CUMULATIVE BEAM BREAKUP WITH RANDOM DISPLACEMENT OF CAVITIES AND FOCUSING ELEMENTS *

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Abstract

We have recently developed an analytical formalism for cumulative beam breakup in linear accelerators with arbitrary beam current profile. The same formalism could be used to investigate the beam breakup-enhanced displacement due to the misalignment of the cavities and the focusing elements. In this paper this analytical formalism is extended and applied to investigate the behavior of beams in misaligned pulsed and cw linear accelerators.

1 INTRODUCTION

Cumulative beam breakup (BBU) has been extensively investigated in the past by many authors; the present work is a continuation of a previous investigation of BBU in the case of bunches of finite length [1] or arbitrary current profile [2]. We extend here this analysis to the steady-state and transient behavior caused by the random displacements of cavities and focusing elements. This extension is motivated by the increasing interest in pulsed high-current superconducting accelerators, and can also be applied to the analysis of long-range effects in linear colliders.

2 EQUATION AND GENERAL SOLUTION

In a continuum approximation, the transverse motion of a beam under the influence of focusing and BBU in a perfectly aligned accelerator can be modeled by [1]

$$\left[\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left(\beta\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2\right] x(\sigma,\zeta)=\varepsilon\int_{-\infty}^{\zeta} d\zeta_1\, w(\zeta-\zeta_1)F(\zeta_1)x(\sigma,\zeta_1)$$

where β and γ are the usual velocity and energy parameters; $\sigma = s/L$, is the distance from the front of the accelerator normalized to the accelerator length; κ is the normalized focusing wave number; $\zeta=\omega\left(t-\int ds/\beta c\right)$ is the time made dimensionless by the frequency ω and measured after the arrival of the head of the beam at location σ; $F(\zeta)=I(\zeta)/\bar{I}$, the current form factor, is the instantaneous current divided by the average current; $w(\zeta)$ is the wake function, which, in the case of a single dipole mode, is assumed to be $w(\zeta)=\Theta(\zeta)\sin\zeta\, e^{-\zeta/2Q}$; ε is the coupling strength between the beam and the dipole mode, and includes properties of the beam and the deflecting mode of the accelerating structure.

Without loss of generality, we will assume a coasting beam and constant BBU and focusing strengths (ε and κ) along the accelerator. We will also assume that the cavities and the focusing elements are displaced from the reference beam line by $d_c(\sigma)$ and $d_f(\sigma)$ respectively and that the (time-dependent) lateral displacement and angular divergence of the beam at the entrance of the accelerator are $x_0(\zeta)$ and $x_0'(\zeta)$ respectively.

Under these assumptions, the equation of motion becomes

$$\frac{\partial^2}{\partial\sigma^2}x(\sigma,\zeta)+\kappa^2\left[x(\sigma,\zeta)-d_f(\sigma)\right] = \varepsilon\int_{-\infty}^{\zeta} d\zeta_1\, w(\zeta-\zeta_1)\, F(\zeta_1)\left[x(\sigma,\zeta_1)-d_c(\sigma)\right]. \tag{1}$$

Applying to Equation (1) the Laplace transform with respect to the variable σ: $\mathcal{L}\left[x(\sigma,\zeta)\right]=x^{\dagger}(p,\zeta)$, and following the method described in [2] we obtain

$$\begin{aligned} x^{\dagger}(p,\zeta) &= \sum_{n=0}^{\infty}\varepsilon^n\left(p^2+\kappa^2\right)^{-(n+1)}\left[\, g_n(\zeta)+p\, h_n(\zeta)\right] \\ &\quad - d_c^{\dagger}(p)\sum_{n=0}^{\infty}\varepsilon^{n+1}\left(p^2+\kappa^2\right)^{-(n+1)} f_{n+1}(\zeta) \\ &\quad + \kappa^2\, d_f^{\dagger}(p)\sum_{n=0}^{\infty}\varepsilon^{n}\left(p^2+\kappa^2\right)^{-(n+1)} f_{n}(\zeta)\ , \end{aligned} \tag{2}$$

with $f_0(\zeta)=1$, $g_0(\zeta)=x_0'(\zeta)$, $h_0(\zeta)=x_0(\zeta)$, $f_{n+1}(\zeta)=\int_{-\infty}^{\zeta} f_n(\zeta_1)\, w(\zeta-\zeta_1)\, F(\zeta_1)\, d\zeta_1$, and $g_{n+1}(\zeta)$ and $h_{n+1}(\zeta)$ being defined by the same recursion relation.

Applying the inverse Laplace transform gives

$$\begin{aligned} x(\sigma,\zeta) &= \sum_{n=0}^{\infty}\varepsilon^n\left[i_n(\sigma)\, g_n(\zeta)+j_n(\sigma)\, h_n(\zeta)\right] \\ &\quad - \sum_{n=0}^{\infty}\varepsilon^{n+1} f_{n+1}(\zeta)\int_0^{\sigma} du\, i_n(u)\, d_c(\sigma-u) \\ &\quad + \kappa^2\sum_{n=0}^{\infty}\varepsilon^{n} f_{n}(\zeta)\int_0^{\sigma} du\, i_n(u)\, d_f(\sigma-u) \end{aligned} \tag{3}$$

where $i_n(\sigma)=\frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n\frac{1}{\kappa}\sqrt{\frac{\pi\kappa\sigma}{2}}\,J_{n+\frac{1}{2}}(\kappa\sigma)$, and

$$j_n(\sigma)=\frac{d}{d\sigma}i_n(\sigma)=\frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n\sqrt{\frac{\pi\kappa\sigma}{2}}\,J_{n-\frac{1}{2}}(\kappa\sigma) = \frac{\sigma}{2n}i_{n-1}(\sigma).$$

Since in this paper we are concerned with the effect of the offsets $d_c(\sigma)$ and $d_f(\sigma)$, we will assume $x_0(\zeta)= x_0'(\zeta)=0$ for the remainder.

3 RMS DISPLACEMENT

From Equation (3) the statistical properties of the beam displacement $x(\sigma,\zeta)$ can be found from those of the offsets $d_c(\sigma)$ and $d_f(\sigma)$. For example, in the case of displaced cavities, the mean-square displacement of the beam at location σ and time ζ is

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$$\bar{x}^2(\sigma,\zeta) = \varepsilon^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^{m+n}\ y_{mn}(\sigma)\ f_{m+1}(\zeta)\ f_{n+1}(\zeta)$$
$$= \sum_{N=0}^{\infty} \varepsilon^{N+2} \sum_{k=0}^{N} y_{k\ N-k}(\sigma) f_{k+1}(\zeta)\ f_{N+1-k}(\zeta) \qquad (4)$$

with $y_{mn}(\sigma) = \int_0^\sigma \int_0^\sigma du\ dv\ i_m(\sigma-u)\ i_n(\sigma-v)\ R_d(u,v)$

where $R_d(u,v)$ is the autocorrelation function of the cavity displacement.

If, instead, the focusing elements are randomly displaced, then

$$\bar{x}^2(\sigma,\zeta) = \kappa^4 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^{m+n}\ y_{mn}(\sigma)\ f_m(\zeta)\ f_n(\zeta). \qquad (5)$$

The expression for $y_{mn}(\sigma)$ is identical to the previous one except that now $R_d(u,v)$ is the autocorrelation function of the displacement of the focusing elements.

In some applications, such as linear colliders, the cavities and the focusing elements are mounted on the same girders and their displacements are perfectly correlated: $d_c(\sigma) = d_f(\sigma)$. In this case

$$\bar{x}^2(\sigma,\zeta) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^{m+n}\ y_{mn}(\sigma)\left[\kappa^2 f_m(\zeta) - \varepsilon f_{m+1}(\zeta)\right] \times\left[\kappa^2 f_n(\zeta) - \varepsilon f_{n+1}(\zeta)\right]. \qquad (6)$$

A simple but reasonable model for the autocorrelation function is $R_d(u,v) = d_0^2\ \sigma_0\ \delta(u-v)$ where d_0 is the rms displacement of the element, and σ_0 the distance over which it is constant [2]. In that case we have

$$y_{mn}(\sigma) = \frac{d_0^2\sigma_0}{2^{m+n}\kappa^{2m+2n+3} m!n!}\int_0^{\kappa\sigma} dx\ x^{m+n}\left(\frac{\pi x}{2}\right) J_{m+\frac{1}{2}}(x)\ J_{n+\frac{1}{2}}(x).$$

The integrals can, in principle, be calculated for arbitrary m and n and reduce to products of polynomials and circular functions of $\kappa\sigma$.

4 SINGLE VERY SHORT BUNCH

For a constant-current very short bunch (much shorter than the wavelength of the deflecting mode), the wake function can be taken as linear, $w(\zeta) = \zeta$, and $f_n(\zeta)$ can be easily calculated: $f_n(\zeta) = \zeta^{2n}/(2n)!$.

In the case of displaced cavities, the mean-square displacement of the bunch is, from Equation (4)

$$\bar{x}^2(\sigma,\zeta) = \sum_{N=0}^{\infty} \varepsilon^{N+2} \sum_{k=0}^{N} y_{k\ N-k}(\sigma)\ f_{k+1}(\zeta)\ f_{N+1-k}(\zeta)$$
$$= d_0^2\sigma_0 \sum_{N=0}^{\infty} \frac{\left(\varepsilon\zeta^2\right)^{N+2}}{2^N \kappa^{2N+3}} \sum_{k=0}^{N} \frac{\int_0^{\kappa\sigma} dx\ x^N \left(\frac{\pi x}{2}\right) J_{k+\frac{1}{2}}(x)\ J_{N-k+\frac{1}{2}}(x)}{k!(N-k)!(2k+2)!(2N+2-2k)!}$$

Keeping only the highest degree monomials in the integrals gives

$$\bar{x}^2(\sigma,\zeta) = \frac{d_0^2\sigma_0\varepsilon^2\zeta^4\sigma}{8\kappa^4}\left[1 + \frac{1}{12}\frac{\varepsilon\zeta^2}{\kappa^2} + \frac{1}{2880}\left(\frac{\varepsilon\zeta^2}{\kappa^2}\right)^2 (\kappa\sigma)^2 + \cdots\right]$$

This result is similar to that obtained in [3] but with small differences. In [3] the second term in the bracket is missing and the factor 2880 is replaced by 1728.

In the case of displaced focusing elements, the mean-square displacement of the bunch is, from Equation (5)

$$\bar{x}^2(\sigma,\zeta) = \kappa^4 \sum_{N=0}^{\infty} \varepsilon^N \sum_{k=0}^{N} y_{k\ N-k}(\sigma)\ f_k(\zeta)\ f_{N-k}(\zeta)$$
$$= d_0^2\sigma_0\kappa^4 \sum_{N=0}^{\infty} \frac{\left(\varepsilon\zeta^2\right)^N}{2^N \kappa^{2N+3}} \sum_{k=0}^{N} \frac{\int_0^{\kappa\sigma} dx\ x^N \left(\frac{\pi x}{2}\right) J_{k+\frac{1}{2}}(x)\ J_{N-k+\frac{1}{2}}(x)}{k!(N-k)!(2k)!(2N-2k)!}$$

Keeping only the highest degree monomials in the integrals gives

$$\bar{x}^2(\sigma,\zeta) = \frac{d_0^2\sigma_0\kappa^2\sigma}{2}\left[1 + \frac{1}{2}\frac{\varepsilon\zeta^2}{\kappa^2} + \frac{5}{288}\left(\frac{\varepsilon\zeta^2}{\kappa^2}\right)^2 (\kappa\sigma)^2 + \cdots\right].$$

5 STEADY-STATE DC AND DELTA-FUNCTION BEAMS

In the case of a dc beam or a delta-function beam (beam composed of point-like bunches separated by $\omega\tau$) the steady state values of the functions $f_n(\zeta)$ can be easily calculated. For the dc beam we have $\lim_{\zeta\to\infty} f_n(\zeta) = \left[\tilde{w}(0)\right]^n$, where $\tilde{w}(Z)$ is the Fourier transform of the wake function $w(\zeta)$. In the case of the delta-function beam we have $\lim_{M\to\infty} f_n(M\omega\tau) = \left[\tilde{W}(0)\right]^n$ where $\tilde{W}(Z) = \sum_{k=-\infty}^{\infty} \tilde{w}\left(Z - \frac{2\pi}{\omega\tau}k\right)$ [1]. So, in both cases, we have $f_n(\infty) = \left[f_1(\infty)\right]^n \equiv \alpha^n$. In the presence of a single deflecting mode $\tilde{w}(0) = \left(1 + 1/4Q^2\right)^{-1}$, and $\tilde{W}(0) = \frac{\omega\tau\sin\omega\tau}{2}\left(\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau\right)^{-1}$.

Assuming, for now, that only the cavities are displaced, Equation (2) yields

$$x^+(p,\infty) = -d_c^+(p)\sum_{n=0}^{\infty}\left(\frac{\varepsilon\alpha}{p^2+\kappa^2}\right)^{n+1} = -\varepsilon\alpha\frac{d_c^+(p)}{p^2+\kappa^2-\varepsilon\alpha}$$
$$\text{or}\ \ x(\sigma,\infty) = -\frac{\varepsilon\alpha}{\lambda}\int_0^\sigma du\ d_c(\sigma-u)\sin\lambda u$$

where $\lambda^2 = \kappa^2 - \varepsilon\alpha$. If we assume that the autocorrelation function is $R_d(u,v) = d_0^2\ \sigma_0\ \delta(u-v)$, then the mean-square displacement is

$$\bar{x}^2(\sigma,\infty) = \frac{d_0^2\sigma_0}{2}\frac{\varepsilon^2\alpha^2}{\kappa^2-\varepsilon\alpha}\left\{\sigma - \frac{\sin\left[2\sigma\left(\kappa^2-\varepsilon\alpha\right)^{1/2}\right]}{2\left(\kappa^2-\varepsilon\alpha\right)^{1/2}}\right\} \qquad (7)$$

for $\kappa^2 - \varepsilon\alpha > 0$;

$$\bar{x}^2(\sigma,\infty) = \frac{d_0^2\sigma_0}{2}\frac{\varepsilon^2\alpha^2}{\varepsilon\alpha-\kappa^2}\left\{\frac{\sinh\left[2\sigma\left(\varepsilon\alpha-\kappa^2\right)^{1/2}\right]}{2\left(\varepsilon\alpha-\kappa^2\right)^{1/2}} - \sigma\right\}$$

for $\kappa^2 - \varepsilon\alpha < 0$; and

$\bar{x}^2(\sigma,\infty) = d_0^2\sigma_0\varepsilon^2\alpha^2\sigma^3/3$ for $\kappa^2 - \varepsilon\alpha = 0$.

If, instead, the focusing elements are displaced, the solution is identical to the one for displaced cavities but for $\varepsilon^2\alpha^2$ in the prefactors being replaced by κ^4.

In the case of perfectly correlated displacements of cavities and focusing elements, $d_c(\sigma) = d_f(\sigma)$, we obtain

$$\bar{x}^2(\sigma,\infty) = \frac{d_0^2\,\sigma_0}{2}\left(\kappa^2 - \varepsilon\alpha\right)\left\{\sigma - \frac{\sin\left[2\sigma\left(\kappa^2 - \varepsilon\alpha\right)^{1/2}\right]}{2\left(\kappa^2 - \varepsilon\alpha\right)^{1/2}}\right\} \qquad (8)$$

for $\kappa^2 - \varepsilon\alpha > 0$;

$$\bar{x}^2(\sigma,\infty) = \frac{d_0^2\,\sigma_0}{2}\left(\varepsilon\alpha - \kappa^2\right)\left\{\frac{\sinh\left[2\sigma\left(\varepsilon\alpha - \kappa^2\right)^{1/2}\right]}{2\left(\varepsilon\alpha - \kappa^2\right)^{1/2}} - \sigma\right\}$$

for $\kappa^2 - \varepsilon\alpha < 0$; and $\bar{x}^2(\sigma,\infty) = 0$ for $\kappa^2 - \varepsilon\alpha = 0$.

For a beam that would be inherently stable $(\kappa^2 - \varepsilon\alpha > 0)$, the steady-state displacement increases as $\sigma^{1/2}$, whereas for an unstable beam it increases exponentially. For perfectly correlated displacements of cavities and focusing elements $(d_c(\sigma) = d_f(\sigma))$ a circumstance exists where there is no steady-state displacement, although there is a transient one.

6 FINITE DELTA-FUNCTION BEAM

In Section 5 we obtained, in closed form, the steady-state displacement of an infinite train of point bunches. In this section we will analyze the transient displacement experienced by a finite number of bunches.

The transient mean-square displacement is still given by Equations (4), (5), or (6); the task is now to determine the functions $f_n(\zeta = M\,\omega\tau)$ for finite M.

Defining $\breve{w}(z) = \sum_{k=0}^{\infty} w(k\omega\tau)\, z^{-k}$ as the z-transform of $w(\zeta)$, the recursion relation in Equation (2) relates the z-transform of $f_n(\zeta)$ to that of $w(\zeta)$:

$$\breve{f}_n(z) = z(z-1)^{-1}\left[\omega\tau\, \breve{w}(z)\right]^n .$$

The inverse z-transform can then be used to calculate $f_n(M\,\omega\tau) = \frac{1}{2\pi i}\oint dz\, z^{M-1}\, \breve{f}_n(z)$. There is no simple closed-form expression for $f_n(M\,\omega\tau)$ but, in the case of a single deflecting mode where $w(\zeta) = \Theta(\zeta)\, \sin\zeta\, e^{-\zeta/2Q}$, they can easily be calculated using a symbolic manipulation package.

As an example we have calculated, using the first 3 terms, the mean-square displacement for a single deflecting mode due to random offset of the cavities:

$$\bar{x}_M^2(\sigma) = \varepsilon^2\left[\begin{array}{l} y_{00}(\sigma) f_1^2(M\,\omega\tau) + 2\varepsilon y_{10}(\sigma) f_2(M\,\omega\tau) f_1(M\,\omega\tau) \\ +\varepsilon^2\left[2 y_{20}(\sigma) f_3(M\,\omega\tau) f_1(M\,\omega\tau) + y_{11}(\sigma) f_2^2(M\,\omega\tau)\right]\end{array}\right],$$

and of the focusing elements:

$$\bar{x}_M^2(\sigma) = \kappa^4\left[\begin{array}{l} y_{00}(\sigma) f_0^2(M\,\omega\tau) + 2\varepsilon y_{10}(\sigma) f_1(M\,\omega\tau) f_0(M\,\omega\tau) \\ +\varepsilon^2\left[2 y_{20}(\sigma) f_2(M\,\omega\tau) f_0(M\,\omega\tau) + y_{11}(\sigma) f_1^2(M\,\omega\tau)\right]\end{array}\right].$$

The results are shown in Figure 1 for normalized mean-square displacement $\bar{x}_M^2/d_0^2\sigma_0$ at the end of the accelerator ($\sigma = 1$), with $\varepsilon = 0.1$, $\kappa = 100$, $\omega\tau = 4.01\,\pi$, Q=1500.

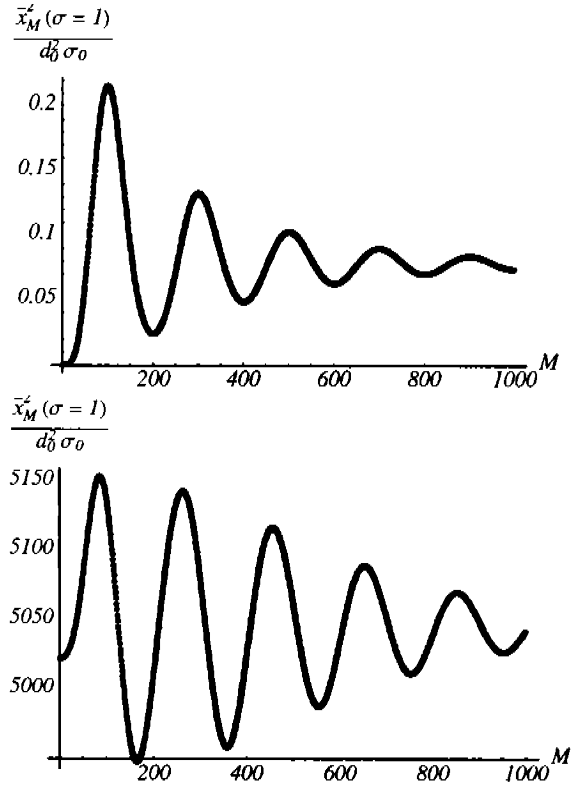

Figure 1: Mean-square displacement as function of bunch number for random offset of cavities (upper) and focusing elements (lower). See text for choice of parameters.

If instead $\bar{x}_M^2(\sigma)/d_0^2\sigma_0$ is plotted as a function of σ, it displays the expected $\bar{x}_M^2(\sigma) \propto \sigma$ dependence. For large M the result is indistinguishable from that given by the steady-state expressions given in Section 5. For much larger ε or for $\omega\tau$ closer to a high-Q resonance a larger number of terms may be needed.

A similar expression for $\bar{x}_M^2(\sigma)$ can be obtained and similar plots can be generated in the case of identical offsets of cavities and focusing elements. They clearly show that, when $\kappa^2 = \varepsilon\frac{\omega\tau\sin\omega\tau}{2}\left(\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau\right)^{-1}$, there is no steady-state $(M = \infty)$ rms displacement for all σ whereas there are large transient ones $(M < \infty)$.

7 REFERENCES

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