TWO-PION EXCHANGE NUCLEON-NUCLEON POTENTIAL: RELATIVISTIC CHIRAL EXPANSION

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We present a relativistic procedure for the chiral expansion of the two-pion exchange component of the NN potential, which emphasizes the role of intermediate πN subamplitudes. The relationship between power counting in πN and NN processes is discussed and results are expressed directly in terms of observable subthreshold coefficients. Interactions are determined by one and two-loop diagrams, involving pions, nucleons and other degrees of freedom, frozen into empirical subthreshold coefficients. The full evaluation of these diagrams produces amplitudes containing many different loop integrals. Their simplification by means of relations among these integrals leads to a set of intermediate results. Subsequent truncation to $\mathcal{O}(q^4)$ yields the relativistic potential, which depends on four loop integrals, representing bubble, triangle, crossed box and box diagrams. The bubble and triangle integrals are the same as in πN scattering and we have shown that they also determine the chiral structures of box and crossed box integrals. Relativistic threshold effects were found to begin to contribute at $\mathcal{O}(q^5)$ only and our results should coincide with those of the standard heavy baryon approach. Checking this explicitly, we found diferences due to the Goldberger-Treiman discrepancy and terms of $\mathcal{O}(q^3)$, possibly associated with the iteration of the OPEP.

I. INTRODUCTION

A considerable refinement in the description of nuclear interactions has ocurred in the last decade, due to the systematic use of chiral symmetry. As the non-Abelian character of QCD prevents low energy calculations, one works with effective theories that mimic, as much as possible, the basic theory. In the case of nuclear processes, where interactions are dominated by the quarks u and d, these theories are required to be Poincaré invariant and to have approximate $SU(2) \times SU(2)$ symmetry. The latter is broken by the small quark masses, which give rise to the pion mass at the effective level.

In the sixties, it became well established that the one-pion exchange potential (OPEP) provides a good description of NN interactions at large distances. When one moves inward, the next class of contributions corresponds to exchanges of two uncorrelated pions [1] and, until recently, there was no consensus in the literature as how to treat this component of the force. An important feature of the two-pion exchange potential (TPEP) is that it is closely related to the pion-nucleon (πN) amplitude, a point stressed more than thirty-five years ago by Cottingham and Vinh Mau [2]. This idea allowed one to overcome the difficulties associated with perturbation theory [3] and led to the construction of the successful Paris potential [4], where the intermediate part of the interaction is obtained by means of dispersion relations. This has the advantages of minimizing the number of unnecessary hipotheses and yielding model independent results, but it does not help in clarifying the role of different dynamical processes, which are always treated in bulk.

Field theory provides an alternative framework for the evaluation of the TPEP. In this case, one uses a Lagrangian, involving the degrees of freedom one considers to be relevant, and calculates amplitudes using Feynman diagrams, which are subsequently transformed into a potential. An important contribution along this line was given in the early seventies by Partovi and Lomon, who considered box and crossed box diagrams, using a Lagrangian containing just pions and nucleons with pseudoscalar (PS) coupling [5]. A study of the same diagrams using a pseudovector (PV) coupling was performed later by Zuilhof and Tjon [6]. The development of this line of research led to the Bonn model for the NN interaction, which included many important degrees of freedom and proved to be effective in reproducing empirical data [7]. On the phenomenological side, accurate potentials also exist, which can reproduce low-energy observables employing parametrized forms of the two-pion exchange component [8].

Nowadays it is widely acknowledged that chiral symmetry provides the best conceptual framework for the construction of nuclear potentials. The importance of this symmetry was pointed out in the early seventies by Brown and Durso [9] and by Chemtob, Durso and Riska [10], who stressed that it constrains the form of the intermediate πN amplitude present in the TPEP.

In the early nineties, the works by Weinberg restating the role of chiral symmetry in nuclear interactions [11] were followed by an effort by Ordóñez and van Kolck [12] and other authors [13,14] to construct the TPEP in that framework. The symmetry was then realized by means of non-linear Lagrangians containing only pions and nucleons. This minimal chiral TPEP is consistent with the requirements of chiral symmetry and reproduces, at the nuclear level, the well known cancellations present in the intermediate πN amplitude [15]. On the other hand, a lagrangian containing just pions and nucleons cannot describe experimental πN data [16] and the corresponding potential missed even the scalar-isoscalar medium range attraction [14].

One needed other degerees of freedom. The Δ contributions were shown to improve predictions by Ordóñez, Ray and van Kolck [17] and other authors [18]. Empirical information about the intermediate πN amplitude at low energies is normally summarized by means of subthreshold coefficients [16,19], which can be used either directly in the construction of the TPEP or to determine unknown coupling constants in chiral lagrangians. This allowed satisfactory descriptions of the asymptotic NN to be produced, with no need of free parameters [23–26].

As far as field theory techniques are concerned, recent calculations of the TPEP were performed using both heavy baryon chiral perturbation theory (HBChPT) and covariant lagrangians. In the former case [12,17,24–27], one uses non-relativistic effective Lagrangians, that include unknown couterterms, and amplitudes are derived in which loop and counterterm contributions are organized in well defined powers of the typical three-momenta exchanged between nucleons. In this approach, relativistic corrections required by precision have to be added separately [28].

The motivation for the heavy baryon formalism was the realization that, for system containing baryons, the dimensional regularization of loop diagrams yielded results that started to contribute at the same order as tree diagrams, spoiling the desired power counting rules [29,30]. To overcome this dificulty Jenkins and Manohar [31] proposed the heavy baryon formalism, which consists essencially of a non-relativistic expansion in inverse of the baryon mass at the level of the lagrangian. The resulting theory recovers the power counting rules of ChPT, but explicit Lorentz invariance is lost.

Recently Becher and Leutwyler [32] showed that the non-relativistic expansion can be avoided by means of the so-called infrared regularization. The basic idea is to separate pole contributions in loop integration located in the low energy domain from those at higher energies in a covariant way. The latter piece spoil the power counting rules, but it is possible to absorb them in the coupling constants of the theory. The resulting formalism gives rise to power counting while preserving Lorentz invariance. The infrared regularized loop integrals contain arbitrary powers of the generic scale q (meson mass, meson four-momentum or baryon three-momentum), which allow us to assess the convergence of the HBChPT series.

Relativistic and heavy baryon calculations were compared in single nucleon systems. In the case of nucleon properties, consistency is possible, provided one uses the nucleon mass as the dimensional regularization scale [33]. In elastic πN scattering, on the other hand, there are important differences. As demonstrated by Becher and Leutwyler [32,34], the amplitude that underlies the scalar form factor cannot be represented by the heavy baryon series around the point that determines its large distance properties. One of the main purposes of the present work is to extend this discussion to the NN interaction.

Our presentation is organized as follows. In section II we present the formal relations between the relativistic TPEP and the intermediate πN subamplitude, whose chiral structure is analysed in section III. In section IV we discuss how power counting in πN is transferred to the TPEP. Three major ingredients of the chiral potential, namely subthreshold coefficients, analytic loop integrals and subtraction of the iterated OPEP are reviewed in sections V, VI and appendix C. The full TPEP, which represents an extension of our earlier works [14,22], is derived in appendix D. This potential is simplified using relations among integrals given in appendix E and the new form is given in appendix F. The truncation of these results gives rise to our $\mathcal{O}(q^4)$ invariant amplitudes, and potential components, presented in sections VII and VIII. In section IX we compare these results with the standard heavy baryon version, using the expansions for loop integrals derived in appendix G. Conclusions are presented in section X, whereas appendices A and B deal with kinematics and loop intgrals.

II. TPEP - FORMALISM

The TPEP is obtained from the T-matrix $\mathcal{T}_{TP}$, which describes the on-shell process $N(p_1)\ N(p_2) \rightarrow N(p_1')\ N(p_2')$ and contains two intermediate pions, as represented in fig.1. In order to derive the corresponding potential, one goes to the center of mass frame and subtracts the iterated OPEP, so as to avoid double counting. The NN interaction is thus closely associated with the off-shell πN amplitude.

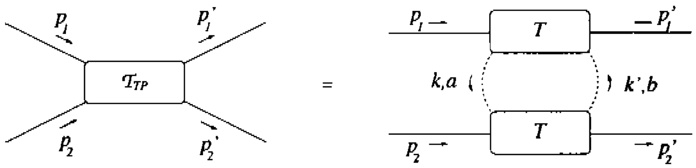

FIG. 1. Two pion exhange amplitude.

The coupling of the two-pion system to a nucleon is described by T, the amplitude for the process $\pi^a(k)N(p) \rightarrow \pi^b(k')N(p')$. It has the isospin structure

$$T_{ba} = \delta_{ab}T^+ + i\epsilon_{bac}\tau_c\, T^- \tag{2.1}$$

and the evaluation of fig.1 yields

$$\mathcal{T}_{TP} = \left[3\ \mathcal{T}^+ + 2\ \boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\ \mathcal{T}^-\right]\ , \tag{2.2}$$

with

$$\mathcal{T}^\pm = -\frac{i}{2!}\int \frac{d^4Q}{(2\pi)^4}\ \frac{[T^\pm]^{(1)}\ [T^\pm]^{(2)}}{[k^2-\mu^2]\ [k'^2-\mu^2]}\ , \tag{2.3}$$

where μ is the pion mass and the factor $1/2!$ accounts for the exchange symmetry of the intermediate pions. The integration variable is $Q = (k' + k)/2$ and we also define $q = (k' - k)$, $t = q^2$ and $\nu_i = (p_i' + p_i)\cdot Q/2m$. Our kinematical variables are fully displayed in appendix A.

For on-shell nucleons, the sub amplitudes $T^\pm$ may be written as

$$T^\pm = \bar{u}(\boldsymbol{p}')\left[A^\pm + \not{Q}\ B^\pm\right]u(\boldsymbol{p}) \tag{2.4}$$

and the functions $A^\pm$ and $B^\pm$ are determined dynamically. An alternative possibility is

$$T^\pm = \bar{u}(\boldsymbol{p}')\left[D^\pm - \frac{i}{2m}\sigma_{\mu\nu}(p'-p)^\mu Q^\nu\ B^\pm\right]u(\boldsymbol{p})\ , \tag{2.5}$$

with $D^\pm = A^\pm + \nu B^\pm$. This second form tends to be more convenient when one is interested in the chiral content of the amplitudes. The information needed about the pion-nucleon sub amplitudes $A^\pm$, $B^\pm$ and $D^\pm$ may be found in the comprehensive review by Höhler [16] and in the recent chiral analysis by Becher and Leutwyler [34].

The intermediate πN subamplitudes $A^\pm$, $B^\pm$ and $D^\pm$ depend on four independent variables, namely k^2, k'^2, ν and t. For physical processes one has $k'^2 = k^2 = \mu^2$, $\nu \geq \mu$ and $t \leq 0$. On the other hand, the conditions of integration in eq.(2.2) are such that the pions are off-shell and the main contributions come from the region $\nu \approx 0$. Physical amplitudes cannot be directly employed in the evaluation of the TPEP and must be continued analytically to the region below threshold, by means of either dispersion relations or field theory. The analytic structure of the πN amplitude plays therefore an important role in the TPEP.

The relativistic spin structure of the TPEP is obtained by using eq.(2.5) into eq.(2.3) and one has, for each isospin channel,

$$\begin{aligned}\mathcal{T} = &[\bar{u}\ u]^{(1)}[\bar{u}\ u]^{(2)}\ \mathcal{I}_{DD} - \frac{i}{2m}[\bar{u}\ u]^{(1)}[\bar{u}\ \sigma_{\mu\lambda}\ (p'-p)^\mu\ u]^{(2)}\ \mathcal{I}_{DB}^\lambda \\ &- \frac{i}{2m}[\bar{u}\ \sigma_{\mu\lambda}(p'-p)^\mu\ u]^{(1)}[\bar{u}\ u]^{(2)}\ \mathcal{I}_{BD}^\lambda \\ &- \frac{1}{4m^2}[\bar{u}\ \sigma_{\mu\lambda}(p'-p)^\mu\ u]^{(1)}[\bar{u}\ \sigma_{\nu\rho}(p'-p)^\nu\ u]^{(2)}\ \mathcal{I}_{BB}^{\lambda\rho}\ ,\end{aligned} \tag{2.6}$$

where

$$\mathcal{I}_{DD} = -i/2 \int [...] \, [D]^{(1)}[D]^{(2)} \,, \tag{2.7}$$

$$\mathcal{I}^{\lambda}_{DB} = -i/2 \int [...] \, [D]^{(1)}[Q^{\lambda} \; B]^{(2)} \,, \tag{2.8}$$

$$\mathcal{I}^{\lambda}_{BD} = -i/2 \int [...] \, [Q^{\lambda} \; B]^{(1)}[D]^{(2)} \,, \tag{2.9}$$

$$\mathcal{I}^{\lambda\rho}_{BB} = -i/2 \int [...] \, [Q^{\lambda} \; B]^{(1)}[Q^{\rho} \; B]^{(2)} \,, \tag{2.10}$$

and

$$\int [...] = \int \frac{d^4Q}{(2\pi)^4} \, \frac{1}{[k^2-\mu^2]\,[k'^2-\mu^2]} \,. \tag{2.11}$$

The Lorentz structure of the integrals $\mathcal{I}$ is realized in terms of the external quantities q, z, W and $g^{\mu\nu}$, defined in appendix A. Terms proportional to q do not contribute and we write

$$\mathcal{I}^{\lambda}_{DB} = \frac{W^{\lambda}}{2m} \, \mathcal{I}^{(w)}_{DB} + \frac{z^{\lambda}}{2m} \, \mathcal{I}^{(z)}_{DB} \,, \tag{2.12}$$

$$\mathcal{I}^{\lambda}_{BD} = \frac{W^{\lambda}}{2m} \, \mathcal{I}^{(w)}_{DB} - \frac{z^{\lambda}}{2m} \, \mathcal{I}^{(z)}_{DB} \,, \tag{2.13}$$

$$\mathcal{I}^{\lambda\rho}_{BB} = g^{\lambda\rho} \, \mathcal{I}^{(g)}_{BB} + \frac{W^{\lambda}W^{\rho}}{4m^2} \, \mathcal{I}^{(w)}_{BB} + \frac{z^{\lambda}z^{\rho}}{4m^2} \, \mathcal{I}^{(z)}_{BB} \,. \tag{2.14}$$

These expressions and the spinor identities (A20) and (A22) yield

$$\begin{aligned}
\mathcal{T} &= [\bar{u} \; u]^{(1)}[\bar{u} \; u]^{(2)} \; \left[\mathcal{I}_{DD} + \frac{q^2}{2m^2} \, \mathcal{I}^{(w)}_{DB} + \frac{q^4}{16m^4} \, \mathcal{I}^{(w)}_{BB} \right] \\
&- \frac{i}{2m} \left\{ [\bar{u} \; u]^{(1)}[\bar{u} \; \sigma_{\mu\lambda}(p'-p)^{\mu} \; u]^{(2)} - [\bar{u} \; \sigma_{\mu\lambda}(p'-p)^{\mu} \; u]^{(1)}[\bar{u} \; u]^{(2)} \right\} \frac{z^{\lambda}}{2m} \\
&\times \left[\mathcal{I}^{(w)}_{DB} + \mathcal{I}^{(z)}_{DB} + \frac{q^2}{4m^2} \, \mathcal{I}^{(w)}_{BB} \right] \\
&- \frac{1}{4m^2} [\bar{u} \; \sigma_{\mu\lambda}(p'-p)^{\mu} \; u]^{(1)}[\bar{u} \; \sigma_{\nu\rho}(p'-p)^{\nu} \; u]^{(2)} \left[g^{\lambda\rho} \, \mathcal{I}^{(g)}_{BB} + \frac{z^{\lambda}z^{\rho}}{4m^2} \left(-\mathcal{I}^{(w)}_{BB} + \mathcal{I}^{(z)}_{BB} \right) \right] \,.
\end{aligned} \tag{2.15}$$

In order to display the ordinary spin content of this amplitude, we go to the center of mass frame and use the identities (A32-A35), which allow one to rewrite $\mathcal{T}_{TP}$, without approximations, in terms of the (2×2) identity matrix and the operators[1]

$$\boldsymbol{\Omega}_{SS} = \boldsymbol{q}^2 \, \boldsymbol{\sigma}^{(1)}\!\cdot\!\boldsymbol{\sigma}^{(2)} \,, \boldsymbol{\Omega}_{T} = -\boldsymbol{q}^2 \, (3\boldsymbol{\sigma}^{(1)}\!\cdot\!\hat{\boldsymbol{q}}\boldsymbol{\sigma}^{(2)}\!\cdot\!\hat{\boldsymbol{q}} - \boldsymbol{\sigma}^{(1)}\!\cdot\!\boldsymbol{\sigma}^{(2)}) \,,$$

$$\boldsymbol{\Omega}_{LS} = i \, (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\!\cdot\!\boldsymbol{q} \times \boldsymbol{z}/4 \,, \boldsymbol{\Omega}_{Q} = \boldsymbol{\sigma}^{(1)}\!\cdot\!\boldsymbol{q}\times\boldsymbol{z} \, \boldsymbol{\sigma}^{(2)}\!\cdot\!\boldsymbol{q}\times\boldsymbol{z} \,.$$

The two-component momentum space amplitude in the CM is derived by dividing $\mathcal{T}$ by the factor $(4E^2)$, present in the relativistic normalization, and introducing back the isospin coefficients as in eq.(2.2). We then have the decomposition

$$t^{\pm}_{cm} \equiv \tau^{\pm} \, \frac{\mathcal{T}^{\pm}_{cm}}{4E^2} = t^{\pm}_{C} + \frac{\boldsymbol{\Omega}_{SS}}{m^2} \, t^{\pm}_{SS} + \frac{\boldsymbol{\Omega}_{T}}{m^2} \, t^{\pm}_{T} + \frac{\boldsymbol{\Omega}_{LS}}{m^2} \, t^{\pm}_{LS} + \frac{\boldsymbol{\Omega}_{Q}}{m^4} \, t^{\pm}_{Q} \,, \tag{2.16}$$

with $\tau^+ = 3$ and $\tau^- = 2$. The momentum space potential, denoted by $\hat{t}^{\pm}$, is obtained by subtracting the iterated OPEP from this expression, so as to avoid double counting.

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[1] We use here the notation and results from Partovi and Lomon [5], eqs.(4.26-4.28).

III. INTERMEDIATE πN AMPLITUDE

The theoretical soundness of the TPEP relies heavily on the description adopted for the intermediate πN amplitude. In this work we employ the relativistic chiral representation produced by the Bern group and collaborators [32,34,29], which incorporates the correct analytic structure. For the sake of completeness, in this section we summarize some of their results.

At low and intermediate energies the πN amplitude is given by the nucleon pole contribution, superimposed to a smooth background. Chiral symmetry is realized differently in these two sectors and it is useful to disentangle the pseudovector Born term (pv) from a remainder (R). We then write

$$T^{\pm} = T^{\pm}_{pv} + T^{\pm}_{R} \; . \tag{3.1}$$

The pv contribution involves two observables, namely the nucleon mass m and the πN coupling constant g, as prescribed by the Ward-Takahashi identity [35]. Depending on the chiral order one is working with, the calculation of these quantities may involve different numbers of loops and several coupling constants[2]. Nevertheless, at the end, all this structure must be organized in such a way as to reproduce the physical values of both m and g [36]. Following Höhler [16] and the Bern group, [29,34] in their treatments of the Born term, we use the constant g in these equations, instead of (g_A/f_π). The motivation for this choice is that the πN coupling constant is indeed the observable determined by the residue of the nucleon pole. We write

$$D^{+}_{pv} = \frac{g^2}{2m}\left(\frac{k'\cdot k}{s-m^2} + \frac{k'\cdot k}{u-m^2}\right) \to \mathcal{O}(q^2) \; , \tag{3.2}$$

$$B^{+}_{pv} = -g^2\left(\frac{1}{s-m^2} - \frac{1}{u-m^2}\right) \to \mathcal{O}(q^{-1}) \; , \tag{3.3}$$

$$D^{-}_{pv} = \frac{g^2}{2m}\left(\frac{k\cdot k'}{s-m^2} - \frac{k\cdot k'}{u-m^2} - \frac{\nu}{m}\right) \to \mathcal{O}(q) \; , \tag{3.4}$$

$$B^{-}_{pv} = -g^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2} + \frac{1}{2m^2}\right) \to \mathcal{O}(q^0) \; . \tag{3.5}$$

The arrows after the equations indicate their chiral orders, estimated by using $s-m^2 \sim W\cdot Q$ and $u-m^2 \sim -W\cdot Q$, with $W = p_1 + p_2 = p'_1 + p'_2$. When the relative sign between the s and u poles is negative, these contributions add up and we have $[1/(s-m^2) - 1/(u-m^2)] \to \mathcal{O}(q)$. On the other hand, when the relative sign is positive, the leading contributions cancell out and we obtain $[1/(s-m^2) + 1/(u-m^2)] \to \mathcal{O}(q^2)$.

In ChPT, the structure of the amplitudes $T^{\pm}_{R}$ involves both tree and loop contributions. The former can be read directly from the basic lagrangians and correspond to polynomials in ν and t, with coefficients given by the renormalized coupling constants of the theory. The calculation of the latter is more complex and results may be expressed in terms of Feynman integrals. In the description of πN processes below threshold, it is useful to approximate these contributions by polynomials, using

$$X_R = \sum x_{mn} \nu^{2m} t^n \; , \tag{3.6}$$

where X_R stands for D^{+}_{R}, B^{+}_{R}/ν, D^{-}_{R}/ν or B^{-}_{R}. The values of the coefficients x_{mn} can be determined empirically, by using dispersion relations in order to extrapolate physical scattering information to the subthreshold region [16,19]. As such, they become a rather important source of information about the coupling constants of the chiral lagrangian.

The isospin odd subthreshold coefficients include leading order contributions, which yield the predictions made by Weinberg [37] and Tomozawa [38] for πN scattering lengths, given by

$$D^{-}_{WT} = \frac{\nu}{2f_\pi^2} \to \mathcal{O}(q) \; , \tag{3.7}$$

$$B^{-}_{WT} = \frac{1}{2f_\pi^2} \to \mathcal{O}(q^0) \; . \tag{3.8}$$

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[2] For instance, up to $\mathcal{O}(q^4)$ $T^{\pm}_{pv}$ receives contributions from tree graphs of $\mathcal{L}^{(1)} \cdots \mathcal{L}^{(4)}$ and 1 loop graphs from $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$, expressed in terms of its bare coupling constants.

Sometime ago, we developed a chiral description of the TPEP based on the empirical values of the subthreshold coefficients, which could reproduce asymptotic NN data [22,23]. As we discuss in the sequence, that description has to be improved when one goes beyond $\mathcal{O}(q^3)$. In nuclear interactions, the ranges of the various processes are associated with the variable t and must be accurately described. In particular, the pion cloud of the nucleon gives rise to scalar and vector form factors [29] which correspond, in configuration space, to structures extended well beyond 1 fm [39]. On the other hand, the representation of an amplitude by means of a power series, as in eq.(3.6), amounts to a zero-range expansion, for its Fourier transform yieds only δ-functions and its derivatives. So, this kind of representation is not the best suited for describing extended objects.

FIG. 2. Long range contributions to the scalar and vector form factors.

In the work of Becher and Leutwyler [34] we can check that the only sources of NN medium range effects are their diagrams k and l, reproduced in our figure 2, which contain two pions propagating in the t-channel. Here we consider explicitly their full contributions and our amplitudes $A_R^{\pm}$ and $B_R^{\pm}$ are written as

$$D_R^+ = \left[\bar{d}_{00}^+ + d_{10}^+\nu^2 + \bar{d}_{01}^+ t\right]_{(2)} + \left[d_{20}^+\nu^4 + d_{11}^+\nu^2 t + \bar{d}_{02}^+ t^2\right]_{(3)} + D_{mr}^+(t)\ , \tag{3.9}$$

$$B_R^+ = \left[b_{00}^+\nu\right]_{(1)} + B_{mr}^+(t)\ , \tag{3.10}$$

$$D_R^- = \left[\nu/(2f_\pi^2)\right]_{(1)} + \left[\bar{d}_{00}^-\nu + d_{10}^-\nu^3 + \bar{d}_{01}^-\nu t\right]_{(3)} + D_{mr}^-(t)\ , \tag{3.11}$$

$$B_R^- = \left[1/(2f_\pi^2) + \bar{b}_{00}^-\right]_{(0)} + \left[b_{10}^-\nu^2 + \bar{b}_{01}^- t\right]_{(1)} + B_{mr}^+(t)\ . \tag{3.12}$$

In these expressions, the labels (n) outside the brackets indicate the presence of leading terms of $\mathcal{O}(q^n)$, whereas the label mr denotes the contribution from the medium range diagrams of fig.2. This decomposition implies the redefinition of some subthreshold coefficients, indicated by a bar over the appropriate symbol. Their explicit forms will be displayed in the sequence.

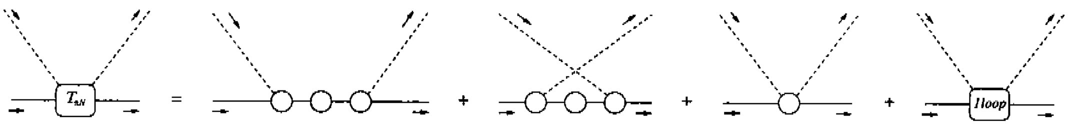

FIG. 3. Dynamical structure of the $\mathcal{O}(q^4)$ πN amplitude; the blobs represent terms coming directly from the effective lagrangians.

The dynamical content of the $\mathcal{O}(q^4)$ $T_{\pi N}$ amplitude derived in [34] is shown in fig.3 and our approximation, in fig.4. In the latter, the first two diagrams correspond to the direct and crossed PV Born amplitudes, with physical masses and coupling constants. The third one represents the contact interaction associated with the Weinberg-Tomozawa vertex whereas the next two describe the medium range effects associated with the scalar and vector form factors. Finally, the last diagram summarizes the terms within square brackets in eqs.(3.9-3.12).

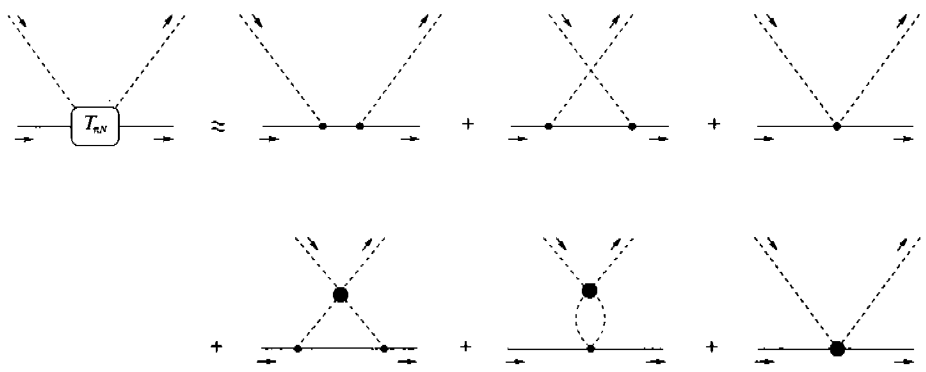

FIG. 4. Dynamical content of the approximate πN amplitude.

IV. POWER COUNTING

One begins the expansion of the TPEP to a given chiral order by recasting the explicitly covariant $\mathcal{T}_{TP}$ into the two-component form of eq.(2.16). This procedure involves no approximations and one finds, in the CM frame,

$$t_C^{\pm} = \tau^{\pm}\,\frac{m^2}{E^2}\left[\left(1+\boldsymbol{q}^2/\lambda^2\right)^2 \mathcal{I}_{DD}^{\pm} - \frac{\boldsymbol{q}^2}{2m^2}\left(1+\boldsymbol{q}^2/\lambda^2\right)\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)\mathcal{I}_{DB}^{(w)\pm} \right.$$
$$+\,\frac{\boldsymbol{q}^4}{16m^4}\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)^2 \mathcal{I}_{BB}^{(w)\pm} + \frac{\boldsymbol{q}^4}{16m^4}\left(1+4m^2\boldsymbol{z}^2/\lambda^4\right)\mathcal{I}_{BB}^{(g)\pm}$$
$$\left. -\,\frac{\boldsymbol{q}^2\boldsymbol{z}^2}{2m^2\lambda^2}\left(1+\boldsymbol{q}^2/\lambda^2\right)\mathcal{I}_{DB}^{(z)\pm} - \frac{\boldsymbol{q}^4\boldsymbol{z}^4}{16m^4\lambda^4}\,\mathcal{I}_{BB}^{(z)\pm}\right], \tag{4.1}$$

$$t_{SS}^{\pm} = \tau^{\pm}\,\frac{m^2}{E^2}\left[-\frac{1}{6}\,\mathcal{I}_{BB}^{(g)\pm}\right], \tag{4.2}$$

$$t_T^{\pm} = \tau^{\pm}\,\frac{m^2}{E^2}\left[-\frac{1}{12}\,\mathcal{I}_{BB}^{(g)\pm}\right], \tag{4.3}$$

$$t_{LS}^{\pm} = \tau^{\pm}\,\frac{m^2}{E^2}\left[-\frac{4m^2}{\lambda^2}\left(1+\boldsymbol{q}^2/\lambda^2\right)\mathcal{I}_{DD}^{\pm} + \left(1+2\boldsymbol{q}^2/\lambda^2\right)\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)\mathcal{I}_{DB}^{(w)\pm}\right.$$
$$-\,\frac{\boldsymbol{q}^2}{4m^2}\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)^2\mathcal{I}_{BB}^{(w)\pm} - \frac{\boldsymbol{q}^2}{4m^2}\left(1+4m^2/\lambda^2+4m^2\boldsymbol{z}^2/\lambda^4\right)\mathcal{I}_{BB}^{(g)\pm}$$
$$\left. +\,\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2+2\boldsymbol{q}^2\boldsymbol{z}^2/\lambda^4\right)\mathcal{I}_{DB}^{(z)\pm} + \frac{\boldsymbol{q}^2\boldsymbol{z}^2}{4m^2\lambda^2}\left(1+\boldsymbol{z}^2/\lambda^2\right)\mathcal{I}_{BB}^{(z)\pm}\right], \tag{4.4}$$

$$t_Q^{\pm} = \tau^{\pm}\,\frac{m^2}{E^2}\left[-\frac{m^4}{\lambda^4}\,\mathcal{I}_{DD}^{\pm} + \frac{m^2}{2\lambda^2}\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)\mathcal{I}_{DB}^{(w)\pm}\right.$$
$$-\,\frac{1}{16}\left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2\right)^2\mathcal{I}_{BB}^{(w)\pm} - \frac{1}{16}\left(1+8m^2/\lambda^2+4m^2\boldsymbol{z}^2/\lambda^4\right)\mathcal{I}_{BB}^{(g)\pm}$$
$$\left. +\,\frac{m^2}{2\lambda^2}\left(1+\boldsymbol{z}^2/\lambda^2\right)\mathcal{I}_{DB}^{(z)\pm} + \frac{1}{16}\left(1+\boldsymbol{z}^2/\lambda^2\right)^2\mathcal{I}_{BB}^{(z)\pm}\right], \tag{4.5}$$

with $\boldsymbol{q}=\boldsymbol{p}'-\boldsymbol{p}$, $\boldsymbol{z}=\boldsymbol{p}'+\boldsymbol{p}$ and $\lambda^2=4m(E+m)$.

The potential to order $\mathcal{O}(q^n)$ is determined by $t_C^{\pm}\to\mathcal{O}(q^n)$, $\{t_{SS}^{\pm}, t_T^{\pm}, t_{LS}^{\pm}\}\to\mathcal{O}(q^{n-2})$ and $t_Q^{\pm}\to\mathcal{O}(q^{n-4})$. This means that one needs $\mathcal{I}_{DD}^{\pm}\to\mathcal{O}(q^n)$, $\{\mathcal{I}_{DB}^{(w)\pm}, \mathcal{I}_{DB}^{(z)\pm}, \mathcal{I}_{BB}^{(g)\pm}\}\to\mathcal{O}(q^{n-2})$ and $\{\mathcal{I}_{BB}^{(w)\pm}, \mathcal{I}_{BB}^{(z)\pm}\}\to\mathcal{O}(q^{n-4})$. We now discuss how the chiral powers in these functions are related with those in the basic πN amplitude. This relationship involves a subtlety, associated with the fact that D_{pv}^{+} and B_{pv}^{-} contain chiral cancellations.

A generic subamplitude $\mathcal{I}^{\pm}_{XY}$ is given by the product of the corresponding πN contributions and we have

$$\mathcal{I}^{\pm}_{XY} = \int [\cdots] \left\{ \left[X^{\pm}_{pv}\right]^{(1)} \left[Y^{\pm}_{pv}\right]^{(2)} + \left[X^{\pm}_{pv}\right]^{(1)} \left[Y^{\pm}_{R}\right]^{(2)} + \left[X^{\pm}_{R}\right]^{(1)} \left[Y^{\pm}_{pv}\right]^{(2)} + \left[X^{\pm}_{R}\right]^{(1)} \left[Y^{\pm}_{R}\right]^{(2)} \right\} . \tag{4.6}$$

The loop integral and the two pion propagators, as given by eq.(2.11), do not interfere with the counting of powers, since $\int[\cdots] \to \mathcal{O}(q^0)$. The loop integration is symmetric under the operation $Q \to -Q$, which gives rise to the exchange $s \leftrightarrow u$ in the Born terms. In the case of $[X^{\pm}_{pv}]^{(1)}\ [Y^{\pm}_{pv}]^{(2)}$, one is allowed to use

$$\left(\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right)^{(i)} \left(\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right)^{(j)} \to 2\left(\frac{1}{s-m^2}\right)^{(i)} \left(\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right)^{(j)} \tag{4.7}$$

within the integrand. For the specific components this yields

$$\begin{aligned}
\left[D^{+}_{pv}\right]^{(i)} \left[D^{+}_{pv}\right]^{(j)} &\to \mathcal{O}(q^3), \qquad & \left[D^{-}_{pv}\right]^{(i)} \left[D^{-}_{pv}\right]^{(j)} &\to \mathcal{O}(q^2), \\
\left[D^{+}_{pv}\right]^{(i)} \left[QB^{+}_{pv}\right]^{(j)} &\to \mathcal{O}(q), \qquad & \left[D^{-}_{pv}\right]^{(i)} \left[QB^{-}_{pv}\right]^{(j)} &\to \mathcal{O}(q), \\
\left[QB^{+}_{pv}\right]^{(i)} \left[QB^{+}_{pv}\right]^{(j)} &\to \mathcal{O}(q^0), \qquad & \left[QB^{-}_{pv}\right]^{(i)} \left[QB^{-}_{pv}\right]^{(j)} &\to \mathcal{O}(q).
\end{aligned}$$

These results show that, inside the integral, D^{+}_{pv} and B^{-}_{pv} cannot be always counted as $\mathcal{O}(q^2)$ and $\mathcal{O}(q^{-1})$ respectively. For the products $[X^{\pm}_{pv}]^{(i)}\ [Y^{\pm}_{R}]^{(j)}$ and $[X^{\pm}_{R}]^{(i)}\ [Y^{\pm}_{pv}]^{(j)}$, one uses

$$\left(\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right)^{(i)} \to 2\left(\frac{1}{s-m^2}\right)^{(i)} \tag{4.8}$$

and has $D^{\pm}_{pv} \to \mathcal{O}(q)$ and $B^{\pm}_{pv} \to \mathcal{O}(q^{-1})$. Assuming $[X^{\pm}_{R}]^{(i)}, [Y^{\pm}_{R}]^{(j)} \to \mathcal{O}(q^r)$, one gets

$$\begin{aligned}
\left[D^{\pm}_{pv}\right]^{(i)} \left[D^{\pm}_{R}\right]^{(j)} &\to \mathcal{O}(q^{1+r}), \qquad & \left[D^{\pm}_{pv}\right]^{(i)} \left[QB^{\pm}_{R}\right]^{(j)} &\to \mathcal{O}(q^{2+r}), \\
\left[D^{\pm}_{R}\right]^{(i)} \left[QB^{\pm}_{pv}\right]^{(j)} &\to \mathcal{O}(q^{r}), \qquad & \left[QB^{\pm}_{pv}\right]^{(i)} \left[QB^{\pm}_{R}\right]^{(j)} &\to \mathcal{O}(q^{1+r}).
\end{aligned}$$

Finally, in the case of $[X^{\pm}_{R}]^{(i)}[Y^{\pm}_{R}]^{(j)}$, one just adds the corresponding powers.

In this work we consider the expansion of the potential to $\mathcal{O}(q^4)$ and need $\mathcal{I}^{\pm}_{DD} \to \mathcal{O}(q^4)$, $\{\mathcal{I}^{(w)\pm}_{DB}, \mathcal{I}^{(z)\pm}_{DB}, \mathcal{I}^{(g)\pm}_{BB}\} \to \mathcal{O}(q^2)$, and $\{\mathcal{I}^{(w)\pm}_{BB}, \mathcal{I}^{(z)\pm}_{BB}\} \to \mathcal{O}(q^0)$. This means that, in the intermediate πN amplitude, we must consider $D^{\pm}_{R}$ to $\mathcal{O}(q^3)$ and $B^{\pm}_{R}$ to $\mathcal{O}(q)$.

V. SUBTHRESHOLD COEFFICIENTS

The polynomial parts of the amplitudes $T^{\pm}_{R}$ to order $\mathcal{O}(q^3)$, as given by eqs.(3.7-3.10), are determined by the subthreshold coefficients of ref. [34], which we reproduce below

$$d^{+}_{00} = -\frac{2\,(2c_1 - c_3)\,\mu^2}{f_\pi^2} + \frac{8\,g_A^4\,\mu^3}{64\,\pi\,f_\pi^4} + \left[\frac{3\,g_A^2\,\mu^3}{64\,\pi\,f_\pi^4}\right]_{mr} , \tag{5.1}$$

$$d^{+}_{10} = \frac{2\,c_2}{f_\pi^2} - \frac{(4 + 5\,g_A^4)\,\mu}{32\,\pi\,f_\pi^4} , \tag{5.2}$$

$$d^{+}_{01} = -\frac{c_3}{f_\pi^2} - \frac{48\,g_A^4\,\mu}{768\,\pi\,f_\pi^4} - \left[\frac{77\,g_A^2\,\mu}{768\,\pi\,f_\pi^4}\right]_{mr} , \tag{5.3}$$

$$d^{+}_{20} = \frac{12 + 5\,g_A^4}{192\,\pi\,f_\pi^4\,\mu} , \tag{5.4}$$

$$d^{+}_{11} = \frac{g_A^4}{64\,\pi\,f_\pi^4\,\mu} , \tag{5.5}$$

$$d^{+}_{02} = \left[\frac{193\,g_A^2}{15360\,\pi\,f_\pi^4\,\mu}\right]_{mr} , \tag{5.6}$$

$$b^{+}_{00} = \frac{4\,m\,(\tilde{d}_{14} - \tilde{d}_{15})}{f_\pi^2} - \frac{g_A^4\,m}{8\,\pi^2\,f_\pi^4} , \tag{5.7}$$

$$d^{-}_{00} = \left[\frac{1}{2\,f_\pi^2}\right]_{WT} + \frac{4\,(\tilde{d}_1 + \tilde{d}_2 + 2\,\tilde{d}_5)\,\mu^2}{f_\pi^2} + \frac{g_A^2\,(-3 + g_A^2)\,\mu^2}{48\,\pi^2\,f_\pi^4} + \left[\frac{3\,g_A^2\,\mu^2}{48\,\pi^2\,f_\pi^4}\right]_{mr} , \tag{5.8}$$

$$d_{10}^- = \frac{4\,\tilde{d}_3}{f_\pi^2} - \frac{15 + 7\,g_A^4}{240\,\pi^2\,f_\pi^4}\,, \tag{5.9}$$

$$d_{01}^- = -\frac{2\,(\tilde{d}_1 + \tilde{d}_2)}{f_\pi^2} - \frac{2\,g_A^4}{192\,\pi^2\,f_\pi^4} - \left[\frac{1 + 7\,g_A^2}{192\,\pi^2\,f_\pi^4}\right]_{mr}\,, \tag{5.10}$$

$$b_{00}^- = \left[\frac{1}{2\,f_\pi^2}\right]_{WT} + \frac{2\,c_4\,m}{f_\pi^2} - \frac{g_A^4\,m\,\mu}{8\,\pi\,f_\pi^4} - \left[\frac{g_A^2\,m\,\mu}{8\,\pi\,f_\pi^4}\right]_{mr}\,, \tag{5.11}$$

$$b_{10}^- = \frac{g_A^4\,m}{32\,\pi\,f_\pi^4\,\mu}\,, \tag{5.12}$$

$$b_{01}^- = \left[\frac{g_A^2\,m}{96\,\pi\,f_\pi^4\,\mu}\right]_{mr}\,, \tag{5.13}$$

where the parameters c_i and $\tilde{d}_i$ are the usual renormalized coupling constants of the chiral lagrangians of order 2 and 3 respectively [33]. The terms within square brackets labelled (mr) in some of these results are due to the medium range diagrams shown in fig.2 and must be neglected[3], because we include their contributions in $D_{mr}^{\pm}$ and $B_{mr}^{\pm}$. The terms bearing the (WT) label must also be excluded, for they were explicitly considered in eqs.(3.9-3.12). This corresponds to the redefinition mentioned at the end of section III.

TABLE I. experimental values for the subthreshold coefficients and medium range (mr) contributions in μ^{-n} units; experimental results are taken from ref. [16].

	d_{00}^+	d_{10}^+	d_{01}^+	d_{20}^+	d_{11}^+	d_{02}^+
exp	-1.46±0.10	1.12±0.02	1.14±0.02	0.200±0.005	0.17±0.01	0.036±0.003
mr	0.12	-	-0.25	-	-	0.032
	b_{00}^+					
exp	-3.54±0.06					
	d_{00}^-	d_{10}^-	d_{01}^-			
exp	1.53±0.02	-0.167±0.005	-0.134±0.005			
$WT + mr$	1.18	-	-0.032			
	b_{00}^-	b_{10}^-	b_{01}^-			
exp	10.36±0.10	1.08 ±0.05	0.24±0.01			
$WT + mr$	-0.99	-	0.18			

[3] In ref. [34], the contribution of the triangle diagram to d_{00}^+ includes both short and medium range terms and only the latter must be excluded.

The values of the subthreshold coefficients are determined from πN scattering data and, in a chiral expansion to $\mathcal{O}(q^3)$, they are used to fix the otherwise indetermined parameters c_i and $\tilde{d}_i$. In our formulation of the TPEP, we bypass the use of these unknown parameters, for the redefined subthreshold coefficients are already the dynamical ingredients that determine the strength of the various interactions. This allows the potential to be expressed directly in terms of observable quantities.

In table I we show the experimental values of the subtheshold coefficients determined in ref. [16] and the sum of (WT) and (mr) contributions. The redefined values are obtained by just subtracting the latter from the former[4]. It is worth noting that the values of $\bar{d}_{02}^+$ and $\bar{b}_{01}^-$ are compatible with zero.

When writing the results for the TPEP, it is very convenient to display explicitly the chiral scales of the various contributions. With this purpose in mind, we will employ the dimensionless subthreshold constants defined in table II.

TABLE II. dimensionless subthreshold coefficients.

	δ_{00}^+	δ_{10}^+	δ_{01}^+	β_{00}^+
definition	$mf_\pi^2\, \bar{d}_{00}^+/\mu^2$	$mf_\pi^2\, \bar{d}_{10}^+$	$mf_\pi^2\, \bar{d}_{01}^+$	$mf_\pi^2\, \bar{b}_{00}^+$
value	-4.72	3.34	4.15	-10.57
	δ_{00}^-	δ_{10}^-	δ_{01}^-	β_{00}^-
definition	$m^2 f_\pi^2\, \bar{d}_{00}^-/\mu^2$	$m^2 f_\pi^2\, \bar{d}_{10}^-$	$m^2 f_\pi^2\, \bar{d}_{01}^-$	$f_\pi^2\, \bar{b}_{00}^-$
value	7.02	-3.35	-2.05	5.04

[4]We use $g_A = 1.25$, $f_\pi = 93$ MeV, $\mu = 139.57$ MeV and $m = 938.28$ MeV.

VI. BASIC INTEGRALS

The covariantly expanded TPEP is expressed in terms of a few functions, associated with loop integrals. Before displaying our results, we introduce the functions Π_ℓ, Π_t, $\Pi_\times$, Π_b and $\tilde{\Pi}_b$, that occur in this expansion. Their mathematical forms are discussed in detail in appendices B and C where, for technical reasons, they are called $\Pi_{cc}^{(000)}$, $\Pi_{sc}^{(000)}$, $\Pi_{ss}^{(000)}$, $\Pi_{reg}^{(000)}$ and $\Pi_{reg}^{(010)}$ respectively.

The function Π_ℓ represents the *bubble* diagram and is given by

$$I_{cc} = \int \frac{d^4Q}{(2\pi)^4} \; \frac{1}{[(Q-q/2)^2-\mu^2][(Q+q/2)^2-\mu^2]} = \frac{i}{(4\pi)^2}\,\Pi_\ell \,. \tag{6.1}$$

This integral can be performed analitically[5] and its regular part may be written as

$$\Pi_\ell = -2\,\frac{\sqrt{1-t/4\mu^2}}{\sqrt{-t/4\mu^2}}\;\ln\left(\sqrt{1-t/4\mu^2}+\sqrt{-t/4\mu^2}\right) . \tag{6.2}$$

The function Π_t, associated with the *triangle* diagram, is expressed by

$$I_{sc} = \int \frac{d^4Q}{(2\pi)^4} \; \frac{1}{[(Q-q/2)^2-\mu^2][(Q+q/2)^2-\mu^2]} \; \frac{2m\mu}{[Q^2+Q\cdot(W+z)-t/4]} = \frac{i}{(4\pi)^2}\,\Pi_t \tag{6.3}$$

and contributes to the scalar form factor of the nucleon. Its properties were discussed in [32], where the following spectral representation[6] can be found

$$\Pi_t = -2m\mu \int_{4\mu^2}^{\infty} dt' \; \frac{1}{(t'-t)}\; G(t') \,, \tag{6.4}$$

with

$$G(m;t') = \frac{2}{\sqrt{t'(4m^2-t')}}\tan^{-1}\left[\frac{\sqrt{(t'-4\mu^2)(4m^2-t')}}{t'-2\mu^2}\right] , \tag{6.5}$$

which has the correct analytic structure around the point $t'=4\mu^2$. Keeping terms up to $\mathcal{O}(q^2)$, we have

$$G(m;t') = \left(1+\frac{t}{8m^2}\right)\frac{1}{m\sqrt{t'}}\tan^{-1}\left[\frac{2m\sqrt{t'-4\mu^2}}{t'-2\mu^2}\right] . \tag{6.6}$$

In BL one also learns that it is possible to write accurately

$$G(m;t') \cong \frac{1}{m\sqrt{t'}}\left\{\left[\frac{\pi}{2}\right]_{HB} - \frac{\mu}{2m}\left[\frac{(t'-2\mu^2)}{\mu\sqrt{t'-4\mu^2}}\right]_{N\!L} + \frac{\mu^2}{4m^2}\,\frac{\pi}{2}\,\frac{t}{2\mu^2} \right.$$
$$\left. + \frac{\mu^2}{4m^2}\left[\frac{2m\sqrt{t'}}{\mu\sqrt{t'-4\mu^2}} - \frac{2m^2\sqrt{t'}}{\mu^3}\tan^{-1}\frac{m}{\sqrt{t'-4\mu^2}}\right]_{th}\right\} , \tag{6.7}$$

The first two terms in this expression correspond to the standard leading heavy-baryon contribution (HB) and next-to-leading correction ($N\!L$), whereas the last one (th) implements the correct analytic behaviour around $t'=4\mu^2$. The use of this result in eq.(6.4) suggests the decomposition

$$\Pi_t = \Pi_a + \frac{\mu}{2m}\,\Pi_t^{N\!L} + \frac{\mu^2}{4m^2}\left[\frac{t}{2\mu^2}\,\Pi_a + \Pi_t^{th}\right] . \tag{6.8}$$

It is important to note that this decomposition represents the ***definition*** of the function Π_t^{th} and *not* a heavy baryon expansion in powers of μ/m. The first two terms are given by[7]

---

[5] The function Π_ℓ is related to the $L(q)$ used in ref. [24] by $\Pi_\ell = -2L(q)$ and to the $J(t)$ of [34] by $\Pi_\ell = (4\pi)^2\,J - 1$.
[6] BL denote it by $\gamma(t)$ and, in our notation, one has $\Pi_t = -2m\mu(4\pi)^2\gamma(t)$.
[7] The function Π_a is related to the $A(q)$ of ref. [24] by $\Pi_a = -4\pi\mu A(q)$.

$$\Pi_a = -\frac{\pi}{\sqrt{-t/4\mu^2}} \tan^{-1}\sqrt{-t/4\mu^2}\;, \tag{6.9}$$

$$\Pi_t^{NL} = (1-t/2\mu^2)\, \Pi'_\ell\;, \tag{6.10}$$

with $\Pi' = \mu\, (d\Pi/d\mu)$.

The function Π_t^{th} may be evaluated either numerically or by means of the approximate expression of BL

$$\Pi_t^{th} \simeq \frac{2\pi m^2}{\mu^2}\left[-\frac{\mu}{m\sqrt{1-t/4\mu^2}} + 2\ln\left(1+\frac{\mu}{2m\sqrt{1-t/4\mu^2}}\right)\right] = -\frac{\pi/2}{1-t/4\mu^2}\;. \tag{6.11}$$

This function is the signature of the covariant formalism, since it is not present in the ordinary heavy baryon series. In that case, the last term of eq.(6.8) would become

$$[\cdots]_{th} \rightarrow \left[\frac{\mu}{6m}\left(\frac{t'-2\mu^2}{\mu\sqrt{t'-4\mu^2}}\right)^3\right]_{NNL} \tag{6.12}$$

and Π_t^{th} would be replaced with

$$\Pi_t^{NNL} = \frac{\mu}{3m}\left\{\left[-2-t/\mu^2+\frac{3}{1-t/4\mu^2}-\frac{1/2}{(1-t/4\mu^2)^2}\right]\Pi_\ell \right.$$

$$\left. +\frac{1}{1-t/4\mu^2} - \frac{1}{1-t/4\mu^2}\sqrt{\frac{-t/4\mu^2}{1-t/4\mu^2}}\;\lim_{u\rightarrow 0}\frac{1}{u}\right\}\;. \tag{6.13}$$

This result shows that Π_t^{NNL} is different from Π_t^{th}. The former contains an extra factor (μ/m) and a non-polynomial singular term, due to the behaviour of eq.(6.12) at threshold.

The functions $\Pi_\times$ Π_b, and $\tilde{\Pi}_b$ are associated with *crossed box* and *box* diagrams and their relativistic expansions, derived in appendix G, read

$$\Pi_\times = -\Pi'_\ell - \left[\frac{\mu}{m}\right]\frac{\pi/2}{(1-t/4\mu^2)} - \left[\frac{\mu}{m}\right]^2\frac{1}{4}\;\left[(1-t/2\mu^2)^2\left(2\,\Pi'_\ell-\Pi''_\ell\right)+(2z^2/3\mu^2)\,\Pi'_\ell\right]+\cdots \tag{6.14}$$

$$\Pi_b = \Pi_\times + \left[\frac{\mu}{m}\right]\frac{\pi/4}{(1-t/4\mu^2)} + \left[\frac{\mu}{m}\right]^2\frac{1}{6}\;\left[(1-t/2\mu^2)^2\left(2\,\Pi'_\ell-\Pi''_\ell\right)\right]+\cdots \tag{6.15}$$

$$\tilde{\Pi}_b = -\frac{1}{2}\,\Pi_a - \left[\frac{\mu}{m}\right]\frac{1}{3}\,(1-t/2\mu^2)\,\Pi'_\ell - \left[\frac{\mu}{m}\right]^2\frac{3}{16}\,\Pi_t^{th}+\cdots\;. \tag{6.16}$$

In the expansion of the potential, the following results are useful

$$\Pi'_\ell = 2+\Pi_\ell/(1-t/4\mu^2)\;, \tag{6.17}$$

$$\Pi''_\ell = 2/(1-t/4\mu^2) + \left[2/(1-t/4\mu^2)-1/(1-t/4\mu^2)^2\right]\Pi_\ell\;, \tag{6.18}$$

$$\Pi'_a = \Pi_a - \pi\, t/(1-t/4\mu^2)\;. \tag{6.19}$$

VII. DYNAMICS

The chiral two-pion exchange potential is determined by the processes depicted in fig.5, derived from the basic πN subamplitude and organized into three different families. The first one corresponds to the minimal realization of chiral symmetry [14], includes the subtraction of the iterated OPEP and involves only pion-nucleon interactions with a single loop, associated with the constants g and f_π. The same constants also determine the two-loop processes of the second family. The last family includes chiral corrections associated with subthreshold coefficients, representing either higher order processes or other degrees of freedom.

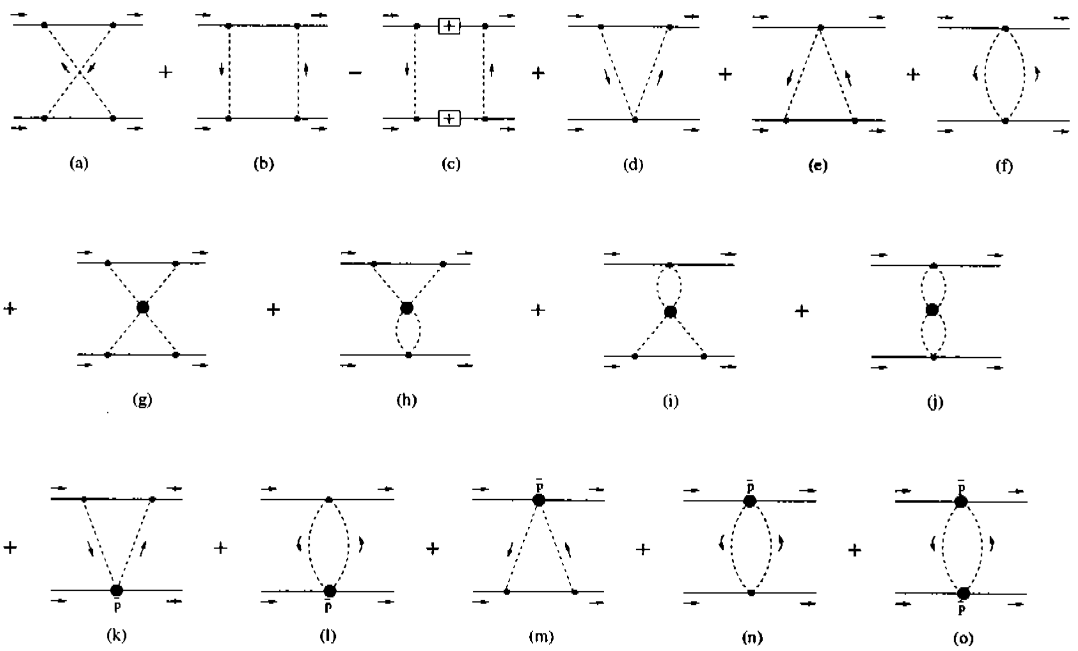

FIG. 5. Dynamical structure of the TPEP. The first two diagrams correspond to the products of Born πN amplitudes, the third one represents the iteration of the OPEP whereas the next three involve contact interactions associated with the Weinberg-Tomozawa vertex; the diagrams on the second line describe medium range effects associated with scalar and vector form factors; the remaining interactions are *triangles* and *bubbles* involving subthreshold coefficients.

The first two diagrams of fig.5, known respectively as *crossed box* and *box*, come from the products of the πN PV Born amplitudes, given by eqs.(3.2-3.5) and involve the propagations of two pions and two nucleons. The third one represents the iteration of the OPEP and gives rise to an amplitude denoted by $\mathcal{T}_{it}$. This contribution is derived after the work of Partovi and Lomon [5] and discussed in detail in appendix C. The remaining interactions corespond to *triangle* and *bubble* diagrams, which contain a single or no nucleon propagators, besides those of two pions.

The construction of the TPEP begins with the determination of the relativistic profile functions, eqs.(2.7-2.10), using the πN subamplitudes $D^{\pm}$ and $B^{\pm}$ discussed in section III. Resuts are then expressed terms of the one-loop Feynman integrals presented in appendices B and C, that may involve two, three or four propagators. The evaluation and manipulation of these integrals represent an important aspect of the present work and it is worth discussing the notation employed.

Momentum space integrals are denoted by Π and labelled in such a way as to recall their dynamical origins. We use lower labels, corresponding to nucleons 1 and 2, with the following meanings: $c \rightarrow$ contact interaction, $s \rightarrow$ s-channel nucleon propagation and $u \rightarrow$ u-channel nucleon propagation. This means that functions carrying the subscripts (cc), (sc), (ss) and (us) correspond, respectively, to *bubble, triangle, crossed box* and *box* diagrams. The last class of integrals includes the OPEP cut, that needs to be subtracted. This subtraction is implemented by replacing the (us) integrals by regular ones, represented by the subscript (reg). Upper labels, on the other hand, indicate the rank of the integral in the external kinematical variables q, z and W. For instance, the rank 2 *crossed box* integral is written as

$$\int \frac{d^4Q}{(2\pi)^2} \left(\frac{Q^\mu Q^\nu}{\mu^2}\right) \frac{2m\mu}{s_1 - m^2} \frac{2m\mu}{s_2 - m^2}$$

$$= \frac{i}{(4\pi)^2} \left[\frac{q^\mu q^\nu}{\mu^2} \Pi_{ss}^{(200)} + \frac{z^\mu z^\nu}{4m^2} \Pi_{ss}^{(020)} + \frac{W^\mu W^\nu}{4m^2} \Pi_{ss}^{(002)} + g^{\mu\nu} \bar{\Pi}_{ss}^{(000)}\right] .$$

All integrals are dimensionless and include suitable powers of pion and nucleon masses, so as to make them relatively stable upon wide variations of the latter. We have studied these integrals numerically and, typically, they change by 30% when one moves the nucleon mass from its empirical value to infinity. This feature is rather useful in discussing chiral scales and heavy baryon limits. At present the infrared regularization techniques are still being developed for the case of two nucleon system [40] and we have used dimensional regularization whenever appropriate.

The final expressions for the potential are written in terms of the axial constant g_A, related to the πN coupling constant by $g = (1+\Delta_{GT})\; g_A m/f_\pi$ where Δ_{GT}, the Goldberger-Treiman discrepancy[8], is proportional to μ^2.

The direct reading of the Feynman diagrams of fig.5 gives rise to our **full results** for the relativistic profile functions, displayed in appendix D. These are the functions which the chiral expansion must converge to and hence they allow one to assess the series directly. On the other hand, they do not exhibit explicitly the chiral scales of the various components of the potential, since their net values are the outcome of several cancellations.

In order to display these scales, in appendix E we derive several relations among integrals, which are used to transform the full results of appendix D into the forms listed in appendix F. The relations given in appendix E are, in principle, exact, provided one keeps *heavy* integrals, that contain a single or no pion propagators. In fact we neglect those *heavy* contributions, because they are short ranged[9]. The importance of this approximation was checked by comparing numerically the Fourier transforms of the various amplitudes of appendices D and F. In all cases, agreement is much better than 1% for distances larger than 1 fm, except for $\mathcal{I}_{DD}^{+}$, where the difference is 4% at 1.5 fm and falls below 1% beyond 2.5 fm. This has very little influence over the full potential.

The truncation of the expressions of appendix F to the orders in q discussed at the end of section IV lead to the following results for the profile functions:

$$\bullet\, \mathcal{I}_{DD}^{+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]^2 \left\{ \frac{g_A^4}{16}\,(1-t/2\mu^2)^2\,(\Pi_\times - \Pi_b) + \left[\frac{\mu}{m}\right]\frac{g_A^2}{8}\,(1-t/2\mu^2)\left[-g_A^2\,\Pi_a + 4\left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\,t/\mu^2\right)\Pi_t\right] + \left[\frac{\mu}{m}\right]^2\left[-\frac{g_A^2}{4}\,\delta_{10}^{+}\,(1-t/2\mu^2)^2 \right.\right.$$
$$\left. + \frac{1}{2}\left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\,t/\mu^2 + \frac{1}{3}\,\delta_{10}^{+}\,(1-t/4\mu^2)\right)^2 + \frac{2}{45}\,(\delta_{10}^{+})^2\,(1-t/4\mu^2)^2\right]\Pi_\ell$$
$$\left. - \left[\frac{\mu}{m}\right]^2 \frac{m^2}{256\,\pi^2 f_\pi^2}\, g_A^4\,(1-2t/\mu^2)\left[(1-t/2\mu^2)\,\Pi_t - 2\pi\right]^2 \right\}, \tag{7.1}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]\left\{-\frac{g_A^4}{8}\,(1-t/2\mu^2)\,\Pi_t + \left[\frac{\mu}{m}\right]\left[\frac{g_A^4}{16}\,(1-t/2\mu^2)^2\,\Pi_\times \right.\right.$$
$$\left.\left. -\frac{g_A^2}{2}\left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\,t/\mu^2 + \frac{1}{3}\,\delta_{10}^{+}\,(1-t/4\mu^2)\right)\Pi_\ell\right]\right\}, \tag{7.2}$$

$$\bullet\, \mathcal{I}_{DB}^{(z)+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]\left\{\frac{g_A^4}{8}\left[(1-t/2\mu^2)\,\tilde{\Pi}_b - (3/2-5t/8\mu^2)\,\Pi_a\right]\right.$$
$$\left. + \left[\frac{\mu}{m}\right]\frac{g_A^2}{2}\left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\,t/\mu^2 - \frac{1}{3}\,\delta_{10}^{+}\,(1-t/4\mu^2)\right)\Pi_\ell\right\}, \tag{7.3}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\left\{\frac{g_A^4}{4}\,(1-t/4\mu^2)\,(\Pi_\times + \Pi_b) + \left[\frac{\mu}{m}\right]\frac{g_A^4}{8}\left[(1-t/2\mu^2)\left(\Pi_t - \tilde{\Pi}_b\right)\right.\right.$$
$$\left.\left. +(1-t/4\mu^2)\,\Pi_a\right] - \left[\frac{\mu}{m}\right]^2 g_A^2\left[\frac{g_A^2}{16}\,(1-t/2\mu^2)^2\,\Pi_\times + \frac{1}{3}\,\beta_{00}^{+}\,(1-t/4\mu^2)\,\Pi_\ell\right]\right\}, \tag{7.4}$$

$$\bullet\, \mathcal{I}_{BB}^{(w)+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\, g_A^4\,\Pi_\ell\,, \tag{7.5}$$

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[8]The G-T discrepancy may be written [34] as $\Delta_{GT} = -2d_{18}\mu^2/g + \mathcal{O}(q^4)$.

[9]It would be very easy to keep those terms, but this would produce longer equations.

$$\bullet\, \mathcal{I}_{BB}^{(z)+} = \frac{m^2}{16\,\pi^2 f_\pi^4}\,\frac{g_A^4}{3}\,\Pi_\ell\,, \tag{7.6}$$

and

$$\bullet\, \mathcal{I}_{DD}^{-} = \frac{m^2}{16\,\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]^2\Bigg\{\frac{g_A^4}{16}\,(1-t/2\mu^2)^2\,(\Pi_\times+\Pi_b) - \frac{g_A^2}{4}\,(g_A^2-1)\,(1-t/2\mu^2)\,\Pi_\ell$$

$$+\frac{1}{24}\,(g_A^2-1)^2(1-t/4\mu^2)\,\Pi_\ell + \left[\frac{\mu}{m}\right]\frac{g_A^2}{8}\,(1-t/2\mu^2)\left[g_A^2\,\Pi_a + (g_A^2-1)\,(1-t/2\mu^2)\,\Pi_t\right]$$

$$+\left[\frac{\mu}{m}\right]^2\Bigg[\frac{g_A^2}{2}\,(1-t/2\mu^2)\left((g_A^2-1)\,z^2/4\mu^2 + \bar{\delta}_{00}^- + \bar{\delta}_{01}^-\,t/\mu^2 + \frac{1}{3}\,\delta_{10}^-\,(1-t/4\mu^2)\right)$$

$$-\frac{(g_A^2-1)}{6}\,(1-t/4\mu^2)\left((g_A^2-1)\,(t/16\mu^2+z^2/8\mu^2) + \bar{\delta}_{00}^- + \bar{\delta}_{01}^-\,t/\mu^2 + \frac{3}{5}\,\delta_{10}^-\,(1-t/4\mu^2)\right)\Bigg]\,\Pi_\ell$$

$$-\left[\frac{\mu}{m}\right]^2\frac{m^2}{64\,\pi^2 f_\pi^2}\Big[-g_A^2\left((1-t/2\mu^2)\,\Pi_\ell + 1 - t/3\mu^2\right)$$

$$+\frac{1}{3}\,(g_A^2-1)\left((1-t/4\mu^2)\,\Pi_\ell + 2 - t/4\mu^2\right)\Big]^2$$

$$+\left[\frac{\mu}{m}\right]^2\frac{m^2}{64\,\pi^2 f_\pi^2}(z^2/4\mu^2)\,g_A^4\left[(1-t/4\mu^2)\Pi_t - \pi - t\pi/12\mu^2\right]^2$$

$$+\Delta_{GT}\Bigg[\left(\frac{1}{6}(1-t/4\mu^2)\,g_A^2(g_A^2-1) - (1-t/2\mu^2)(g_A^2-1/2)\right)\Pi_\ell$$

$$+\frac{g_A^4}{4}\,(1-t/2\mu^2)^2\,\left(\Pi_\times + \Pi_b\right)\Bigg]\Bigg\}\,, \tag{7.7}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)-} = \frac{m^2}{16\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]\Bigg\{\left[-\frac{g_A^4}{8}\,(1-t/2\mu^2)\,\Pi_t\right]$$

$$+\left[\frac{\mu}{m}\right]\left[\frac{1}{24}\,(g_A^2-1)(g_A^2-1-2\bar{\beta}_{00}^-)(1-t/4\mu^2) - \frac{g_A^2}{4}\,(g_A^2-1-\bar{\beta}_{00}^-)\,(1-t/2\mu^2)\right]\Pi_\ell$$

$$+\left[\frac{\mu}{m}\right]\left[\frac{g_A^4}{16}\,(1-t/2\mu^2)^2\Pi_\times\right]\Bigg\}\,, \tag{7.8}$$

$$\bullet\, \mathcal{I}_{DB}^{(z)-} = \frac{m^2}{16\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]\Bigg\{\Bigg[\frac{g_A^2}{4}\,(g_A^2-1-\bar{\beta}_{00}^-)\,(1-t/4\mu^2)\,\Pi_t - \frac{g_A^4}{8}\,\Big((1-t/2\mu^2)\,\tilde{\Pi}_b -$$

$$-(3/2-5t/8\mu^2)\,\Pi_a\Big)\Bigg] + \left[\frac{\mu}{m}\right]\Bigg[\frac{1}{24}\,(g_A^2-1)(g_A^2-1-2\bar{\beta}_{00}^-)\,(1-t/4\mu^2)$$

$$+\frac{g_A^2}{8}\,(g_A^2-1-\bar{\beta}_{00}^-)\,(1-t/2\mu^2)\Bigg]\,\Pi_\ell$$

$$-\left[\frac{\mu}{m}\right]\frac{m^2}{64\pi^2 f_\pi^2}\,g_A^4\left[(1-t/4\mu^2)^2\Pi_t - \pi - t\pi/12\mu^2\right]^2\Bigg\}\,, \tag{7.9}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)-} = \frac{m^2}{16\pi^2 f_\pi^4}\left[\frac{\mu}{m}\right]\Bigg\{\frac{g_A^2}{4}\,(g_A^2-1-2\bar{\beta}_{00}^-)\,(1-t/4\mu^2)\,\Pi_t - \frac{g_A^4}{8}\,\Big[(1-t/2\mu^2)\,\tilde{\Pi}_b$$

$$+(1-t/4\mu^2)\,\Pi_a\Big] + \left[\frac{\mu}{m}\right]\Bigg[\frac{1}{24}\,(g_A^2-1-2\bar{\beta}_{00}^-)^2\,(1-t/4\mu^2)$$

$$+\frac{g_A^2}{8}\,(g_A^2-1-2\bar{\beta}_{00}^-)\,(1-t/2\mu^2)\Bigg]\,\Pi_\ell$$

$$- \left[\frac{\mu}{m}\right] \frac{m^2}{64\pi^2 f_\pi^2}\; g_A^4 \left[(1-t/4\mu^2)\Pi_t - \pi - t\pi/12\mu^2\right]^2 \Bigg\} \;, \tag{7.10}$$

$$\bullet\, \mathcal{I}_{BB}^{(w)-} \simeq \mathcal{I}_{BB}^{(z)-} \simeq 0 \;. \tag{7.11}$$

The leading contributions in $\Pi_\times$ and Π_b are identical and cancel out in $\mathcal{I}_{DD}^{+}$, yielding a result of $\mathcal{O}(q^3)$, in agreement with the counting rules of section IV. The results for the basic subamplitudes presented in this section are closely related to the underlying πN dynamics and, in many cases, this relationship can be directly perceived in the final forms of our expressions. For instance, reorganizing the contributions proportional to Π_t in eq.(7.10), one has

$$\mathcal{I}_{BB}^{(g)-} = \frac{m^2/f_\pi^4}{(4\pi)^2} \left[\frac{\mu}{m}\right] \Bigg\{ \Bigg[\frac{g_A^2}{4} \left(g_A^2 - 1 - \frac{g_A^2}{f_\pi^2}\, \frac{\mu m}{(4\pi)^2} \left((1-t/4\mu^2)\,\Pi_t - \pi\right) - 2\bar{\beta}_{00}^{-} \right)$$

$$\times \left((1-t/4\mu^2)\,\Pi_t - \pi\right) + \cdots \Bigg] \Bigg\} \;. \tag{7.12}$$

The terms within the parenthesis represent the contributions from fig.4, which read: (a) Born terms, proportional to g_A^2; (b) Weinberg-Tomozawa term; (c) two-loops medium range interactions; (d) other degrees of freedom plus two-loops short range interactions. The organization of the last three terms may be better understood by noting that, around the point $t=0$, the following expansion holds $(1-t/4\mu^2)\,\Pi_t \to -\pi + t\,\pi/6\mu^2$ and the content of the parenthesis of eq.(7.12) may be written as

$$\left\{ \frac{g^2}{2m^2} - \left[\frac{1}{2f_\pi^2} + \bar{b}_{00}^{-} + \frac{1}{2} \left(-\frac{g_A^2\, m\, \mu}{8\,\pi\, f_\pi^4} + t\, \frac{g_A^2\, m}{96\,\pi\, f_\pi^4\, \mu} \right) \right] \right\} \;. \tag{7.13}$$

This shows that the structure of eq.(3.12) is recovered, except for the medium range contribution, which is divided by a factor 2, characteristic of the topology of Feynman diagrams.

VIII. TPEP

The relativistic profile functions derived in the previous section give rise to the potential, which is expressed in terms of five basic functions (section VI) and empirical subthreshold coefficients (section V). The various components, obtained by means of eqs.(4.1-4.5) are listed below.

$$t_C^+ = \frac{m^2}{E^2}\, \frac{3m^2}{256\pi^2 f_\pi^4} \left[\frac{\mu}{m}\right]^2 \Big\{ g_A^4\, (1-t/2\mu^2)^2\, (\Pi_\times - \Pi_b)$$

$$+ \left[\frac{\mu}{m}\right] g_A^2\, (1-t/2\mu^2) \left[-g_A^2 \left(2\Pi_a + \Pi_t\, t/\mu^2\right) + 8\left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\, t/\mu^2\right) \Pi_t \right]$$

$$+ \left[\frac{\mu}{m}\right]^2 \left[-\frac{m^2 g_A^4}{16\pi^2 f_\pi^2}\, (1-2t/\mu^2) \left((1-t/2\mu^2)^2 \Pi_t - 2\pi\right)^2 + \frac{g_A^4}{4}\, \frac{t}{\mu^2} (\Pi_\times + \Pi_b) \right]$$

$$+ \left[\frac{\mu}{m}\right]^2 \Bigg[g_A^4\, \frac{t^2}{\mu^4} - 4 g_A^2 \left((\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\, t/\mu^2) t/\mu^2 + \delta_{10}^{+} (1 - 2\,t/3\mu^2 + t^2/6\mu^2) \right)$$

$$+8 \left(\bar{\delta}_{00}^{+} + \bar{\delta}_{01}^{+}\, t/\mu^2 + (\delta_{10}^{+}/3)(1 - t/4\mu^2) \right)^2 + \frac{32}{45} (\delta_{10}^{+})^2 (1-t/4\mu^2)^2 \Bigg] \Pi_\ell \Big\} \;, \tag{8.1}$$

$$t_T^+ = t_{SS}^+/2 = \frac{m^2}{E^2}\, \frac{m^2 g_A^2}{256\pi^2 f_\pi^4} \left\{ -g_A^2\, (1-t/4\mu^2) \left[\Pi_\times + \Pi_b\right] \right.$$

$$- \left[\frac{\mu}{m}\right] \frac{g_A^2}{2} \left[(1 - t/2\mu^2)(\Pi_t - \bar{\Pi}_b) + (1 - t/4\mu^2)\Pi_a \right]$$

$$\left. + \left[\frac{\mu}{m}\right]^2 \left[\frac{g_A^2}{4}\, (1-t/2\mu^2)^2\, \Pi_\times + \frac{4}{3}\, \beta_{00}^{+} (1 - t/4\mu^2) \Pi_\ell \right] \right\} \;, \tag{8.2}$$

$$t_{LS}^+ = \frac{m^2}{E^2}\, \frac{3m^2 g_A^2}{128\pi^2 f_\pi^4} \left[\frac{\mu}{m}\right] \Big\{ g_A^2 \left[(1 - t/2\mu^2)(\bar{\Pi}_b - \Pi_t) - (3/2 - 5\,t/8\mu^2)\, \Pi_a \right]$$

$$+\left[\frac{\mu}{m}\right]\left[\frac{g_A^2}{4}\left(\Pi_\times+\Pi_b\right)+\left(g_A^2 t/\mu^2-\frac{8}{3}\,\delta_{10}^{+}\,(1-t/4\mu^2)\right)\Pi_\ell\right]\Bigg\}\ , \tag{8.3}$$

$$t_Q^+=\frac{m^2}{E^2}\frac{m^2 g_A^4}{256\pi^2 f_\pi^4}\,\Pi_\ell\ . \tag{8.4}$$

and

$$t_C^-=\frac{m^2}{E^2}\frac{\mu^2}{8\pi^2 f_\pi^2}\left\{\frac{g_A^4}{16}\left(1-\frac{t}{2\mu^2}\right)^2\left(\Pi_\times+\Pi_b\right)-\frac{g_A^2}{4}\left(g_A^2-1\right)\left(1-\frac{t}{2\mu^2}\right)\Pi_\ell\right.$$

$$+\frac{1}{24}\left(g_A^2-1\right)^2\left(1-\frac{t}{4\mu^2}\right)\Pi_\ell$$

$$+\left[\frac{\mu}{m}\right]\frac{g_A^2}{8}\left(1-\frac{t}{2\mu^2}\right)\left[g_A^2\left(\Pi_a-\frac{t}{2\mu^2}\,\Pi_t\right)+\left(g_A^2-1\right)\left(1-\frac{t}{2\mu^2}\right)\Pi_t\right]$$

$$+\left[\frac{\mu}{m}\right]^2\left\{\frac{g_A^2}{2}\left(1-\frac{t}{2\mu^2}\right)\left[\left(g_A^2-1\right)\left(-\frac{t}{8\mu^2}+\frac{z^2}{4\mu^2}\right)+\bar{\delta}_{00}^-+\bar{\delta}_{01}^-\,\frac{t}{\mu^2}+\frac{1}{3}\,\delta_{10}^-\left(1-\frac{t}{4\mu^2}\right)+\bar{\beta}_{00}^-\,\frac{t}{4\mu^2}\right]\right.$$

$$\left.-\frac{(g_A^2-1)}{6}\left(1-\frac{t}{4\mu^2}\right)\left[\left(g_A^2-1\right)\frac{z^2}{8\mu^2}+\bar{\delta}_{00}^-+\bar{\delta}_{01}^-\,\frac{t}{\mu^2}+\frac{3}{5}\,\delta_{10}^-\left(1-\frac{t}{4\mu^2}\right)+\bar{\beta}_{00}^-\,\frac{t}{4\mu^2}\right]\right\}\Pi_\ell$$

$$-\left[\frac{\mu}{m}\right]^2\frac{m^2}{64\pi^2 f_\pi^2}\left[-g_A^2\left(1-\frac{t}{2\mu^2}\right)\left(\Pi_\ell+1-\frac{t}{3\mu^2}\right)+\frac{1}{3}(g_A^2-1)\left(1-\frac{t}{4\mu^2}\right)\left(\Pi_\ell+2-\frac{t}{4\mu^2}\right)\right]^2$$

$$\left.+\left[\frac{\mu}{m}\right]^2\frac{m^2}{256\pi^2 f_\pi^2}\,\frac{z^2}{\mu^2}\,g_A^2\left[\left(1-\frac{t}{4\mu^2}\right)\Pi_a-\pi\right]^2\right\} \tag{8.5}$$

$$t_T^-=t_{SS}^-/2=\frac{m^2}{E^2}\,\frac{m^2}{768\pi^2 f_\pi^4}\,\left[\frac{\mu}{m}\right]\left\{g_A^4\left[(1-t/2\mu^2)\,\tilde{\Pi}_b+(1-t/4\mu^2)\,\Pi_a\right]\right.$$

$$-2\,g_A^2(g_A^2-1-2\,\bar{\beta}_{00}^-)(1-t/4\mu^2)\,\Pi_t$$

$$+\left[\frac{\mu}{m}\right]\left[-g_A^2(g_A^2-1-2\,\bar{\beta}_{00}^-)(1-t/2\mu^2)-\frac{1}{3}\,(g_A^2-1-2\,\bar{\beta}_{00}^-)^2(1-t/4\mu^2)\right]\Pi_\ell$$

$$\left.+\frac{m^2}{8\pi^2 f_\pi^2}\left[(1-t/4\mu^2)\,\Pi_t-\pi\right]^2\right\}\ , \tag{8.6}$$

$$t_{LS}^-=\frac{m^2}{E^2}\,\frac{m^2}{64\pi^2 f_\pi^4}\,\left[\frac{\mu}{m}\right]\left\{g_A^4\left[\left(3/2-5\,t/8\mu^2\right)\Pi_a-(1-t/2\mu^2)(\Pi_t+\tilde{\Pi}_b)\right]\right.$$

$$+2g_A^2(g_A^2-1-\bar{\beta}_{00}^-)(1-t/4\mu^2)\Pi_t$$

$$+\left[\frac{\mu}{m}\right]\left[\frac{1}{2}(g_A^2-1)^2(1-t/4\mu^2)+g_A^2\,\bar{\beta}_{00}^-\,(1-t/2\mu^2)-\frac{4}{3}\,(g_A^2-1)\bar{\beta}_{00}^-(1-t/4\mu^2)\right]\Pi_\ell$$

$$\left.-\frac{m^2 g_A^4}{8\pi^2 f_\pi^2}\left[(1-t/4\mu^2)\Pi_t-\pi\right]^2\right\}\ , \tag{8.7}$$

$$t_Q^-\simeq 0\ . \tag{8.8}$$

This potential, which is ready to be used as an input in other calculations, is the main result of this work. To order $\mathcal{O}(q^3)$ and, in the framework of dimensional regularization, it coincides numerically with that derived earlier by ourselves [22]. As far as $\mathcal{O}(q^4)$ terms are concerned, the only difference is due to the explicit treatment of medium range contributions. In our previous study we have shown that diagrams (k-o) of fig.5 strongly dominate the potential. In the above expressions, this terms are represented by products of g_A^2 by subthreshold coefficients. About 70% of the isoscalar potential t_C^+ comes from the term proportional to $(\bar{\delta}_{00}^+ + \bar{\delta}_{01}^+ \, t/\mu^2)$, which is related to the scalar form factor of the nucleon [39], given by

$$\sigma(t) = \frac{3\mu^3 g_A^2}{64\pi^2 f_\pi^2} \,(1 - t/2\mu^2)\, \Pi_t \,. \tag{8.9}$$

The leading contribution to t_C^+ then reads

$$t_C^+ \sim 2 \, \frac{(\bar{\delta}_{00}^+ + \bar{\delta}_{01}^+ \, t/\mu^2)}{m f_\pi^2} \, \sigma(t) \sim \frac{4}{f_\pi^2} \left[-2\, c_1 - c_3 \,(1 - t/2\mu^2)\right] \sigma(t) \,. \tag{8.10}$$

As the scalar form factor represents the part of the nucleon mass associated with its pion cloud, the leading term of the NN potential corresponds to a picture in which one of the nucleons, acting as a scalar source, disturbs the pion cloud of the other. A rather puzzling aspect of this problem is that the largest term in a $\mathcal{O}(q^2)$ potential is of $\mathcal{O}(q^3)$.

IX. COMPARISON WITH OTHER WORKS

The potential of the preceding section involves functions and subthreshold coefficients that can be reexpressed in terms of explicit powers of μ/m. For the latter, one uses the results of ref. [34], summarized in section V. For the functions, one uses the results of section VI, derived using covariant techniques. Up to $\mathcal{O}(q^4)$, expressions coincide with those produced by means of heavy baryon techniques. In this section we display the full μ/m dependence of our potential, without including terms due to the common factor m^2/E^2.

We reproduce below the results of refs. [24,27,28], which include relativistic corrections and were elaborated further by Entem and Machleidt [41]. The few terms which are only present in our potential are indicated by $[\cdots]^*$.

$$\begin{aligned}
\bullet\, V_C = t_C^+ = {} & \frac{3g_A^2}{16\pi f_\pi^4} \left\{ - \frac{g_A^2\, \mu^5}{16m(4\mu^2 + \boldsymbol{q}^2)} + \left[2\mu^2(2c_1 - c_3) - \boldsymbol{q}^2 c_3\right] (2\mu^2 + \boldsymbol{q}^2)\, A(q) \right. \\
& \left. + \frac{g_A^2 (2\mu^2 + \boldsymbol{q}^2)\, A(q)}{16m} \left[-3\boldsymbol{q}^2 + (4\mu^2 + \boldsymbol{q}^2)^*\right] \right\} \\
& + \frac{g_A^2\, L(q)}{32\pi^2 f_\pi^4 m} \left\{ \frac{24\mu^6}{4\mu^2 + \boldsymbol{q}^2}\, (2c_1 + c_3) + 6\mu^4 (c_2 - 2c_3) + 4\mu^2 \boldsymbol{q}^2 (6c_1 + c_2 - 3c_3) + \boldsymbol{q}^4 (c_2 - 6c_3) \right\} \\
& - \frac{3L(q)}{16\pi^2 f_\pi^4} \left\{ \left[-4\mu^2 c_1 + c_3(2\mu^2 + \boldsymbol{q}^2) + c_2(4\mu^2 + \boldsymbol{q}^2)/6\right]^2 + \frac{1}{45}\, (c_2)^2 (4\mu^2 + \boldsymbol{q}^2)^2 \right\} \\
& + \frac{g_A^4}{32\pi^2 f_\pi^4 m^2} \left\{ L(q) \left[\frac{2\mu^8}{(4\mu^2 + \boldsymbol{q}^2)^2} + \frac{8\mu^6}{(4\mu^2 + \boldsymbol{q}^2)} - 2\mu^4 - \boldsymbol{q}^4 \right] + \frac{\mu^6/2}{(4\mu^2 + \boldsymbol{q}^2)} \right\} \\
& - \frac{3g_A^4 [A(q)]^2}{1024\pi^2 f_\pi^6}\, (\mu^2 + 2\boldsymbol{q}^2)\, (2\mu^2 + \boldsymbol{q}^2)^2 \\
& - \frac{3g_A^4 (2\mu^2 + \boldsymbol{q}^2)\, A(q)}{1024\pi^2 f_\pi^6} \left\{ 4\mu\, g_A^2\, (2\mu^2 + \boldsymbol{q}^2) + 2\mu\, (\mu^2 + 2\boldsymbol{q}^2)] \right\} \,,
\end{aligned} \tag{9.1}$$

$$\begin{aligned}
\bullet\, V_T = - \frac{3\, t_T^+}{m^2} = {} & \frac{3g_A^4\, L(q)}{64\pi^2 f_\pi^4} - \frac{g_A^4\, A(q)}{512\pi f_\pi^4 m} \left[9(2\mu^2 + \boldsymbol{q}^2) + 3(4\mu^2 + \boldsymbol{q}^2)^*\right] \\
& - \frac{g_A^4\, L(q)}{32\pi^2 f_\pi^4 m^2} \left[\boldsymbol{z}^2/4 + [5/8 - (3/8)^*]\boldsymbol{q}^2 + \frac{\mu^4}{4\mu^2 + \boldsymbol{q}^2} \right] \\
& + \frac{g_A^2\, (4\mu^2 + \boldsymbol{q}^2)\, L(q)}{32\pi^2 f_\pi^4} \left[(\tilde{d}_{14} - \tilde{d}_{15}) - \left(g_A^4/32\, \pi^2 f_\pi^2 \right)^* \right] \,,
\end{aligned} \tag{9.2}$$

$$\bullet\, V_{LS} = -\frac{t^+_{LS}}{m^2} = -\frac{3g_A^4\, A(q)}{32\pi f_\pi^4 m}\, \left[(2\mu^2 + \boldsymbol{q}^2) + (\mu^2 + 3\boldsymbol{q}^2/8)^*\right]$$

$$-\frac{g_A^4\, L(q)}{4\pi^2 f_\pi^4 m^2}\left[\frac{\mu^4}{4\mu^2+\boldsymbol{q}^2} + \frac{11}{32}\,\boldsymbol{q}^2\right]$$

$$-\frac{g_A^2\,[1-(1/2)^*]\,c_2\,L(q)}{8\pi^2 f_\pi^4 m}\,(4\mu^2 + \boldsymbol{q}^2)\,, \tag{9.3}$$

$$\bullet\, V_{\sigma L} = \frac{4\,t^+_Q}{m^4} = -\frac{g_A^4\,L(q)}{32\pi^2 f_\pi^4 m^2}\,, \tag{9.4}$$

and

$$\bullet\, W_C = t^-_C = \frac{L(q)}{384\pi^2 f_\pi^4}\left[4\mu^2\left(5g_A^4 - 4g_A^2 - 1\right) + \boldsymbol{q}^2\left(23g_A^4 - 10g_A^2 - 1\right) + \frac{48g_A^4\mu^4}{4\mu^2+\boldsymbol{q}^2}\right]$$

$$-\frac{g_A^2}{128\pi f_\pi^4 m}\left\{3g_A^2\,\frac{\mu^5}{4\mu^2+\boldsymbol{q}^2} + A(q)\,(2\mu^2+\boldsymbol{q}^2)\left[g_A^2\,(4\mu^2+3\boldsymbol{q}^2) - 2\,(2\mu^2+\boldsymbol{q}^2)\right.\right.$$

$$\left.\left.+g_A^2\,(4\mu^2+\boldsymbol{q}^2)^*\right]\right\} + \frac{\boldsymbol{q}^2\,c_4\,L(q)}{192\pi^2 f_\pi^4 m}\left[g_A^2(8\mu^2 + 5\boldsymbol{q}^2) + (4\mu^2 + \boldsymbol{q}^2)\right]$$

$$-\frac{L(q)}{768\pi^2 f_\pi^4 m^2}\left\{(4\mu^2+\boldsymbol{q}^2)\boldsymbol{z}^2 + g_A^2\left[\frac{48\mu^6}{4\mu^2+\boldsymbol{q}^2} - 24\mu^4 - 12\,(2\mu^2+\boldsymbol{q}^2)\,\boldsymbol{q}^2 + (16\mu^2+10\boldsymbol{q}^2)\boldsymbol{z}^2\right]\right.$$

$$+g_A^4\left[\boldsymbol{z}^2\left(\frac{16\mu^4}{4\mu^2+\boldsymbol{q}^2} - 7\boldsymbol{q}^2 - 20\mu^2\right)\right.$$

$$\left.\left.-\frac{64\mu^8}{(4\mu^2+\boldsymbol{q}^2)^2} - \frac{48\mu^6}{4\mu^2+\boldsymbol{q}^2} + \frac{[16-(24)^*]\mu^4\boldsymbol{q}^2}{4\mu^2+\boldsymbol{q}^2} + [20-(6)^*]\boldsymbol{q}^4 + 24\mu^2\boldsymbol{q}^2 + 24\mu^4\right]\right\}$$

$$+\frac{16g_A^4\mu^6}{768\pi^2 f_\pi^4 m^2}\,\frac{1}{4\mu^2+\boldsymbol{q}^2}$$

$$-\frac{L(q)}{18432\pi^4 f_\pi^6}\left\{192\pi^2 f_\pi^2(4\mu^2+\boldsymbol{q}^2)\,\bar{d}_3\left[2g_A^2(2\mu^2+\boldsymbol{q}^2) - 3/5(g_A^2-1)(4\mu^2+\boldsymbol{q}^2)\right]\right.$$

$$+\left[6g_A^2(2\mu^2+\boldsymbol{q}^2) - (g_A^2-1)\,(4\mu^2+\boldsymbol{q}^2)\right]\left[384\pi^2 f_\pi^2\left((2\mu^2+\boldsymbol{q}^2)\,(\bar{d}_1+\bar{d}_2) + 4\mu^2\,\bar{d}_5\right)\right.$$

$$+L(q)\left(4\mu^2(1+2g_A^2) + \boldsymbol{q}^2(1+5g_A^2)\right) - \left(\frac{\boldsymbol{q}^2}{3}\,(5+13g_A^2) + 8\mu^2(1+2g_A^2)\right)$$

$$\left.+\left(2g_A^4(2\mu^2+\boldsymbol{q}^2) + \frac{2}{3}\,\boldsymbol{q}^2(1+2g_A^2)\right)^*\right]$$

$$\left.-\frac{1}{25}\left(4\mu^2+\boldsymbol{q}^2\right)\left(15+7g_A^4\right)\left[10g_A^2(2\mu^2+\boldsymbol{q}^2) - 3(g_A^2-1)\,(4\mu^2+\boldsymbol{q}^2)\right]^*\right\}$$

$$-\frac{\boldsymbol{z}^2 g_A^4}{2048\pi^2 f_\pi^6}\left\{\left[(4\mu^2+\boldsymbol{q}^2)A(q)\right]^2 + 2\mu(4\mu^2+\boldsymbol{q}^2)A(q)\right\}^*$$

$$+\Delta_{GT}\,\frac{g_A^2}{96\pi^2 f_\pi^2}\left[g_A^2\left(\frac{48\mu^4}{4\mu^2+\boldsymbol{q}^2} + 20\mu^2 + 23\boldsymbol{q}^2\right) - 8\mu^2 - 5\boldsymbol{q}^2\right]^*\,, \tag{9.5}$$

$$\bullet\, W_T = -\frac{3}{m^2}\,t^-_T = \frac{g_A^2 A(q)}{32\pi f_\pi^4}\left[\left(c_4 + \frac{1}{4m}\right)(4\mu^2+\boldsymbol{q}^2) - \frac{g_A^2}{8m}\left[10\mu^2 + 3\boldsymbol{q}^2 - (4\mu^2+\boldsymbol{q}^2)^*\right]\right]$$

$$-\frac{c_4^2\,L(q)}{96\pi^2 f_\pi^4}(4\mu^2+\boldsymbol{q}^2) + \frac{c_4\,L(q)}{192\pi^2 f_\pi^4 m}\left[g_A^2(16\mu^2 + 7\boldsymbol{q}^2) - (4\mu^2+\boldsymbol{q}^2)\right]$$

$$-\frac{L(q)}{1536\pi^2 f_\pi^4 m^2}\left[g_A^4\left(28\mu^2+17\boldsymbol{q}^2+\frac{16\,\mu^4}{4\mu^2+\boldsymbol{q}^2}\right)-g_A^2(32\mu^2+14\boldsymbol{q}^2)+(4\mu^2+\boldsymbol{q}^2)\right]$$

$$-\frac{[A(q)]^2 g_A^4\,(4\mu^2+\boldsymbol{q}^2)^2}{2048\pi^2 f_\pi^6}-\frac{A(q)g_A^4\,(4\mu^2+\boldsymbol{q}^2)}{1024\pi^2 f_\pi^6}\,\mu(1+2g_A^2)\,, \tag{9.6}$$

$$\bullet\, W_{LS}=-\frac{1}{m^2}\,t_{LS}^-=\frac{A(q)}{32\pi f_\pi^4 m}\left[g_A^2\left[(g_A^2-1)-(2mc_4)^*\right]\,(4\mu^2+\boldsymbol{q}^2)\right.$$

$$\left.+g_A^4\,(2\mu^2+3\boldsymbol{q}^2/4)^*\right]+\frac{c_4\,L(q)}{48\pi^2 m f_\pi^4}\left[g_A^2\left(8\mu^2+5\boldsymbol{q}^2-6(2\mu^2+\boldsymbol{q}^2)^*\right)+(4\mu^2+\boldsymbol{q}^2)\right]$$

$$+\frac{L(q)}{256\pi^2 m^2 f_\pi^4}\left[(4\mu^2+\boldsymbol{q}^2)-16g_A^2\,(\mu^2+3\boldsymbol{q}^2/8)+\frac{4g_A^4}{3}\left(9\mu^2+11\boldsymbol{q}^2/4-\frac{4\mu^4}{4\mu^2+\boldsymbol{q}^2}\right)\right]$$

$$+\frac{g_A^4}{512\pi^2 f_\pi^6}\left\{\left[(4\mu^2+\boldsymbol{q}^2)A(q)\right]\left[(4\mu^2+\boldsymbol{q}^2)A(q)+2\mu(g_A^2+1)\right]\right\}^*\,, \tag{9.7}$$

$$\bullet\, W_{\sigma L}^R\simeq W_{\sigma L}^{HB}\simeq 0\,. \tag{9.8}$$

X. SUMMARY AND CONCLUSIONS

We have presented a relativistic procedure for the chiral expansion of the two-pion eschange component of the NN potential, based on that derived by Becher and Leutwyler [32,34] for elastic πN scattering. The basic dynamics is given by three families of diagrams, corresponding to the minimal realization of chiral symmetry, two-loop interactions in the t-channel and processes involving πN subthreshold coefficients, which represent frozen degrees of freedom. The calculation begins with the full evaluation of these diagrams. Results are then projected into a relativistic spin basis and expressed in terms of many different loop integrals (appendix D). At this stage the chiral structure of the problem is not evident. However, chiral scales emerge when these crude amplitudes are simplified by means of relations among loop integrals. This gives rise to our intermediate results (appendix F), which involve no truncations and preserve the numerical content of the various subamplitudes. The truncation of these intermediate results to $\mathcal{O}(q^4)$ yields directly the relativistic potential (section VIII), which is ready to be used in momentum space calculations of NN observables.

Our treatment of the NN interaction emphasizes the role of the intermediate πN subamplitudes and, in this sense, it is akin to that used in the Paris potential. We discuss how power counting in πN and NN processes are related (section IV) and results are expressed directly in terms of observable subthreshold coefficients. The low energy coupling constants c and d remain hidden inside these coefficients, grouped together with two-loop short range contributions.

If the potential presented here were truncated at order $\mathcal{O}(q^3)$, one would recover numerically the results derived by ourselves sometime ago [22]. However, processes involving two loops in the t-channel do show up at $\mathcal{O}(q^4)$ and results begin to depart at this order.

The dependence of the potential on the external variables is incorporated into four loop integrals, representing bubble, triangle, crossed box and box diagrams. Only the first of them can be evaluated analytically. The other ones, in the spirit of the relativistic expansion, are not homogeneous functions of pion masses and external three-momenta. The triangle integral is the same entering the scalar form factor of the nucleon and can be represented as a sum of three terms, scaled by powers of (μ/m) (section VI). The two leading terms coincide with those arising in the heavy baryon expansion, whereas the last one implements the correct analytic behaviour at threshold [32]. We have shown that this kind of representation can also be used to disclose the chiral structures of box and crossed box integrals(appendix G) and found out that threshold effects are proportional to $(\mu/m)^3$. As the leading term of the potential is of $\mathcal{O}(q^2)$, these threshold effects only begin to contribute at $\mathcal{O}(q^5)$.

This means that to $\mathcal{O}(q^4)$, our results should coincide with those of the standard heavy baryon approach, corrected by relativity [24,27,28,41].

In order to check this, we have reexpanded both our basic loop integrals and the subthreshold coefficients (section IX). Comparing the results of both calculations, we find two systematic diferences, apart from some minor scattered ones. The first of them is due to the Goldberger-Treiman discrepancy, which corrects the isospin odd central potential. The other one concerns terms of $\mathcal{O}(q^3)$, whose origin is less certain. However, the fact that they occur at the same order as the iteration of the OPEP suggests that there may be an important dependence on the procedure adopted for subtracting this contribution. This aspect of the problem is rather relevant in numerical applications of the potential and deserves being clarified.

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APPENDIX A: KINEMATICS

The initial and final nucleon momenta are denoted by p and p', whereas k and k' are the momenta of the exchanged pions, as in fig.1. We define the variables

$$W = p_1 + p_2 = p_1' + p_2' \;, \tag{A1}$$

$$z = [(p_1 + p_1') - (p_2 + p_2')]/2 \;, \tag{A2}$$

$$q = k' - k = p_1' - p_1 = p_2 - p_2' \;, \tag{A3}$$

$$Q = (k + k')/2 \;. \tag{A4}$$

The external nucleons are on shell and the following constraints hold

$$m^2 = (W^2 + z^2 + q^2)/4 \;, \tag{A5}$$

$$W \cdot z = W \cdot q = z \cdot q = 0 \;. \tag{A6}$$

For the Mandelstam variables one has

$$t = q^2 \;, \tag{A7}$$

$$s_1 = [Q^2 + Q\cdot(W+z) - t/4 + m^2] \;, \tag{A8}$$

$$u_1 = [Q^2 - Q\cdot(W+z) - t/4 + m^2] \;, \tag{A9}$$

$$\nu_1 = (W+z)\cdot Q/2m \;, \tag{A10}$$

$$s_2 = [Q^2 + Q\cdot(W-z) - t/4 + m^2] \;, \tag{A11}$$

$$u_2 = [Q^2 - Q\cdot(W-z) - t/4 + m^2] \;. \tag{A12}$$

$$\nu_2 = (W-z)\cdot Q/2m \;, \tag{A13}$$

Sometimes it is useful to write

$$Q^2 = (k^2 - \mu^2)/2 + (k'^2 - \mu^2)/2 + \left(\mu^2 - t/4\right) \;, \tag{A14}$$

$$Q \cdot q = (k'^2 - \mu^2)/2 - (k^2 - \mu^2)/2 \;. \tag{A15}$$

For free spinors, the following results hold

$$[\bar{u}(\boldsymbol{p}')\; \not{q}\; u(\boldsymbol{p})]^{(1)} = [\bar{u}(\boldsymbol{p}')\; \not{q}\; u(\boldsymbol{p})]^{(2)} = 0 \;, \tag{A16}$$

$$[\bar{u}(\boldsymbol{p}')\; (\not{W} + \not{z})\; u(\boldsymbol{p})]^{(1)} = 2m\; [\bar{u}(\boldsymbol{p}')\; u(\boldsymbol{p})]^{(1)} \;, \tag{A17}$$

$$[\bar{u}(\boldsymbol{p}')\; (\not{W} - \not{z})\; u(\boldsymbol{p})]^{(2)} = 2m\; [\bar{u}(\boldsymbol{p}')\; u(\boldsymbol{p})]^{(2)} \;, \tag{A18}$$

and also

$$\{\bar{u}\gamma_\lambda u\}^{(1)} = \{(W+z)_\lambda/2m[\bar{u}\; u] - i/2m[\bar{u}\; \sigma_{\mu\lambda}(p'-p)^\mu\; u]\}^{(1)} \;, \tag{A19}$$

$$(q^2/4m^2)\{\bar{u}\; u\}^{(1)} = \{-i/2m[\bar{u}\; \sigma_{\mu\lambda}(p'-p)^\mu\; u](W+z)^\lambda/2m\}^{(1)} \;, \tag{A20}$$

$$\{\bar{u}\gamma_\rho u\}^{(2)} = \{(W-z)_\rho/2m[\bar{u}\; u] - i/2m[\bar{u}\; \sigma_{\nu\rho}(p'-p)^\nu\; u]\}^{(2)} \;, \tag{A21}$$

$$(q^2/4m^2)\{\bar{u}\; u\}^{(2)} = \{-i/2m[\bar{u}\; \sigma_{\nu\rho}(p'-p)^\nu\; u](W-z)^\rho/2m\}^{(2)} \;. \tag{A22}$$

In the center of mass (CM) one has

$$p_1 = (E; \boldsymbol{p}) \ , \ \ p_1' = (E; \boldsymbol{p}') \ , \tag{A23}$$

$$p_2 = (E; -\boldsymbol{p}) \ , \ \ p_2' = (E; -\boldsymbol{p}') \ , \tag{A24}$$

$$W = (2E; 0) \ , \tag{A25}$$

$$q = (0; \boldsymbol{p}' - \boldsymbol{p}) \ , \tag{A26}$$

$$z = (0; \boldsymbol{p}' + \boldsymbol{p}) \tag{A27}$$

and the on shell condition for nucleons reads

$$E^2 = m^2 + \boldsymbol{q}^2/4 + \boldsymbol{z}^2/4 \ . \tag{A28}$$

In the CM frame, the nucleon spin functions may be expressed in terms of two component matrices as

$$\left\{\bar{u}(\boldsymbol{p}')\, u(\boldsymbol{p})\right\}^{(i)} = \chi^\dagger \left[2m + \frac{1}{2(E+m)}\left(\boldsymbol{q}^2 - i\, \boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{z}\right)\right]\chi \ , \tag{A29}$$

$$\left\{\frac{i}{2m}\bar{u}(\boldsymbol{p}')\, \sigma_{\mu 0}(p'-p)^\mu\, u(\boldsymbol{p})\right\}^{(i)} = \chi^\dagger \left[\frac{1}{2m}\left(\boldsymbol{q}^2 - i\, \boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{z}\right)\right]\chi \ , \tag{A30}$$

$$\left\{\frac{i}{2m}\bar{u}(\boldsymbol{p}')\, \sigma_{\mu j}(p'-p)^\mu\, u(\boldsymbol{p})\right\}^{(i)}$$

$$= s(i)\, \chi^\dagger \left[-i\, \boldsymbol{\sigma}\times(\boldsymbol{p}'-\boldsymbol{p}) + \left(\boldsymbol{q}^2 - i\, \boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{z}\right)\frac{(\boldsymbol{p}'+\boldsymbol{p})}{4m(E+m)}\right]_j \chi \ , \tag{A31}$$

where $s(i) = (1, -1)$ for $i = (1, 2)$. These results, which contain no approximations, allow one to write the identities

$$[\bar{u}\; u]^{(1)}[\bar{u}\; u]^{(2)} = 4m^2\left[\left(1+\boldsymbol{q}^2/\lambda^2\right)^2 - 4\left(1+\boldsymbol{q}^2/\lambda^2\right)\frac{\boldsymbol{\Omega}_{LS}}{\lambda^2} - \frac{\boldsymbol{\Omega}_Q}{\lambda^4}\right] \ , \tag{A32}$$

$$-\frac{i}{2m}\left\{[\bar{u}\; u]^{(1)}[\bar{u}\; \sigma_{\mu\lambda}\, (p'-p)^\mu\, u]^{(2)} - (1\leftrightarrow 2)\right\}\frac{z^\lambda}{2m} = 4m^2\left[-\left(1+\boldsymbol{q}^2/\lambda^2\right)\frac{\boldsymbol{z}^2\boldsymbol{q}^2}{2m^2\lambda^2}\right.$$

$$\left. + \left(1+\boldsymbol{q}^2/\lambda^2+\boldsymbol{z}^2/\lambda^2 + 2\boldsymbol{q}^2\boldsymbol{z}^2/\lambda^4\right)\frac{\boldsymbol{\Omega}_{LS}}{m^2} + \left(1+\boldsymbol{z}^2/\lambda^2\right)\frac{\boldsymbol{\Omega}_Q}{2m^2\lambda^2}\right] \ , \tag{A33}$$

$$-\frac{1}{4m^2}[\bar{u}\; \sigma_{\mu\lambda}(p'-p)^\mu\; u]^{(1)}[\bar{u}\; \sigma_{\nu\rho}(p'-p)^\nu\; u]^{(2)} g^{\lambda\rho} = 4m^2\left[\left(1+4m^2\boldsymbol{z}^2/\lambda^4\right)\frac{\boldsymbol{q}^4}{16m^4} - \frac{\Omega_{SS}}{6m^2}\right.$$

$$\left. -\frac{\Omega_T}{12m^2} - \left(1+4m^2/\lambda^2+4m^2\boldsymbol{z}^2/\lambda^4\right)\frac{\boldsymbol{q}^2\, \Omega_{LS}}{4m^4} - \left(1+8m^2/\lambda^2+4m^2\boldsymbol{z}^2/\lambda^4\right)\frac{\Omega_Q}{16m^4}\right] \ , \tag{A34}$$

$$-\frac{1}{4m^2}[\bar{u}\; \sigma_{\mu\lambda}(p'-p)^\mu\; u]^{(1)}[\bar{u}\; \sigma_{\nu\rho}(p'-p)^\nu\; u]^{(2)}\frac{z^\lambda z^\rho}{4m^2} = 4m^2\left[-\frac{\boldsymbol{q}^4\boldsymbol{z}^4}{16m^4\lambda^4}\right.$$

$$\left. + \left(1+\boldsymbol{z}^2/\lambda^2\right)\frac{\boldsymbol{q}^2\boldsymbol{z}^2\, \Omega_{LS}}{4m^4\lambda^2} + \left(1+\boldsymbol{z}^2/\lambda^2\right)^2\frac{\Omega_Q}{16m^4}\right] \ , \tag{A35}$$

where the two-component spin operators $\boldsymbol{\Omega}$ were defined in section II and $\lambda^2 = 4m(E+m)$.

APPENDIX B: LOOP INTEGRALS

The basic loop integrals needed in this work are

$$I_{cc}^{\mu\cdots} = \int[\cdots]\left(\frac{Q^\mu}{\mu}\cdots\right) \ , \tag{B1}$$

$$I_{sc}^{\mu\cdots} = \int[\cdots]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{2m\mu}{[Q^2+Q\cdot(W+z)-t/4]} \ , \tag{B2}$$

$$I_{ss}^{\mu\cdots} = \int[\cdots]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{2m\mu}{[Q^2+Q\cdot(W+z)-t/4]}\,\frac{2m\mu}{[Q^2+Q\cdot(W-z)-t/4]} \ , \tag{B3}$$

with

$$\int [\cdots] = \int \frac{d^4Q}{(2\pi)^4} \; \frac{1}{[(Q-q/2)^2-\mu^2][(Q+q/2)^2-\mu^2]} \,. \tag{B4}$$

All denominators are symmetric under $q \to -q$ and results cannot contain odd powers of this variable. The integrals are dimensionless and have the following tensor structure

$$I_{cc} = \frac{i}{(4\pi)^2} \left\{ \Pi_{cc}^{(000)} \right\} \,, \tag{B5}$$

$$I_{cc}^{\mu\nu} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\mu^2} \left[q^\mu q^\nu \, \Pi_{cc}^{(200)} \right] + g^{\mu\nu} \, \bar{\Pi}_{cc}^{(000)} \right\} \,, \tag{B6}$$

$$I_{cc}^{\mu\nu\lambda\rho} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\mu^4} \left[q^\mu q^\nu q^\lambda q^\rho \, \Pi_{cc}^{(400)} \right] + \frac{1}{\mu^2} \left[\left(g^{\mu\nu} q^\lambda q^\rho + g^{\nu\lambda} q^\rho q^\mu + g^{\lambda\rho} q^\mu q^\nu + g^{\mu\lambda} q^\nu q^\rho \right.\right.\right.$$
$$\left.\left.\left. + g^{\mu\rho} q^\lambda q^\nu + g^{\nu\rho} q^\mu q^\lambda \right) \bar{\Pi}_{cc}^{(200)} \right] + \left[\left(g^{\mu\nu} g^{\lambda\rho} + g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda} \right) \bar{\bar{\Pi}}_{cc}^{(000)} \right] \right\} \,, \tag{B7}$$

$$I_{sc} = \frac{i}{(4\pi)^2} \left\{ \Pi_{sc}^{(000)} \right\} \,, \tag{B8}$$

$$I_{sc}^{\mu} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{2m} \left[(z^\mu + W^\mu) \, \Pi_{sc}^{(001)} \right] \right\} \,, \tag{B9}$$

$$I_{sc}^{\mu\nu} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\mu^2} \left[q^\mu q^\nu \, \Pi_{sc}^{(200)} \right] + \frac{1}{4m^2} \left[(z^\mu z^\nu + W^\mu W^\nu) \, \Pi_{sc}^{(002)} \right] + g^{\mu\nu} \, \bar{\Pi}_{sc}^{(000)} \right\} \,, \tag{B10}$$

$$I_{ss} = \frac{i}{(4\pi)^2} \left\{ \Pi_{ss}^{(000)} \right\} \,, \tag{B11}$$

$$I_{ss}^{\mu} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{2m} \left[z^\mu \, \Pi_{ss}^{(010)} + W^\mu \, \Pi_{ss}^{(001)} \right] \right\} \,, \tag{B12}$$

$$I_{ss}^{\mu\nu} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\mu^2} \left[q^\mu q^\nu \, \Pi_{ss}^{(200)} \right] + \frac{1}{4m^2} \left[z^\mu z^\nu \, \Pi_{ss}^{(020)} + W^\mu W^\nu \, \Pi_{ss}^{(002)} \right] + g^{\mu\nu} \, \bar{\Pi}_{ss}^{(000)} \right\} \,, \tag{B13}$$

The usual Feynman techniques for loop integration allow us to write

$$\Pi_{cc}^{(k00)} = \int_0^1 da \, (-C_a)^k \left[\rho_0 - \ln\left(\frac{D_{cc}}{\mu^2} \right) \right] \,, \tag{B14}$$

$$\bar{\Pi}_{cc}^{(k00)} = -\frac{1}{2} \int_0^1 da \, (-C_a)^k \, \frac{D_{cc}}{\mu^2} \left[-\rho_1 + \ln\left(\frac{D_{cc}}{\mu^2} \right) \right] \,, \tag{B15}$$

$$\bar{\bar{\Pi}}_{cc}^{(000)} = \frac{1}{8} \int_0^1 da \, \frac{D_{cc}^2}{\mu^4} \left[\rho_2 - \ln\left(\frac{D_{cc}}{\mu^2} \right) \right] \,, \tag{B16}$$

$$\Pi_{sc}^{(kmn)} = \left(-\frac{2m}{\mu} \right)^{m+n+1} \int_0^1 da \; a \int_0^1 db \; \frac{\mu^2 \, (-C_q)^k (C_b)^{m+n}}{D_{sc}} \,, \tag{B17}$$

$$\bar{\Pi}_{sc}^{(000)} = -\left(\frac{2m}{\mu} \right) \frac{1}{2} \int_0^1 da \; a \int_0^1 db \; \left[-\rho_0 + \ln\left(\frac{D_{sc}}{\mu^2} \right) \right] \,, \tag{B18}$$

$$\Pi_{ss}^{(kmn)} = \left(-\frac{2m}{\mu} \right)^{m+n+2} \int_0^1 da \; a^2 \int_0^1 db \; b \int_0^1 dc \; \frac{\mu^4 \, (-C_q)^k \, (C_c)^m \, (C_b)^n}{D_{ss}^2} \,, \tag{B19}$$

$$\bar{\Pi}_{ss}^{(000)} = -\left(\frac{2m}{\mu} \right)^2 \frac{1}{2} \int_0^1 da \; a^2 \int_0^1 db \; b \int_0^1 dc \; \frac{\mu^2}{D_{ss}} \,. \tag{B20}$$

with

$$C_a = a - 1/2 \,, \tag{B21}$$

$$\Sigma_{cc}^2 = -q^2/4 + \mu^2 \,, \tag{B22}$$

$$D_{cc} = P_{cc}^2 + \Sigma_{cc}^2 = C_a^2 \, q^2 + \Sigma_{cc}^2 \,, \tag{B23}$$

$$C_b = ab/2 \tag{B24}$$

$$C_q = C_a - C_b \; , \tag{B25}$$

$$\Sigma^2_{m,c} = -(1-2ab)\; q^2/4 + (1-ab)\; \mu^2 \; , \tag{B26}$$

$$D_{sc} = P^2_{s,c} + \Sigma^2_{m,c} = C_q^2\; q^2 + C_b^2\; \left(z^2 + W^2\right) + \Sigma^2_{m,c} \; , \tag{B27}$$

$$C_c = abc/2 \; , \tag{B28}$$

$$\Sigma^2_{m,m} = \Sigma^2_{m,c} \; , \tag{B29}$$

$$D_{ss} = P^2_{s,s} + \Sigma^2_{m,m} = C_q^2\; q^2 + C_c^2\; z^2 + C_b^2\; W^2 + \Sigma^2_{m,m} \; . \tag{B30}$$

The case (cs) is obtained from (sc) by making $z^\mu \to -z^\mu$. The case (us) is obtained from (ss) by making $C_b \leftrightarrow -C_c$.

APPENDIX C: OPEP ITERATION

The iteration of the OPEP has to be subtracted from the elastic scattering amplitude, in order to avoid double counting in the potential. In this work we adopt the procedure used by Partovi and Lomon [5], based on a prescription developed by Blankenbecler and Sugar [42]. In this appendix we adapt their expressions to our relativistic notation and also simplify some of the results.

The iterated OPEP is contained in the box diagram, corresponding to the amplitude

$$\mathcal{T}_{box} = \left[3 - 2\; \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}\right]\; \mathcal{T}_{us} \; , \tag{C1}$$

where

$$\mathcal{T}_{us} = i \left[\frac{g}{m}\right]^4 \frac{m^2}{4} \int [\cdots] \frac{Q^\mu Q^\nu}{\mu^2}\; \left[\frac{2m\mu}{u - m^2}\; \bar{u}\gamma_\mu u\right]^{(1)} \left[\frac{2m\mu}{s - m^2}\; \bar{u}\gamma_\nu u\right]^{(2)} \; . \tag{C2}$$

Evaluating this integral using the results of appendix B, one recovers the spin structure of eq.(2.6) with

$$\mathcal{I}_{DD}\big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[-\frac{z^4}{16m^4}\; \Pi^{(020)}_{us} + \frac{W^4}{16m^4}\; \Pi^{(002)}_{us} + \frac{W^2 - z^2}{4m^2}\; \bar{\Pi}^{(000)}_{us}\right] \; , \tag{C3}$$

$$\mathcal{I}^{(w)}_{DB}\Big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[\frac{W^2}{4m^2}\; \Pi^{(002)}_{us} + \bar{\Pi}^{(000)}_{us}\right] \; , \tag{C4}$$

$$\mathcal{I}^{(z)}_{DB}\Big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[\frac{z^2}{4m^2}\; \Pi^{(020)}_{us} + \bar{\Pi}^{(000)}_{us}\right] \; , \tag{C5}$$

$$\mathcal{I}^{(g)}_{BB}\Big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[\bar{\Pi}^{(000)}_{us}\right] \; , \tag{C6}$$

$$\mathcal{I}^{(w)}_{BB}\Big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[\Pi^{(002)}_{us}\right] \; , \tag{C7}$$

$$\mathcal{I}^{(z)}_{BB}\Big]_{us} = -\frac{m^2/4}{(4\pi)^2} \left[\frac{g}{m}\right]^4 \left[\Pi^{(020)}_{us}\right] \; . \tag{C8}$$

The iterated amplitude is denoted by $\mathcal{T}_\pi$ and given by

$$\mathcal{T}_\pi = \left[3 - 2\; \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}\right]\; \mathcal{T}_{it} \; , \tag{C9}$$

with

$$\mathcal{T}_{it} = -i \left[\frac{g}{m}\right]^4 \frac{m^2}{4} \left\{ (\bar{u}u)^{(1)} (\bar{u}u)^{(2)}\; (I_B - 2\; I_C) \right.$$

$$- \left[(\bar{u}u)^{(1)} (\bar{u}\; \gamma_i\; u)^{(2)} - (\bar{u}\; \gamma_i\; u)^{(1)} (\bar{u}u)^{(2)}\right] \left[\frac{\mu}{m}\; I_C^i + \frac{z^i}{2m}\; (I_B - 2\; I_C)\right]$$

$$- (\bar{u}\,\gamma_i\,u)^{(1)}(\bar{u}\,\gamma_j\,u)^{(2)} \left[\frac{\mu^2}{m^2}\left(I_A^{ij} - I_C^{ij}\right) + \frac{\mu}{m}\left(\frac{z^i}{2m}\,I_C^j + I_C^i\,\frac{z^j}{2m}\right)\right.$$

$$\left.\left. + \frac{z^i z^j}{4m^2}\,(I_B - 2\,I_C)\right]\right\} . \tag{C10}$$

The functions I_i are three-dimensional loop integrals, defined as

$$I_A^{i\cdots} = i\int(\cdots)\left(\frac{Q^i}{\mu}\cdots\right)\frac{m^3}{E[E_Q^2 - E^2]}\,, \tag{C11}$$

$$I_B = i\int(\cdots)\,\frac{m^3}{E^2 E_Q}\,, \tag{C12}$$

$$I_C^{i\cdots} = I_A^{i\cdots} - I_d^{i\cdots}\,, \tag{C13}$$

$$I_D^{i\cdots} = i\int(\cdots)\left(\frac{Q^i}{\mu}\cdots\right)\frac{m^3}{E_Q[E_Q^2 - E^2]}\,, \tag{C14}$$

where $E_Q = \sqrt{m^2 + (\boldsymbol{Q} - \boldsymbol{z}/2)^2}$ and

$$\int(\cdots) = \int\frac{d^3Q}{(2\pi)^3}\,\frac{m}{[(\boldsymbol{Q} - \boldsymbol{q}/2)^2 + \mu^2][(\boldsymbol{Q} + \boldsymbol{q}/2)^2 + \mu^2]}\,. \tag{C15}$$

The usual Feynman parametrization techniques, the representation

$$\frac{m}{E_Q} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{mE\,d\epsilon}{[(\boldsymbol{Q} - \boldsymbol{z}/2)^2 + m^2 + \epsilon^2 E^2]}\,, \tag{C16}$$

and the tensor decomposition

$$I_x = \frac{i}{(4\pi)^2}\left\{\Pi_x^{(000)}\right\}\,, \tag{C17}$$

$$I_x^i = \frac{i}{(4\pi)^2}\left\{\frac{z^i}{2m}\,\Pi_x^{(010)}\right\}\,, \tag{C18}$$

$$I_x^{ij} = \frac{i}{(4\pi)^2}\left\{\frac{q^i q^j}{\mu^2}\,\Pi_x^{(200)} + \frac{z^i z^j}{4m^2}\,\Pi_x^{(020)} + g^{ij}\,\bar{\Pi}_x^{(000)}\right\}\,, \tag{C19}$$

(for $x = a, b, c$) yield

$$\mathcal{I}_{DD}]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\frac{\mu^2}{m^2}\left(\frac{z^4}{16m^4}\,\Pi_A^{(020)} + \frac{z^2}{4m^2}\,\bar{\Pi}_A^{(000)}\right) + \left(1 - \frac{z^2}{4m^2}\right)^2\left(\Pi_B^{(000)} - 2\,\Pi_C^{(000)}\right)\right.$$

$$\left. - \frac{z^2}{4m^2}\left(\frac{\mu^2}{m^2}\,\bar{\Pi}_C^{(000)} + \frac{2\mu}{m}\,\Pi_C^{(010)}\right) - \frac{z^4}{16m^4}\left(\frac{\mu^2}{m^2}\,\Pi_C^{(020)} - \frac{2\mu}{m}\,\Pi_C^{(010)}\right)\right\}\,, \tag{C20}$$

$$\left.\mathcal{I}_{DB}^{(z)}\right]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{-\frac{\mu^2}{m^2}\left(\frac{z^2}{4m^2}\,\Pi_A^{(020)} + \bar{\Pi}_A^{(000)}\right) + \frac{\mu^2}{m^2}\,\bar{\Pi}_C^{(000)} + \frac{\mu}{m}\,\Pi_C^{(010)}\right.$$

$$\left. + \left(1 - \frac{z^2}{4m^2}\right)\left(\Pi_B^{(000)} - 2\,\Pi_C^{(000)}\right) + \frac{z^2}{4m^2}\left(\frac{\mu^2}{m^2}\,\Pi_C^{(020)} - \frac{2\mu}{m}\,\Pi_C^{(010)}\right)\right\}\,, \tag{C21}$$

$$\left.\mathcal{I}_{BB}^{(g)}\right]_{it} = -\frac{W^2}{4m^2}\left.\mathcal{I}_{BB}^{(w)}\right]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\frac{\mu^2}{m^2}\left[-\bar{\Pi}_A^{(000)} + \bar{\Pi}_C^{(000)}\right]\right\}\,, \tag{C22}$$

$$\left.\mathcal{I}_{BB}^{(z)}\right]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{-\frac{\mu^2}{m^2}\,\Pi_A^{(020)} + \frac{\mu^2}{m^2}\,\Pi_C^{(020)} - \frac{2\mu}{m}\,\Pi_C^{(010)} - \Pi_B^{(000)} + 2\,\Pi_C^{(000)}\right\}\,. \tag{C23}$$

The functions Π and $\bar{\Pi}$ are written as

$$\Pi_A^{(020)} = \left(\frac{2m}{\mu}\right)^2 \frac{4m^4}{E} \int_0^1 da\; a \int_0^1 db \int_0^\infty dQ \; \frac{(C_b)^2}{[Q^2+\Sigma_A^2-\boldsymbol{P}_I^2]^2} \;, \tag{C24}$$

$$\bar{\Pi}_A^{(000)} = -\frac{2m^4}{\mu^2 E} \int_0^1 da\; a \int_0^1 db \int_0^\infty dQ \; \frac{1}{[Q^2+\Sigma_A^2-\boldsymbol{P}_I^2]} \;, \tag{C25}$$

$$\Pi_B^{(000)} = \frac{4m^4}{\pi E} \int_0^1 da\; a \int_0^1 db \int_{-\infty}^\infty d\epsilon \int_0^\infty dQ \; \frac{1}{[Q^2+\Sigma_B^2-\boldsymbol{P}_I^2]^2} \;, \tag{C26}$$

$$\Pi_C^{(0n0)} = \left(\frac{2m}{\mu}\right)^n \frac{4m^4}{\pi E} \int_0^1 da\; a \int_0^1 db \int_{-\infty}^\infty \frac{d\epsilon}{1+\epsilon^2} \int_0^\infty dQ \; \frac{(C_b)^n}{[Q^2+\Sigma_B^2-\boldsymbol{P}_I^2]^2} \;, \tag{C27}$$

$$\bar{\Pi}_C^{(000)} = -\frac{2m^4}{\pi\mu^2 E} \int_0^1 da\; a \int_0^1 db \int_{-\infty}^\infty \frac{d\epsilon}{1+\epsilon^2} \int_0^\infty dQ \; \frac{1}{[Q^2+\Sigma_B^2-\boldsymbol{P}_I^2]} \;. \tag{C28}$$

where

$$\boldsymbol{P}_I = C_q\, \boldsymbol{q} - C_b\, \boldsymbol{z} \;, \tag{C29}$$
$$\Sigma_A^2 = \Sigma_{m,c}^2 \;, \tag{C30}$$
$$\Sigma_B^2 = \Sigma_{m,c}^2 + ab\,(1+\epsilon^2)\, E^2 \;, \tag{C31}$$
$$\Sigma_D^2 = \Sigma_{m,c}^2 + ab\lambda\,(1+\epsilon^2)\, E^2 \;. \tag{C32}$$

The contribution from the OPEP cut in the functions Π_{us} is cancelled by the integrals Π_A. One parametrizes the loop momentum in those integrals as $Q = (abc\; W/2) = (-C_c\; W)$ and have $[Q^2+\Sigma_A^2-\boldsymbol{P}_I^2] = D_{us}$ and write

$$\frac{\mu^2}{m^2}\, \Pi_A^{(020)} \equiv \Pi_{it}^{(020)} \;, \qquad \frac{\mu^2}{m^2}\, \bar{\Pi}_A^{(000)} \equiv \bar{\Pi}_{it}^{(000)} \;, \tag{C33}$$

with

$$\Pi_{it}^{(kmn)} = \left(-\frac{2m}{\mu}\right)^{m+n+2} \int_0^1 da\; a^2 \int_0^1 db\; b \int_0^\infty dc \; \frac{\mu^4\,(C_q)^k(-C_b)^m(-C_c)^n}{D_{us}^2} \;, \tag{C34}$$

$$\bar{\Pi}_{it}^{(000)} = -\left(\frac{2m}{\mu}\right)^2 \frac{1}{2} \int_0^1 da\; a^2 \int_0^1 db\; b \int_0^\infty dc \; \frac{\mu^2}{D_{us}} \;. \tag{C35}$$

The integrals Π_B and Π_C can also be simplified, by adopting the new variables c and θ, defined by the relations $\epsilon = \sqrt{a^2b^2c^2-ab}\; \cos\theta/\sqrt{ab}$, $Q = E\sqrt{a^2b^2c^2-ab}\; \sin\theta$. Performing the angular integrations, we have

$$\Pi_B^{(000)} = \left(\frac{2m}{\mu}\right)^4 \int_0^1 da\; a^2 \int_0^1 db\; b \int_{\sqrt{ab}}^\infty dc \; \frac{\mu^4\sqrt{ab}\; c}{D_{us}^2} \;, \tag{C36}$$

$$\Pi_C^{(0m0)} = \left(\frac{2m}{\mu}\right)^{m+4} \int_0^1 da\; a^2 \int_0^1 db\; b \int_{\sqrt{ab}}^\infty dc \; \frac{\mu^4 (C_b)^m}{D_{us}^2} \;, \tag{C37}$$

$$\bar{\Pi}_C^{(000)} = -\left(\frac{2m}{\mu}\right)^4 \frac{1}{2} \int_0^1 da\; a^2 \int_0^1 db\; b \int_{\sqrt{ab}}^\infty dc \; \frac{\mu^2}{D_{us}} \;. \tag{C38}$$

The results presented so far in this appendix correspond just to a reorganization of those obtained by Partovi and Lomon [5]. They may be further simplified by noting that

$$\begin{aligned} I_B &= i \int (\cdots)\, \frac{m^3}{E^2 E_Q} \\ &\simeq i\, \frac{m^3}{E^3} \int (\cdots) \left[1 - (\boldsymbol{Q}^2 - \boldsymbol{q}^2/4 - \boldsymbol{Q}\cdot\boldsymbol{z})/2E^2 + 3(\boldsymbol{Q}^2 - \boldsymbol{q}^2/4 - \boldsymbol{Q}\cdot\boldsymbol{z})^2/8E^4\right] \;, \end{aligned} \tag{C39}$$

$$\begin{aligned} I_C^{i\cdots} &= i \int (\cdots) \left(\frac{Q^i}{\mu}\cdots\right) \frac{m^3}{E E_Q (E+E_Q)} \\ &\simeq i\, \frac{m^3}{2E^3} \int (\cdots) \left[1 - 3(\boldsymbol{Q}^2 - \boldsymbol{q}^2/4 - \boldsymbol{Q}\cdot\boldsymbol{z})/4E^2 + 5(\boldsymbol{Q}^2 - \boldsymbol{q}^2/4 - \boldsymbol{Q}\cdot\boldsymbol{z})^2/8E^4\right] \;. \end{aligned} \tag{C40}$$

The integrals $\int(\cdots)$ can be performed analytically and we have

$$\int(\cdots) = -\frac{1}{(4\pi)^2}\,\frac{2m}{\mu}\,\Pi_a\ , \tag{C41}$$

$$\int(\cdots)\left(\frac{Q_iQ_j}{\mu^2}\right) = \frac{1}{(4\pi)^2}\,\frac{m}{\mu}\left(\delta_{ij}-\frac{q_iq_j}{q^2}\right)\left(1+\frac{q^2}{4\mu^2}\right)\Pi_a\ , \tag{C42}$$

where Π_a is the function given in eq.(6.9).

Thus

$$\Pi_B^{(000)} = -\frac{2m}{\mu}\,\frac{m^3}{E^3}\Big[1+\frac{1}{2m^2}(\mu^2-q^2/2)+\frac{1}{8m^4}(q^2+z^2)(\mu^2-q^2/2)$$

$$+\frac{3}{8m^4}(\mu^2-q^2/2)^2+\frac{3}{16m^4}\,z^2(\mu^2-q^2/4)\Big]\,\Pi_a\ , \tag{C43}$$

$$\Pi_C^{(000)} = -\frac{m}{\mu}\,\frac{m^3}{E^3}\Big[1+\frac{3}{4m^2}(\mu^2-q^2/2)+\frac{3}{16m^4}(q^2+z^2)(\mu^2-q^2/2)$$

$$+\frac{5}{8m^4}(\mu^2-q^2/2)^2+\frac{5}{16m^4}\,z^2(\mu^2-q^2/4)\Big]\,\Pi_a\ , \tag{C44}$$

$$\Pi_C^{(010)} = \frac{3}{4}\left(1-\frac{q^2}{4\mu^2}\right)\Pi_a\ , \tag{C45}$$

$$\Pi_C^{(020)} = 0\ , \tag{C46}$$

$$\bar{\Pi}_C^{(000)} = -\frac{m}{2\mu}\left(1-\frac{q^2}{4\mu^2}\right)\Pi_a\ . \tag{C47}$$

The results presented in eqs.(C3-C8), (C20-C23), (C33-C35) and (C43-C47) allow one to write

$$\mathcal{I}_{DD}\big]_{us} - \mathcal{I}_{DD}\big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\Big\{-\frac{z^4}{16m^4}\,\Pi_{reg}^{(020)}+\frac{W^4}{16m^4}\,\Pi_{reg}^{(002)}+\frac{W^2-z^2}{4m^2}\,\bar{\Pi}_{reg}^{(000)}$$

$$-\frac{\mu}{2m}\left(1-\frac{q^2}{2\mu^2}\right)\left(1+\frac{\mu^2}{m^2}+\frac{q^2}{8m^2}+\frac{z^2}{8m^2}\right)\Pi_a\Big\}\ , \tag{C48}$$

$$\mathcal{I}_{DB}^{(w)}\Big]_{us} - \mathcal{I}_{DB}^{(w)}\Big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\frac{W^2}{4m^2}\,\Pi_{reg}^{(002)}+\bar{\Pi}_{reg}^{(000)}\right\}\ , \tag{C49}$$

$$\mathcal{I}_{DB}^{(z)}\Big]_{us} - \mathcal{I}_{DB}^{(z)}\Big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\frac{z^2}{4m^2}\,\Pi_{reg}^{(020)}+\bar{\Pi}_{reg}^{(000)}-\frac{\mu}{2m}\left(\frac{3}{2}-\frac{5q^2}{8\mu^2}\right)\Pi_a\right\}\ , \tag{C50}$$

$$\mathcal{I}_{BB}^{(g)}\Big]_{us} - \mathcal{I}_{BB}^{(g)}\Big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\bar{\Pi}_{reg}^{(000)}+\frac{\mu}{2m}\left(1-\frac{q^2}{4\mu^2}\right)\Pi_a\right\}\ , \tag{C51}$$

$$\mathcal{I}_{BB}^{(w)}\Big]_{us} - \mathcal{I}_{BB}^{(w)}\Big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\Pi_{reg}^{(002)}\right\}\ , \tag{C52}$$

$$\mathcal{I}_{BB}^{(z)}\Big]_{us} - \mathcal{I}_{BB}^{(z)}\Big]_{it} = \frac{m^2/4}{(4\pi)^2}\left[\frac{g}{m}\right]^4\left\{\Pi_{reg}^{(020)}\right\}\ , \tag{C53}$$

where the integrals $\Pi_{reg} \equiv \Pi_{it} - \Pi_{us}$ are regular and given by

$$\Pi_{reg}^{(kmn)} = \left(-\frac{2m}{\mu}\right)^{m+n+2}\int_0^1 da\,a^2\int_0^1 db\,b\int_1^\infty dc\,\frac{\mu^4\,(C_q)^k(-C_b)^m(-C_c)^n}{D_{us}^2}\ , \tag{C54}$$

$$\bar{\Pi}_{reg}^{(000)} = -\left(\frac{2m}{\mu}\right)^2\frac{1}{2}\int_0^1 da\,a^2\int_0^1 db\,b\int_1^\infty dc\,\frac{\mu^2}{D_{us}}\ . \tag{C55}$$

APPENDIX D: FULL RESULTS

In this appendix we list the results for the amplitudes that enter eq.(2.15), obtained by reading the diagrams of fig.5 and representing loop integrals by means of the functions displayed in appendices B and C.

<u>**family 1**</u> (diagrams a+b+c+d+e+f)

$$\bullet\, \mathcal{I}_{DD}^{+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{2\,\Pi_{cc}^{(000)} - 4\,\frac{W^2+z^2}{4m^2}\,\Pi_{sc}^{(001)} - \frac{z^4}{16m^4}\Pi_{ss}^{(020)} + \frac{W^4}{16m^4}\Pi_{ss}^{(002)}\right.$$
$$+\frac{W^2-z^2}{4m^2}\bar{\Pi}_{ss}^{(000)} - \frac{z^4}{16m^4}\Pi_{reg}^{(020)} + \frac{W^4}{16m^4}\Pi_{reg}^{(002)} + \frac{W^2-z^2}{4m^2}\bar{\Pi}_{reg}^{(000)}$$
$$\left. -\frac{\mu}{2m}\left(1-\frac{q^2}{2\mu^2}\right)\left(1+\frac{\mu^2}{m^2}+\frac{q^2}{8m^2}+\frac{z^2}{8m^2}\right)\Pi_a\right\}\,, \tag{D1}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{-2\,\Pi_{sc}^{(001)} + \frac{W^2}{4m^2}\Pi_{ss}^{(002)} + \bar{\Pi}_{ss}^{(000)} + \frac{W^2}{4m^2}\Pi_{reg}^{(002)} + \bar{\Pi}_{reg}^{(000)}\right\}\,, \tag{D2}$$

$$\bullet\, \mathcal{I}_{DB}^{(z)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{2\,\Pi_{sc}^{(001)} + \frac{z^2}{4m^2}\Pi_{ss}^{(020)} + \bar{\Pi}_{ss}^{(000)} + \frac{z^2}{4m^2}\Pi_{reg}^{(020)} + \bar{\Pi}_{reg}^{(000)}\right.$$
$$\left. -\frac{\mu}{2m}\left(\frac{3}{2}-\frac{5q^2}{8\mu^2}\right)\Pi_a\right\}\,, \tag{D3}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{\bar{\Pi}_{ss}^{(000)} + \bar{\Pi}_{reg}^{(000)} + \frac{\mu}{2m}\left(1-\frac{q^2}{4\mu^2}\right)\Pi_a\right\}\,, \tag{D4}$$

$$\bullet\, \mathcal{I}_{BB}^{(w)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{\Pi_{ss}^{(002)} + \Pi_{reg}^{(002)}\right\}\,, \tag{D5}$$

$$\bullet\, \mathcal{I}_{BB}^{(z)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{\Pi_{ss}^{(020)} + \Pi_{reg}^{(020)}\right\}\,, \tag{D6}$$

and

$$\bullet\, \mathcal{I}_{DD}^{-} = \frac{m^2/4}{(4\pi)^2}\left\{\frac{\mu^2}{2m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)^2\frac{W^2-z^2}{4m^2}\,\bar{\Pi}_{cc}^{(000)}\right.$$
$$+\frac{2\mu}{m}\,\frac{g^2}{m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\frac{W^2-z^2}{4m^2}\left[\frac{W^2+z^2}{4m^2}\,\Pi_{sc}^{(002)} + \bar{\Pi}_{sc}^{(000)}\right]$$
$$+\frac{g^4}{m^4}\left[-\frac{z^4}{16m^4}\Pi_{ss}^{(020)} + \frac{W^4}{16m^4}\Pi_{ss}^{(002)} + \frac{W^2-z^2}{4m^2}\bar{\Pi}_{ss}^{(000)}\right.$$
$$+\frac{z^4}{16m^4}\Pi_{reg}^{(020)} - \frac{W^4}{16m^4}\Pi_{reg}^{(002)} - \frac{W^2-z^2}{4m^2}\bar{\Pi}_{reg}^{(000)}$$
$$\left.\left. +\frac{\mu}{2m}\left(1-\frac{q^2}{2\mu^2}\right)\left(1+\frac{\mu^2}{m^2}+\frac{q^2}{8m^2}+\frac{z^2}{8m^2}\right)\Pi_a\right]\right\}\,, \tag{D7}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)-} = \frac{m^2/4}{(4\pi)^2}\left\{\frac{\mu^2}{2m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)^2\bar{\Pi}_{cc}^{(000)} + \frac{2\mu}{m}\,\frac{g^2}{m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\left[\frac{W^2}{4m^2}\,\Pi_{sc}^{(002)} + \bar{\Pi}_{sc}^{(000)}\right]\right.$$
$$\left. +\frac{g^4}{m^4}\left[\frac{W^2}{4m^2}\Pi_{ss}^{(002)} + \bar{\Pi}_{ss}^{(000)} - \frac{W^2}{4m^2}\Pi_{reg}^{(002)} - \bar{\Pi}_{reg}^{(000)}\right]\right\}\,, \tag{D8}$$

$$\bullet\, \mathcal{I}^{(z)-}_{DB} = \frac{m^2/4}{(4\pi)^2}\left\{\frac{\mu^2}{2m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)^2\bar{\Pi}^{(000)}_{cc} + \frac{2\mu}{m}\,\frac{g^2}{m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\left[\frac{z^2}{4m^2}\,\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right]\right.$$

$$\left. + \frac{g^4}{m^4}\left[\frac{z^2}{4m^2}\Pi^{(020)}_{ss} + \bar{\Pi}^{(000)}_{ss} - \frac{z^2}{4m^2}\Pi^{(020)}_{reg} - \bar{\Pi}^{(000)}_{reg} + \frac{\mu}{2m}\left(\frac{3}{2}-\frac{5q^2}{8\mu^2}\right)\Pi_a\right]\right\}, \tag{D9}$$

$$\bullet\, \mathcal{I}^{(g)-}_{BB} = \frac{m^2/4}{(4\pi)^2}\left\{\frac{\mu^2}{2m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)^2\bar{\Pi}^{(000)}_{cc} + \frac{2\mu}{m}\,\frac{g^2}{m^2}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{\Pi}^{(000)}_{sc}\right.$$

$$\left. + \frac{g^4}{m^4}\left[\bar{\Pi}^{(000)}_{ss} - \bar{\Pi}^{(000)}_{reg} - \frac{\mu}{2m}\left(1-\frac{q^2}{4\mu^2}\right)\Pi_a\right]\right\}, \tag{D10}$$

$$\bullet\, \mathcal{I}^{(w)-}_{BB} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\frac{2\mu}{m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\Pi^{(002)}_{sc} + \frac{g^2}{m^2}\left[\Pi^{(002)}_{ss} - \Pi^{(002)}_{reg}\right]\right\}, \tag{D11}$$

$$\bullet\, \mathcal{I}^{(z)-}_{BB} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\frac{2\mu}{m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\Pi^{(020)}_{sc} + \frac{g^2}{m^2}\left[\Pi^{(020)}_{ss} - \Pi^{(020)}_{reg}\right]\right\}. \tag{D12}$$

<u>family 2</u> (diagrams g+h+i+j)

$$\bullet\, \mathcal{I}^{+}_{DD} = -\frac{\mu^2/4f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\,(1-2q^2/\mu^2)\left[\Pi^{(000)}_{cc} - \Pi^{(001)}_{sc} + 1\right]^2, \tag{D13}$$

and

$$\bullet\, \mathcal{I}^{-}_{DD} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,m^2\left\{\frac{W^2}{4m^2}\left[\frac{g^2}{m^2}\left(\frac{W^2}{4m^2}\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right) + \frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]^2\right.$$

$$\left. - \frac{z^2}{4m^2}\left[\frac{g^2}{m^2}\left(\frac{z^2}{4m^2}\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right) + \frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]^2\right\}, \tag{D14}$$

$$\bullet\, \mathcal{I}^{(w)-}_{DB} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\left[\frac{W^2}{4m^2}\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc} + \frac{\mu}{2m}\left(1-\frac{m^2}{g^2f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]^2, \tag{D15}$$

$$\bullet\, \mathcal{I}^{(z)-}_{DB} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\left[\frac{z^2}{4m^2}\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc} + \frac{\mu}{2m}\left(1-\frac{m^2}{g^2f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]^2, \tag{D16}$$

$$\bullet\, \mathcal{I}^{(g)-}_{BB} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\left[\bar{\Pi}^{(000)}_{sc} + \frac{\mu}{2m}\left(1-\frac{m^2}{g^2f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]^2, \tag{D17}$$

$$\bullet\, \mathcal{I}^{(w)-}_{BB} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\left\{\frac{W^2}{4m^2}\left[\Pi^{(002)}_{sc}\right]^2\right.$$

$$\left. + 2\,\Pi^{(002)}_{sc}\left[\bar{\Pi}^{(000)}_{sc} + \frac{\mu}{2m}\left(1-\frac{m^2}{g^2f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]\right\}, \tag{D18}$$

$$\bullet\, \mathcal{I}^{(z)-}_{BB} = -\frac{\mu^2/f_\pi^2}{(4\pi)^4}\,\frac{g^4}{m^2}\left\{\frac{z^2}{4m^2}\left[\Pi^{(002)}_{sc}\right]^2\right.$$

$$\left. + 2\,\Pi^{(002)}_{sc}\left[\bar{\Pi}^{(000)}_{sc} + \frac{\mu}{2m}\left(1-\frac{m^2}{g^2f_\pi^2}\right)\bar{\Pi}^{(000)}_{cc}\right]\right\}, \tag{D19}$$

<u>family 3</u> (diagrams (k+l+m+n+o)

$$\bullet\, \mathcal{I}^{+}_{DD} = \frac{1}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{m\left(\bar{d}^{+}_{00}+q^2 d^{+}_{01}\right)\left[\Pi^{(000)}_{cc} - \frac{W^2+z^2}{4m^2}\,\Pi^{(001)}_{sc}\right]\right.$$

$$\left. +\frac{\mu^3}{2}\left(1-q^2/2\mu^2\right)\left(d^{+}_{10}+q^2 d^{+}_{11}\right)\left[\frac{W^4+z^4}{16m^4}\Pi^{(002)}_{sc} + \frac{W^2+z^2}{4m^2}\bar{\Pi}^{(000)}_{sc}\right]\right\}$$

$$+\frac{1/2}{(4\pi)^2}\left\{\left(\bar{d}^{+}_{00}+q^2 d^{+}_{01}\right)^2\Pi^{(000)}_{cc} + 2\mu^2\left(\bar{d}^{+}_{00}+q^2 d^{+}_{01}\right)d^{+}_{10}\,\bar{\Pi}^{(000)}_{cc} + 3\mu^4\left(d^{+}_{10}\right)^2\bar{\bar{\Pi}}^{(000)}_{cc}\right\}\ , \tag{D20}$$

$$\bullet\, \mathcal{I}^{(w)+}_{DB} = -\frac{m/2}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\left(\bar{d}^{+}_{00}+q^2 d^{+}_{01}\right)\Pi^{(001)}_{sc} + \mu^2\left(d^{+}_{10}+q^2 d^{+}_{11}\right)\bar{\Pi}^{(000)}_{cc}\right\}\ , \tag{D21}$$

$$\bullet\, \mathcal{I}^{(z)+}_{DB} = \frac{m/2}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\left(\bar{d}^{+}_{00}+q^2 d^{+}_{01}\right)\Pi^{(001)}_{sc} - \mu^2\left(d^{+}_{10}+q^2 d^{+}_{11}\right)\bar{\Pi}^{(000)}_{cc}\right\}\ , \tag{D22}$$

$$\bullet\, \mathcal{I}^{(g)+}_{BB} = -\frac{\mu^2 m}{(4\pi)^2}\,\frac{g^2}{m^2}\,b^{+}_{00}\,\bar{\Pi}^{(000)}_{cc}\ , \tag{D23}$$

and

$$\bullet\, \mathcal{I}^{-}_{DD} = -\frac{\mu m}{(4\pi)^2}\,\frac{W^2-z^2}{4m^2}\left(\bar{d}^{-}_{00}+q^2\bar{d}^{-}_{01}\right)\left\{\frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}-\bar{d}^{-}_{00}-q^2\bar{d}^{-}_{01}\right)\bar{\Pi}^{(000)}_{cc}\right.$$

$$\left. +\frac{g^2}{m^2}\left[\frac{W^2+z^2}{4m^2}\,\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right]\right\}$$

$$+\frac{\mu^4/2}{(4\pi)^2}\left\{d^{-}_{10}\left[-3\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{\bar{\Pi}}^{(000)}_{cc} + \frac{g^2}{m^2}\left(1-q^2/2\mu^2\right)\bar{\Pi}^{(000)}_{cc}\right]\right\}\ , \tag{D24}$$

$$\bullet\, \mathcal{I}^{(w)-}_{DB} = -\frac{\mu m/2}{(4\pi)^2}\left\{\frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{b}^{-}_{00}\,\bar{\Pi}^{(000)}_{cc} + \frac{g^2}{m^2}\,\bar{b}^{-}_{00}\left[\frac{W^2}{4m^2}\,\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right]\right\}\ , \tag{D25}$$

$$\bullet\, \mathcal{I}^{(z)-}_{DB} = -\frac{\mu m/2}{(4\pi)^2}\left\{\frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}\right)\bar{b}^{-}_{00}\,\bar{\Pi}^{(000)}_{cc} + \frac{g^2}{m^2}\,\bar{b}^{-}_{00}\left[\frac{z^2}{4m^2}\,\Pi^{(002)}_{sc} + \bar{\Pi}^{(000)}_{sc}\right]\right\}\ , \tag{D26}$$

$$\bullet\, \mathcal{I}^{(g)-}_{BB} = -\frac{\mu m}{(4\pi)^2}\,\bar{b}^{-}_{00}\left\{\frac{\mu}{2m}\left(\frac{g^2}{m^2}-\frac{1}{f_\pi^2}-\bar{b}^{-}_{00}\right)\bar{\Pi}^{(000)}_{cc} + \frac{g^2}{m^2}\,\bar{\Pi}^{(000)}_{sc}\right\}\ . \tag{D27}$$

APPENDIX E: RELATIONS AMONG INTEGRALS

We derive here the relations among integrals needed for the chiral expansion of the potential. Results may also involve short range integrals, wich contain just one or no pion propagators, and are given by

$$H^{\mu\cdots}_{cc} \equiv \int[...]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{(k^2-\mu^2)}{\mu^2}\ , \tag{E1}$$

$$H^{\mu\cdots}_{sc} \equiv \int[...]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{2m/\mu\ \ (k^2-\mu^2)}{[s_1-m^2]}\ , \tag{E2}$$

$$H^{\mu\cdots}_{ss} \equiv \int[...]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{4m^2\ \ (k^2-\mu^2)}{[s_1-m^2][s_2-m^2]}\ , \tag{E3}$$

$$H^{\mu\cdots}_{us} \equiv \int[...]\left(\frac{Q^\mu}{\mu}\cdots\right)\frac{4m^2\ \ (k^2-\mu^2)}{[u_1-m^2][s_2-m^2]}\ , \tag{E4}$$

$$H_A^{i\cdots} \equiv \int [...] \left(\frac{Q^i}{\mu}\right) \frac{m^3}{\mu^2} \frac{(\kappa^2+\mu^2)}{E[E_Q^2 - E^2]} \,. \tag{E5}$$

In the sequence we use several times the results (A14) and (A15) in order to cancel pion propagators and obtain short range integrals.

◇1. Using eq.(B1) and the tensor decompositions of appendix B, we write

$$\frac{q^\mu}{\mu} I_{cc}^{\mu\nu} = \int [...] \left(\frac{Q^\nu}{\mu}\right) \frac{Q\cdot q}{\mu^2} = H_{cc}^\mu \to \quad \frac{q^2}{\mu^2}\, \Pi_{cc}^{(200)} + \bar{\Pi}_{cc}^{(000)} = \cdots \,, \tag{E6}$$

$$g_{\mu\nu}\, I_{cc}^{\mu\nu} = \int [...] \frac{Q^2}{\mu^2} = \left(1 - \frac{t}{4\mu^2}\right) I_{cc} + H_{cc}$$

$$\to \quad \frac{q^2}{\mu^2}\, \Pi_{cc}^{(200)} + 4\, \bar{\Pi}_{cc}^{(000)} = \left(1 - \frac{t}{4\mu^2}\right) \Pi_{cc}^{(000)} + \cdots \tag{E7}$$

where the ellipsis indicate that short range contributions were discarded. The combination of both results produces

$$\bullet \quad \bar{\Pi}_{cc}^{(000)} = \frac{1}{3}\left(1 - \frac{t}{4\mu^2}\right) \Pi_{cc}^{(000)} + \cdots \tag{E8}$$

◇2. Eq.(B1) also yields

$$\frac{q^\mu}{\mu} I_{cc}^{\mu\nu\lambda\rho} = \int [...] \left(\frac{Q^\nu Q^\lambda Q^\rho}{\mu^3}\right) \frac{Q\cdot q}{\mu^2} = H_{cc}^{\mu\lambda\rho} \to \quad \frac{q^2}{\mu^2}\, \bar{\Pi}_{cc}^{(200)} + \bar{\bar{\Pi}}_{cc}^{(000)} = \cdots \,, \tag{E9}$$

$$g_{\lambda\rho}\, I_{cc}^{\mu\nu\lambda\rho} = \int [...] \left(\frac{Q^\mu Q^\nu}{\mu^2}\right) \frac{Q^2}{\mu^2} = \left(1 - \frac{t}{4\mu^2}\right) I_{cc}^{\mu\nu} + H_{cc}^{\mu\nu}$$

$$\to \quad \bar{\bar{\Pi}}_{cc}^{(000)} = \frac{1}{5}\left(1 - \frac{t}{4\mu^2}\right) \bar{\Pi}_{cc}^{(000)} + \cdots \to \bullet \quad \bar{\bar{\Pi}}_{cc}^{(000)} = \frac{1}{15}\left(1 - \frac{t}{4\mu^2}\right)^2 \Pi_{cc}^{(000)} + \cdots \tag{E10}$$

◇3. Using eq.(B2) we get

$$\frac{(W+z)_\mu}{2m} I_{sc}^\mu = \int [...] \frac{Q\cdot(W+z)}{(s_1-m^2)} = \int [...] \left[1 - \frac{(Q^2-q^2/4)}{(s_1-m^2)}\right]$$

$$= \frac{(W-z)_\mu}{2m} I_{cs}^\mu = I_{cc} - \frac{\mu}{2m}\left(1 - \frac{t}{2\mu^2}\right) I_{sc} - \frac{\mu}{2m} H_{sc} \,. \tag{E11}$$

$$\to \bullet \quad \frac{W^2+z^2}{4m^2}\, \Pi_{sc}^{(001)} = \Pi_{cc}^{(000)} - \frac{\mu}{2m}\left(1 - \frac{t}{2\mu^2}\right) \Pi_{sc}^{(000)} + \cdots \tag{E12}$$

◇4. Using eq.(B2) again, we obtain

$$\frac{q^\mu}{2m} I_{sc}^{\mu\nu} = \int [...] \left(\frac{Q^\nu}{\mu}\right) \frac{Q\cdot q}{(s_1-m^2)} = \frac{\mu}{2m} H_{sc}^\nu \to \quad \frac{q^2}{\mu^2}\, \Pi_{sc}^{(200)} + \bar{\Pi}_{sc}^{(000)} = \cdots \,, \tag{E13}$$

$$\frac{(W+z)_\mu}{2m} I_{sc}^{\mu\nu} = \int [...] \frac{Q\cdot(W+z)\, Q^\nu/\mu}{(s_1-m^2)} = -\frac{\mu}{2m}\left(1 - \frac{t}{2\mu^2}\right) I_{sc}^\nu - \frac{\mu}{2m} H_{sc}^\nu$$

$$\to \bullet \quad \frac{W^2+z^2}{4m^2}\, \Pi_{sc}^{(002)} + \bar{\Pi}_{sc}^{(000)} = -\frac{\mu}{2m}\left(1 - \frac{t}{2\mu^2}\right) \Pi_{sc}^{(001)} + \cdots \,, \tag{E14}$$

$$g_{\mu\nu}\, I_{sc}^{\mu\nu} = \int [...] \frac{2m\, Q^2}{\mu(s_1-m^2)} = \left(1 - \frac{t}{4\mu^2}\right) I_{sc} + H_{sc}$$

$$\to \quad \frac{q^2}{\mu^2}\, \Pi_{sc}^{(200)} + \frac{W^2+z^2}{4m^2}\, \Pi_{sc}^{(002)} + 4\, \bar{\Pi}_{sc}^{(000)} = \left(1 - \frac{t}{4\mu^2}\right) \Pi_{sc}^{(000)} + \cdots \,. \tag{E15}$$

These results then give rise to

$$\bullet \quad \bar{\Pi}_{sc}^{(000)} = \frac{1}{2}\left(1-\frac{t}{4\mu^2}\right)\Pi_{sc}^{(000)} + \frac{\mu}{4m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{sc}^{(001)} + \cdots \, . \tag{E16}$$

◇5. From eq.(B3) we get

$$\frac{W_\mu}{2m}\, I_{ss}^{\mu} = \int [...]\, \frac{2m\mu\, Q\cdot W}{(s_1-m^2)(s_2-m^2)} = \int [...] \left[\frac{m\mu}{(s_1-m^2)} + \frac{m\mu}{(s_2-m^2)} - \frac{2m\mu\,(Q^2-t/4)}{(s_1-m^2)(s_2-m^2)}\right]$$

$$= \frac{1}{2}\,(I_{cs}+I_{sc}) - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right) I_{ss} - \frac{\mu}{2m} H_{ss}$$

$$\rightarrow \bullet \quad \frac{W^2}{4m^2}\,\Pi_{ss}^{(001)} = \Pi_{sc}^{(000)} - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{ss}^{(000)} + \cdots \, . \tag{E17}$$

◇6. Eq.(B3) also yields

$$\frac{q_\mu}{\mu}\, I_{ss}^{\mu\nu} = \int [...]\, \frac{2m\;\, Q\cdot q\;\, Q^\nu}{(s_1-m^2)(s_2-m^2)} = H_{ss}^{\mu} \rightarrow \quad \frac{q^2}{\mu^2}\,\Pi_{ss}^{(200)} + \bar{\Pi}_{ss}^{(000)} = \cdots \, , \tag{E18}$$

$$\frac{z_\mu}{2m}\, I_{ss}^{\mu\nu} = \int [...]\, \frac{2m\;\, Q\cdot z\;\, Q^\nu}{(s_1-m^2)(s_2-m^2)} = \int [...] \left[\frac{m\;\, Q^\nu}{(s_2-m^2)} - \frac{m\;\, Q^\nu}{(s_1-m^2)}\right]$$

$$= \frac{1}{2}\,(I_{cs}^{\nu} - I_{sc}^{\nu}) \rightarrow \bullet \quad \frac{z^2}{4m^2}\,\Pi_{ss}^{(020)} + \bar{\Pi}_{ss}^{(000)} = -\Pi_{sc}^{(001)} \, , \tag{E19}$$

$$\frac{W_\mu}{2m}\, I_{ss}^{\mu\nu} = \int [...]\, \frac{2m\, Q\cdot W\;\, Q^\nu}{(s_1-m^2)(s_2-m^2)} = \frac{1}{2}\,(I_{cs}^{\nu}+I_{sc}^{\nu}) - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right) I_{ss}^{\nu} - \frac{\mu}{2m} H_{ss}^{\nu}$$

$$\rightarrow \bullet \quad \frac{W^2}{4m^2}\,\Pi_{ss}^{(002)} + \bar{\Pi}_{ss}^{(000)} = \Pi_{sc}^{(001)} - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{ss}^{(001)} + \cdots \, . \tag{E20}$$

$$g_{\mu\nu}\, I_{ss}^{\mu\nu} = \int [...]\, \frac{4m^2\, Q^2}{(s_1-m^2)(s_2-m^2)} = \left(1-\frac{t}{4\mu^2}\right) I_{ss} + H_{ss}$$

$$\rightarrow \quad \frac{q^2}{\mu^2}\,\Pi_{ss}^{(200)} + \frac{z^2}{4m^2}\,\Pi_{ss}^{(020)} + \frac{W^2}{4m^2}\,\Pi_{ss}^{(002)} + 4\,\bar{\Pi}_{ss}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\Pi_{ss}^{(000)} + \cdots \, . \tag{E21}$$

Combining these results, we find

$$\bullet \quad \frac{W^4}{16m^4}\,\Pi_{ss}^{(002)} + \frac{W^2}{4m^2}\bar{\Pi}_{ss}^{(000)} =$$

$$\frac{\mu^2}{4m^2}\left(1-\frac{t}{2\mu^2}\right)^2 \Pi_{ss}^{(000)} + \frac{W^2}{4m^2}\,\Pi_{sc}^{(001)} - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\,\Pi_{sc}^{(000)} + \cdots \, , \tag{E22}$$

$$\bullet \quad \bar{\Pi}_{ss}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\Pi_{ss}^{(000)} + \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{ss}^{(001)} \, . \tag{E23}$$

◇7. Another relation involving $\Pi_{ss}^{(000)}$ may be obtained by deriving eq.(B20) with respect to μ and using eq.(B30):

$$\bullet \quad \mu\, \frac{d\,\bar{\Pi}_{ss}^{(000)}}{d\mu} = \Pi_{ss}^{(000)} + \frac{\mu}{m}\,\Pi_{ss}^{(001)} \, . \tag{E24}$$

◇8. Relations among the integrals Π_{us} can be obtained form those corresponding to Π_{ss} by making $W \leftrightarrow -z$ and we obtain

$$\frac{z^2}{4m^2}\,\Pi_{us}^{(010)} = -\Pi_{sc}^{(000)} + \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{us}^{(000)} + \cdots\,, \tag{E25}$$

$$\frac{q^2}{\mu^2}\,\Pi_{us}^{(200)} + \bar{\Pi}_{us}^{(000)} = \cdots\,, \tag{E26}$$

$$\frac{z^2}{4m^2}\,\Pi_{us}^{(020)} + \bar{\Pi}_{us}^{(000)} = \Pi_{sc}^{(001)} + \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{us}^{(010)} + \cdots\,, \tag{E27}$$

$$\frac{W^2}{4m^2}\,\Pi_{us}^{(002)} + \bar{\Pi}_{us}^{(000)} = -\Pi_{sc}^{(001)}\,, \tag{E28}$$

$$\frac{q^2}{\mu^2}\,\Pi_{us}^{(200)} + \frac{z^2}{4m^2}\,\Pi_{us}^{(020)} + \frac{W^2}{4m^2}\,\Pi_{us}^{(002)} + 4\,\bar{\Pi}_{us}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\Pi_{us}^{(000)} + \cdots\,. \tag{E29}$$

◇9. Using eq.(C11), we have

$$\frac{z_i}{2m}\,I_A^i = i\int(...)\,\frac{m^2\;\boldsymbol{Q}\cdot\boldsymbol{z}}{2\mu E[E_Q^2-E^2]} = i\int(...)\frac{m^2\;[(\boldsymbol{Q}^2-\boldsymbol{q}^2/4)-(E_Q^2-E^2)]}{2\mu\;E[E_Q^2-E^2]}$$

$$= -\frac{\mu}{2m}\left(1+\frac{\boldsymbol{q}^2}{2\mu^2}\right)I_A + \frac{\mu}{2m}H_A + \frac{i}{(4\pi)^2}\,\frac{m^3}{\mu^2 E}\,\Pi_a$$

$$\rightarrow \frac{z^2}{4m^2}\,\Pi_{it}^{(010)} = \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{it}^{(000)} - \frac{2m}{W}\,\Pi_a + \cdots\,. \tag{E30}$$

Using eq.(E25) we get

$$\bullet\quad \frac{z^2}{4m^2}\,\Pi_{reg}^{(010)} = \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{reg}^{(000)} + \Pi_{sc}^{(000)} - \frac{2m}{W}\,\Pi_a + \cdots\,. \tag{E31}$$

◇10. Integrating eq.(C55) by parts, we find

$$\frac{W^2}{4m^2}\,\Pi_{it}^{(002)} + \bar{\Pi}_{it}^{(000)} = 0 \tag{E32}$$

and eq.(E28) yields

$$\bullet\quad \frac{W^2}{4m^2}\,\Pi_{reg}^{(002)} + \bar{\Pi}_{reg}^{(000)} = \Pi_{sc}^{(001)}\,. \tag{E33}$$

◇11. Eq.(C11) also allows one to write

$$\frac{q_i}{\mu}\,I_A^{ij} = i\int(...)\,\frac{m^3\;\boldsymbol{Q}\cdot\boldsymbol{q}\;Q^j}{\mu^3 E[E_Q^2-E^2]} = -H_A^j \rightarrow \frac{q^2}{\mu^2}\,\Pi_{it}^{(200)} + \bar{\Pi}_{it}^{(000)} = \cdots\,, \tag{E34}$$

$$\frac{z_i}{2m}\,I_A^{ij} = i\int(...)\,\frac{m^2\;\boldsymbol{Q}\cdot\boldsymbol{z}\;Q^j}{2\mu^2 E[E_Q^2-E^2]} = -\frac{\mu}{2m}\left(1+\frac{\boldsymbol{q}^2}{2\mu^2}\right)I_A^j + \frac{\mu}{2m}H_A^j$$

$$\rightarrow \frac{z^2}{4m^2}\,\Pi_{it}^{(020)} + \bar{\Pi}_{it}^{(000)} = \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{it}^{(010)} + \cdots\,, \tag{E35}$$

$$\delta_{ij}\,I_A^{ij} = i\int(...)\,\frac{m^3\;\boldsymbol{Q}^2}{\mu^2 E[E_Q^2-E^2]} = i\int(...)\frac{m^3\;[\boldsymbol{q}^2/4+\boldsymbol{Q}\cdot\boldsymbol{z}+(E_Q^2-E^2)]}{\mu^2 E[E_Q^2-E^2]}$$

$$= \frac{\boldsymbol{q}^2}{4\mu^2}\,I_A + \frac{z_i}{\mu}\,I_A^i + i\frac{m^3}{\mu^2 E}\int(...) \rightarrow \frac{q^2}{\mu^2}\,\Pi_{it}^{(200)} + \frac{z^2}{4m^2}\,\Pi_{it}^{(020)} + 3\,\bar{\Pi}_{it}^{(000)}$$

$$= \frac{\boldsymbol{q}^2}{4\mu^2}\,\Pi_{it}^{(000)} + \frac{z^2}{2m\mu}\,\Pi_{it}^{(010)} + \frac{4m^2}{\mu W}\,\Pi_a\,. \tag{E36}$$

Using eq.(E30) and (E32) into the last expression, we find

$$\frac{q^2}{\mu^2}\,\Pi_{it}^{(200)} + \frac{z^2}{4m^2}\,\Pi_{it}^{(020)} + \frac{W^2}{4m^2}\,\Pi_{it}^{(002)} + 4\,\bar{\Pi}_{it}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\,\Pi_{it}^{(000)}\,. \tag{E37}$$

Combining these results with eqs.(E26-E29), we get

$$\frac{q^2}{\mu^2}\,\Pi_{reg}^{(200)} + \bar{\Pi}_{reg}^{(000)} = \cdots\,, \tag{E38}$$

$$\bullet\;\frac{z^2}{4m^2}\,\Pi_{reg}^{(020)} + \bar{\Pi}_{reg}^{(000)} = \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{reg}^{(010)} - \Pi_{sc}^{(001)} + \cdots\,, \tag{E39}$$

$$\frac{q^2}{\mu^2}\,\Pi_{reg}^{(200)} + \frac{z^2}{4m^2}\,\Pi_{reg}^{(020)} + \frac{W^2}{4m^2}\,\Pi_{reg}^{(002)} + 4\,\bar{\Pi}_{reg}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\Pi_{reg}^{(000)} + \cdots\,. \tag{E40}$$

Using eqs.(E31) and (E33), we also find

$$\bullet\;\frac{z^4}{16m^4}\,\Pi_{reg}^{(020)} + \frac{z^2}{4m^2}\,\bar{\Pi}_{reg}^{(000)} = \frac{\mu^2}{4m^2}\left(1-\frac{t}{2\mu^2}\right)^2\Pi_{reg}^{(000)} - \frac{z^2}{4m^2}\,\Pi_{sc}^{(001)}$$

$$+\frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\left(\Pi_{sc}^{(000)} - \frac{2m}{W}\,\Pi_a\right) + \cdots\,. \tag{E41}$$

$$\bullet\;\bar{\Pi}_{reg}^{(000)} = \left(1-\frac{t}{4\mu^2}\right)\Pi_{reg}^{(000)} - \frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_{reg}^{(010)}\,. \tag{E42}$$

◊12. Deriving eq.(C55) with respect to μ we find

$$\bullet\;\mu\,\frac{d\,\bar{\Pi}_{reg}^{(000)}}{d\mu} = \Pi_{reg}^{(000)} - \frac{\mu}{m}\,\Pi_{reg}^{(010)}\,. \tag{E43}$$

APPENDIX F: INTERMEDIATE RESULTS

The results presented here for the TPEP were obtained by using the relations among integrals of the previous appendix into the full expressions of appendix D. In this procedure we just neglected short range integrals and both sets of equations are equivalent for distances larger than 1 fm. In family 3, we did not keep contributions larger than $\mathcal{O}(q^4)$, in order to avoid unnecessarily long equations.

<u>**family 1**</u> (diagrams a+b+c+d+e+f)

$$\bullet\,\mathcal{I}_{DD}^{+} = \frac{\mu^2/8}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{\frac{1}{2}\,(1-t/2\mu^2)^2\,\left[\Pi_{ss}^{(000)} - \Pi_{reg}^{(000)}\right] - \frac{\mu}{m}\,(1-t/2\mu^2)\,\Pi_a\right\}\,, \tag{F1}$$

$$\bullet\,\mathcal{I}_{DB}^{(w)+} = -\,\frac{\mu m/8}{(4\pi)^2}\,\frac{g^4}{m^4}\,(1-t/2\mu^2)\,\Pi_{ss}^{(001)}\,, \tag{F2}$$

$$\bullet\,\mathcal{I}_{DB}^{(z)+} = \frac{\mu m/8}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{(1-t/2\mu^2)\,\Pi_{reg}^{(010)} - (3/2 - 5t/8\mu^2)\,\Pi_a\right\}\,, \tag{F3}$$

$$\bullet\,\mathcal{I}_{BB}^{(g)+} = \frac{m^2/4}{(4\pi)^2}\,\frac{g^4}{m^4}\left\{(1-t/4\mu^2)\left(\Pi_{ss}^{(000)} + \Pi_{reg}^{(000)}\right)\right.$$

$$\left.+\frac{\mu}{2m}\left[(1-t/2\mu^2)\left(\Pi_{ss}^{(001)} - \Pi_{reg}^{(010)}\right) + (1-t/4\mu^2)\,\Pi_a\right]\right\}\,, \tag{F4}$$

$$\bullet\, \mathcal{I}_{BB}^{(w)+} = \frac{m^2/4}{(4\pi)^2}\, \frac{g^4}{m^4} \left\{\Pi_{ss}^{(002)} + \Pi_{reg}^{(002)}\right\} , \tag{F5}$$

$$\bullet\, \mathcal{I}_{BB}^{(z)+} = \frac{m^2/4}{(4\pi)^2}\, \frac{g^4}{m^4} \left\{\Pi_{ss}^{(020)} + \Pi_{reg}^{(020)}\right\} , \tag{F6}$$

and

$$\bullet\, \mathcal{I}_{DD}^{-} = \frac{\mu^2/8}{(4\pi)^2} \left\{ \frac{1}{3}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)^2 \left[1 - \frac{\mu^2}{m^2}(t/4\mu^2 + z^2/2\mu^2)\right] (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$

$$- 2\,\frac{g^2}{m^2}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right) \left[1 - \frac{\mu^2}{m^2}(t/4\mu^2 + z^2/2\mu^2)\right] (1-t/2\mu^2)\, \Pi_{sc}^{(001)}$$

$$\left. + \frac{g^4}{m^4}\, \left[\frac{1}{2}\,(1-t/2\mu^2)^2 \left(\Pi_{ss}^{(000)} + \Pi_{reg}^{(000)}\right) + \frac{\mu}{m}(1-t/2\mu^2)\, \Pi_a\right]\right\} , \tag{F7}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)-} = \frac{\mu m/8}{(4\pi)^2} \left\{ \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)^2 (1-t/4\mu^2)\, \Pi_{cc}^{(000)} - \frac{g^2}{m^2}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)\right.$$

$$\left. \times \frac{\mu}{m}\left[2\,(1-t/2\mu^2)\, \Pi_{sc}^{(001)} + \frac{\mu}{m}\,(z^2/\mu^2)\, \Pi_{sc}^{(002)}\right] - \frac{g^4}{m^4}\,(1-t/2\mu^2)\, \Pi_{ss}^{(001)} \right\} , \tag{F8}$$

$$\bullet\, \mathcal{I}_{DB}^{(z)-} = \frac{\mu m/8}{(4\pi)^2} \left\{ \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)^2 (1-t/4\mu^2)\, \Pi_{cc}^{(000)} + \frac{g^2}{m^2}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)\right.$$

$$\times \left[2\,(1-t/4\mu^2)\, \Pi_{sc}^{(000)} + \frac{\mu}{m}\,(1-t/2\mu^2)\, \Pi_{sc}^{(001)} + \frac{\mu^2}{m^2}\,(z^2/\mu^2)\, \Pi_{sc}^{(002)}\right]$$

$$\left. - \frac{g^4}{m^4}\left[(1-t/2\mu^2)\, \Pi_{reg}^{(010)} - (3/2 - 5t/8\mu^2)\Pi_a\right] \right\} , \tag{F9}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)-} = \frac{\mu m/8}{(4\pi)^2} \left\{ \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)^2 (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$

$$+ \frac{g^2}{m^2}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right) \left[2\,(1-t/4\mu^2)\, \Pi_{sc}^{(000)} + \frac{\mu}{m}\,(1-t/2\mu^2)\, \Pi_{sc}^{(001)}\right]$$

$$\left. - \frac{g^4}{m^4}\left[(1-t/2\mu^2)\, \Pi_{reg}^{(010)} + (1-t/4\mu^2)\, \Pi_a + \frac{\mu}{m}\,(z^2/2\mu^2)\,\left(\Pi_{ss}^{(020)} - \Pi_{reg}^{(020)}\right)\right]\right\} , \tag{F10}$$

$$\bullet\, \mathcal{I}_{BB}^{(w)-} = \frac{m^2/4}{(4\pi)^2}\, \frac{g^2}{m^2} \left\{ \frac{2\mu}{m}\left(\frac{g^2}{f_\pi^2} - \frac{1}{f_\pi^2}\right) \Pi_{sc}^{(002)} + \frac{g^2}{m^2}\left[\Pi_{ss}^{(002)} - \Pi_{reg}^{(002)}\right]\right\} , \tag{F11}$$

$$\bullet\, \mathcal{I}_{BB}^{(z)-} = \frac{m^2/4}{(4\pi)^2}\, \frac{g^2}{m^2} \left\{ \frac{2\mu}{m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right) \Pi_{sc}^{(020)} + \frac{g^2}{m^2}\left[\Pi_{ss}^{(020)} - \Pi_{reg}^{(020)}\right]\right\} . \tag{F12}$$

family 2 (diagrams g+h+i+j)

$$\bullet\, \mathcal{I}_{DD}^{+} = -\frac{\mu^4/16 f_\pi^2}{(4\pi)^4}\, \frac{g^4}{m^4}\, (1 - 2t/\mu^2)\left[(1 - t/2\mu^2)\Pi_{sc}^{(000)} - 2\pi\right]^2 , \tag{F13}$$

and

$$\bullet\, \mathcal{I}^{-}_{DD} = -\frac{\mu^4/4f_\pi^2}{(4\pi)^4}\left\{\frac{W^2}{4m^2}\left[-\frac{g^2}{m^2}\left((1-t/2\mu^2)\,\Pi^{(001)}_{sc} + \frac{\mu}{m}\,(z^2/2\mu^2)\,\Pi^{(002)}_{sc} + 1 - t/3\mu^2\right)\right.\right.$$

$$\left. + \frac{1}{3}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)\left((1-t/4\mu^2)\,\Pi^{(000)}_{cc} + 2 - t/4\mu^2\right)\right]^2$$

$$-\frac{z^2}{4\mu^2}\left[\frac{g^2}{m^2}\left((1-t/4\mu^2)\,\Pi^{(000)}_{sc} + \frac{\mu}{2m}\,(1-t/2\mu^2)\,\Pi^{(001)}_{sc} + \frac{\mu^2}{m^2}\,(z^2/2\mu^2)\,\Pi^{(002)}_{sc} - \pi\right)\right.$$

$$\left.\left. + \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right]^2\right\}\,, \tag{F14}$$

$$\bullet\, \mathcal{I}^{(w)-}_{DB} = -\frac{\mu^4/4f_\pi^2}{(4\pi)^4}\left[-\frac{g^2}{m^2}\left((1-t/2\mu^2)\,\Pi^{(001)}_{sc} + \frac{\mu}{m}\,(z^2/2\mu^2)\,\Pi^{(002)}_{sc} + 1 - t/3\mu^2\right)\right.$$

$$\left. + \frac{1}{3}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)\left((1-t/4\mu^2)\,\Pi^{(000)}_{cc} + 2 - t/4\mu^2\right)\right]^2\,, \tag{F15}$$

$$\bullet\, \mathcal{I}^{(z)-}_{DB} = -\frac{\mu^2 m^2/4f_\pi^2}{(4\pi)^4}\left[\frac{g^2}{m^2}\left((1-t/4\mu^2)\,\Pi^{(000)}_{sc} + \frac{\mu}{2m}\,(1-t/2\mu^2)\,\Pi^{(001)}_{sc}\right.\right.$$

$$\left.\left. + \frac{\mu^2}{m^2}\,(z^2/2\mu^2)\,\Pi^{(002)}_{sc} - \pi\right) + \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right]^2\,, \tag{F16}$$

$$\bullet\, \mathcal{I}^{(g)-}_{BB} = -\frac{\mu^2 m^2/4f_\pi^2}{(4\pi)^4}\left[\frac{g^2}{m^2}\left((1-t/4\mu^2)\,\Pi^{(000)}_{sc} + \frac{\mu}{2m}\,(1-t/2\mu^2)\,\Pi^{(001)}_{sc} - \pi\right)\right.$$

$$\left. + \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right]^2\,, \tag{F17}$$

$$\bullet\, \mathcal{I}^{(w)-}_{BB} = -\frac{\mu^2 m^2/2f_\pi^2}{(4\pi)^4}\,\frac{g^2}{m^2}\,\Pi^{(002)}_{sc}\left[\frac{g^2}{m^2}\left((1-t/4\mu^2)\,\Pi^{(000)}_{sc} - \frac{\mu}{2m}\,(1-t/2\mu^2)\,\Pi^{(001)}_{sc}\right.\right.$$

$$\left.\left. - \frac{\mu^2}{m^2}\,(z^2/\mu^2)\,\Pi^{(002)}_{sc}\right) + \frac{2\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right]\,, \tag{F18}$$

$$\bullet\, \mathcal{I}^{(z)-}_{BB} = -\frac{\mu^2 m^2/f_\pi^2}{(4\pi)^4}\,\frac{g^2}{m^2}\,\Pi^{(002)}_{sc}\left[\frac{g^2}{m^2}\left((1-t/4\mu^2)\,\Pi^{(000)}_{sc} + \frac{\mu}{2m}\,(1-t/2\mu^2)\,\Pi^{(001)}_{sc}\right.\right.$$

$$\left.\left. + \frac{\mu^2}{m^2}\,(z^2/4\mu^2)\,\Pi^{(002)}_{sc}\right) + \frac{\mu}{3m}\left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2}\right)(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right]\,. \tag{F19}$$

__family 3__ (diagrams k+l+m+n+o)

$$\bullet\, \mathcal{I}^{+}_{DD} = \frac{\mu^3/2mf_\pi^2}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\left(\bar{\delta}^+_{00} + t/\mu^2\,\delta^+_{01}\right)\,(1-t/2\mu^2)\,\Pi^{(000)}_{sc}\right.$$

$$\left. -\frac{\mu}{2m}\,\delta^+_{10}\,(1-t/2\mu^2)^2\,\Pi^{(001)}_{sc}\right\} + \frac{\mu^4/2m^2}{(4\pi)^2}\,\frac{1}{f_\pi^4}\left\{\left(\bar{\delta}^+_{00} + t/\mu^2\,\delta^+_{01}\right)^2\right.$$

$$\left. + \frac{2}{3}\left(\bar{\delta}^+_{00} + t/\mu^2\,\delta^+_{01}\right)\delta^+_{10}\,(1-t/4\mu^2) + \frac{1}{5}\left(\delta^+_{10}\right)^2\,(1-t/4\mu^2)^2\right\}\Pi^{(000)}_{cc}\,, \tag{F20}$$

$$\bullet\, \mathcal{I}^{(w)+}_{DB} = -\frac{\mu^2/2f_\pi^2}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\left[\bar{\delta}^+_{00} + (t/\mu^2)\,\delta^+_{01}\right]\Pi^{(001)}_{sc} + \frac{1}{3}\,\delta^+_{10}\,(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right\}\,, \tag{F21}$$

$$\bullet\, \mathcal{I}^{(z)+}_{DB} = \frac{\mu^2/2f_\pi^2}{(4\pi)^2}\,\frac{g^2}{m^2}\left\{\left[\bar{\delta}^+_{00} + (t/\mu^2)\,\delta^+_{01}\right]\Pi^{(001)}_{sc} - \frac{1}{3}\,\delta^+_{10}\,(1-t/4\mu^2)\,\Pi^{(000)}_{cc}\right\}\,, \tag{F22}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)+} = -\frac{\mu^2/f_\pi^2}{(4\pi)^2}\, \frac{g^2}{m^2}\, \frac{1}{3}\, \beta_{00}^{+}\, (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \;, \tag{F23}$$

and

$$\bullet\, \mathcal{I}_{DD}^{-} = -\frac{\mu^4/2m^2 f_\pi^2}{(4\pi)^2} \left\{ \frac{1}{3} \left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2} \right) \left[\bar{\delta}_{00}^{-} + (t/\mu^2)\, \bar{\delta}_{01}^{-} + \frac{3}{5}\, (1-t/4\mu^2)\, \delta_{10}^{-} \right] (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$
$$\left. - \frac{g^2}{m^2}\, (1-t/2\mu^2) \left[\left(\bar{\delta}_{00}^{-} + (t/\mu^2)\, \bar{\delta}_{01}^{-} \right) \Pi_{sc}^{(001)} + \frac{1}{3}\, \delta_{10}^{-}\, (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right] \right\} \;, \tag{F24}$$

$$\bullet\, \mathcal{I}_{DB}^{(w)-} = -\frac{\mu^2/4f_\pi^2}{(4\pi)^2}\, \bar{\beta}_{00}^{-} \left\{ \frac{1}{3} \left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2} \right) (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$
$$\left. - \frac{g^2}{m^2} \left[(1-t/2\mu^2)\, \Pi_{sc}^{(001)} + \frac{\mu}{2m}\, (z^2/\mu^2)\, \Pi_{sc}^{(002)} \right] \right\} \;, \tag{F25}$$

$$\bullet\, \mathcal{I}_{DB}^{(z)-} = -\frac{\mu m/4f_\pi^2}{(4\pi)^2}\, \bar{\beta}_{00}^{-} \left\{ \frac{\mu}{3m} \left(\frac{g^2}{m^2} - \frac{1}{f_\pi^2} \right) (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$
$$\left. + \frac{g^2}{m^2} \left[(1-t/4\mu^2)\, \Pi_{sc}^{(000)} + \frac{\mu}{2m}\, (1-t/2\mu^2)\, \Pi_{sc}^{(001)} + \frac{\mu^2}{m^2}\, (z^2/2\mu^2)\, \Pi_{sc}^{(002)} \right] \right\} \;, \tag{F26}$$

$$\bullet\, \mathcal{I}_{BB}^{(g)-} = -\frac{\mu m/2f_\pi^2}{(4\pi)^2}\, \bar{\beta}_{00}^{-} \left\{ \frac{\mu}{3m} \left(\frac{g^2}{m^2} - \frac{1+\bar{\beta}_{00}^{-}}{f_\pi^2} \right) (1-t/4\mu^2)\, \Pi_{cc}^{(000)} \right.$$
$$\left. + \frac{g^2}{m^2} \left[(1-t/4\mu^2)\, \Pi_{sc}^{(000)} + \frac{\mu}{2m}\, (1-t/2\mu^2)\, \Pi_{sc}^{(001)} \right] \right\} \;. \tag{F27}$$

APPENDIX G: RELATIVISTIC EXPANSIONS

In section VI we have discussed the relativistic expansion of the function $\Pi_{sc}^{(000)}$ derived by Becher and Leutwyler, which does not coincide with the usual heavy baryon expansion. In this appendix we show how their results cam be used to produce relativistic expansions for box and crossed box integrals.

The triangle, crossed box and regularized box integrals given respectively by eqs.(B17), (B19) and (C54) can be written as

$$\bar{\Pi}_{ss}^{(000)} = -\int_0^1 dc\, \Pi_{sc}^{(001)}(\mathcal{M}_{ss}) \;, \tag{G1}$$

$$\bar{\Pi}_{reg}^{(000)} = -\int_1^\infty dc\, \Pi_{sc}^{(001)}(\mathcal{M}_{us}) \;, \tag{G2}$$

where $\Pi_{sc}^{(00n)}(\mathcal{M})$ is a generalized triangle integral, given by

$$\Pi_{sc}^{(00n)}(\mathcal{M}) = \left(-\frac{2m}{\mu} \right)^{n+1} \int_0^1 da\, a \int_0^1 db\, \frac{\mu^2\, (ab/2)^n}{D(\mathcal{M})} \tag{G3}$$

and the denominator $D(\mathcal{M})$ is

$$D(\mathcal{M}) = \mathcal{M}^2 a^2 b^2 - a(1-a)(1-b)\, q^2 + (1-ab)\, \mu^2 \;. \tag{G4}$$

When $\mathcal{M}_{sc} = m$, one recovers the triangle integral defined in eq.(B17). On the other hand, the values $\mathcal{M}_{ss}^2 = (W^2 + q^2 + c^2 z^2)/4$ and $\mathcal{M}_{us}^2 = (c^2 W^2 + q^2 + z^2)/4$ yield eqs.(G1) and (G2).

Performing explicitly the b integration in eq.(G4), we obtain the generalization of eq.(E12), that reads

$$\left(1-t/4\mathcal{M}^2\right)\ \Pi_{sc}^{(001)}(\mathcal{M})=\frac{m^2}{\mathcal{M}^2}\left[\Pi_{cc}^{(000)}-\frac{\mu}{2m}\left(1-t/2\mu^2\right)\Pi_{sc}^{(000)}(\mathcal{M})+\Pi_L(\mathcal{M})\right]\ , \tag{G5}$$

with

$$\Pi_L(\mathcal{M})=\left(1-\frac{\mu^2}{\mathcal{M}^2}\right)\left(\ln\frac{\mathcal{M}^2}{\mu^2}-2\right)+2\ \frac{\mu}{\mathcal{M}}\sqrt{1-\frac{\mu^2}{4\mathcal{M}^2}}\ \tan^{-1}\left(\frac{\mathcal{M}}{\mu}\ \sqrt{\frac{1-\mu/2\mathcal{M}}{1+\mu/2\mathcal{M}}}\right)\ . \tag{G6}$$

In all cases $\mathcal{M}$ is a large parameter and we can use the relativistic expansion of the triangle integral discussed in section VI which, in the present case, is given by

$$\Pi_{sc}^{(000)}(\mathcal{M})=\frac{m}{\mathcal{M}}\left[\Pi_a+\frac{\mu}{2\mathcal{M}}\ \Pi_t^{NL}+\frac{\mu^2}{4\mathcal{M}^2}\left(\frac{t}{2\mu^2}\ \Pi_a+\Pi_t^{th}(\mathcal{M})\right)\right]\ , \tag{G7}$$

with Π_a and Π_t^{NL} given by eqs.(6.9,6.10) and

$$\begin{aligned}\Pi_t^{th}(\mathcal{M})&=\frac{2\pi\mathcal{M}^2}{\mu^2}\left[-\frac{\mu}{\mathcal{M}\sqrt{1-t/4\mu^2}}+2\ \ln\left(1+\frac{\mu}{2\mathcal{M}\sqrt{1-t/4\mu^2}}\right)\right]\\&=-\ \frac{\pi/2}{1-t/4\mu^2}\ .\end{aligned} \tag{G8}$$

Inserting these results into eqs.(G1) and (G2), we obtain

$$\begin{aligned}\bar{\Pi}^{(000)}=&-\int dc\ \frac{m^2}{\mathcal{M}^2-t/4}\ \{\Pi_b+\Pi_L(\mathcal{M})\\&-\frac{\mu}{2\mathcal{M}}\left(1-t/2\mu^2\right)\left[\Pi_a+\frac{\mu}{2\mathcal{M}}\ \Pi_t^{NL}+\frac{\mu^2}{4\mathcal{M}^2}\left(\frac{t}{2\mu^2}\ \Pi_a+\Pi_t^{th}(\mathcal{M})\right)\right]\right\}\ .\end{aligned} \tag{G9}$$

In the chiral limit $\mu\rightarrow 0$, we have

$$\bar{\Pi}_{ss}^{(000)}=\bar{\Pi}_{reg}^{(000)}\rightarrow-(1+z^2/6m^2)\ \Pi_\ell \tag{G10}$$

and, using eqs.(E17,E19,E20,E24) and (E31,E33,E39,E43), one finds the following relationships valid in that limit

$$\Pi_{ss}^{(001)}=-2\ \Pi_{reg}^{(001)}\rightarrow\Pi_a\ , \tag{G11}$$

$$\Pi_{ss}^{(020)}=\Pi_{reg}^{(020)}\rightarrow 2\ \Pi_\ell/3\ , \tag{G12}$$

$$\Pi_{ss}^{(002)}=\Pi_{reg}^{(002)}\rightarrow 2\ \Pi_\ell\ , \tag{G13}$$

$$\Pi_{ss}^{(000)}=\Pi_{reg}^{(000)}\rightarrow-\ \Pi_\ell'\ . \tag{G14}$$

These results may also be combined with those preseented in appendix E, in order to produce relativistic $\mathcal{O}(q^2)$ expansions for box and crossed box interals. Eqs.(E19,G14,E12) yield

- $$\bar{\Pi}_{ss}^{(000)}=-\left(1+\frac{t}{4m^2}+\frac{z^2}{6m^2}\right)\Pi_\ell+\frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\Pi_t \tag{G15}$$

and, using eqs.(E24,E17), one has

$$\Pi_{ss}^{(000)}=-\left(1+\frac{\mu^2}{2m^2}+\frac{z^2}{6m^2}\right)\ \Pi_\ell'-\frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\left[\Pi_t-\Pi_t'\right]\ . \tag{G16}$$

The results of section VI then produce

- $$\Pi_{ss}^{(000)}=-\Pi_\ell'-\frac{\mu}{m}\ \frac{\pi/2}{(1-t/4\mu^2)}-\frac{\mu^2}{4m^2}\left[(1-t/2\mu^2)^2\left(2\ \Pi_t'-\Pi_\ell''\right)+(2z^2/3\mu^2)\ \Pi_\ell'\right]+\cdots \tag{G17}$$

where the ellipsis represent polynomials in t.

For the box integrals we evaluate eq.(G19) directly and otain

- $\bar{\Pi}_{reg}^{(000)} = -\left(1+\frac{t}{4m^2}+\frac{z^2}{6m^2}\right)\Pi_\ell$

$$+\frac{\mu}{2m}\left(1-\frac{t}{2\mu^2}\right)\left[\frac{1}{2}\,\Pi_a+\frac{\mu}{6m}\,\Pi_t^{NL}+\frac{\mu^2}{16m^2}\,\Pi_t^{th}\right]\,. \tag{G18}$$

Comparing with eq.(E39) and using eq.(E12), we find

- $$\Pi_{reg}^{(010)} = -\,\frac{1}{2}\left[\Pi_a+\frac{2\mu}{3m}\,\Pi_t^{NL}+\frac{3\mu^2}{8m^2}\,\Pi_t^{th}\right]\,. \tag{G19}$$

Evaluating eq.(E43) we have

- $$\Pi_{reg}^{(000)} = -\Pi'_\ell-\frac{\mu}{m}\,\frac{\pi/4}{(1-t/4\mu^2)}-\frac{\mu^2}{12m^2}\left[(1-t/2\mu^2)^2\left(2\,\Pi'_\ell-\Pi''_\ell\right)+(2z^2/\mu^2)\,\Pi'_\ell\right]+\cdots \tag{G20}$$

Finally, we note that eq.(E31) allows one to write

- $$\left(1-\frac{t}{2\mu^2}\right)\Pi_{reg}^{(000)} = -\Pi_t^{NL}-\frac{\mu}{2m}\Pi_t^{th}-\frac{z^2}{6m^2}\,\Pi_t^{NL}\,. \tag{G21}$$

Using eq.(G21) and comparing with this result we obtain the *independent* relation for Π_t^{th}

$$\Pi_t^{th} = -\frac{\pi/2}{1-t/4\mu^2}+\frac{\mu}{6m}(1-t/2\mu^2)^3\left(2\,\Pi'_\ell-\Pi''_\ell\right)+\cdots\,, \tag{G22}$$

which is fully consistent with that derived by BL and discussed in section VI of the main text.

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