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BETATRON MOTION WITH COUPLING OF HORIZONTAL AND VERTICAL DEGREES OF FREEDOM*

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The Courant-Snyder[1] parametrization of one-dimensional linear betatron motion is generalized to two-dimensional coupled linear motion. To represent the 4×4 symplectic transfer matrix the following ten parameters were chosen: four beta-functions, four alpha-functions and two betatron phase advances. The beta-functions have a meaning similar to the Courant-Snyder parameterization, and the definition of alpha-functions coincides with the standard one at regions with zero longitudinal magnetic field, where they are equal to negative half-derivatives of the beta-functions. Such a parameterization can be useful for analysis of coupled betatron motion in circular machines and transfer lines.

Introduction

In many applications analysis of coupled betatron motion is an important part of the machine design. Although many studies of the coupled motion have been performed over the last 30 years[2,3,4,5,6,7], there is still no complete, in our opinion, representation of coupled betatron motion, which would be as elegant as the Courant-Snyder parameterization for one-dimensional case. This article attempts to introduce such a representation. To parametrize a 4x4 symplectic transfer matrix, the following ten parameters were chosen: four beta-functions, four alpha-functions and two betatron phase advances. The beta-functions have similar meaning to the Courant-Snyder parameterization, and the definition of alpha-functions coincides with the standard one at regions with zero longitudinal magnetic field, where they are equal to negative half-derivatives of the beta-functions.

Further implementation of this parameterization was included in a computer code developed by one of the authors♦. The resulting numerical study of a model lattice has proven to be useful for both theoretical understanding of coupled betatron motion and further code development. It has also assured us of the correctness of the analytical results.

1. Equations of Motion and Condition of Symplecticity

Two-dimensional linear motion of a particle in a focusing lattice structure can be described by the following set of equations:

$$\begin{aligned} &x'' + \left(K_x^{\ 2} + k\right)x + \left(N - \frac{1}{2}R'\right)y - Ry' = 0 \quad , \\ &y'' + \left(K_y^{\ 2} - k\right)y + \left(N + \frac{1}{2}R'\right)x + Rx' = 0 \quad . \end{aligned} \tag{1.1}$$

Here x and y are the horizontal and vertical particle displacements from the ideal orbit; the

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derivatives are calculated along the longitudinal coordinate *s*; $K_{x,y} = eB_{y,x} / Pc$; $k = eG / Pc$; $N = eG_s / Pc$; $R = eB_s / Pc$; B_x, B_y and B_s are the corresponding components of the magnetic field; *G* is the normal component of the magnetic field gradient; and G_s is the skew component of the magnetic field gradient (a quad tilted by 45 deg. around *s* axis in the right-handed coordinate system).

The Hamiltonian corresponding to Eq. (1.1) is equal to

$$H = \frac{p_x^{\,2} + p_y^{\,2}}{2} + \left(K_x^{\,2} + k + \frac{R^2}{4} \right) \frac{x^2}{2} + \left(K_y^{\,2} - k + \frac{R^2}{4} \right) \frac{y^2}{2} + Nxy + \frac{R}{2} \left(y p_x - x p_y \right) \ , \tag{1.2}$$

and the corresponding canonical momenta are

$$\begin{aligned} p_x &= x' - \frac{R}{2} y \ , \\ p_y &= y' + \frac{R}{2} x \ . \end{aligned} \tag{1.3}$$

Rewriting Eq.(1.3) in matrix form we obtain the relation between the canonical, $\hat{\mathbf{x}}$, and the geometric coordinates, **x**,

$$\hat{\mathbf{x}} = \mathbf{R}\mathbf{x} \ , \tag{1.4}$$

where

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} \ , \quad \mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix} \ , \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix} \ , \tag{1.5}$$

$\theta_x = x'$ and $\theta_y = y'$.

Introducing matrix **H**,

$$\mathbf{H} = \begin{bmatrix} K_x^{\,2} + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^{\,2} - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix} \ , \tag{1.6}$$

we can rewrite Eqs. (1.1) and (1.2) in the matrix form:

$$\frac{d}{ds}\hat{\mathbf{x}} = \mathbf{UH}\hat{\mathbf{x}} \quad , \tag{1.7}$$

$$H = \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}\hat{\mathbf{x}} \quad , \tag{1.8}$$

where the unit symplectic matrix $\mathbf{U}$ is introduced as follows,

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad . \tag{1.9}$$

For any two solutions of Eq. (1.7), $\hat{\mathbf{x}}_1(s)$ and $\hat{\mathbf{x}}_2(s)$, one can write that

$$\frac{d}{ds}\left(\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2\right) = \frac{d\hat{\mathbf{x}}_1{}^T}{ds}\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{U}\frac{d\hat{\mathbf{x}}_2}{ds} = \hat{\mathbf{x}}_1{}^T\mathbf{H}^T\mathbf{U}^T\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{UUH}\hat{\mathbf{x}}_2 = 0 \quad , \tag{1.10}$$

and, consequently,

$$\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2 = \text{const} \quad . \tag{1.11}$$

This motion integral is called the Lagrange invariant. Above in Eq. (1.10) the following properties of the unit symplectic matrix were employed: $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{UU} = -\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Introducing the transfer matrix from coordinate 0 to coordinate s, $\mathbf{x}(s) = \mathbf{M}(0,s)\mathbf{x}_0$, and the corresponding transfer matrix for the canonical variables, $\hat{\mathbf{x}}(s) = \hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0$, one finds using Eq.(1.4) that

$$\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(s) = \mathbf{R}(s)\mathbf{x}(s) = \mathbf{R}(s)\mathbf{M}(0,s)\mathbf{x}(0) = \mathbf{R}(s)\mathbf{M}(0,s)\mathbf{R}^{-1}(0)\hat{\mathbf{x}}_0 \quad , \tag{1.12}$$

which yields that the matrices are bound up by the following equation:

$$\hat{\mathbf{M}}(0,s) = \mathbf{R}(s)\mathbf{M}(0,s)\mathbf{R}(0)^{-1} \quad . \tag{1.13}$$

Here and below we put a cap above transfer matrices and vectors related to the canonical variables. It is also useful to note that the inverse of matrix $\mathbf{R}$ is

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & R/2 & 0 \\ 0 & 0 & 1 & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix} \quad . \tag{1.14}$$

Taking into account that the invariant of Eq.(1.11) does not change during motion we can write that

$$\hat{\mathbf{x}}_0{}^T\mathbf{U}\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0{}^T\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0 = \mathrm{const} \quad . \tag{1.15}$$

As the above equation is sutisfied for any $\hat{\mathbf{x}}$ it yields

$$\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s) = \mathbf{U} \quad . \tag{1.16}$$

Eq. (1.16) expresses the symplecticity condition for particle motion. It is equivalent to n^2=16 scalar equations, but taking into account that the matrix $\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)$ is antisymmetric, only six ((n^2–n)/2=6) of these equations are independent. Consequently, only 10 of 16 elements of the transfer matrix are independent. As one can see, the symplecticity condition imposes more severe limitations than Liouville's theorem, which imposes only one condition, det(**M**)=1, and leaves 15 independent parameters.

2. Eigen-vectors and Conditions of Orthogonality

Consider a circular accelerator with the total transfer matrix $\hat{\mathbf{M}}$. The transfer matrix has four eigen-values, λ_i, and four corresponding eigen-vectors, $\hat{\mathbf{v}}_i$ (i = 1, 2, 3, 4). It follows from the matrix symplecticity that the eigen-values are reciprocal. This can be proven as follows; let λ_i and λ_j ($i \neq j$) be two eigen-values,

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_i = \lambda_i\hat{\mathbf{v}}_i \quad , \text{ and } \qquad \hat{\mathbf{M}}\hat{\mathbf{v}}_j = \lambda_j\hat{\mathbf{v}}_j \ . \tag{2.1}$$

Then, using Eqs.(2.1) and Eq.(1.16) one can write the identity

$$0 = \lambda_j\hat{\mathbf{v}}_j{}^T\mathbf{U}\left(\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_i\hat{\mathbf{v}}_i\right) = \left(\hat{\mathbf{M}}\hat{\mathbf{v}}_j\right)^T\mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_j\hat{\mathbf{v}}_j{}^T\mathbf{U}\lambda_i\hat{\mathbf{v}}_i = \left(1-\lambda_j\lambda_i\right)\hat{\mathbf{v}}_j{}^T\mathbf{U}\hat{\mathbf{v}}_i \quad . \tag{2.2}$$

Any vector in the phase-space can be represented as a linear combination of four eigen-vectors $\hat{\mathbf{v}}_i$, and, consequently, these four vectors are linearly independent. The vector $\mathbf{U}\hat{\mathbf{v}}_i$ appearing in the last part of Eq. (2.2) cannot be simultaneously orthogonal to all four eigen-vectors $\hat{\mathbf{v}}_j$, and, hence, there is at least one vector $\hat{\mathbf{v}}_j$ for which $\lambda_j\lambda_i = 1$. That proves that the four eigen-values always appear in two reciprocal pairs.

Below, we will consider the case of a stable betatron motion, meaning all four eigen-values are confined to a unit circle and, consequently, the four eigen-values split into two complex conjugate pairs. We will denote them as λ_1, λ_1^*, λ_2 and λ_2^*, and the corresponding eigen-vectors as $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_1^*$, $\hat{\mathbf{v}}_2$ and $\hat{\mathbf{v}}_2^*$, where * denotes the complex conjugate value.

For the case of $\lambda_1 \neq \lambda_2$, we obtain the following set of orthogonality conditions:

$$\hat{\mathbf{v}}_1^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 \neq 0 \,,$$
$$\hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_2 \neq 0 \,, \qquad . \tag{2.3}$$
$$\hat{\mathbf{v}}_i^{\,T}\mathbf{U}\hat{\mathbf{v}}_j = 0 \,, \quad - \quad \text{otherwise},$$

where $\hat{\mathbf{v}}^+ = \hat{\mathbf{v}}^{*T}$. The values in the two top lines of Eq.(2.3) are purely imaginary, indeed:

$$\left(\hat{\mathbf{v}}^+\mathbf{U}\hat{\mathbf{v}}\right)^* = \left(\hat{\mathbf{v}}^+\mathbf{U}\hat{\mathbf{v}}\right)^+ = \hat{\mathbf{v}}^+\mathbf{U}^+\hat{\mathbf{v}} = -\hat{\mathbf{v}}^+\mathbf{U}\hat{\mathbf{v}} \quad . \tag{2.4}$$

Therefore one can normalize the eigen-vectors as follows:

$$\begin{aligned} &\hat{\mathbf{v}}_1^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = -2i \;, && \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_2 = -2i \;, \\ &\hat{\mathbf{v}}_1^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \;, && \hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_2 = 0 \;, \\ &\hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \;, && \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \;. \end{aligned} \tag{2.5}$$

Other combinations can be obtained by applying the transposition and (or) the complex conjugation to Eqs. (2.5).

3. Relation between Eigen-vectors and Emittance Ellipsoid in 4D Phase Space

The turn-by-turn particle positions and angles (at the beginning of the lattice) can be represented as a linear combination of four independent solutions,

$$\begin{aligned} \hat{\mathbf{x}} &= \mathrm{Re}\left(A_1 e^{-i\psi_1}\hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2}\hat{\mathbf{v}}_2\right) = A_1 \,\mathrm{Re}\left(\left(\cos\psi_1 - i\sin\psi_1\right)\left(\hat{\mathbf{v}}_1' + i\hat{\mathbf{v}}_1''\right)\right) \\ &\quad + A_2\,\mathrm{Re}\left(\left(\cos\psi_2 - i\sin\psi_2\right)\left(\hat{\mathbf{v}}_2' + i\hat{\mathbf{v}}_2''\right)\right) \\ &= A_1\left(\hat{\mathbf{v}}_1'\cos\psi_1 + \hat{\mathbf{v}}_1''\sin\psi_1\right) + A_2\left(\hat{\mathbf{v}}_2'\cos\psi_2 + \hat{\mathbf{v}}_1''\sin\psi_2\right) \quad , \end{aligned} \tag{3.1}$$

where four real parameters, A_1, A_2, ψ_1 and ψ_2, represent the betatron amplitudes and phases. The amplitudes remain constant in the course of betatron motion while the phases change after each turn.

Let us introduce the following real matrix

$$\mathbf{V} = \left[\hat{\mathbf{v}}_1', -\hat{\mathbf{v}}_1'', \hat{\mathbf{v}}_2', -\hat{\mathbf{v}}_2''\right] \quad . \tag{3.2}$$

This allows one to rewrite Eq. (3.1) in the compact form

$$\hat{\mathbf{x}} = \mathbf{V}\vec{\xi}_A \quad , \tag{3.3}$$

where

$$\vec{\xi}_A = \begin{bmatrix} A_1 \cos\psi_1 \\ -A_1 \sin\psi_1 \\ A_2 \cos\psi_2 \\ -A_2 \sin\psi_2 \end{bmatrix} \quad . \tag{3.4}$$

Applying orthogonality conditions given by Eq.(2.5), one can prove that matrix **V** is a symplectic matrix. It can be seen explicitly as follows:

$$\mathbf{V}^T\mathbf{U}\mathbf{V} = \left[\frac{\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1^*}{2}, -\frac{\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_1^*}{2i}, \frac{\hat{\mathbf{v}}_2 + \hat{\mathbf{v}}_2^*}{2}, -\frac{\hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_2^*}{2i}\right]^T \mathbf{U} \cdot$$
$$\left[\frac{\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1^*}{2}, -\frac{\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_1^*}{2i}, \frac{\hat{\mathbf{v}}_2 + \hat{\mathbf{v}}_2^*}{2}, -\frac{\hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_2^*}{2i}\right] = \mathbf{U} \quad . \tag{3.5}$$

Here we took into account that every matrix element in matrix $\mathbf{V}^T\mathbf{U}\mathbf{V}$ can be calculated using vector multiplication of Eqs. (2.5).

Let us consider an ensemble of particles, whose motion (at the beginning of lattice) is contained in a 4D ellipsoid. A 3D surface of this ellipsoid is determined by particles with extreme betatron amplitudes. For any of these particles, Eqs. (3.3) and (3.4) describe the 2D-subspace of single particle motion, which is a subspace of the 3D surface of the ellipsoid, described by the bilinear form, $\boldsymbol{\Xi}$, as follows:

$$\hat{\mathbf{x}}^T \boldsymbol{\Xi} \hat{\mathbf{x}} = 1 \quad . \tag{3.6}$$

This ellipsoid confines the motion of all particles. To describe a 3D surface we introduce the third parameter ψ_3 so that the vector $\vec{\xi}$ would describe a 3D sphere with a unit radius, according to the equation

$$\left(\vec{\xi}, \vec{\xi}\right) = 1 \quad , \tag{3.7}$$

where

$$\vec{\xi} = \begin{bmatrix} \cos\psi_1 \cos\psi_3 \\ -\sin\psi_1 \cos\psi_3 \\ \cos\psi_2 \sin\psi_3 \\ -\sin\psi_2 \sin\psi_3 \end{bmatrix} \quad . \tag{3.8}$$

Then, combining Eqs. (3.3), (3.4) and (3.8) we can write down the equation,

$$\hat{\mathbf{x}} = \mathbf{V}\mathbf{A}\vec{\xi} \ , \tag{3.9}$$

which describes a 3D subspace confining all particles of the beam. Here the amplitude matrix **A** is

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \quad . \tag{3.10}$$

Substituting Eq. (3.9) into Eq. (3.7) one obtains the quadratic form describing a 4D ellipsoid containing all particles

$$\hat{\mathbf{x}}^T \left((\mathbf{VA})^{-1} \right)^T (\mathbf{VA})^{-1} \hat{\mathbf{x}} = 1 \quad . \tag{3.11}$$

Comparing Eqs. (3.6) and (3.11) one can reduce it to the following simple form,

$$\boldsymbol{\Xi} = \left((\mathbf{VA})^{-1} \right)^T (\mathbf{VA})^{-1} \ , \tag{3.12}$$

where Ξ is a symmetric matrix depending on two amplitudes, A_1 and A_2.

To determine the beam emittance (volume of the occupied 4D phase-space) described by Eq. (3.11), one can perform an orthogonal transformation, **T**, which reduces Ξ to a diagonal form, according to the equation

$$\boldsymbol{\Xi}' = \mathbf{T}^T \boldsymbol{\Xi} \mathbf{T} \quad , \tag{3.13}$$

where det(T) =1. Then, in the new coordinate frame the 3D ellipsoid enclosing total 4D phase-space of the beam can be described by the following equation

$$\Xi'_{11} x'^2 + \Xi'_{22} p'^2_x + \Xi'_{33} y'^2 + \Xi'_{44} p'^2_y = 1 \ . \tag{3.14}$$

It is natural to define the beam emittance as a product of the ellipsoid axes (omitting a factor correcting for the real 4D volume of ellipsoid) so that

$$\varepsilon_{4D} = \frac{1}{\sqrt{\Xi'_{11}\Xi'_{22}\Xi'_{33}\Xi'_{44}}} = \frac{1}{\sqrt{\det(\Xi')}} = \frac{1}{\sqrt{\det(\Xi)}} \quad . \tag{3.15}$$

Calculation of the determinant using Eq. (3.12) yields

$$\varepsilon_{4D} = \frac{1}{\sqrt{\det(\Xi)}} = (A_1 A_2)^2 |\det(\mathbf{V})| = (A_1 A_2)^2 \quad , \tag{3.16}$$

where we used the condition det(**V**) =1 following from the matrix **V** symplecticity.

The squares of amplitudes A_1 and A_2 can be considered as the corresponding 2D emittances, which coincide with the horizontal and vertical emittances of uncoupled motion.

Furthermore, the symplecticity of matrix **V** yields the following useful expression for the inverse matrix, $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1} = -\mathbf{U}\mathbf{V}^T\mathbf{U} \quad , \tag{3.17}$$

where we took into account that $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{U}\mathbf{U} = -\mathbf{I}$.

One can finally rewrite Eq. (3.12) in the following compact form

$$\boldsymbol{\Xi} = \mathbf{U}^T\mathbf{V}\begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}\mathbf{V}^T\mathbf{U} \quad , \tag{3.18}$$

where ε_1 and ε_2 are the 2D emittances corresponding to the eigen-vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$.

4. Beta-functions for Coupled Motion

Employing previously introduced notation, one can describe a single particle phase-space trajectory along the beam orbit as

$$\begin{aligned}\hat{\mathbf{x}}(s) &= \hat{\mathbf{M}}(0,s)\,\mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1 e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2 e^{-i\psi_2}\right) \\ &= \mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s) e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s) e^{-i(\psi_2+\mu_2(s))}\right) \quad ,\end{aligned} \tag{4.1}$$

where the vectors $\hat{\mathbf{v}}_1(s) \equiv e^{i\mu_1(s)}\hat{\mathbf{M}}(0,s)\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2(s) \equiv e^{i\mu_1(s)}\hat{\mathbf{M}}(0,s)\hat{\mathbf{v}}_2$ are the eigen-vectors of the matrix $\hat{\mathbf{M}}(0,s)\hat{\mathbf{M}}\hat{\mathbf{M}}(0,s)^{-1}$, ψ_1 and ψ_2 are the initial phases of betatron motion and $\hat{\mathbf{M}} = \hat{\mathbf{M}}(0,L)$ is the transfer matrix of the entire ring. One can notice that the terms $e^{-i\mu_1(s)}$ and $e^{-i\mu_2(s)}$ are introduced to bring the eigen-vectors to the standard form (see Eq. (4.2) below). Similar to the case of uncoupled motion we can rewrite Eq. (4.1) in the form

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1}\begin{bmatrix} \sqrt{\beta_{1x}(s)} \\ -\dfrac{iu_1(s)+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} \\ \sqrt{\beta_{1y}(s)}e^{i\nu_1(s)} \\ -\dfrac{iu_2(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)} \end{bmatrix} e^{-i(\mu_1(s)+\psi_1)} + \sqrt{\varepsilon_2}\begin{bmatrix} \sqrt{\beta_{2x}(s)}e^{i\nu_2(s)} \\ -\dfrac{iu_3(s)+\alpha_{2x}(s)}{\sqrt{\beta_{2x}(s)}}e^{i\nu_2(s)} \\ \sqrt{\beta_{2y}(s)} \\ -\dfrac{iu_4(s)+\alpha_{2y}(s)}{\sqrt{\beta_{2y}(s)}} \end{bmatrix} e^{-(i\mu_2(s)+\psi_2)}\right) \quad , \tag{4.2}$$

where $\beta_{1x}(s)$, $\beta_{1y}(s)$, $\beta_{2x}(s)$ and $\beta_{2y}(s)$ are the beta-functions, $\alpha_{1x}(s)$, $\alpha_{1y}(s)$, $\alpha_{2x}(s)$ and $\alpha_{2y}(s)$ are the alpha-functions which, as will be shown in the next section, coincide with the beta-functions negative half-derivatives at regions with zero longitudinal magnetic field, $\mu_1(s)$ and

$\mu_2(s)$ are the phase advances of betatron motion, and six real functions $u_1(s)$, $u_2(s)$, $u_3(s)$, $u_4(s)$, $\nu_1(s)$ and $\nu_2(s)$ are determined by the orthogonality condition of Eq.(2.5). Below we will call ten functions $\beta_{1x}(s)$, $\beta_{1y}(s)$, $\beta_{2x}(s)$, $\beta_{2y}(s)$, $\alpha_{1x}(s)$, $\alpha_{1y}(s)$, $\alpha_{2x}(s)$, $\alpha_{2y}(s)$, $\mu_1(s)$ and $\mu_2(s)$ as the generalized Twiss functions.

The first orthogonality condition,

$$\left(\hat{\mathbf{v}}_1^{\,+}\mathbf{U}\hat{\mathbf{v}}_1\right)=\begin{bmatrix}\sqrt{\beta_{1x}}\\ -\dfrac{iu_1+\alpha_{1x}}{\sqrt{\beta_{1x}}}\\ \sqrt{\beta_{1y}}e^{i\nu_1}\\ -\dfrac{iu_2+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1}\end{bmatrix}^{+}\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}\sqrt{\beta_{1x}}\\ -\dfrac{iu_1+\alpha_{1x}}{\sqrt{\beta_{1x}}}\\ \sqrt{\beta_{1y}}e^{i\nu_1}\\ -\dfrac{iu_2+\alpha_{1y}}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1}\end{bmatrix}=-2i\left(u_1+u_2\right)=-2i\quad, \tag{4.3}$$

yields $u_1 = 1 - u_2$, and similarly for the second eigen-vector, $u_4 = 1 - u_3$. The next two equations, $\hat{\mathbf{v}}_1^{\,T}\mathbf{U}\hat{\mathbf{v}}_1=0$ and $\hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_2=0$, are identities. For the rest two non-trivial orthogonality conditions, taking into account the above relations for u_1 and u_4 one can write,

$$\left(\hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_1\right)=\begin{bmatrix}\sqrt{\beta_{2x}}e^{i\nu_2}\\ \dfrac{-iu_3-\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2}\\ \sqrt{\beta_{2y}}\\ \dfrac{i(u_3-1)-\alpha_{2y}}{\sqrt{\beta_{2y}}}\end{bmatrix}^{+}\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}\sqrt{\beta_{1x}}\\ \dfrac{i(u_2-1)-\alpha_{1x}}{\sqrt{\beta_{1x}}}\\ \sqrt{\beta_{1y}}e^{i\nu_1}\\ \dfrac{-iu_2-\alpha_{1y}}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1}\end{bmatrix}=$$

$$\begin{aligned}&-\left(\sqrt{\frac{\beta_{2x}}{\beta_{1x}}}\left(i(1-u_2)+\alpha_{1x}\right)+\sqrt{\frac{\beta_{1x}}{\beta_{2x}}}\left(iu_3-\alpha_{2x}\right)\right)e^{-i\nu_2}\\ &-\left(\sqrt{\frac{\beta_{1y}}{\beta_{2y}}}\left(i(1-u_3)-\alpha_{2y}\right)+\sqrt{\frac{\beta_{2y}}{\beta_{1y}}}\left(iu_2+\alpha_{1y}\right)\right)e^{i\nu_1}=0\quad,\end{aligned} \tag{4.4}$$

$$\left(\hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_1\right)=\begin{bmatrix}\sqrt{\beta_{2x}}e^{i\nu_2}\\ \dfrac{-iu_3-\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2}\\ \sqrt{\beta_{2y}}\\ \dfrac{i(u_3-1)-\alpha_{2y}}{\sqrt{\beta_{2y}}}\end{bmatrix}^{T}\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}\sqrt{\beta_{1x}}\\ \dfrac{i(u_2-1)-\alpha_{1x}}{\sqrt{\beta_{1x}}}\\ \sqrt{\beta_{1y}}e^{i\nu_1}\\ \dfrac{-iu_2-\alpha_{1y}}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1}\end{bmatrix}=$$

$$-\left(\sqrt{\frac{\beta_{2x}}{\beta_{1x}}}\left(i(1-u_2)+\alpha_{1x}\right)-\sqrt{\frac{\beta_{1x}}{\beta_{2x}}}\left(iu_3+\alpha_{2x}\right)\right)e^{i\nu_2}$$
$$-\left(\sqrt{\frac{\beta_{1y}}{\beta_{2y}}}\left(i(u_3-1)-\alpha_{2y}\right)+\sqrt{\frac{\beta_{2y}}{\beta_{1y}}}\left(iu_2+\alpha_{1y}\right)\right)e^{i\nu_1}=0 \quad . \tag{4.5}$$

Multiplying both terms in Eq.(4.4) and Eq.(4.5) by their complex conjugate values one obtains

$$A_x^{\ 2}+\left(\kappa_x(1-u_2)+\kappa_x^{\ -1}u_3)\right)^2=A_y^{\ 2}+\left(\kappa_y(1-u_3)+\kappa_y^{\ -1}u_2\right)^2 \quad ,$$
$$A_x^{\ 2}+\left(\kappa_x(1-u_2)-\kappa_x^{\ -1}u_3)\right)^2=A_y^{\ 2}+\left(\kappa_y(1-u_3)-\kappa_y^{\ -1}u_2\right)^2 \quad , \tag{4.6}$$

where

$$A_x=\kappa_x\alpha_{1x}-\kappa_x^{\ -1}\alpha_{2x} \ ,$$
$$A_y=\kappa_y\alpha_{2y}-\kappa_y^{\ -1}\alpha_{1y} \ ,$$
$$\kappa_x=\sqrt{\frac{\beta_{2x}}{\beta_{1x}}}, \quad \kappa_y=\sqrt{\frac{\beta_{1y}}{\beta_{2y}}} \ . \tag{4.7}$$

Subtracting Eqs. (4.6) yields $u_2=u_3$. Substituting $u_2=u_3=u$. into Eqs.(4.4) and Eq.(4.5) one obtains the following expression for u:

$$u=-i\frac{\left(A_x+i\kappa_x\right)e^{-i\nu_2}-\left(A_y-i\kappa_y\right)e^{i\nu_1}}{\left(\kappa_x-\kappa_x^{\ -1}\right)e^{-i\nu_2}+\left(\kappa_y-\kappa_y^{\ -1}\right)e^{i\nu_1}} \quad ,$$
$$u=-i\frac{\left(A_x+i\kappa_x\right)e^{i\nu_2}-\left(A_y+i\kappa_y\right)e^{i\nu_1}}{\left(\kappa_x+\kappa_x^{\ -1}\right)e^{i\nu_2}-\left(\kappa_y+\kappa_y^{\ -1}\right)e^{i\nu_1}} \quad . \tag{4.8}$$

Solutions of the above two complex equations determine the last three real unknown parameters. To solve Eqs. (4.8) one takes into account that the imaginary parts of the right-hand sides of both Eqs. (4.8) are equal to zero. That yields the following pair of equations:

$$A_+\cos(\nu_1+\nu_2)-B\sin(\nu_1+\nu_2)+C_+=0 \quad ,$$
$$A_-\cos(\nu_1-\nu_2)-B\sin(\nu_1-\nu_2)+C_-=0 \quad , \tag{4.9}$$

where

$$
\begin{aligned}
A_+ &= A_x\left(\kappa_y - \kappa_y^{-1}\right) - A_y\left(\kappa_x - \kappa_x^{-1}\right) \quad , \\
A_- &= -A_x\left(\kappa_y + \kappa_y^{-1}\right) - A_y\left(\kappa_x + \kappa_x^{-1}\right) \quad , \\
B &= \kappa_x \kappa_y^{-1} - \kappa_x^{-1}\kappa_y \quad , \\
C_+ &= A_x\left(\kappa_x - \kappa_x^{-1}\right) - A_y\left(\kappa_y - \kappa_y^{-1}\right) \quad , \\
C_- &= A_x\left(\kappa_x + \kappa_x^{-1}\right) + A_y\left(\kappa_y + \kappa_y^{-1}\right) \quad .
\end{aligned}
\tag{4.10}
$$

Each of Eqs. (4.9) has two roots:

$$
\begin{aligned}
\nu_{+_{1,2}} &\equiv (\nu_1 + \nu_2)_{1,2} = \begin{cases} \pi - \operatorname{asin}\left(\dfrac{C_+}{\sqrt{A_+^{\,2} + B^2}}\right) + \arg(B + iA_+) \quad , \\ \operatorname{asin}\left(\dfrac{C_+}{\sqrt{A_+^{\,2} + B^2}}\right) + \arg(B + iA_+) \quad , \end{cases} \\
\nu_{-_{1,2}} &\equiv (\nu_2 - \nu_1)_{1,2} = \begin{cases} \pi + \operatorname{asin}\left(\dfrac{C_-}{\sqrt{A_-^{\,2} + B^2}}\right) - \arg(B + iA_-) \quad , \\ -\operatorname{asin}\left(\dfrac{C_-}{\sqrt{A_-^{\,2} + B^2}}\right) - \arg(B + iA_-) \quad . \end{cases}
\end{aligned}
\tag{4.11}
$$

Combining expressions for ν_+ and ν_- one obtains solutions for ν_1 and ν_2 as follows:

$$
\begin{aligned}
\nu_1 &= n\pi + \frac{1}{2}(\nu_+ - \nu_-) \quad , \\
\nu_2 &= n\pi + \frac{1}{2}(\nu_+ + \nu_-) \quad .
\end{aligned}
\tag{4.12}
$$

Here we took into account that phases ν_- and ν_+ are determined modulo 2π, which yields that ν_1 and ν_2 are determined modulo π. This results in four different solutions for ν_1 and ν_2 (see below). To find u we can use any of Eqs. (4.8). Using the top equation and substituting Eq.(4.11) to it we obtain two roots for u,

$$
u_{1,2} = \frac{\kappa_x^{\,2} + \kappa_y^{\,2} - 2 + \kappa_{+_{xy}} \cos\left(\nu_{+_{1,2}}\right) - \left(A_x\left(\kappa_y - \kappa_y^{-1}\right) + A_y\left(\kappa_x - \kappa_x^{-1}\right)\right)\sin\left(\nu_{+_{1,2}}\right)}{\left(\kappa_x - \kappa_x^{-1}\right)^2 + \left(\kappa_y - \kappa_y^{-1}\right)^2 + 2\left(\kappa_x - \kappa_x^{-1}\right)\left(\kappa_y - \kappa_y^{-1}\right)\cos\left(\nu_{+_{1,2}}\right)} \quad , \tag{4.13}
$$

where $\kappa_{+_{xy}} = 2\kappa_x\kappa_y - \kappa_x^{-1}\kappa_y - \kappa_x\kappa_y^{-1}$. As one can verify, using the bottom equation in Eq. (4.8) yields the same value for u. The roots $u_{1,2}$ have opposite signs and close absolute values. Finally, we can write down the following set of four different solutions for u, ν_1 and ν_2:

$$
(u,\nu_1,\nu_2)=\begin{cases}\left(u_1,\ (\nu_{+_1}-\nu_{-_1})/2,\ (\nu_{+_1}+\nu_{-_1})/2\right) \ , \\ \left(u_1,\ \pi+(\nu_{+_1}-\nu_{-_1})/2,\ \pi+(\nu_{+_1}+\nu_{-_1})/2\right) \ , \\ \left(u_2,\ (\nu_{+_2}-\nu_{-_2})/2,\ (\nu_{+_2}+\nu_{-_2})/2\right) \ , \\ \left(u_2,\ \pi+(\nu_{+_2}-\nu_{-_2})/2,\ \pi+(\nu_{+_2}+\nu_{-_2})/2\right) \ . \end{cases} \qquad (4.14)
$$

These four solutions can be separated into two pairs; with the same values of u and phases, ν_1 and ν_2, different by π. This property originates from the fact that the mirror reflection with respect to x or y axis does not change β's and α's but only changes relative signs for x and y components of the eigen-vectors[1], with subsequent change of $\nu_1(s)$ and $\nu_2(s)$ by π.

Finally we can rewrite Eq. (4.2) in the following compact form

$$
\hat{\mathbf{x}}(s)=\mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s)e^{-i(\mu_1(s)+\psi_1)}+\sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s)e^{-(i\mu_2(s)+\psi_2)}\right) \ , \qquad (4.15)
$$

where the eigen-vectors, $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$, are given explicitly as follows:

$$
\hat{\mathbf{v}}_1(s)=\begin{bmatrix}\sqrt{\beta_{1x}(s)} \\ -\dfrac{i(1-u(s))+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} \\ \sqrt{\beta_{1y}(s)}e^{i\nu_1(s)} \\ -\dfrac{iu(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)}\end{bmatrix} \ , \qquad \hat{\mathbf{v}}_2(s)=\begin{bmatrix}\sqrt{\beta_{2x}(s)}e^{i\nu_2(s)} \\ -\dfrac{iu(s)+\alpha_{2x}(s)}{\sqrt{\beta_{2x}(s)}}e^{i\nu_2(s)} \\ \sqrt{\beta_{2y}(s)} \\ -\dfrac{i(1-u(s))+\alpha_{2y}(s)}{\sqrt{\beta_{2y}(s)}}\end{bmatrix} \ . \qquad (4.16)
$$

Here $\nu_1(s)$ and $\nu_2(s)$ and $u(s)$ are determined by the beta- and alpha-functions from Eqs. (4.11), (4.12) and (4.13).

5. Differential Equation for Beam Envelopes

Let us consider the relations between the beta-and alpha-functions. A differential trajectory displacement related to the first eigen-vector can be expressed as follows:

$$
x(s+\Delta s)=x(s)+x'(s)\Delta s+O(\Delta s^2)=x(s)+\left(p_x(s)+\frac{R}{2}y\right)\Delta s+O(\Delta s^2)=
$$
$$
\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)}+\left(-\frac{i(1-u(s))+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}}+\frac{R}{2}\sqrt{\beta_{1y}(s)}\,e^{i\nu_1(s)}\right)\Delta s\right)e^{-i(\mu_1(s)+\psi_1)}\right)+O(\Delta s^2) \ . \qquad (5.1)
$$

---

[1] It can also be achieved by a change of coupling sign (like simultaneous sign change for gradients of all skew quads and magnetic fields of all solenoids), which does not change the beta-functions but does change the ν-functions by π.

Alternatively, one can express particle position through the beta-functions at the new coordinate $s + \Delta s$:

$$x(s+\Delta s) = \mathrm{Re}\left(\sqrt{\varepsilon_1 \beta_x (s+\Delta s)} e^{-i(\mu_1(s+\Delta s)+\psi)}\right) =$$
$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)} + \frac{d\beta_{1x}}{2\sqrt{\beta_{1x}(s)}} - i\sqrt{\beta_{1x}(s)}d\mu\right)e^{-i(\mu_1(s)+\psi)}\right) + O(\Delta s^2) \quad . \tag{5.2}$$

Comparing both the imaginary and real parts of Eqs. (5.1) and (5.2) one obtains:

$$\begin{aligned} \frac{d\beta_{1x}}{ds} &= -2\alpha_{1x} + R\sqrt{\beta_{1x}(s)\beta_{1y}(s)}\cos(\nu_1(s)) \quad , \\ \frac{d\mu_1}{ds} &= \frac{1-u(s)}{\beta_{1x}(s)} - \frac{R}{2}\sqrt{\frac{\beta_{1y}(s)}{\beta_{1x}(s)}}\sin(\nu_1(s)) \quad . \end{aligned} \tag{5.3}$$

Similarly, one can write down equivalent expressions for the vertical displacement,

$$y(s+ds) = y(s) + y'(s)ds = y(s) + \left(p_y(s) - \frac{R}{2}x\right)ds =$$
$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1y}(s)}\,e^{i\nu_1(s)} - \left(\frac{iu(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)} + \frac{R}{2}\sqrt{\beta_{1x}(s)}\right)ds\right)e^{-i(\mu_1(s)+\psi_1)}\right) \quad , \tag{5.4}$$

and

$$x(s+ds) = \sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1y}(s)} + \frac{d\beta_{1y}}{2\sqrt{\beta_{1y}(s)}} + i\sqrt{\beta_{1y}(s)}\,(d\nu_1 - d\mu_1)\right)e^{-i(\mu_1(s)+\psi-\nu_1(s))}\right), \tag{5.5}$$

which yields:

$$\begin{aligned} \frac{d\beta_{1y}}{ds} &= -2\alpha_{1y} - R\sqrt{\beta_{1x}(s)\beta_{1y}(s)}\cos(\nu_1(s)) \quad , \\ \frac{d\nu_1}{ds} &= \frac{1-u(s)}{\beta_{1x}(s)} + \frac{u(s)}{\beta_{1y}(s)} - \frac{R}{2}\left(\sqrt{\frac{\beta_{1y}(s)}{\beta_{1x}(s)}} - \sqrt{\frac{\beta_{1x}(s)}{\beta_{1y}(s)}}\right)\sin(\nu_1(s)) \quad . \end{aligned} \tag{5.6}$$

Similar calculations carried out for the second eigen-vector yield,

$$
\begin{aligned}
&\frac{d\beta_{2y}}{ds} = -2\alpha_{2y} - R\sqrt{\beta_{2x}(s)\beta_{2y}(s)}\cos(\nu_2(s)) \quad , \\
&\frac{d\mu_2}{ds} = \frac{1-u(s)}{\beta_{2y}(s)} + \frac{R}{2}\sqrt{\frac{\beta_{2x}(s)}{\beta_{2y}(s)}}\sin(\nu_2(s)) \quad , \\
&\frac{d\beta_{2x}}{ds} = -2\alpha_{2x} + R\sqrt{\beta_{2x}(s)\beta_{2y}(s)}\cos(\nu_2(s)) \quad , \\
&\frac{d\nu_2}{ds} = \frac{1-u(s)}{\beta_{2y}(s)} - \frac{u(s)}{\beta_{2x}(s)} - \frac{R}{2}\left(\sqrt{\frac{\beta_{2y}(s)}{\beta_{2x}(s)}} - \sqrt{\frac{\beta_{2x}(s)}{\beta_{2y}(s)}}\right)\sin(\nu_2(s)) \quad .
\end{aligned}
\tag{5.7}
$$

To obtain the equation for the betatron functions (the generalized Floquet envelope equation) one needs to substitute the phase-space coordinates expressed through the eigen-vector into Eq. (1.7):

$$
\frac{d}{ds}\left(\hat{\mathbf{v}}_k e^{-\mu_k(s)}\right) = \mathbf{UH}\hat{\mathbf{v}}_k e^{-\mu_k(s)} \quad , \quad k = 1,2. \tag{5.8}
$$

Performing differentiation one obtains

$$
\frac{d\hat{\mathbf{v}}_k}{ds} = \left(\mathbf{UH} + i\frac{d\mu_k}{ds}\right)\hat{\mathbf{v}}_k \quad . \tag{5.9}
$$

Using Eqs. (5.3) and (5.7) $d\mu_k/ds$ can be expressed through the eigen-vector components as follows:

$$
\begin{aligned}
&\frac{d\mu_1}{ds} = -\frac{1}{\mathrm{v}_{1_1}}\mathrm{Im}\left(\mathrm{v}_{1_2} + \frac{R}{2}\mathrm{v}_{1_3}\right) \quad , \\
&\frac{d\mu_2}{ds} = -\frac{1}{\mathrm{v}_{2_3}}\mathrm{Im}\left(\mathrm{v}_{2_4} - \frac{R}{2}\mathrm{v}_{2_1}\right) \quad .
\end{aligned}
\tag{5.10}
$$

Knowing components of the eigen-vectors, $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$, and employing Eq.(4.16) one finally obtains the generalized betatron functions.

6. Explicit Representation of Transfer Matrix in Terms of Generalized Twiss Functions

One can derive a useful representation of the one-turn transfer matrix **M** in terms of the ten generalized Twiss functions. Using an explicit definition of the matrix **V**(s) (see Eq.(3.2)) in terms of eigen-vectors $\hat{\mathbf{v}}_1(s)$ and $\hat{\mathbf{v}}_2(s)$ given by Eq.(4.16), one can express it as follows

$$
\mathbf{V}=\begin{bmatrix}
\sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\
-\dfrac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \dfrac{1-u}{\sqrt{\beta_{1x}}} & \dfrac{u\sin\nu_2-\alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \dfrac{u\cos\nu_2+\alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\
\sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\
\dfrac{u\sin\nu_1-\alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \dfrac{u\cos\nu_1+\alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\dfrac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \dfrac{1-u}{\sqrt{\beta_{2y}}}
\end{bmatrix} . \quad (6.1)
$$

Using Eq.(2.1) one can derive the useful identity

$$
\mathbf{M}\,\mathbf{V}(s)=\mathbf{V}(s)\,\mathbf{S} \quad , \qquad (6.2)
$$

where the matrix **S** is given explicitly as follows

$$
\mathbf{S}=\begin{bmatrix}
\cos\mu_1 & \sin\mu_1 & 0 & 0 \\
-\sin\mu_1 & \cos\mu_1 & 0 & 0 \\
0 & 0 & \cos\mu_2 & \sin\mu_2 \\
0 & 0 & -\sin\mu_2 & \cos\mu_2
\end{bmatrix} . \qquad (6.3)
$$

Multiplying both sides of Eq.(6.2) by the inverse matrix, $\mathbf{V}^{-1}=-\mathbf{U}\mathbf{V}^T\mathbf{U}$, as given by Eq.(3.17), allows one to express the transfer matrix, $\mathbf{M}$, in the form

$$
\mathbf{M}=-\mathbf{V}\mathbf{S}\mathbf{U}\mathbf{V}^{\mathrm{T}}\mathbf{U} \quad . \qquad (6.4)
$$

Matrix multiplication in Eq.(6.4) was carried out using a symbolic math program. The resulting 16 transfer matrix elements as a function of generalized Twiss functions are summarized in Appendix A.

7. Summary

The article introduces a new representation of two-dimensional coupled betatron motion. This approach is based on a novel parametrization of the 4×4 symplectic transfer matrix by introducing the following ten functions: 4 beta-functions, 4 alpha-functions and 2 betatron phase advances which we call the generalized Twiss functions. The beta-functions have similar meaning to the Courant-Snyder parameterization, and the definition of alpha-functions coincides with the standard one at regions with zero longitudinal magnetic field, where they are equal to negative half-derivatives of the beta-functions. Furthermore, one can easily obtain the generalized betatron functions, knowing components of the eigen-vectors, $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$, and employing Eq.(4.15). A useful representation of transfer matrix **M** in terms of the generalized Twiss functions is also introduced (see Appendix A).

A definition of 4D emittance is introduced for an ensemble of particles, whose motion is contained in a 4D ellipsoid. A 3D surface of this ellipsoid is determined by particles with extreme betatron amplitudes. For any of these particles, Eqs. (3.3) and (3.4) describe a 2D-subspace of single particle motion, which is a subspace of the 3D surface of the ellipsoid. An explicit expression for the 4D phase-space volume enclosed by the 3D ellipsoid is derived. It reduces to usual 2D case in the absence of coupling (see Appendix B).

The presented parameterization has been proven very useful for both analytic and numerical analysis of coupled betatron motion in circular machines and transfer lines. It is important to note that although the canonical coordinates were used through the article it usually does not bring complications in practical applications of the developed formalism because the canonical and geometric coordinates coincide at regions with zero longitudinal magnetic field. The above-mentioned software developed for study of coupled betatron motion always uses transfer matrices which start and end at points with zero longitudinal magnetic field and, thus, canonical and geometric coordinates always coincide. Appendix C shows an example of analysis of how the strongly coupled motion for the Fermilab electron cooling project can be analyzed with the developed formalism.

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Appendix A. Explicit Representation of Transfer Matrix in Terms of Generalized TWISS Functions

Performing matrix multiplication given by Eq.(6.4) and using explicit expressions for matrices **V** and **S**, Eqs.(6.1)-(6.3) yield the following expressions for all 16 elements of the transfer matrix **M**:

$$M_{11} = (1-u)\cos\mu_1 + \alpha_{1x}\sin\mu_1 + u\cos\mu_1 + \alpha_{2x}\sin\mu_2 \ , \tag{A.1}$$

$$M_{12} = \beta_{1x}\sin\mu_1 + u\cos\mu_1 + \beta_{2x}\sin\mu_2 \ , \tag{A.2}$$

$$M_{13} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[\alpha_{1y}\sin(\mu_1-\nu_1) + u\cos(\mu_1-\nu_1)\right] + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[\alpha_{2y}\sin(\mu_2-\nu_2) + (1-u)\cos(\mu_1-\nu_1)\right] \ , \tag{A.3}$$

$$M_{14} = \sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1+\nu_1) + \sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2-\nu_2) \ , \tag{A.4}$$

$$M_{21} = -\frac{1}{\beta_{1x}}\left[(1-u^2) + \alpha_{1x}^2\right]\sin\mu_1 - \frac{1}{\beta_{2x}}\left[u^2 + \alpha_{2x}^2\right]\sin\mu_2 \ , \tag{A.5}$$

$$M_{22} = (1-u)\cos\mu_1 - \alpha_{1x}\sin\mu_1 - \alpha_{2x}\sin\mu_2 \ , \tag{A.6}$$

$$\begin{aligned} M_{23} = & \frac{1}{\sqrt{\beta_{1x}\beta_{1y}}}\left\{\left[(1-u)\alpha_{1y} - u\alpha_{1x}\right]\cos(\mu_1+\nu_1) - \left[\alpha_{1x}\alpha_{1y} + u(u-1)\right]\sin(\mu_1+\nu_1)\right\} + \\ & \frac{1}{\sqrt{\beta_{2x}\beta_{2y}}}\left\{\left[u\alpha_{2y} - (1-u)\alpha_{2x}\right]\cos(\mu_2-\nu_2) - \left[\alpha_{2x}\alpha_{2y} + u(u-1)\right]\sin(\mu_2-\nu_2)\right\} \ , \end{aligned} \tag{A.7}$$

$$M_{24}=\sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[-\alpha_{1x}\sin(\mu_1+\nu_1)+(1-u)\cos(\mu_1+\nu_1)\right]+\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[-\alpha_{2x}\sin(\mu_2-\nu_2)+u\cos(\mu_2-\nu_2)\right] \quad , \qquad (A.8)$$

$$M_{31}=\sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[\alpha_{1x}\sin(\mu_1-\nu_1)+(1-u)\cos(\mu_1-\nu_1)\right]+\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[\alpha_{2x}\sin(\mu_2+\nu_2)+u\cos(\mu_2+\nu_2)\right] \quad , \qquad (A.9)$$

$$M_{32}=\sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1-\nu_1)+\sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2+\nu_2) \quad , \qquad (A.10)$$

$$M_{33}=(1-u)\cos\mu_2+\alpha_{2x}\sin\mu_2+\alpha_{1y}\sin\mu_1 \quad , \qquad (A.11)$$

$$M_{34}=\beta_{1y}\sin\mu_1+\beta_{2y}\sin\mu_2 \quad , \qquad (A.12)$$

$$M_{41}=\frac{1}{\sqrt{\beta_{1x}\beta_{1y}}}\left\{(1-u)\left[u\cos(\mu_1+\nu_1)-\alpha_{1y}\sin(\mu_1+\nu_1)\right]-\alpha_{1x}\left[u\cos(\mu_1-\nu_1)-\alpha_{1y}\sin(\mu_1-\nu_1)\right]\right\}+$$

$$\frac{1}{\sqrt{\beta_{2x}\beta_{2y}}}\left\{(1-u)\left[\alpha_{2x}\cos(\mu_2+\nu_2)-u\sin(\mu_2+\nu_2)\right]-\alpha_{2y}\left[u\cos(\mu_2+\nu_2)+\alpha_{2x}\sin(\mu_2+\nu_2)\right]\right\} \quad , \qquad (A.13)$$

$$M_{42}=\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[-\alpha_{1y}\sin(\mu_1-\nu_1)+u\cos(\mu_1-\nu_1)\right]-\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[\alpha_{2y}\sin(\mu_2+\nu_2)+(1-u)\cos(\mu_2+\nu_2)\right] \quad , \qquad (A.14)$$

$$M_{43}=-\frac{1}{\beta_{1y}}\left[u^2+\alpha_{1y}^2\right]\sin\mu_1-\frac{1}{\beta_{2y}}\left[(1-u^2)+\alpha_{2y}^2\right]\sin\mu_2 \quad , \qquad (A.15)$$

$$M_{44}=(1-u)\cos\mu_2-\alpha_{1y}\sin\mu_1+u\cos\mu_1-\alpha_{2y}\sin\mu_2 \quad . \qquad (A.16)$$

Appendix B. 1D Betatron Motion – 2D Phase-space Formalism

In the case of one-dimensional betatron motion, our formalism reduces to standard Courant-Snyder parametrization. The phase-space trajectory can be represented as

$$\begin{bmatrix} x \\ p_x \end{bmatrix} = A\begin{bmatrix} \sqrt{\beta} \\ -\dfrac{\alpha+i}{\sqrt{\beta}} \end{bmatrix} e^{-i\psi} \, , \quad A=\sqrt{\varepsilon} \quad , \qquad (B.1)$$

where ε is the beam emittance, β is the beta-function and α is its negative half derivative. Then matrix **V** (see Eq.(3.2)) can be written in the simple form

$$\mathbf{V} = \begin{bmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix} \quad , \tag{B.2}$$

One can see that its determinant is equal to 1. The inverse matrix is given by

$$\mathbf{V}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{bmatrix} . \tag{B.3}$$

Finally, the matrix Ξ (see Eq.(3.6)) can be expressed as

$$\boldsymbol{\Xi} = \left((\mathbf{AV})^{-1}\right)^{T} (\mathbf{AV})^{-1} = \frac{1}{A^2} \begin{bmatrix} \frac{1+\alpha^2}{\beta} & \alpha \\ \alpha & \beta \end{bmatrix} \quad , \tag{B.4}$$

and the beam emittance (compare Eq.(3.16)) is equal to

$$\varepsilon_{2D} = \frac{1}{\sqrt{\det(\Xi)}} = A^2 = \varepsilon \quad . \tag{B.5}$$

Appendix C. Generalized Twiss Functions for Axisymmetric Distribution Function

To increase Tevatron luminosity Fermilab is developing a high energy electron cooling device for cooling of antiprotons[8]. Because of high energy of the electron beam (~5 MeV), it is impractical to use the standard choice used in electron cooling devices for the beam transport where the beam moves in the longitudinal magnetic field along the entire way from the electron gun to the collector. Nevertheless the longitudinal magnetic field is still used for beam focusing in the cooling section to cancel the beam defocusing due to the electron beam space charge and more important to reach collective stability of the electron beam. To neutralize the rotational motion of particles in the cooling section the beam is produced in the electron gun immersed in the longitudinal magnetic field. Consequently, the beam transport is going to be quite sophisticated with a large number of bends and focusing elements. Taking into account that the space charge effects are comparatively small everywhere except the gun and the collector it looks attractive to use the developed formalism for beam transport analysis. The beam motion in the gun should be analyzed by a specialized code, which calculates beam parameters at exit of the electrostatic accelerator. Then, the beam transport can be analyzed with the generalized Twiss functions, the initial values of which we calculate in this appendix.

At the exit of electrostatic accelerator the electron beam distribution is axially symmetric, and before the beam leaves the magnetic field its distribution function can be described by the bilinear form

$$\Xi_B = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0 & \alpha_0 & 0 & 0 \\ \alpha_0 & \beta_0 & 0 & 0 \\ 0 & 0 & \gamma_0 & \alpha_0 \\ 0 & 0 & \alpha_0 & \beta_0 \end{bmatrix} \quad , \qquad \text{(C.1)}$$

where $\varepsilon_T = r_c\sqrt{mkT_c} / P_0$ is the thermal emittance of the beam, r_c is the cathode radius, T_c is the cathode temperature, P_0 and m are the particle momentum and mass, $\beta_0 = a^2 / \varepsilon_T$, $\alpha_0 = -\sqrt{\beta / \varepsilon_T}(da/ds)$, and $\gamma_0 = \left(1+\alpha_0{}^2\right)/\beta_0$ are the initial Twiss functions, and a is the beam radius at the electrostatic accelerator exit. After exiting from the magnetic field the electrons will acquire angular momentum proportional to the radius, and the distribution can be characterized by the bilinear form:

$$\Xi_{in} = \Phi^T \,\Xi_B\, \Phi = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0+\Phi^2\beta_0 & \alpha_0 & 0 & -\Phi\beta_0 \\ \alpha_0 & \beta_0 & \Phi\beta_0 & 0 \\ 0 & \Phi\beta_0 & \gamma_0+\Phi^2\beta_0 & \alpha_0 \\ -\Phi\beta_0 & 0 & \alpha_0 & \beta_0 \end{bmatrix} \quad , \qquad \text{(C.2)}$$

where

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Phi & 0 \\ 0 & 0 & 1 & 0 \\ -\Phi & 0 & 0 & 1 \end{bmatrix} \quad , \qquad \text{(C.3)}$$

$\Phi = eB / 2P_0c$ is the rotational focusing strength of the solenoid edge, B is the solenoid magnetic field.

To choose initial values for generalized Twiss functions we use the axial symmetry of the electron distribution function. It implies that the horizontal and vertical alpha- and beta-functions are equal, and we obtain for the eigen-vectors:

$$\hat{\mathbf{v}}_1(s) = \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta}e^{i\nu_1} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}}e^{i\nu_1} \end{bmatrix} \quad , \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta}e^{i\nu_2} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}}e^{i\nu_2} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \quad . \qquad \text{(C.4)}$$

In this case the coefficients of Eq. (4.7) are

$$\kappa_x = \kappa_y = 1 \qquad \text{and} \qquad A_x = A_y = 0 \ , \tag{C.5}$$

which creates uncertainty in Eq. (4.11) for ν_1 and ν_2. To avoid this uncertainty we will use initial Eqs. (4.4) and (4.5). Substituting Eq. (C.5) into these equations we obtain,

$$\begin{cases} e^{-i\nu_2} = e^{i\nu_2} & , \\ (1-2u)\left(e^{i\nu_2} - e^{i\nu_2}\right) = 0 & . \end{cases} \tag{C.6}$$

The solutions of these equations are: u=1/2 and $\nu_1 = -\ \nu_2 + 2\pi n$. As one can see in this case we have an unlimited number of solutions for ν_1 and ν_2. We will choose $\nu_1 = -\ \nu_2 = \pi/2$ to reach a better symmetry for the eigen-vectors. Then, the matrix **V** is equal to:

$$\mathbf{V} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} & -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} \end{bmatrix} \ . \tag{C.7}$$

Using Eq. (3.18) we obtain the bilinear form,

$$\mathbf{\Xi} = \begin{bmatrix} \dfrac{1+4\alpha^2}{4\beta}\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & \alpha\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & 0 & \dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1}-\dfrac{1}{\varepsilon_2}\right) \\ \alpha\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & \beta\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & -\dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1}-\dfrac{1}{\varepsilon_2}\right) & 0 \\ 0 & -\dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1}-\dfrac{1}{\varepsilon_2}\right) & \dfrac{1+4\alpha^2}{4\beta}\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & \alpha\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1}-\dfrac{1}{\varepsilon_2}\right) & 0 & \alpha\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) & \beta\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}\right) \end{bmatrix} \ . \tag{C.8}$$

Comparing Eqs. (C.2) and (C.8) one can express generalized Twiss functions through the Twiss parameters of the beam distribution function in the magnetic field:

$$\beta = \frac{\beta_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\alpha = \frac{\alpha_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} - \Phi\beta_0} \quad , \qquad \text{(C.9)}$$

$$\varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} + \Phi\beta_0} \quad .$$

One can see that $\varepsilon_1\varepsilon_2 = \varepsilon_T^{\ 2}$, which verifies the conclusions of Section 3.

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