightforward to write down the expansion for an arbitrary scalar quantity with possible isospin breaking but preserving I_3 (as in electromagnetic interactions or masses). Our example is the derivative of the baryon charge (Sachs) form factor $F(q^2)$ at $q^2 = 0$, but note that the same expansion would hold for the whole q^2-dependent form factor, as well as masses [3]:

$$ -6 \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0} = \left\langle c_0 \mathbb{1} + c_1 I_3 + \frac{c_2}{N_c} I^2 + \frac{c_3}{N_c} \{ I_3, I_3 \} + \frac{c_4}{N_c^2} \{ I^2, I_3 \} + \frac{c_5}{N_c^2} \{ I_3, \{ I_3, I_3 \} \} \right\rangle. $$

$$ (3.2) $$

The brackets indicate that the operators are to be evaluated for a particular baryon state; anticommutators are used to remind one that the commutator combinations are reducible, owing to the spin-flavor symmetry. Here, each of the coefficients c_i possesses a $1/N_c$ expansion starting at order N_c^0; they play the role of reduced matrix elements in the Wigner-Eckart theorem. We make the naturalness assumption that any dimensionless coefficient appearing in the analysis is of order unity, unless one can think of a reason it is suppressed (additional symmetry or chance dynamical cancellation) or enhanced (additional dynamics). An example of the first case is the neutron-proton mass difference, where one would find an anomalously small coefficient unless the approximate symmetry of isospin is recognized. An example of the second case is that the neutron-proton scattering lengths are much larger than "natural size," pointing to shallow bound (the deuteron) or nearly bound ($^1S_0$) states.

To illustrate the r-quark picture for the baryons, consider the case of N and Δ masses using the r.h.s. of Eq. (3.2). The operator $\mathbb{1}$ clearly gives a common mass c_0 to each r-quark, while the I_3 term differentiates u and d r-quarks. The remaining operators require interactions of the r-quarks and may be considered matrix elements of the potential. Using Breit-Wigner masses for the Δ states (Note, however, Ref. [19] for a treatment using pole masses), one finds

$$ m_u = c_0 + c_1/2 = 287.6 \text{ MeV}, \quad m_d = c_0 - c_1/2 = 289.6 \text{ MeV}, \qquad (3.3) $$

and the interaction energy terms for nucleons and Δ's amount to about 73 and 366 MeV, respectively. These values for the quark masses are consistent with those used in ordinary constituent quark models. The r-quark masses thus account for the bulk of baryon masses, underscoring the economy of this picture.

It is convenient to rewrite Eq. (3.2) in terms of the global quantum numbers $J(J+1)$ and Q, which equal $I(I+1)$ and $I_3 + 1/2$, respectively, in the two-flavor case. Then the expansion reads

$$ -6 \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0} = d_0 N_c + d_1 Q + \frac{d_2}{N_c} J(J+1) + \frac{d_3}{N_c} Q^2 + \frac{d_4}{N_c^2} Q J(J+1) + \frac{d_5}{N_c^2} Q^3, \qquad (3.4) $$

where again each d_i possesses a $1/N_c$ expansion starting at order N_c^0. Note in either case that there are 6 independent operators, reflecting that there are 6 observables, corresponding to the isodoublet of N's and the isoquartet of Δ's. Equation (3.4) is therefore the most general expansion one can write down, modified only by the $1/N_c$ suppression factors.

For the particular case of the charge form factor, one can go a bit further. Despite their $O(N_c)$ masses, baryons in large N_c nevertheless have a finite size [13], so $d_0 N_c$ in Eq. (3.4)

should actually be replaced by d_0. One can see this by noting that no interaction diagram in the baryon is larger than N_c^1, so that the interaction energy per quark is no larger than N_c^0, and thus the wavefunction of each quark has a spatial extent of $O(N_c^0)$. Thus, the most general expansion based solely upon symmetry and the grossest features of large N_c reads

$$-6 \left.\frac{dF(q^2)}{dq^2}\right|_{q^2=0} = d_0 + d_1 Q + \frac{d_2}{N_c}J(J+1) + \frac{d_3}{N_c}Q^2 + \frac{d_4}{N_c^2}QJ(J+1) + \frac{d_5}{N_c^2}Q^3. \quad (3.5)$$

This operator method lies at one extreme end of possible analyses, in that it includes *only* symmetry information. At the other end lie phenomenological models, in which not only the structure of the individual operators but also their coefficients are provided. As an intermediate choice, one may impose mild physical constraints on the allowed operators; this is the approach of the "general parametrization (GP) method" [12]. It was applied to the case of baryon charge radii [20] in order to check relations appearing in a quark model calculation [11] that includes two-body exchange currents. Here we extend the analysis to arbitrary N_c.

The assumptions of the GP method for charge form factors are quite minimal: All scalar operators are allowed that couple to the quarks (r-quarks in our case) through precisely one factor of the quark charges, which is what one expects from a single photon vertex. Then one has

$$-6 \left.\frac{dF(q^2)}{dq^2}\right|_{q^2=0} = A \sum_i^{N_c} Q_i + \frac{B}{N_c} \sum_{i \neq j}^{N_c} Q_i \, \sigma_i \cdot \sigma_j + \frac{C}{N_c^2} \sum_{i \neq j \neq k}^{N_c} Q_i \, \sigma_j \cdot \sigma_k. \quad (3.6)$$

The rules for assigning $1/N_c$ suppressions in the coefficients are the same as above: n-body operators have a factor $1/N_c^{n-1}$, and A, B, C each possess $1/N_c$ expansions starting at order N_c^0. Note that this expression, unlike Eq. (5) in Ref. [20], has no strange quark term: As discussed above, the N's and Δ's have no strange r-quarks; the $s\bar{s}$ contributions appear as $O(1/N_c)$ corrections to the dynamical coefficients already presented.

It is straightforward to evaluate matrix elements of these three operators. The sums are re-expressed in terms of the Casimirs Q, $\mathbf{J}^2$, $\mathbf{S}_u^2$, and $\mathbf{S}_d^2$. To evaluate the final two Casimirs, note that the spin-flavor wavefunction is completely symmetric. Thus, all of the u quarks, for example, are in a symmetric state, and one then has total u-quark spin $S_u = N_u/2$. After simplifying all terms, one finds

$$\sum_i Q_i = Q,$$

$$\sum_{i \neq j} Q_i \, \sigma_i \cdot \sigma_j = Q(N_c - 1) - [N_c + 2(J+1)][N_c - 2J]/2N_c,$$

$$\sum_{i \neq j \neq k} Q_i \, \sigma_j \cdot \sigma_k = Q[4J(J+1) + 2 - 5N_c] + [N_c + 2(J+1)][N_c - 2J]/N_c. \quad (3.7)$$

The GP expansion then reads

$$-6 \left.\frac{dF(q^2)}{dq^2}\right|_{q^2=0} = AQ + \frac{B}{N_c^2}\left\{ QN_c(N_c - 1) - \frac{1}{2}[N_c + 2(J+1)][N_c - 2J] \right\}$$

$$+ \frac{C}{N_c^3}\left\{ QN_c[4J(J+1) + 2 - 5N_c] + [N_c + 2(J+1)][N_c - 2J] \right\}. \quad (3.8)$$

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The charge radii, defined as

$$r_B^2 = -6 \left.\frac{1}{F(q^2)}\frac{dF(q^2)}{dq^2}\right|_{q^2=0} = -6 \left.\frac{1}{Q}\frac{dF(q^2)}{dq^2}\right|_{q^2=0} \quad (3.9)$$

if $Q \neq 0$, and neglecting the Q factor if $Q = 0$, are presented for the N and Δ states in Table I.

It is interesting to compare the two expressions Eqs. (3.5) and (3.8). First, one sees that the latter is, as it must be, a special case of the most general possible expression Eq. (3.5). Specifically, the two expressions are related by

$$d_0 = -\frac{B}{2} - \frac{1}{N_c}(B - C) + \frac{2C}{N_c^2},$$

$$d_1 = A + B - \frac{1}{N_c}(B + 5C) + \frac{2C}{N_c^2},$$

$$d_2 = \frac{2B}{N_c} - \frac{4C}{N_c^2},$$

$$d_3 = 0,$$

$$d_4 = 4C,$$

$$d_5 = 0,$$

$$(3.10)$$

meaning that in GP the coefficients d_0, d_1, d_4 are independent and of natural [$O(1)$] size, d_2 is dependent and subleading in $1/N_c$, and $d_3 = d_5 = 0$. Note also that the coefficient B can appear at $O(1)$ and C at $O(1/N_c)$, a factor N_c larger than naively expected from Eq. (3.6), a result arising from the combined spin (σ) and flavor (Q_i) structure of the corresponding operators. Since the Q operator, containing a piece transforming as $I = 1$, is the sole source of isospin breaking in the GP, one expects that the $I = 2$ and 3 contributions, first appearing in Q^2 and Q^3 terms, are absent. By the Wigner-Eckart theorem, one can see that these relations involve only Δ states.

IV. RESULTS AND DISCUSSION

We have pointed out that the GP expression Eq. (3.8) is not the most general possible expansion for the charge radius. The other terms in Eq. (3.5) but not (3.8) can appear if subleading effects are taken into account. For example, in the GP expression, the only source of isospin quantum numbers is the quark charge operator Q_i. Explicit isospin breaking due to, say, the u-d quark mass difference introduces factors of the operator $I_3 = Q - 1/2$, which do not conform to the expression (3.8), but appear with an additional small ($\sim 5 \times 10^{-3}$) coefficient. Similar statements are expected for loop corrections; for example, one can see how electromagnetic loop corrections induce a Q^3 and possibly other suppressed terms in the expansion, at the cost of an $\alpha_{EM}/4\pi$ suppression. Inasmuch as these additional effects are dynamically suppressed, the GP expansion should give an excellent expansion for the charge form factors.

One interesting feature of the GP expression Eq. (3.8) is that the terms not proportional to the total baryon charge Q are all proportional to $N_c - 2J$, and in particular, vanish

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for $J = N_c/2$. That is, all charge radii (and other electromagnetic matrix elements) are proportional to Q for $J = N_c/2$, which was pointed out by Coleman [21] for the case $N_c = 3$. The symmetry reason for this feature is not hard to see: The charge operator Q transforms according to the adjoint representation of the spin-flavor group. The $J = N_c/2$ flavor representation, unlike that of $J = 1/2, 3/2, \ldots, N_c/2 - 1$, is completely symmetric, and has the same Young tableau as the spin-flavor representation in Fig. 1. In the product of this representation with its conjugate (relevant to baryon bilinears) there is only one adjoint representation, and since one already has one such operator, Q, its matrix elements must be proportional to the eigenvalue Q. For the flavor representations with $J < N_c/2$ (such as that of spin-3/2 for $N_c > 3$), the corresponding product has two or more adjoints, and exact proportionality to Q no longer holds.

As discussed above, $I = 2$ and 3 terms are absent in Eq. (3.8). The following relations (or any combination thereof) hold in the GP:

$$2r^2_{\Delta^{++}} - r^2_{\Delta^+} - r^2_{\Delta^0} - r^2_{\Delta^-} = 0 \quad (I = 2),$$
$$2r^2_{\Delta^{++}} - 3r^2_{\Delta^+} + 3r^2_{\Delta^0} + r^2_{\Delta^-} = 0 \quad (I = 3). \tag{4.1}$$

One also sees from Eq. (3.8) and Table I that both A and B terms are of leading order (N_c^0) for generic N's and Δ's in large N_c, despite the fact that the former comes from one-body and the latter from two-body operators. This is due to the coherence effect in the two-body operator. Similarly, the three-body operator (C terms) is suppressed only by $1/N_c$. It is only special combinations of the charge radii in which these leading effects cancel. A particularly interesting combination of this type is

$$(r^2_p - r^2_{\Delta^+}) - (r^2_n - r^2_{\Delta^0}) = -12C/N_c^2, \tag{4.2}$$

in which the full one- and two-body terms, as well as the coherent part of the three-body term, cancel for all N_c. Thus, if three-body operators are neglected, one has

$$r^2_p - r^2_{\Delta^+} = r^2_n - r^2_{\Delta^0}, \tag{4.3}$$

for all N_c. The only other such combinations are obtained by adding linear combinations of Eqs. (4.1). If we still demand an $O(1/N_c^2)$ combination but allow a two-body operator (which would serve to distinguish large N_c from the straightforward GP approach), one finds the separate relations

$$r^2_p - r^2_{\Delta^+} = -\frac{6}{N_c^2}\left[B + 2C\left(1 - \frac{1}{N_c}\right)\right],$$
$$r^2_n - r^2_{\Delta^0} = -\frac{6}{N_c^2}\left[B - \frac{2C}{N_c}\right]. \tag{4.4}$$

One may combine these relations with Eqs. (4.1) to predict all the Δ charged radii in terms of $r^2_{p,n}$ good to $O(1/N_c^2)$.

Alternately, if one allows N_c-dependent coefficients, the only relation in addition to Eqs. (4.1) with no corrections in the GP is

$$(N_c + 5)(N_c - 3)\, r^2_n = (N_c + 3)(N_c - 1)\, r^2_{\Delta^0}, \tag{4.5}$$

which is trivial for $N_c = 3$.

For completeness, the isovector and isoscalar charge radii are given by

$$r^2_{I=1} = (r^2_p - r^2_n) = A + B\frac{N_c - 1}{N_c} - 5C\frac{N_c - 1}{N_c^2},$$
$$r^2_{I=0} = (r^2_p + r^2_n) = A - 3B\frac{N_c - 1}{N_c^2} - 3C\frac{(N_c - 1)(N_c - 2)}{N_c^3}. \tag{4.6}$$

The experimental values $r^2_p = 0.792(24)$ fm^2 [22] and $r^2_n = -0.113(3)(4)$ fm^2 [23], together with Table I, suggest that $A/B \approx 5$ if C is neglected. While this is somewhat larger than one would expect from a pure naturalness assumption, dynamical models for A and B must be studied to decide whether this ratio is unnatural. Moreover, using Eqs. (4.1) and (4.4) with these experimental values and estimating $O(1/N_c^2)$ terms to be about $r^2_p/9 \approx 0.09$ fm^2 (which overwhelms statistical uncertainties on $r^2_{p,n}$), one finds

$$r^2_{\Delta^{++}} = r^2_p - \frac{1}{2}r^2_n + \frac{3}{N_c^2}\left[B + 2C\left(2 - \frac{1}{N_c}\right)\right] = 0.85 \pm 0.09\,\text{fm}^2,$$
$$r^2_{\Delta^+} = r^2_p + \frac{6}{N_c^2}\left[B + 2C\left(1 - \frac{1}{N_c}\right)\right] = 0.79 \pm 0.09\,\text{fm}^2,$$
$$r^2_{\Delta^0} = r^2_n + \frac{6}{N_c^2}\left[B - \frac{2C}{N_c}\right] = -0.11 \pm 0.09\,\text{fm}^2,$$
$$r^2_{\Delta^-} = r^2_p - 2r^2_n - \frac{6}{N_c^2}\left[B - 2C\left(1 + \frac{1}{N_c}\right)\right] = 1.02 \pm 0.09\,\text{fm}^2. \tag{4.7}$$

V. CONCLUSIONS

We have seen that a rigorous constituent quark picture for baryons, in that all spin-flavor matrix elements are reproduced by construction, follows from the assumption that the physical baryons are narrow-width eigenstates of distinct spin-flavor representations. Both of these requirements hold in the large-N_c limit. To improve systematically upon these assumptions, baryon strong decay amplitudes and configuration mixing must be accommodated, opening up new possibilities for large-N_c quark models.

The analysis of observables is possible in this simplified scheme. In particular, here we have studied N, Δ charge radii, and showed 1) that the one-body and part of the two-body operator are of leading order in $1/N_c$, and 2) that a number of useful relations follow from a simple parametrization representing the most important physical effects. It will be interesting to test which of these relations are supported by experiment.

Acknowledgments

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TABLES

r_p^2	$A + B \dfrac{(N_c-1)(N_c-3)}{2N_c^2} - C \dfrac{(N_c-1)(4N_c-3)}{N_c^3}$	$A - \frac{2}{3}C$
r_n^2	$-B \dfrac{(N_c-1)(N_c+3)}{2N_c^2} + C \dfrac{(N_c-1)(N_c+3)}{N_c^3}$	$-\frac{2}{3}B + \frac{4}{9}C$
$r_{\Delta^{++}}^2$	$A + B \dfrac{3(N_c^2-2N_c+5)}{4N_c^2} - C \dfrac{3(3N_c^2-12N_c+5)}{2N_c^3}$	$A + \frac{2}{3}B + \frac{2}{9}C$
$r_{\Delta^+}^2$	$A + B \dfrac{N_c^2-4N_c+15}{2N_c^2} - C \dfrac{(N_c-1)(4N_c-15)}{N_c^3}$	$A + \frac{2}{3}B + \frac{2}{9}C$
$r_{\Delta^0}^2$	$-B \dfrac{(N_c-3)(N_c+5)}{2N_c^2} + C \dfrac{(N_c-3)(N_c+5)}{N_c^3}$	0
$r_{\Delta^-}^2$	$A + B \dfrac{3(N_c^2-5)}{2N_c^2} - C \dfrac{3(2N_c^2-5N_c-5)}{N_c^3}$	$A + \frac{2}{3}B + \frac{2}{9}C$

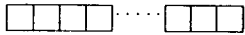TABLE I. Charge radii of N and Δ states as functions of N_c and for $N_c = 3$.

FIG. 1. The completely symmetric spin-flavor N_c-box Young tableau, corresponding to ground-state baryons.
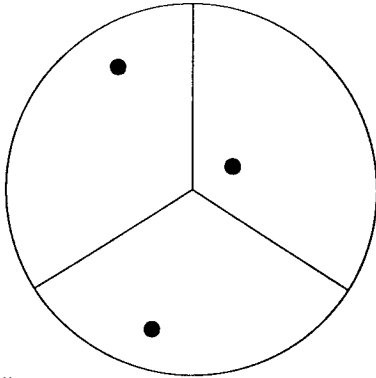

FIG. 2. Qualititative illustration of current quarks (dots) versus r-quarks (wedges) for $N_c = 3$ baryons. Note that the actual division is in spin-flavor, not spatial, coordinates. The entire baryon, including glue, sea quarks, etc., is subsumed into the r-quarks.