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QCD Constraints on Form Factor Shapes^{*1}

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ABSTRACT

This talk presents an introduction to the use of dispersion relations to constrain the shapes of hadronic form factors consistent with QCD. The applications described include methods for studying $|V_{ud}|$ and $|V_{us}|$, the strange quark mass, and the pion charge radius.

1. Introduction and History

Between the mid 1950s and late 1960s, a great deal of theoretical activity focused on attempting to solve (or at least severely constrain) problems of strong interaction physics using dispersion theory. An extensive and elegant body of work was developed to study the analyticity properties of form factors and scattering amplitudes. Eventually, however, when theorists believed they had reached the limits of what could be gleaned from dispersive techniques, their attentions were drawn elsewhere: to the quark-parton model, to current algebra, and eventually to gauge theories, especially QCD.

The appeal of dispersion theory lies in its ability to incorporate in a completely rigorous and model-independent fashion those features shared by all well-defined field theories, namely, causality, unitarity, and crossing symmetry. Moreover, it works equally well in perturbative and nonperturbative regimes of the underlying dynamical theory. However, no specific Lagrangian is demanded by this scheme, and this lack of specificity acts as a double-edged sword: Without dynamical input, one can only deduce those consequences common to all possible dynamics. On the other hand, we now possess QCD, which is the fundamental, albeit unsolved, theory of strong interactions. A combination of the two, in which QCD inputs are inserted directly into dispersion relations, should yield a rich harvest of rigorous and model-independent bounds on hadronic quantities. In this talk we explore the implementation of this idea to the specific cases of weak and electromagnetic hadronic form factors.

Dispersion theory has been with us in particle physics for quite some time. The origin of the name traces directly back to the famous Kramers-Kronig relation¹ in

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electromagnetism, used to describe the dispersion of light in an arbitrary medium. The standard formula reads

$$\text{Re } f(\omega) = \text{Re } f(0) + \frac{\omega^2}{2\pi^2} P \int_0^\infty d\omega' \frac{\sigma_{\text{tot}}(\omega')}{\omega'^2 - \omega^2}, \quad (1)$$

where f , ω , σ_{tot} , and P denote, respectively, the forward scattering amplitude, frequency, total cross section, and principal value prescription to remove the denominator singularity. It follows that the dispersive ($\text{Re } f$) and absorptive ($\text{Im } f \sim \sigma_{\text{tot}}$) amplitudes are intimately connected. This relation is derived by using causality and unitarity, which lead to restrictions on the analyticity properties of f in the space of complex ω . From there, the elegant theorems of complex analysis, especially Cauchy's theorem, provide the identities known as dispersion relations.

Since quantum mechanics and quantum field theory are expressed over the field of complex numbers, it is natural to expect that some variant of the dispersive approach should also exist in particle theory. Indeed, as early as 1951, Gell-Mann, Goldberger, and Thirring² described how causality and unitarity lead to a dispersion relation for the vacuum polarization two-point function of QED. A flurry of other dispersion relations followed in the literature, each presented with more or less rigor, depending upon assumptions about the analytic structure of the quantity under scrutiny; however, the particular dispersion relation used below is nothing more than the QCD version of the one first studied in 1951.

2. Formalism

We begin by defining the vacuum polarization tensor as the two-point current correlator in momentum space:

$$\Pi^{\mu\nu}(q) \equiv i \int d^4x e^{iqx} \langle 0 | T J^\mu(x) J^{\nu\dagger}(0) | 0 \rangle, \quad (2)$$

Here J is some chosen current; since we are working with QCD, we choose it to be a quark bilinear. Moreover, we choose J to be a weak or electromagnetic, rather than gluonic, current, so that the individual (perturbative) current insertions are easier to identify. Suppressing for now the Lorentz indices $\mu\nu$, we would like to use Cauchy's theorem to write an expression

$$\Pi(q^2) = \frac{1}{2\pi i} \int_C dt \frac{\Pi(t)}{(t - q^2)}, \quad (3)$$

which relates Π at two different momentum arguments, q^2 and t . However, in order to do this, we must identify a closed contour C , inside of which Π is analytic in t . In the present case, causality implies that $\Pi(t)$ is analytic in t except on parts of the positive t axis, where J^\dagger can create on-shell hadrons, which generates a discontinuity only in the imaginary (absorptive) part of Π . We choose C to consist of the lower and upper sides of this branch cut, together with the circle with $|t| \rightarrow \infty$; the latter contribution vanishes as long as $\Pi \rightarrow 0$ for large $|t|$. Then

$$\Pi(q^2) = \frac{1}{2\pi} \int_0^\infty dt \frac{\text{Im } \Pi(t + i\epsilon)}{(t - q^2)} + \frac{1}{2\pi} \int_\infty^0 dt \frac{\text{Im } \Pi(t - i\epsilon)}{(t - q^2)}. \quad (4)$$

Using the Schwarz reflection principle ($\Pi(z^*) = \Pi^*(z)$ if Π is real on some segment of the real axis, which is true for $t < 0$ since there are no on-shell thresholds and hence is no imaginary part there), the two terms in (4) are equal:

$$\Pi(q^2) = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im } \Pi(t + i\epsilon)}{(t - q^2)}. \quad (5)$$

If $\Pi(q^2)$ diverges, or the contribution from the circle $|t| \rightarrow \infty$ does not vanish, such terms may be removed through the process called “subtraction”: Since the offending terms appear as coefficients of a polynomial in q^2 , taking a sufficient number n of q^2 derivatives yields a finite result,

$$\frac{\partial^n \Pi(q^2)}{(\partial q^2)^n} = \frac{\partial^n \Pi_{\text{finite}}(q^2)}{(\partial q^2)^n}. \quad (6)$$

Then the expression for the dispersion relation reads

$$\Pi^{(n)}(q^2) \equiv \frac{1}{n!} \frac{\partial^n \Pi(q^2)}{(\partial q^2)^n} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im } \Pi(t + i\epsilon)}{(t - q^2)^{n+1}}. \quad (7)$$

Restoring the Lorentz indices and inserting a complete set of states between J and J^\dagger (unitarity) yields

$$\text{Im } \Pi^{\mu\nu}(t + i\epsilon) = \frac{1}{2} \sum_\Gamma \int d\Phi(\Gamma) (2\pi)^4 \delta^4 \left(t - \sum_\Gamma p \right) \langle 0 | J^\mu | \Gamma \rangle \langle \Gamma | J^{\nu\dagger} | 0 \rangle, \quad (8)$$

where only on-shell states Γ with phase space Φ are included in the sum (a consequence of reducing the step functions in the time ordering). The matrix elements $\langle 0 | J^\mu | \Gamma \rangle$ are nothing more than decay constants and form factors—pure hadronic quantities—while q^2 can be chosen so that $\Pi(q^2)$ can be evaluated directly in the fundamental theory of QCD. In particular, one chooses q^2 to be far from the hadronic (strong coupling) region, and then $\Pi(q^2)$ may be computed using an operator product expansion. A very useful observation due to Meiman³ in 1963 is that the $\mu = \nu$ components of Eq. (8) are positive definite, meaning that each hadronic contribution serves only to saturate further the partonic (perturbative QCD) side of the dispersion relation. In this way one obtains a rigorous inequality between partonic and hadronic physics.

One path from Eqs. (7)–(8) leads to the famous QCD sum rules,⁴ which study the saturation of the equality between the partonic and hadronic sides. We focus also on what this equality tells us about the behavior of matrix elements $\langle 0 | J^\mu | \Gamma \rangle$. The first work to use the Meiman inequality with QCD inputs was by Bourrely, Machet, and de Rafael⁵ in 1981.

As an explicit example, consider the pion electromagnetic form factor:

$$\langle \pi^+(p') | J_{\text{EM}}^\mu | \pi^+(p) \rangle = f(q^2) (p + p')^\mu, \quad (9)$$

where $q = p - p'$. Then Eq. (8) becomes

$$\text{Im } \Pi^\mu(t) \geq \frac{1}{48\pi} (t - 4m_\pi^2)^{3/2} t^{-1/2} |f(t)|^2 \theta(t - 4m_\pi^2), \quad (10)$$

while the partonic side, finite after two subtractions, is computed to be

$$\Pi^{ii(2)}(q^2) = \frac{1}{8\pi(-q^2)} \left\{ 1 + \frac{\alpha_s(q^2)}{\pi} + O\left[\left(\frac{\alpha_s(q^2)}{\pi}\right)^2\right] + n.p. \right\}, \quad (11)$$

where $n.p.$ stands for nonperturbative corrections such as vacuum condensates. The combined inequality reads

$$\frac{1}{8\pi(-q^2)} \left\{ 1 + \frac{\alpha_s(q^2)}{\pi} + O\left[\left(\frac{\alpha_s(q^2)}{\pi}\right)^2\right] + n.p. \right\} \geq \frac{1}{48\pi^2} \int_{4m_\pi^2}^{\infty} dt \frac{(t - 4m_\pi^2)^{3/2}}{t^{1/2}(t - q^2)^3} |f(t)|^2. \quad (12)$$

In general, one obtains an inequality of the form

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \frac{W_F(t) |F(t)|^2}{(t - q^2)^{n+1}} \leq \Pi^{(n)}(q^2), \quad (13)$$

where t_+ is the lowest threshold and W_F is a positive weighting factor arising from phase space and the quantum numbers of the form factor $F(t)$. As discussed by Okubo and Fushih⁶ in 1971, it is very convenient to map the complex t plane with a cut for $t_+ \leq t < +\infty$ to the unit disc using a complex kinematic variable z :

$$z(t; t_s) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}}, \quad (14)$$

which maps the upper (lower) side of the cut to the lower (upper) half of the unit circle. The parameter⁷ $t_s < t_+$ is chosen later for convenience. One then defines a weighting function

$$\phi_F(t; t_s) = \sqrt{\frac{W_F(t)}{\Pi^{(n)}(q^2)(t - q^2)^{n+1} |dz(t; t_s)/dt|}}, \quad (15)$$

which is analytic inside the unit circle, and in terms of which the dispersive bound reads

$$\frac{1}{2\pi i} \oint \frac{dz}{z} |\phi_F(z) F(z)|^2 \leq 1. \quad (16)$$

If any subthreshold poles remain inside the unit circle at points $z = z_p$, they may be removed by means of so-called Blaschke factors,

$$P_F(z) = \prod_p \frac{z - z_p}{1 - z_p^* z}. \quad (17)$$

$P_F(z)$ has the feature that for $|z| = 1$, $|P_F(z)| = 1$, so that the dispersive bound is unchanged,

$$\frac{1}{2\pi i} \oint \frac{dz}{z} |\phi_F(z) P_F(z) F(z)|^2 \leq 1, \quad (18)$$

and $\phi_F(z) P_F(z) F(z)$ is analytic on the whole unit disc. Crossing symmetry relates the form factor in all kinematic regimes by analytic continuation. We have thus isolated the analytic structure of the form factor,

$$F(t) = \frac{1}{P_F(t) \phi_F(t; t_s)} \sum_{n=0}^{\infty} a_n z^n(t; t_s), \quad (19)$$

where the coefficients a_n are unknown; however, inserting Eq. (19) back into (18) gives

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1. \quad (20)$$

Equations (19) and (20) first appeared⁸ in 1995, and re-express in a very compact and explicit notation all of the analyticity, unitarity, and explicit QCD information implicit in Eqs. (7) and (8). Since ϕ_F and P_F are known functions, the form factor is known except for a set of parameters a_n , each of which must be less than unity in magnitude. A randomly chosen shape for a form factor would almost inevitably have some $|a_n| > 1$, and thus would be disallowed by the dispersive bound (20).

One more point that makes the model-independent parameterization (19) useful is that for spacelike and semileptonic processes, the allowed kinematic range for z tends to have $|z| \ll 1$. Indeed, the parameter t_s is chosen to enhance this effect. For example, for $\bar{B} \rightarrow D \ell \bar{\nu}$, $|z| < 0.03$. This means that the convergence of Eq. (19) is geometrically fast, and only the first few a_n 's are relevant to the shape of the form factor. The theoretical uncertainty incurred by ignoring the other, infinite set of a_n 's is called "truncation error,"⁸ and falls off geometrically fast with the number of a_n 's used to parameterize the form factor.

3. A Gallery of Results

1) $|V_{ub}|$ and $|V_{cb}|$. The need for a parameterization describing all solutions of Eqs. (7)–(8) was recognized⁹ in studies of the $\bar{B} \rightarrow \pi \ell \bar{\nu}$ form factor, useful for the extraction of $|V_{ub}|$. There it was seen that the inclusion of each (at that time, hypothetical) form factor data point served to decrease the region allowed by the dispersion relation geometrically fast (Fig. 3). Similar comments apply to using points from a lattice simulation.¹¹

The model-independent form factor parameterization Eq. (19) was first used⁸ to extrapolate measured $\bar{B} \rightarrow D^{(*)} \ell \bar{\nu}$ form factor data to a point where phase space vanishes. In order to extract $|V_{cb}|$ from the form factor* in the differential width

$$\frac{d\Gamma}{dq^2}(\bar{B} \rightarrow D^{(*)} \ell \bar{\nu}) \propto |V_{cb}|^2 |F(q^2)|^2 \sqrt{(M_B + M_{D^*})^2 - q^2}, \quad (21)$$

one must separate $|V_{cb}|$ from $|F(q^2)|$. The normalization of F , namely, $F(q^2 = (M_B - M_{D^*})^2) = 1$ (up to small corrections), is determined by the heavy quark limit.¹⁰ However, Eq. (21) shows that phase space vanishes at exactly this q^2 ; therefore, an extrapolation is needed. Previously, experimental measurements of the form factor used an

*Strictly speaking, 4 form factors contribute; however, in the limit of heavy quarks, each one either vanishes or is proportional to a single "Isgur-Wise" form factor.¹⁰

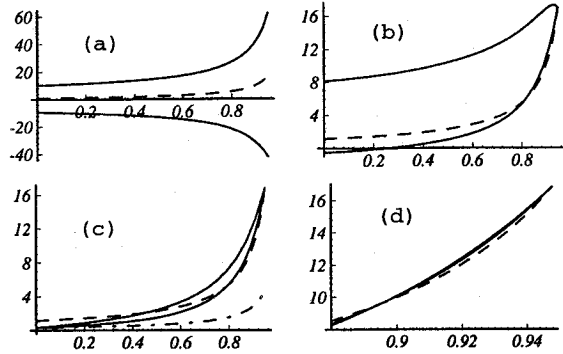


Fig. 1. Bounds on the $\bar{B} \rightarrow \pi \ell \bar{\nu}$ form factor f using the dispersive bound and fixing zero, one, and two points in (a), (b), and (c), respectively. Dashed lines indicate pole dominance models. (d) shows how certain choices of B^* pole parameters can violate the dispersive bounds.

Experiment	Process	$ V_{cb} \cdot 10^3$
CLEO ¹³	$\bar{B} \rightarrow D^* \ell \bar{\nu}$	$36.9^{+2.0}_{-2.1}$
CLEO ¹⁶	$\bar{B} \rightarrow D \ell \bar{\nu}$	44.8 ± 6.1
ALEPH ¹³	$\bar{B} \rightarrow D^* \ell \bar{\nu}$	31.9 ± 2.4
ALEPH ¹³	$\bar{B} \rightarrow D \ell \bar{\nu}$	29.2 ± 7.3
DELPHI ¹⁷	$\bar{B} \rightarrow D^* \ell \bar{\nu}$	38.0 ± 1.3

Table 1. Determinations of $|V_{cb}|$ using Eqs. (19) and (20). Footnotes reference the source of the fit. Uncertainties are statistical plus theoretical.

ad hoc linear or quadratic extrapolation, which implies a theoretical uncertainty of unknown size (see Fig. 2a). Using (19) and (20) removes this uncertainty, and subsequent work^{12,13} refined the analysis to the point that it is used by both theorists and the experimental groups themselves (Table 1).

2) *Strange quark mass.* K_{L3} decays possess two form factors, one of which appears with the coefficient m_ℓ^2 in the rate and is called the scalar form factor. The corresponding $\Pi^{(n)}(q^2)$, evaluated deep in the Euclidean region, is proportional to $(m_s - m_u)^2$, i.e., is sensitive to m_s . One can invert the program of 1), so that a large amount of form factor data, thus delineating its shape, is used¹⁴ to constrain the function $\Pi^{(n)}(q^2)$ and hence m_s . Indeed, one finds that $F(t) \propto m_s a_n$, so that (20) implies a rigorous lower bound on m_s .

Currently, not enough data exists in the world sample for such a determination, although DAΦNE expects to increase the available pool many times over. Until such data exists, one may apply the results of a model, or better, a chiral perturbation theory

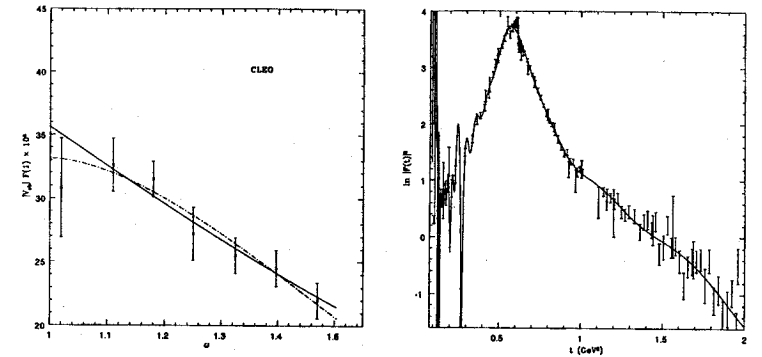


Fig. 2. (a). Fit to CLEO data for the $\bar{B} \rightarrow D^* \ell \bar{\nu}$ form factor using the model-independent parameterization (19) with (solid) and without (dashed) the dispersive constraint Eq. (20). The eye prefers the dashed curve, but it is forbidden by QCD. (b). Fit to timelike pion electromagnetic form factor data. Contrast the smoothness of the fit above $t = 0.4 \text{ GeV}^2$ to the oscillations below.

(χ PT) calculation to compute the required shape parameters a_n . Believing χ PT to a level of 5%, 1%, or 1/20% leads to lower bounds on $m_s^{\overline{\text{MS}}}(1 \text{ GeV})$ of 40, 90, and 140 MeV, respectively.

3) *Pion form factor.* The parameterization (19) exhibits geometric convergence for $t < t_+$ ($|z| < 1$). On the other hand, one often possesses data directly on the cut $t > t_+$ ($|z| = 1$), the timelike region. Does (19) have anything to say about this region? Although one must be much more careful about convergence, the answer¹⁵ appears to be yes. Theorems of complex convergence plus knowledge of asymptotic ($t \rightarrow \infty$) properties of form factors allow one to use (19) even in the timelike region.

For example, for the pion electromagnetic form factor, one obtains the fit of Fig. 2b. The presence of the ρ peak is not put in by hand, but simply emerges from fitting to (19). Note, however, the wild oscillations for $t < 0.4 \text{ GeV}^2$; one can show that these occur due to large gaps in the data for $\theta > \pi/2$ on the unit circle $|z| = 1$. These large oscillations persist when one analytically continues into the spacelike region, where one obtains

$$|F(t=0)| = 2.56 \pm 2.00, \quad \langle r^2 \rangle = 2.66 \pm 3.44 \text{ fm}^2, \quad (22)$$

which are rather loose bounds, considering that, e.g., $|F(t=0)| = 1$ by charge conservation. This points to the well-known problem of the instability of analytic continuation of discrete timelike data to the spacelike region; now, however one can quantify exactly how unstable this continuation is.

One can also proceed directly with spacelike pion form factor data alone, where

the geometric convergence of (19) is restored. Then one finds,¹⁵ using this model-independent parameterization, $\langle r^2 \rangle = 0.480 \pm 0.020 \text{ fm}^2$, a few σ larger than the usual numbers quoted in the literature ($\simeq 0.42 \text{ fm}^2$), which use *ad hoc* parameterizations.

4. Conclusions

Dispersive techniques provide an elegant, rigorous bridge between hadronic and elementary quantities. For semileptonic or electromagnetic decays, they provide a model-independent, rapidly convergent parameterization of form factors. We have seen that a number of different problems have already been studied using this method. Nucleon form factors, the K charge radius, and improvement of timelike form factors are obvious future directions. The reader can doubtless imagine many others.

References

1. R. Kronig, J. Op. Soc. Am., **12**, 547 (1926); H. A. Kramers, Atti. Congr. Intern. Fisici, **2**, 545 Como (1927).
2. M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. **95**, 1612 (1951).
3. N. N. Meiman, Sov. Phys. JETP **17**, 830 (1963).
4. For a collection of seminal works, see M. A. Shifman (editor), *Vacuum Structure and QCD Sum Rules*, North-Holland, New York, 1992.
5. C. Bourrely, B. Machet, and E. de Rafael, Nucl. Phys. B **189**, 157 (1981).
6. S. Okubo and I. Fushih, Phys. Rev. D **4**, 2020 (1971).
7. C. G. Boyd and R. F. Lebed, Nucl. Phys. B **485**, 275 (1997).
8. C. G. Boyd, B. Grinstein, and R. F. Lebed, Phys. Lett. B **353**, 306 (1995).
9. C. G. Boyd, B. Grinstein, and R. F. Lebed, Phys. Rev. Lett. **74**, 4603 (1995).
10. N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); B **237**, 527 (1990).
11. L. Lellouch, Nucl. Phys. B **479**, 353 (1996).
12. C. G. Boyd, B. Grinstein, and R. F. Lebed, Nucl. Phys. B **461**, 493 (1996); I. Caprini, L. Lellouch, and M. Neubert, Nucl. Phys. B **530**, 153 (1998).
13. C. G. Boyd, B. Grinstein, and R. F. Lebed, Phys. Rev. D **56**, 6895 (1997).
14. R. F. Lebed and K. Schilcher, Phys. Lett. B **430**, 341 (1998).
15. W. W. Buck and R. F. Lebed, Phys. Rev. D **58**, 056001 (1998).
16. CLEO Collaboration (J. Bartelt *et al.*), Phys. Rev. Lett. **82**, 3746 (1999).
17. DELPHI Collaboration (M. Margoni and F. Simonetto, spokespersons), Report No. DELPHI 99-107 CONF 294, submitted to the HEP '99 Conference, Tampere, Finland, 15-21 July, 1999.