NLO evolution of color dipole

G. A. Chirilli

JLAB & ODU

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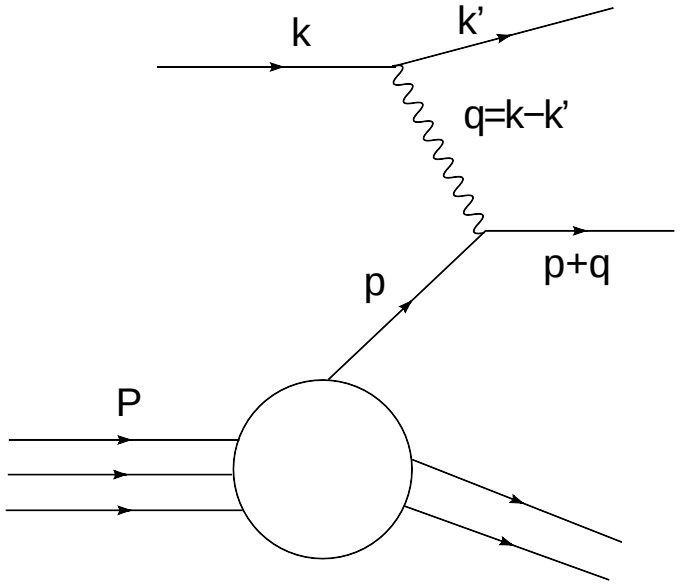
Otuline

- Brief introduction to small-x physics.
- DIS from nucleus at high energy and Wilson line.
- Leading order: BK equation.
- Quark and gluon loop in the shock-wave.
- NLO kernel in QCD.
- Conformal invariance of the leading BK kernel.
- NLO kernel in $\mathcal{N} = 4$ SYM.
- Conclusions.
- Outlook.

Deep Inelastic Scattering at Small-x

$$s = (P+q)^2 \quad Q^2 = -q^2 > 0$$

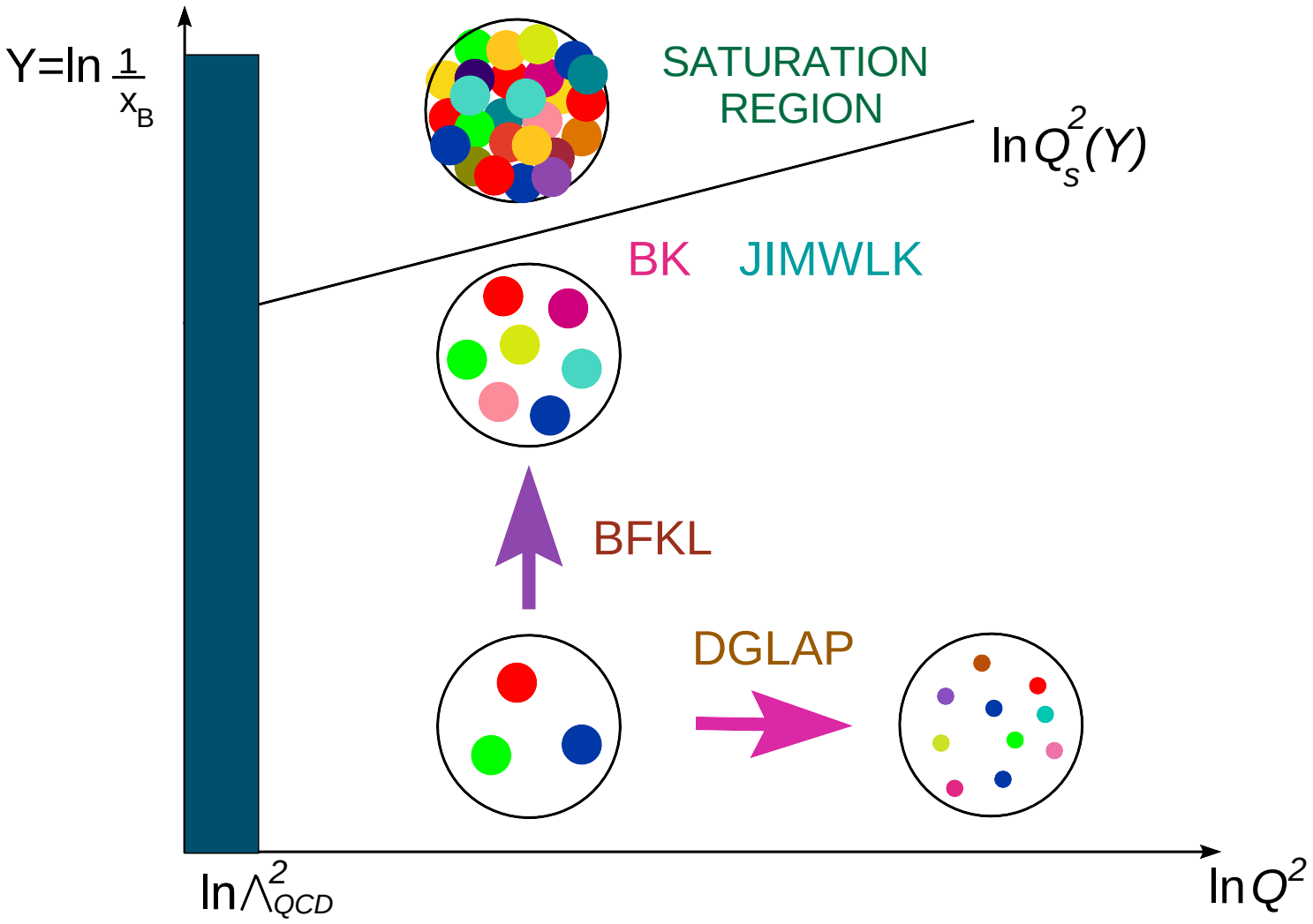


Small-x

- $s >> M^2 \qquad Q^2 >> M^2$
- $x_B \equiv \frac{Q^2}{2P\cdot q} \sim \frac{Q^2}{s}$
- $x << 1 \Leftrightarrow s >> Q^2$

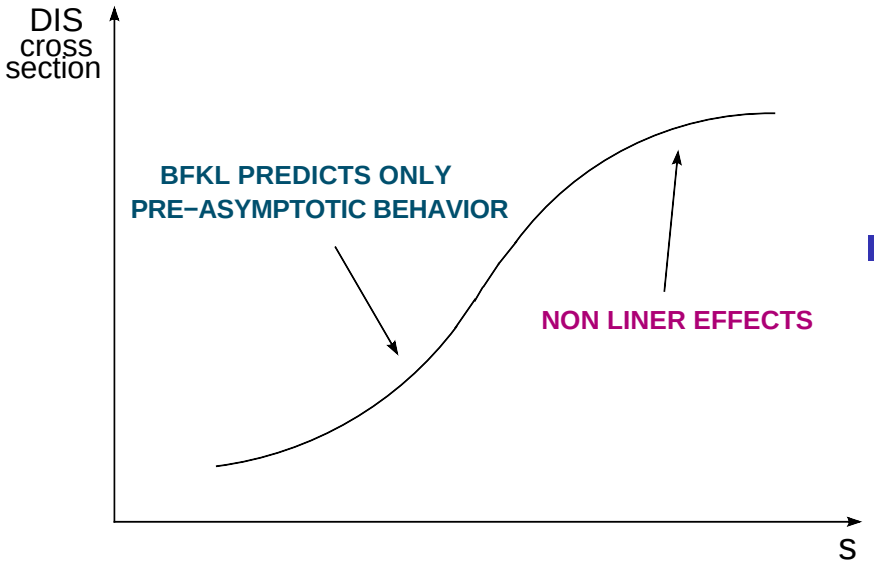
- Because of confining property, quarks and gluons cannot be measured
- DIS processes are used to study the quarks and gluons distributions inside a hadron
- Experiments have shown that the structure functions grow very rapidly with decreasing value of x_B

DGLAP vs. BFKL



DIS cross section

BFKL: Leading Logarithmic Approximation $\alpha_s << 1 \qquad (\alpha_s \ln s)^n \sim 1$



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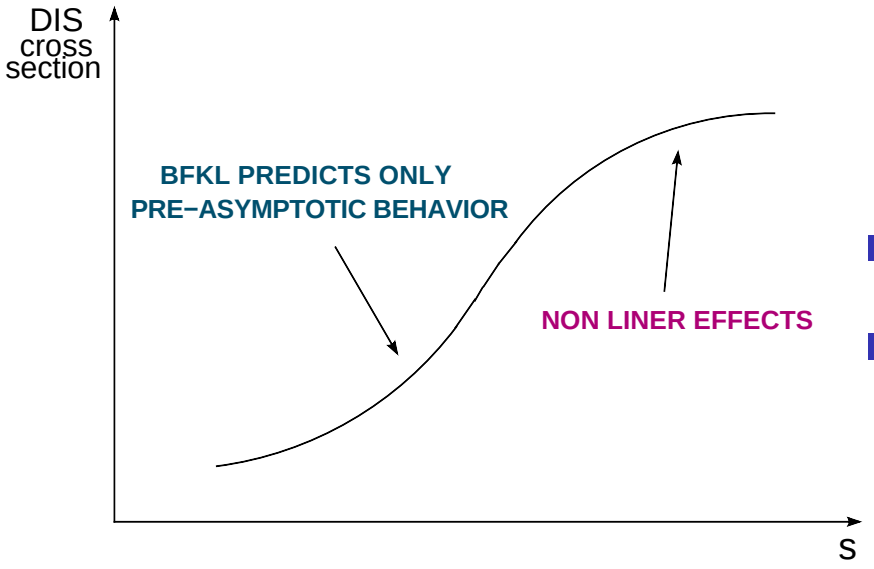

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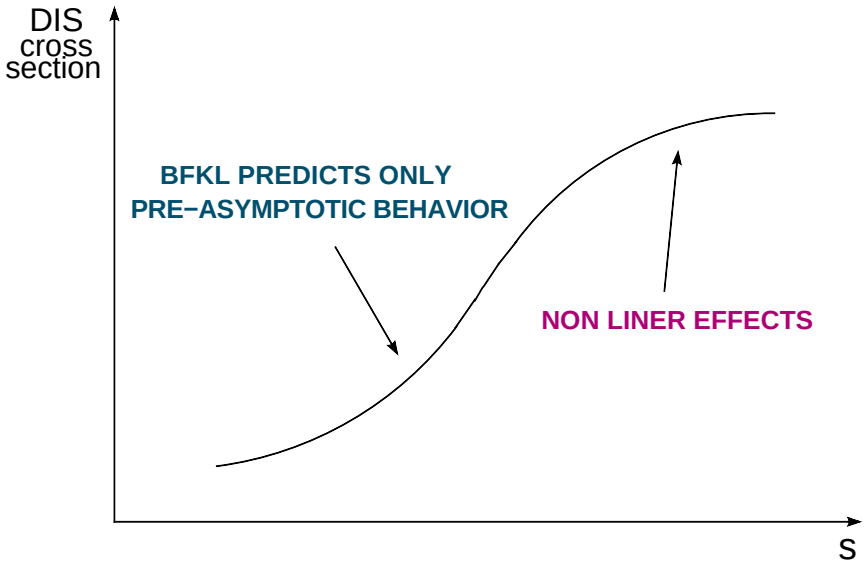

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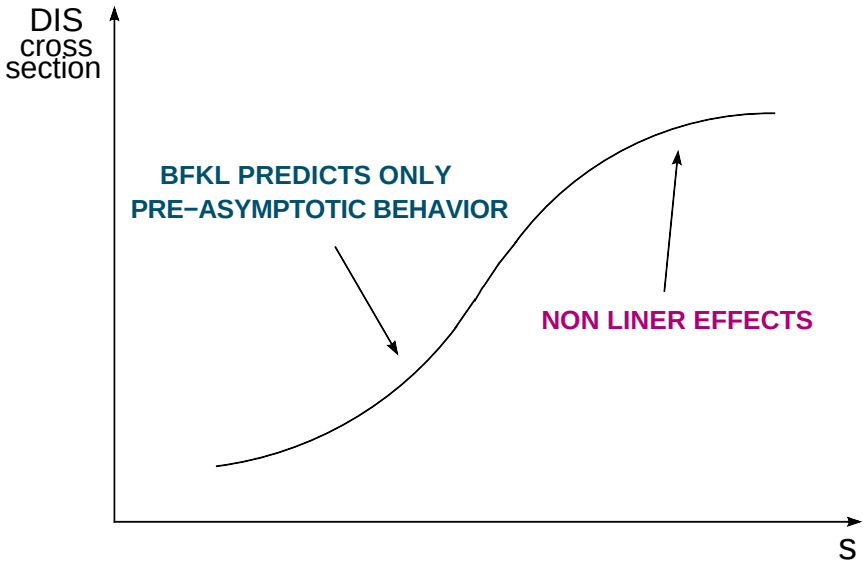

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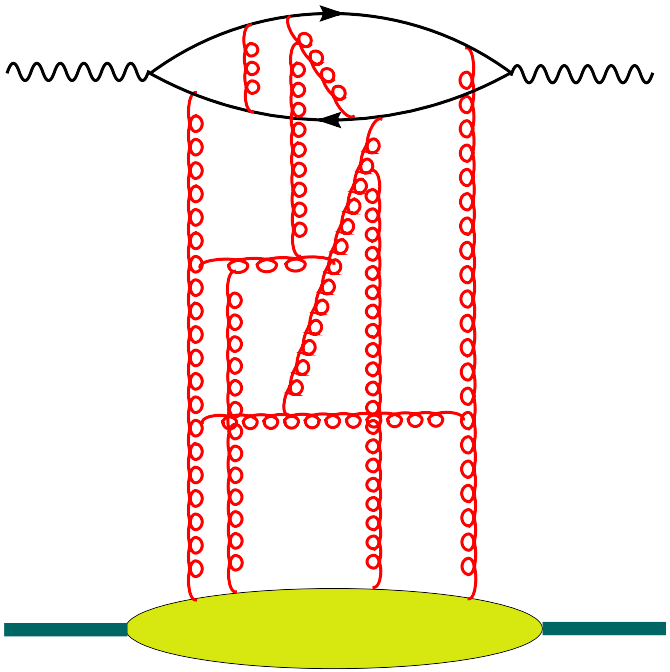
- In order to take in to account recombination of gluons the evolution equation for the structure function has to be non-linear.

DIS at high energy

- Consider the amplitude of $\gamma^* A \to \gamma^* A$



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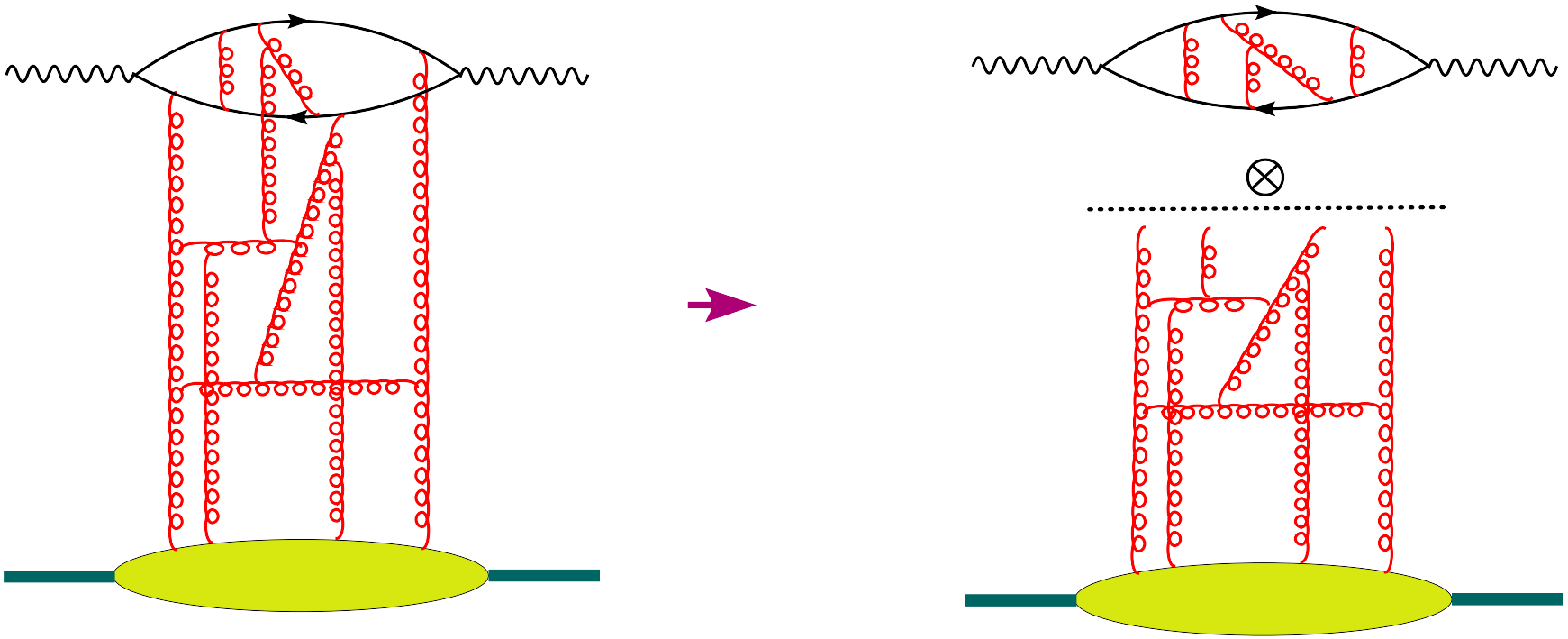

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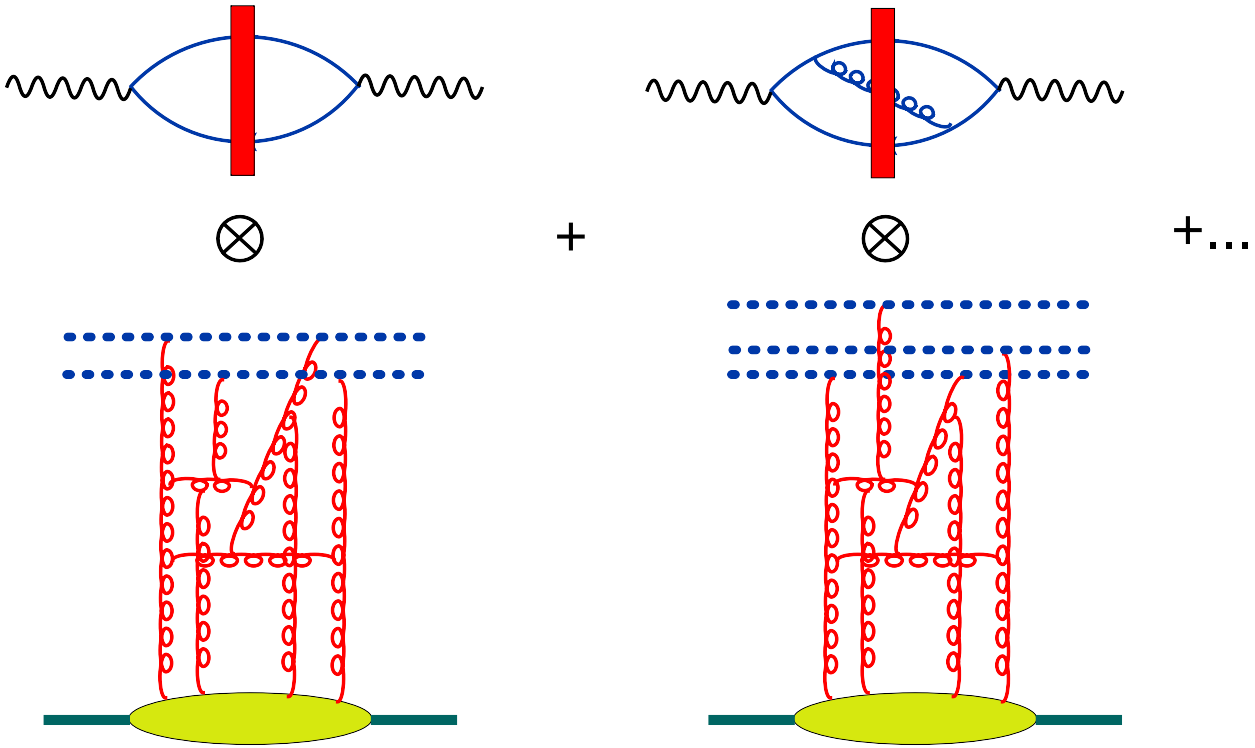

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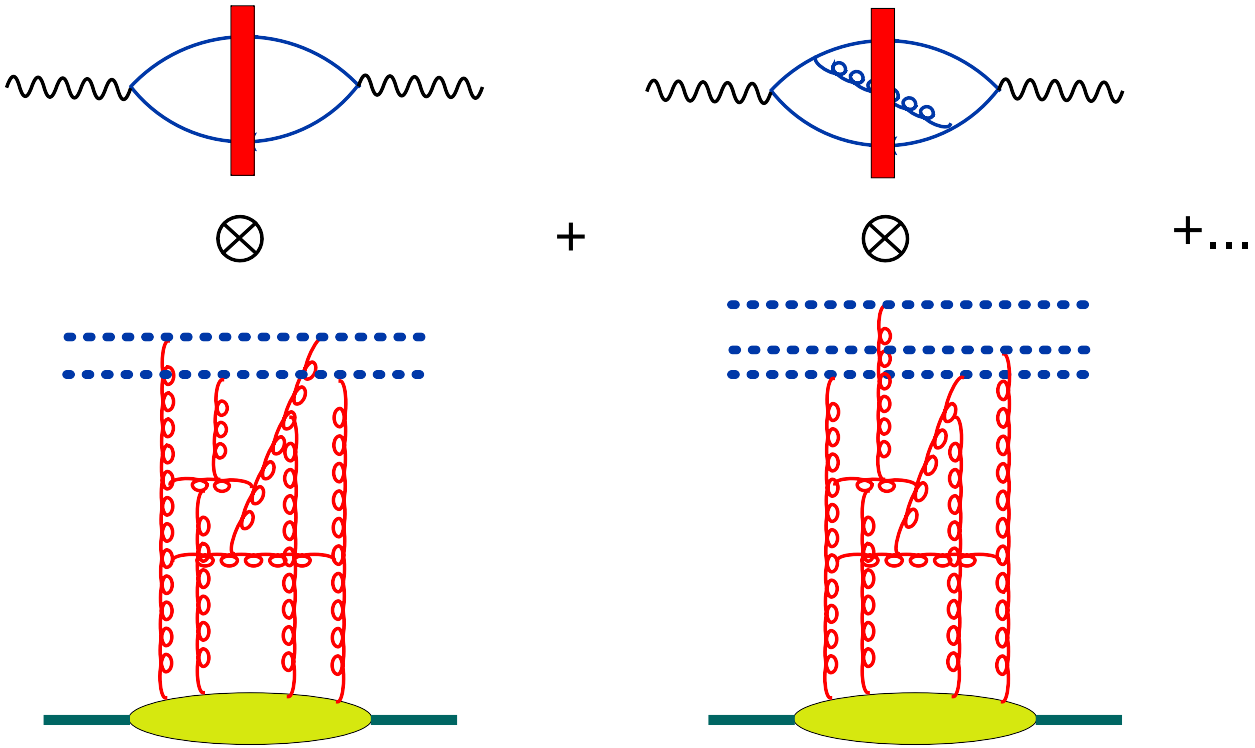

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = P e^{ig \int_{-\infty}^{\infty} du \; n^\mu A_\mu (un + x_\perp)}$$

Wilson line

DIS at high energy

The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{\mathcal{U}}^{\eta}(x_\perp, y_\perp) = 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^{\eta}(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\}$$

switched between the target's states ($N_c = 3$ for QCD). The gluon parton density is approximately:

$$x_B G(x_B, \mu^2 = Q^2) \;\simeq\; \langle p|\, \hat{\mathcal{U}}^{\eta}(x_\perp, 0)|p\rangle\Big|_{x_\perp^2 = Q^{-2}}$$

where $\eta = \ln \frac{1}{x_B}$

Propagation in the shock wave: Wilson line

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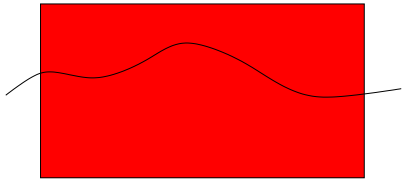

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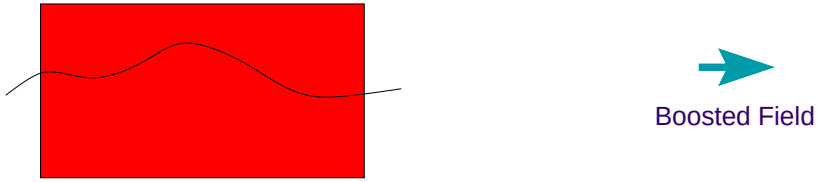

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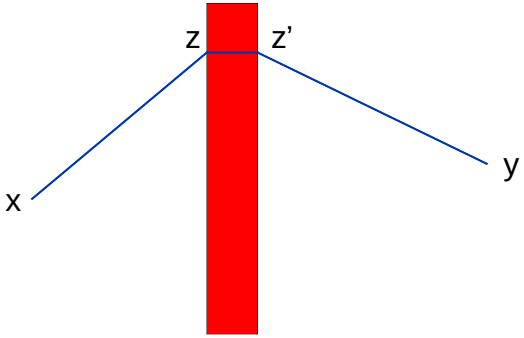

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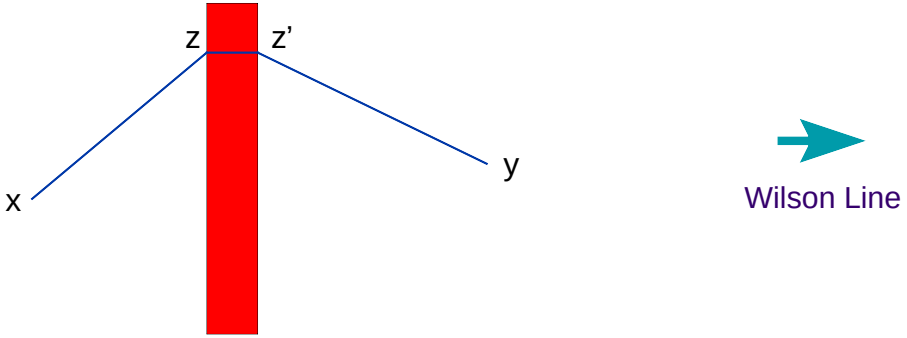

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$$[x,y] = Pe^{ig\int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

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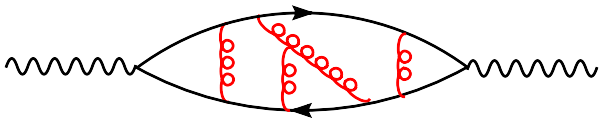

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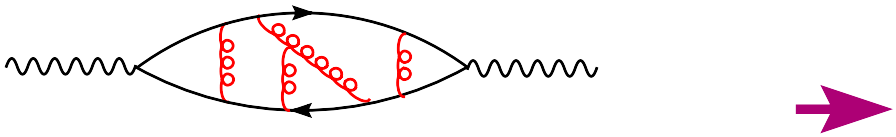

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The interaction with the shock wave is given by the Wilson line in the adjoint representation.

Evolution Equation

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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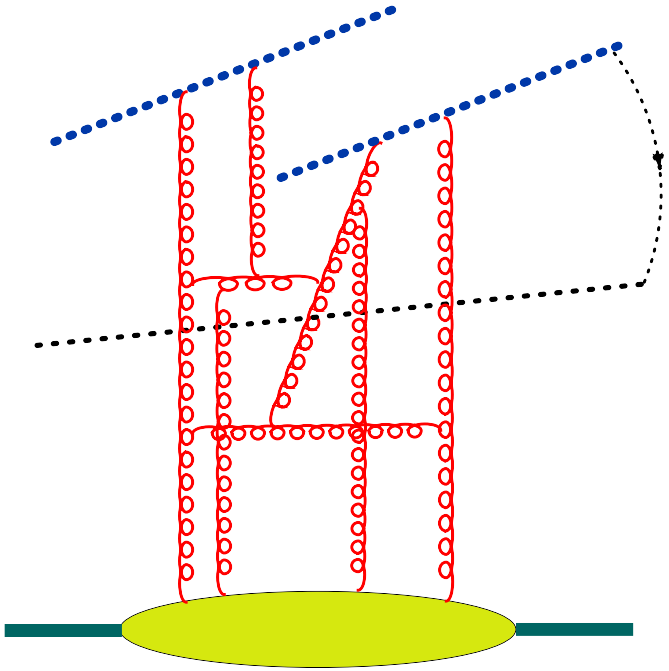

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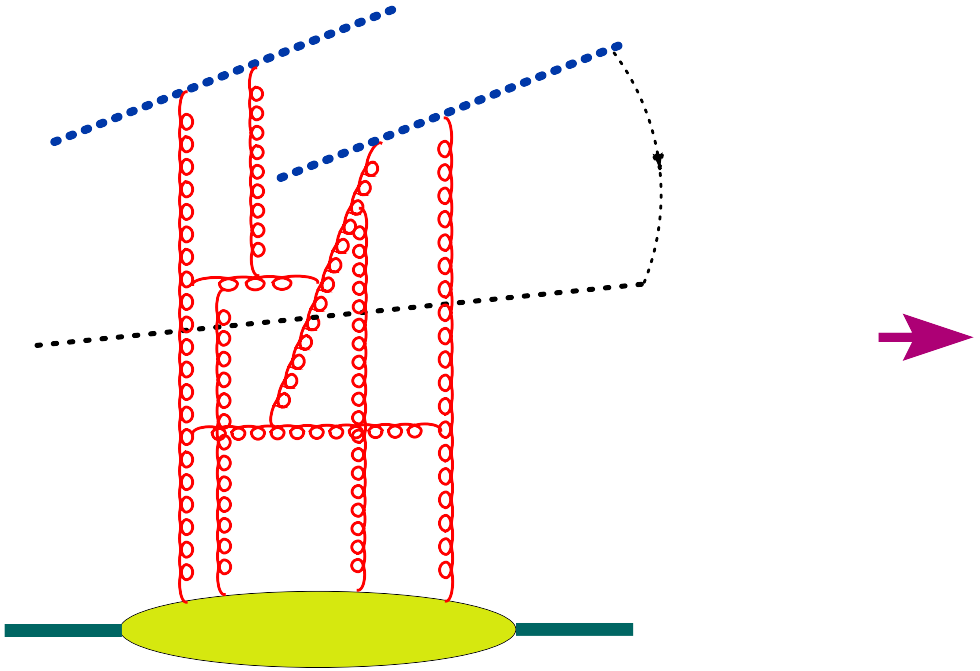

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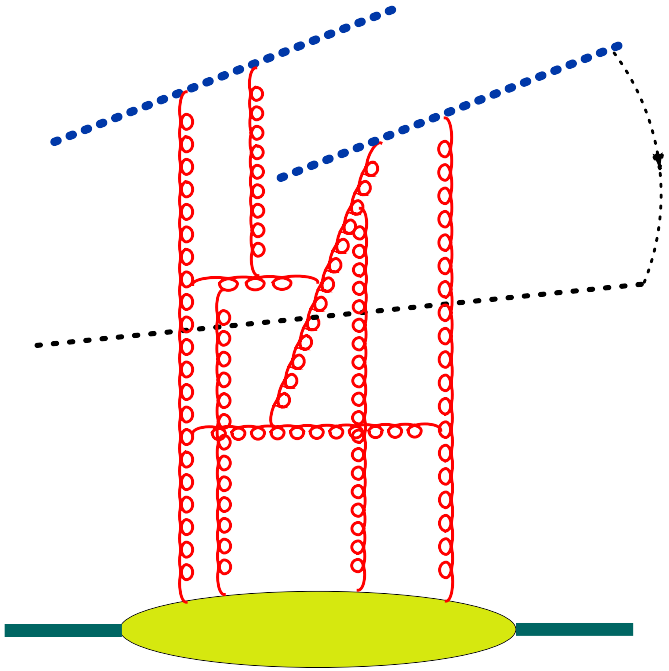
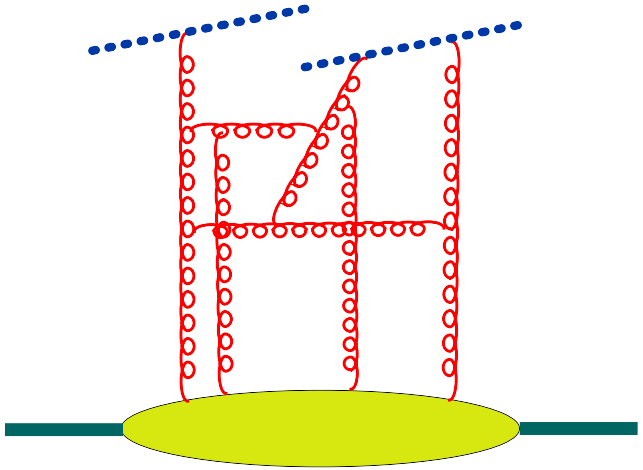


$$\alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \quad \otimes$$



Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$

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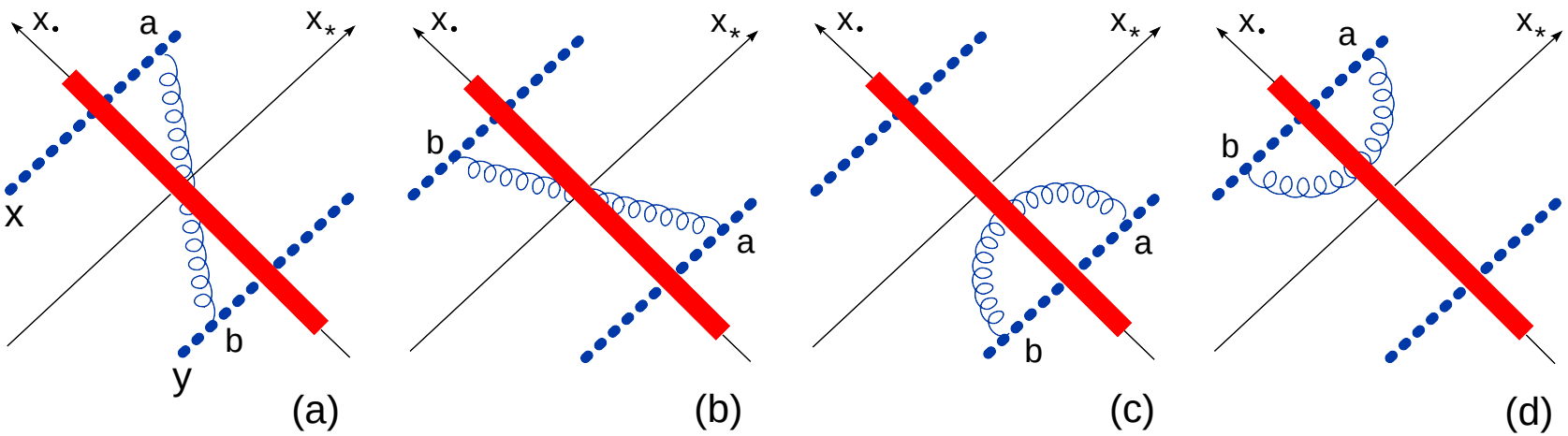

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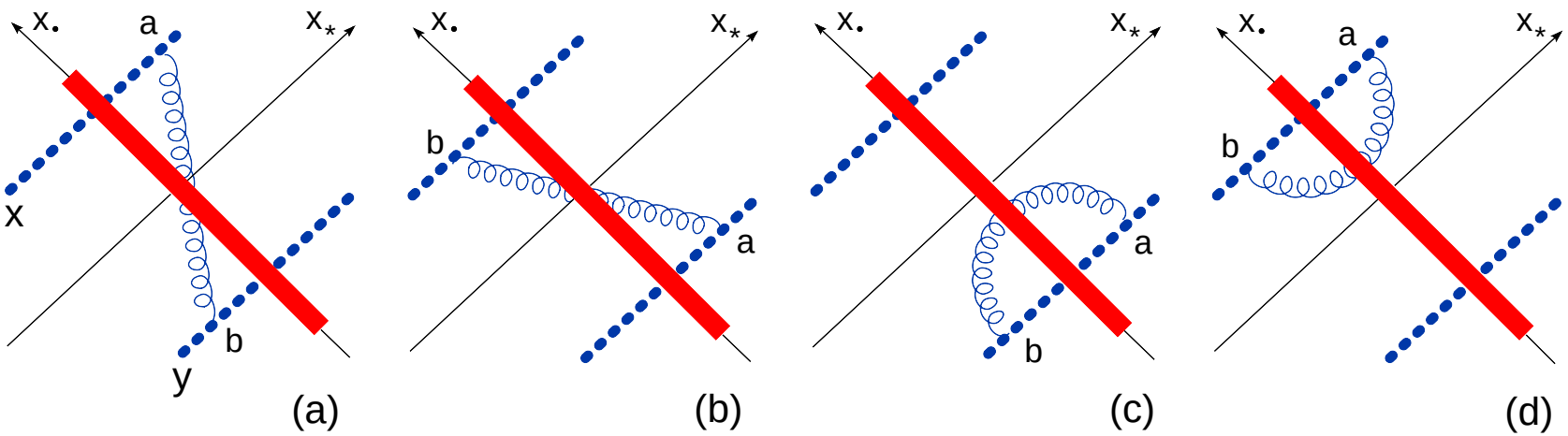

$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non linear evolution equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

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Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semiclassical)

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- To check conformal invariance in $\mathcal{N} = 4$ SYM.

Non linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_x U_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_z U_y^\dagger\}] +$$
$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots$$

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Where $\langle \ldots \rangle$ is evaluated in the background of the shock wave

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Subtraction of BK^2 contribution $\Longrightarrow$ $\left[\frac{1}{u}\right]_+$ prescription

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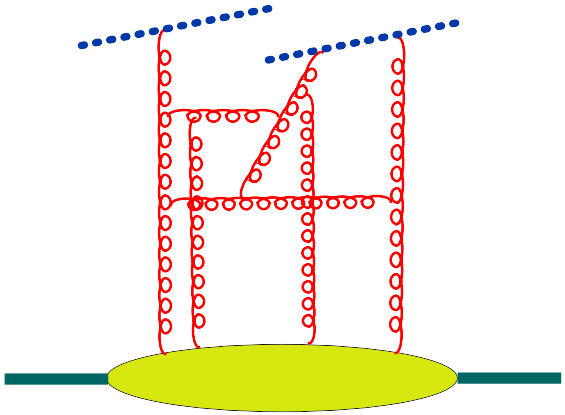
Where $\langle \ldots \rangle$ is evaluated in the background of the shock wave

Subtraction of BK^2 contribution $\Longrightarrow$ $\left[\frac{1}{u}\right]_+$ prescription

$$\int_0^1 du\, f(u)\left[\frac{1}{u}\right]_+ \equiv \int_0^1 du\, \frac{f(u)-f(0)}{u}\,, \qquad \int_0^1 du\, f(u)\left[\frac{1}{\bar{u}}\right]_+ \equiv \int_0^1 du\, \frac{f(u)-f(1)}{\bar{u}}$$

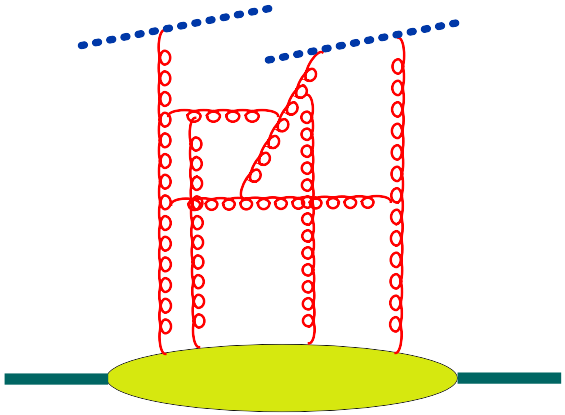
Regularization of the rapididty divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty$$

Regularization of the rapididty divergence



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Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\left\{ ig \int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp) \right\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Regularization of the rapididty divergence

Regularization by:slope

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Regularization by:Rigid cut-off

$$U_x^{\eta} \;=\; \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\; p_1^{\mu} A_{\mu}^{\eta}(up_1 + x_{\perp})\Big]$$

$$A_{\mu}^{\eta}(x) \;=\; \int \frac{d^4k}{(2\pi)^4} \theta(e^{\eta} - |\alpha_k|) e^{-ik\cdot x} A_{\mu}(k)$$

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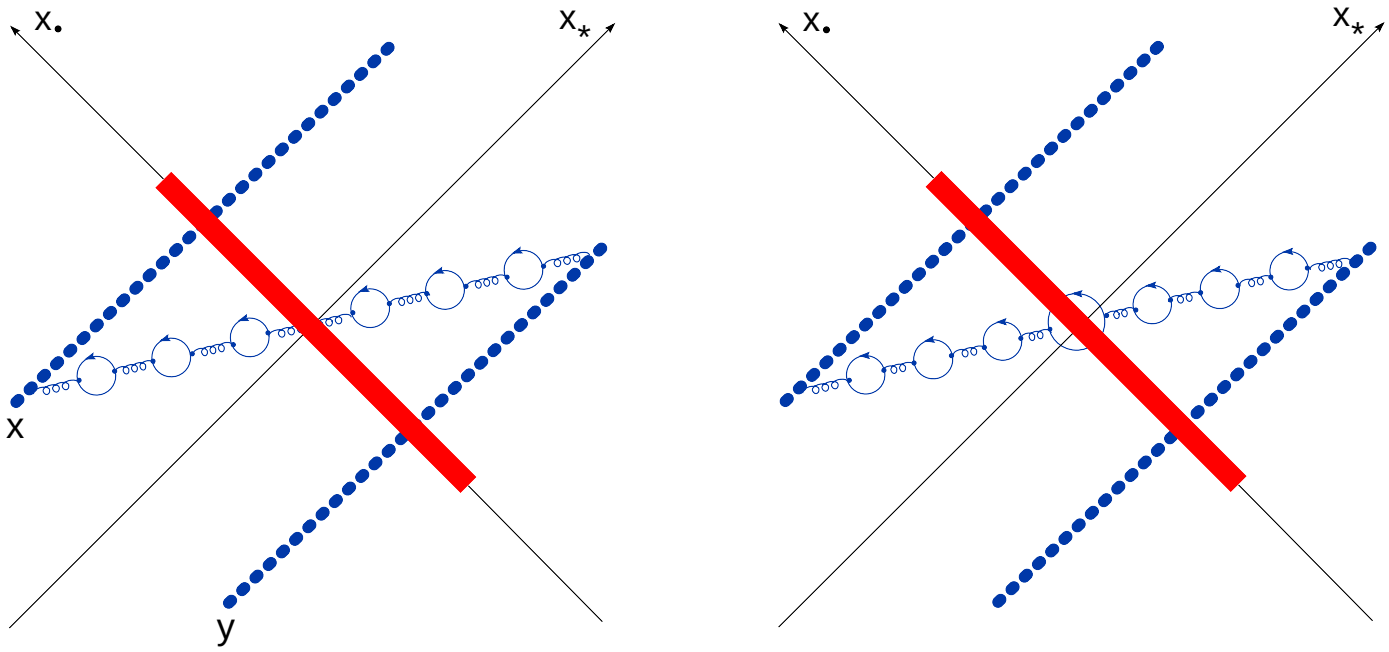
The rigid cut-off leads to (almost) conformal result

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s(?)N_c}{2\pi^2}\int \frac{dz(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2(\vec{z}-\vec{y})^2}\left\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

I. Balitsky (2006) **Renormalon-based approach: summation of quark bubbles**

$$-\frac{2}{3}n_f \to b = \frac{11}{3}N_c - \frac{2}{3}n_f$$



Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$
$$\times\left[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\right) + \frac{1}{Y^2}\left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\right)\right] + \ldots$$

I.Balitsky; Yu. Kovchegov and H. Weigert (2006)

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$
$$\times\left[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\right) + \frac{1}{Y^2}\left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\right)\right] + \ldots$$

I.Balitsky; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2}$	$\lvert x-y\rvert \ll \lvert x-z\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(X)^2)}{2\pi^2X^2}$	$\lvert x-z\rvert \ll \lvert x-y\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2}$	$\lvert y-z\rvert \ll \lvert x-y\rvert, \lvert x-z\rvert$

Argument of coupling constant

Bubble chain sum:

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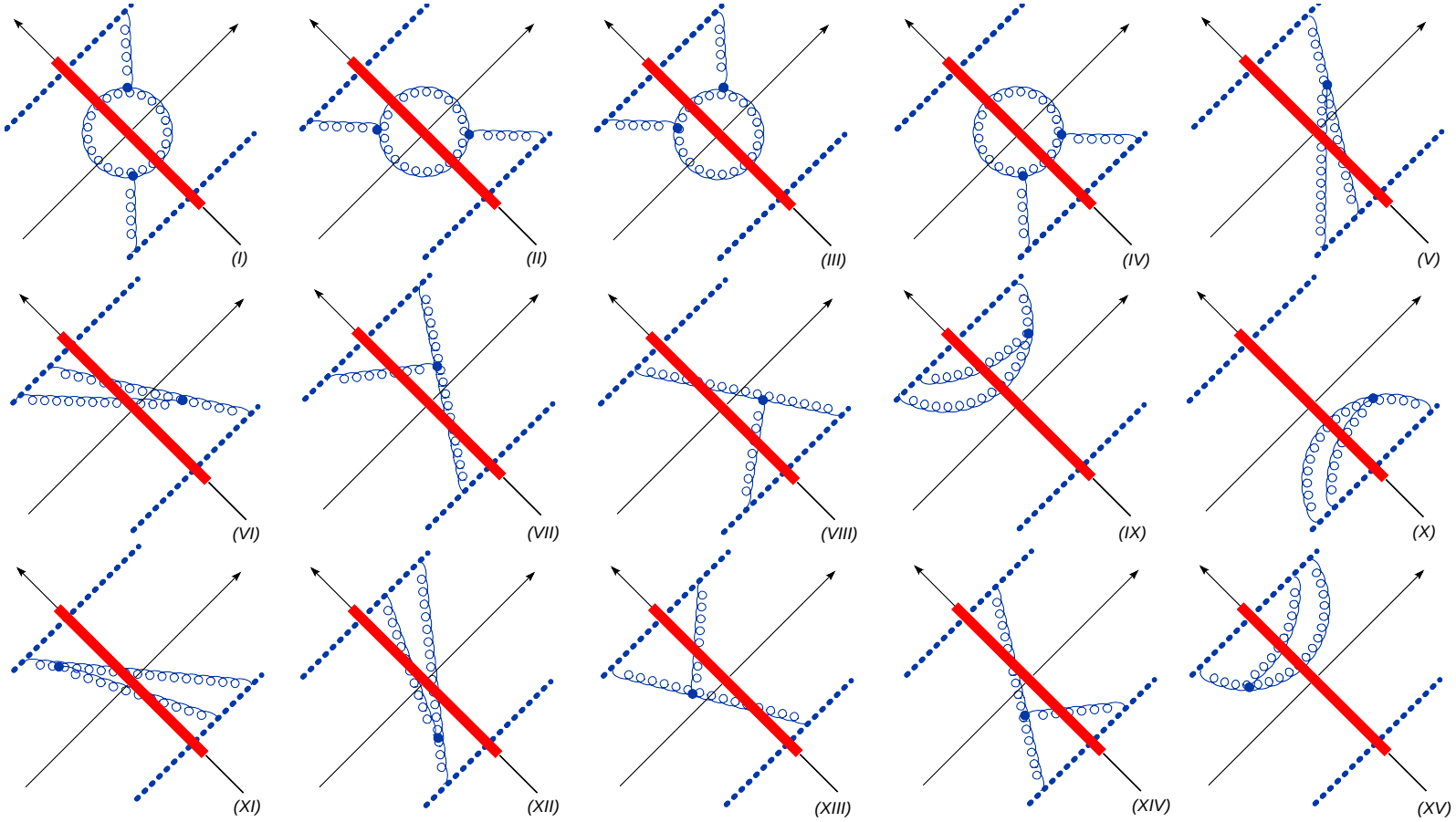
When the sizes of the dipoles are very different the kernel reduces to:

$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2}$	$\lvert x-y\rvert \ll \lvert x-z\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(X)^2)}{2\pi^2X^2}$	$\lvert x-z\rvert \ll \lvert x-y\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2}$	$\lvert y-z\rvert \ll \lvert x-y\rvert, \lvert x-z\rvert$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

Diagrams with 2 gluons interaction



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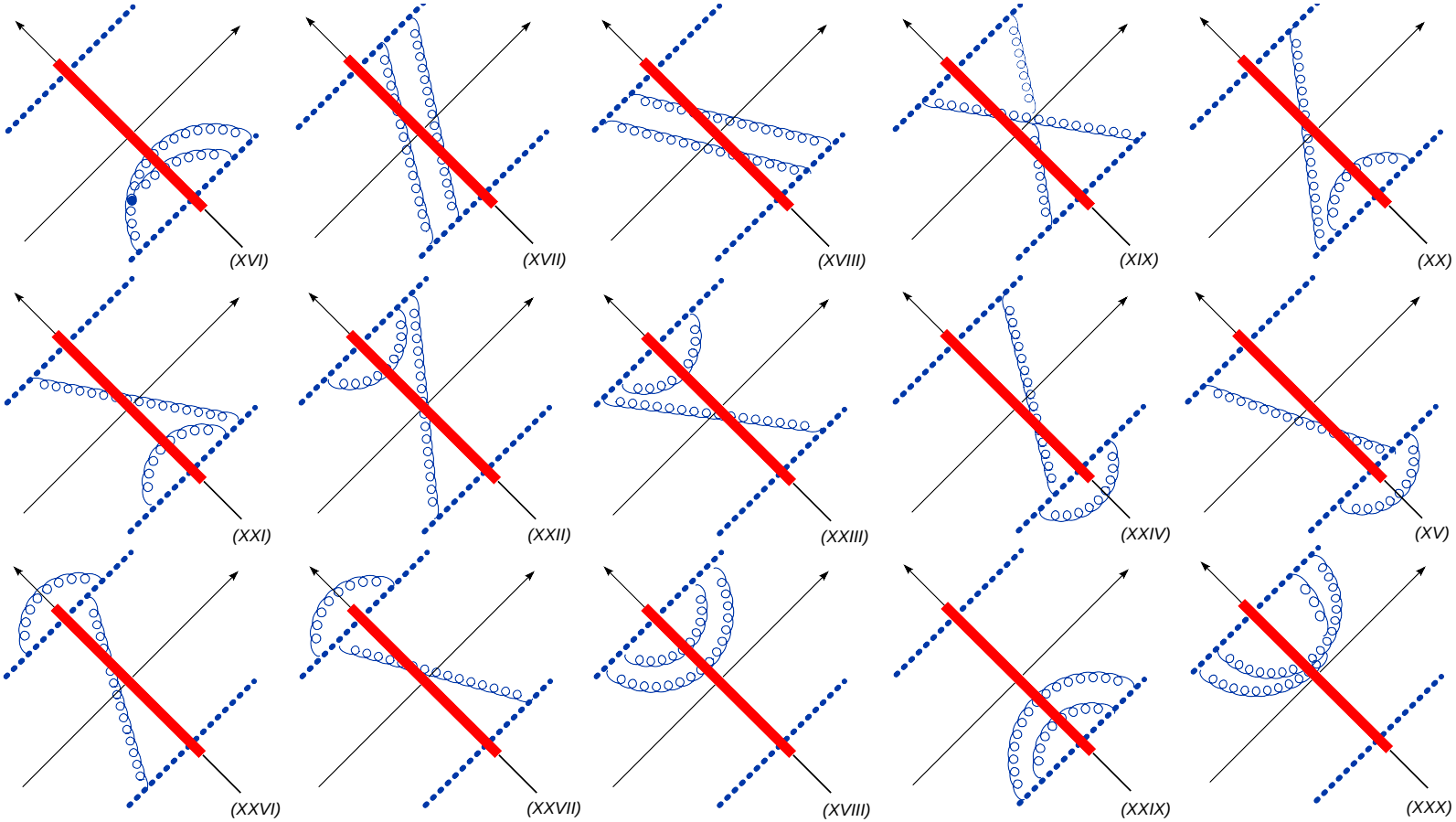

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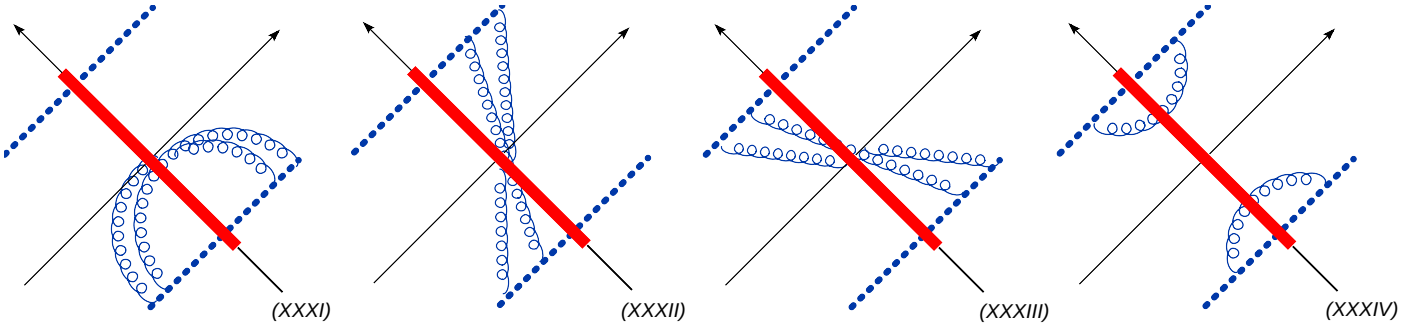
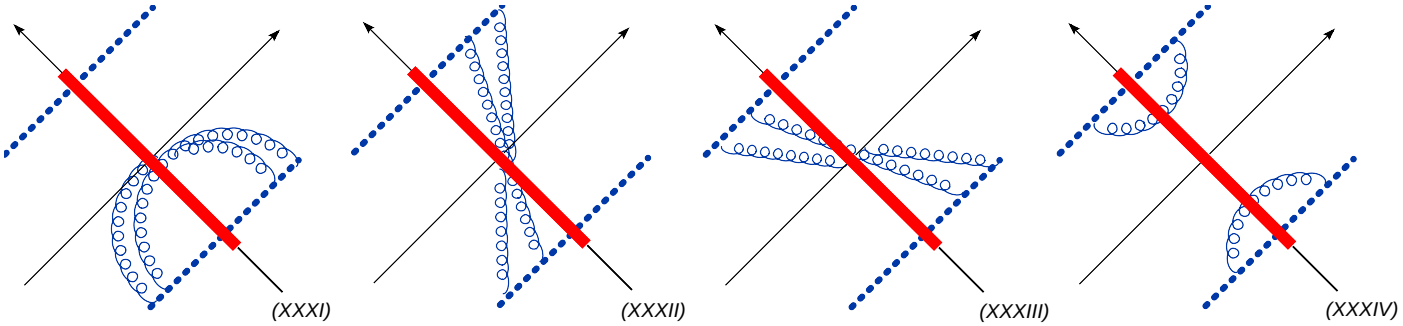


Diagrams with 2 gluons interaction

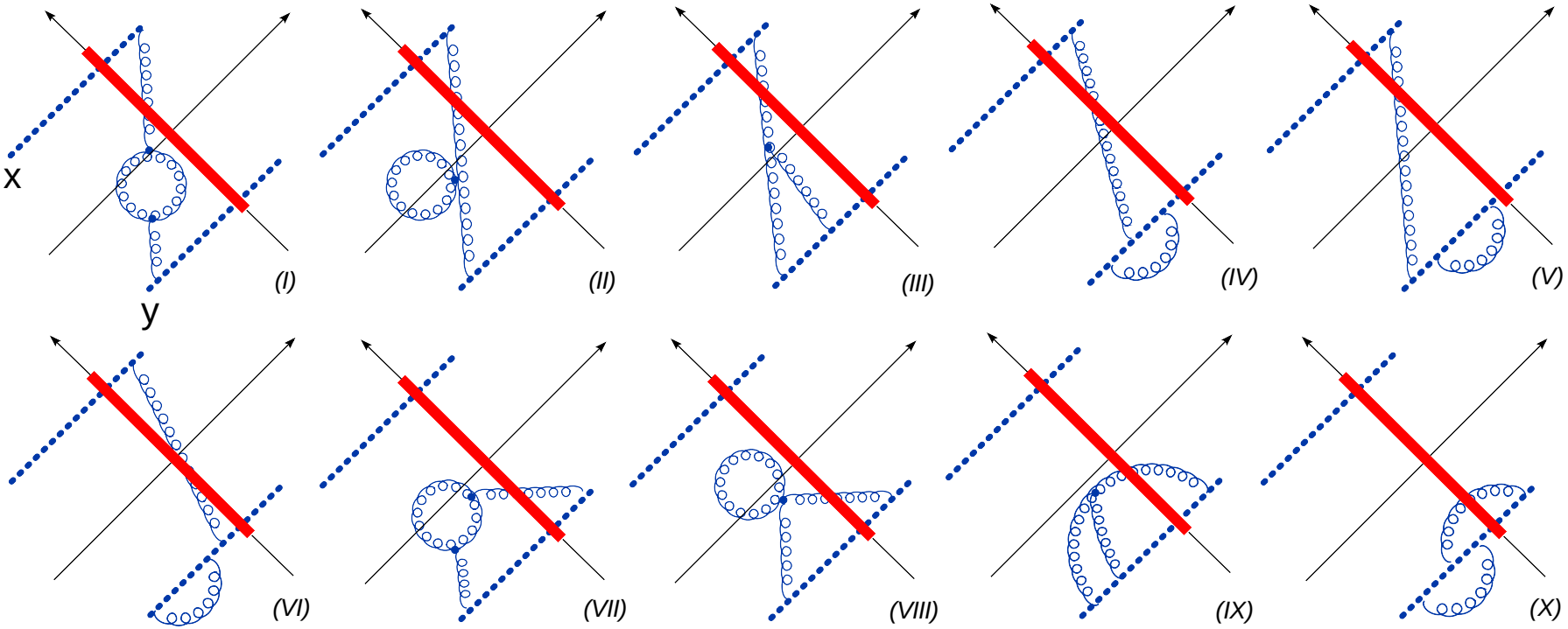


Extracting the UV divergencies

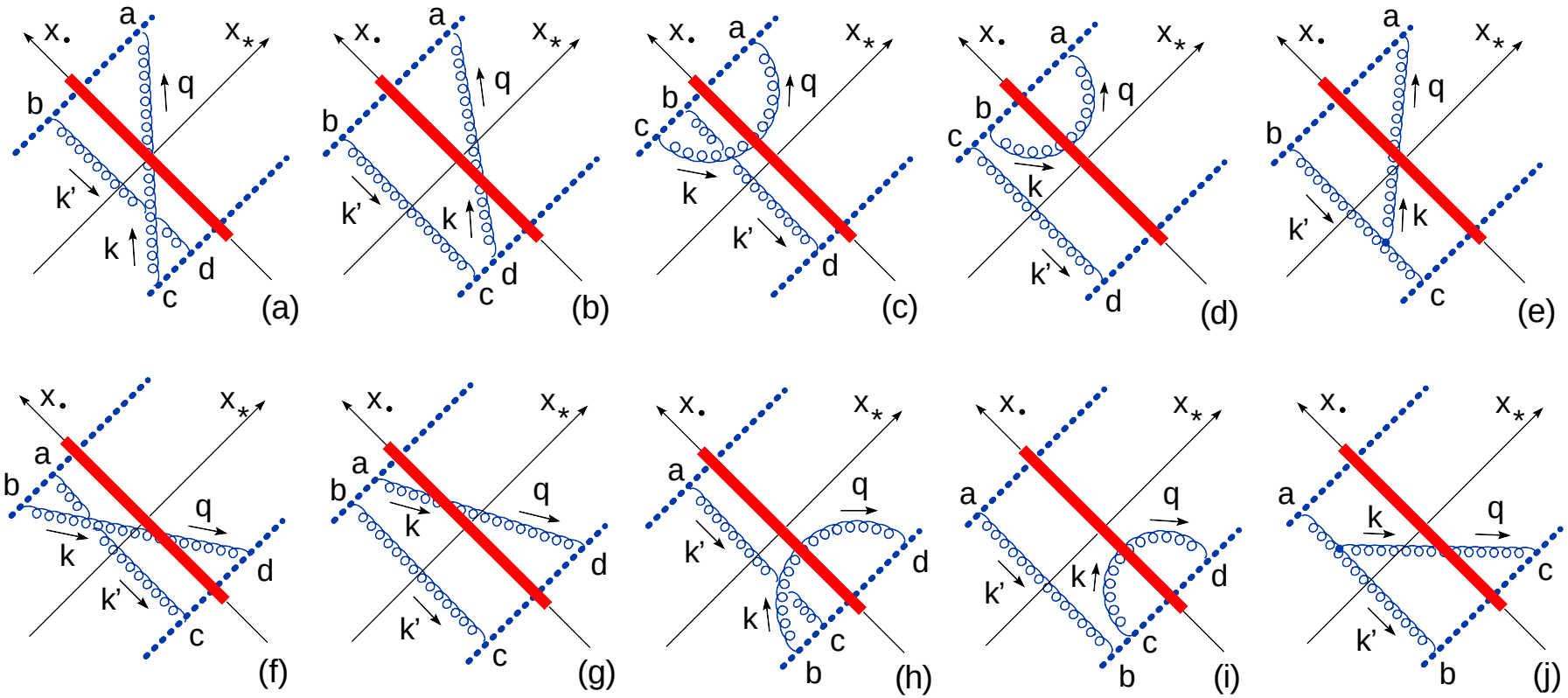
$$\mathrm{Tr}\{t^a U_z t^b U^\dagger_{z'}\} = \mathrm{Tr}\{t^a U_z t^b U^\dagger_{z'} - t^a U_z t^b U^\dagger_z\} + \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\}$$

I.Balitsky 2006; Y. Kovchegov, H. Weigert 2006

"Running coupling" diagrams

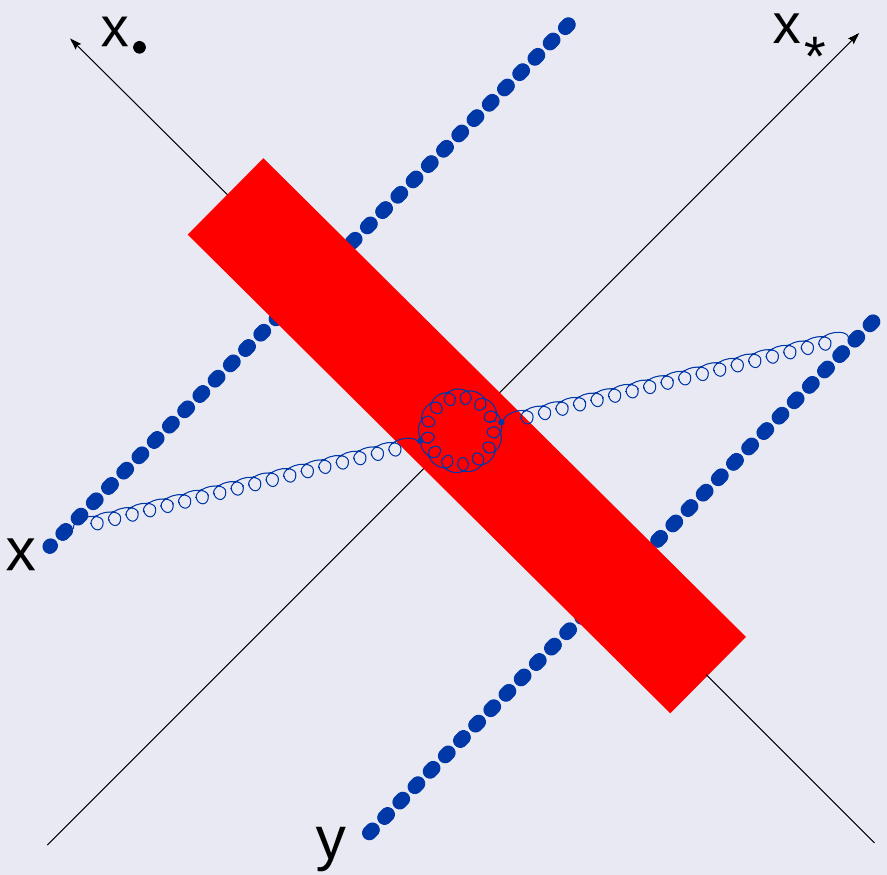


1 → 2 dipole transition diagrams

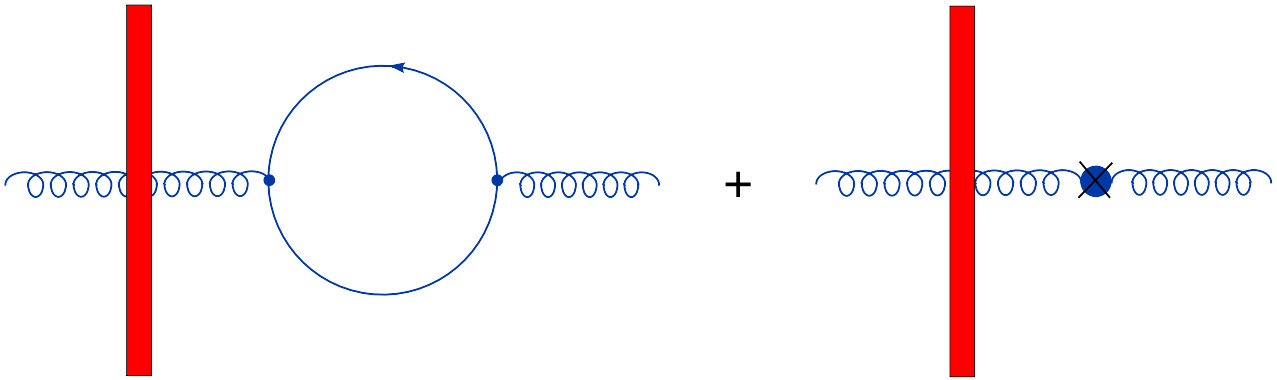


Quark and gluon loop in the shock wave

typical diagram: gluon in the shock wave

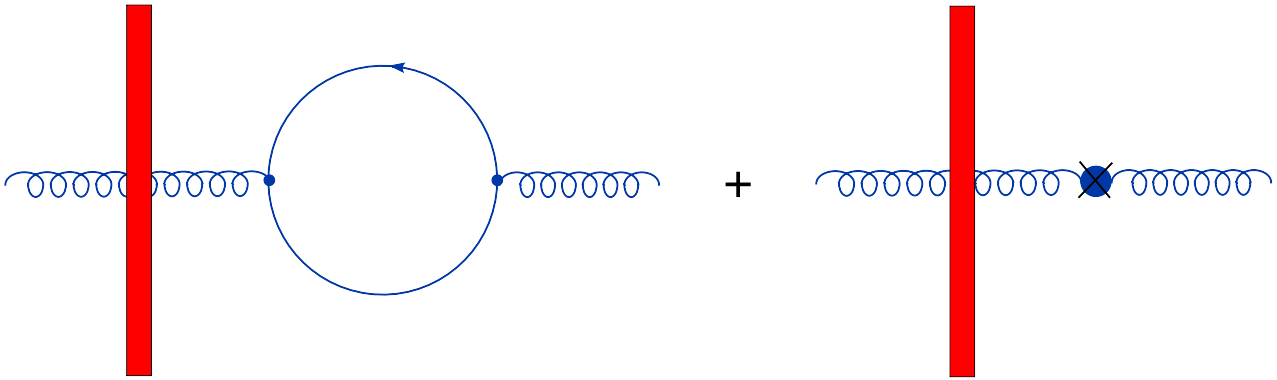


Quark loop in the shock wave



QCD conterterms

Quark loop in the shock wave



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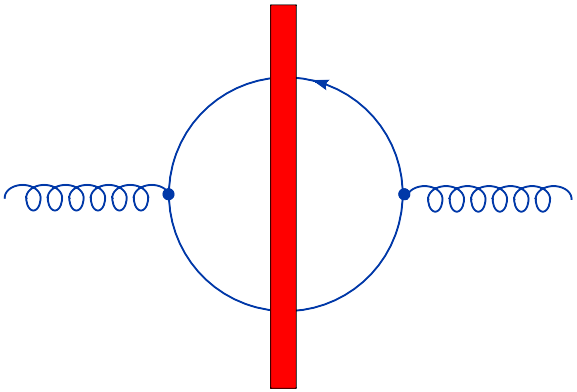
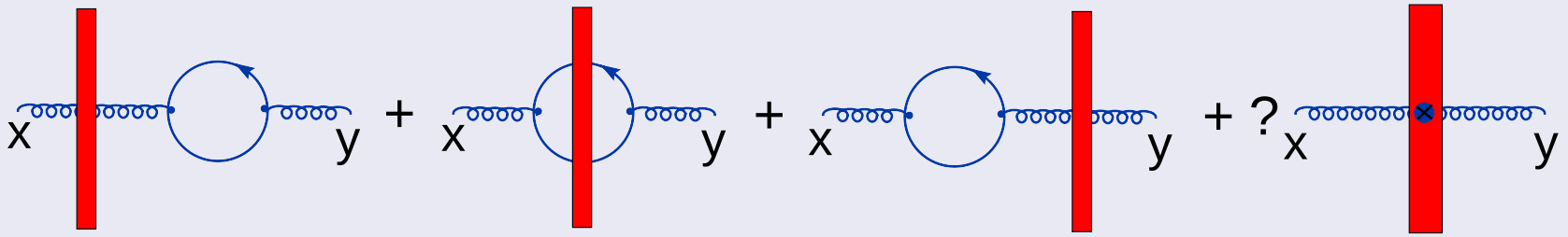


If we simply subtract the divergence using $\overline{MS}$, we could be missing some extra finite terms.

Quark loop in the shock wave

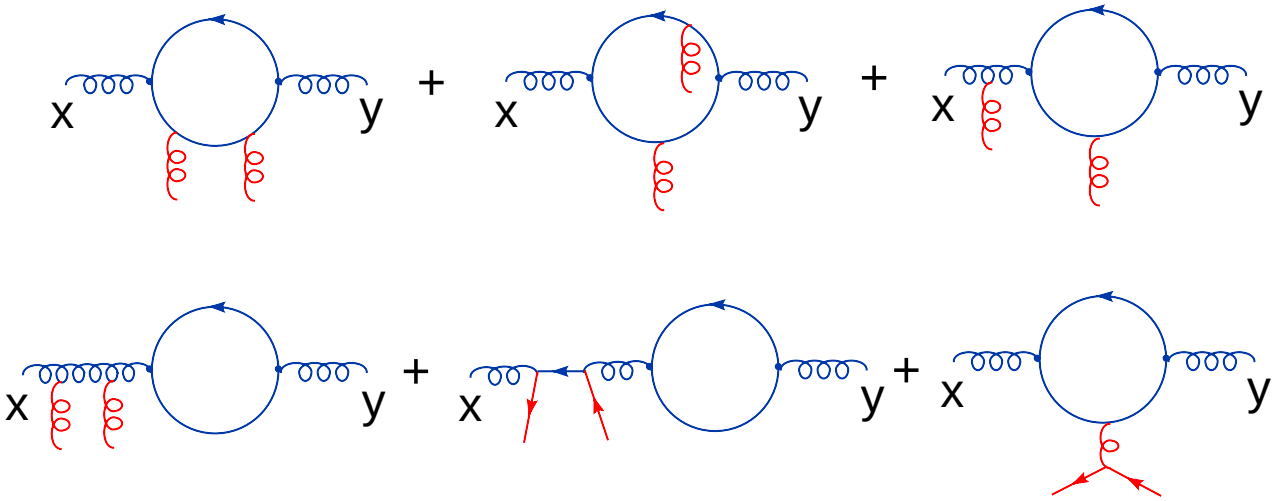
Balitsky 2006

A way to get potential extra terms is to consider the exact calculation of the light-cone expansion of $U_x U_y^\dagger$ at $(x-y)_\perp^2 \to 0$ in QCD and compare it with the expansion of the sum of the diagrams:



light-cone expansion

Typical diagrammatic expression of the light-cone expansion $(x-y)^2_\perp \to 0$ of the quark-loop contribution to the gluon propagator in the external field is



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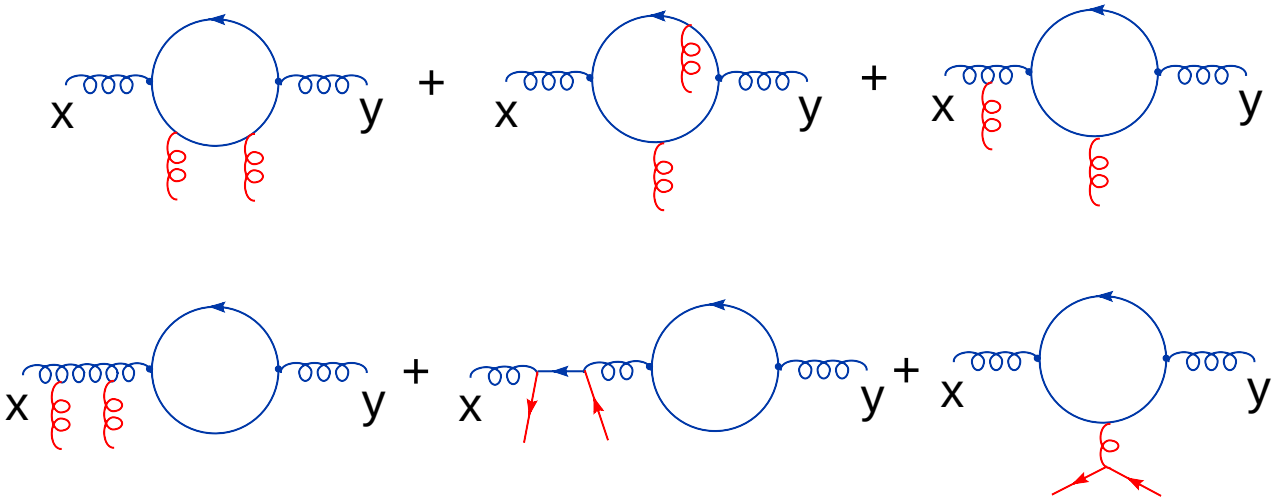
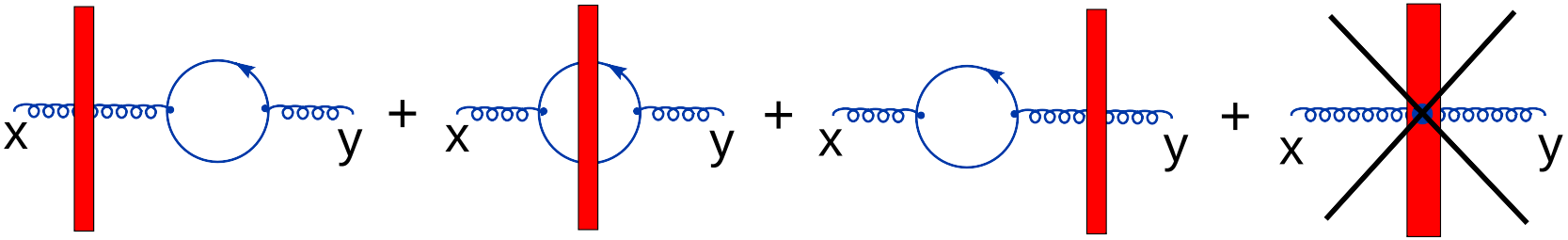


Which coincides with the expansion of



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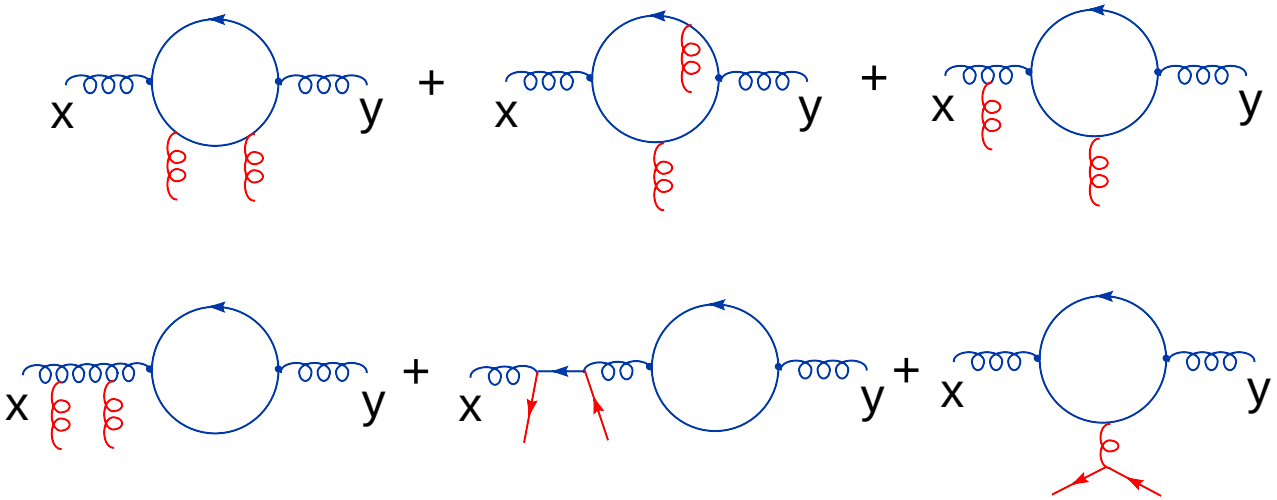

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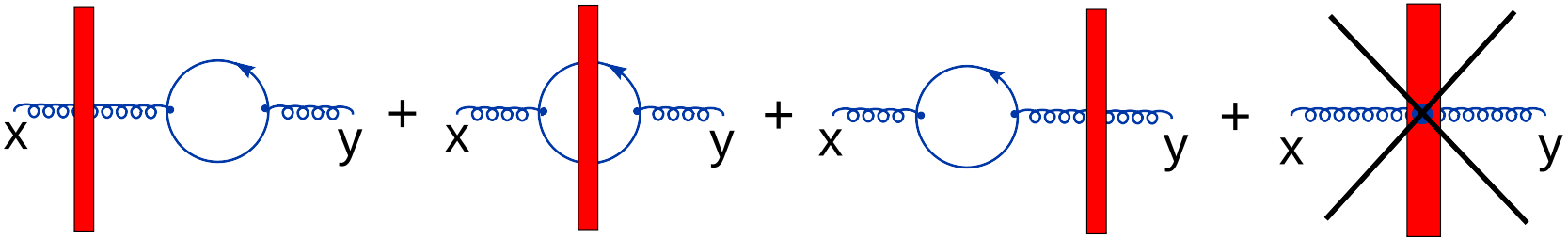

$\Rightarrow$ no additional local operator at one loop level.

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

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& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
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& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part + Non-conformal part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'}U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part $+$ Non-conformal part $+$ Conformal "non-analytic" part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}] \\
& -(z'\to z)\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part $+$ Non-conformal part $+$ Conformal "non-analytic" part $+$ "conformal-analytic" ($\mathcal{N}=4$) part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'}U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_x U_y^\dagger\} =
\end{aligned}
$$

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL

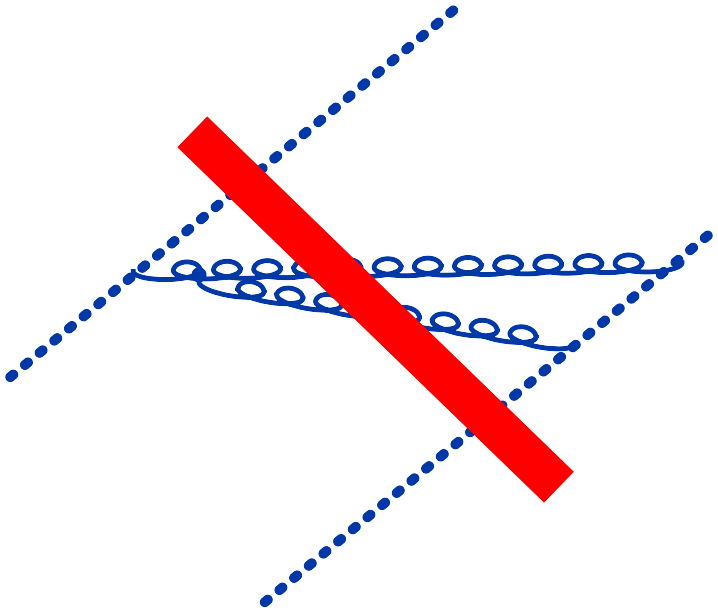
(Comparing the eigenvalue of the forward kernel)

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

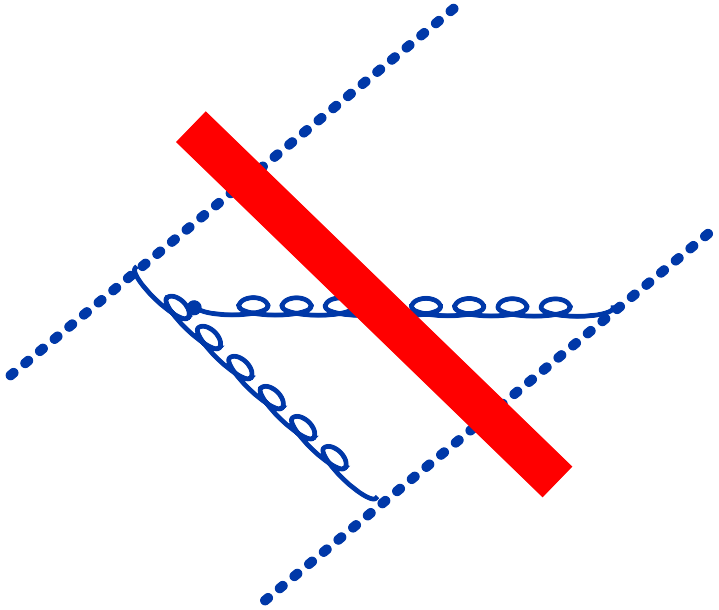
$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'}U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\}
\end{aligned}
$$

However the term $\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\mathrm{U_x}\mathrm{U_y^\dagger}$ contradicts the requirement $\frac{d}{d\eta}U_xU_Y^\dagger = 0$ at $x = y$.

Conformal and non-conformal diagrams



Conformal



Non−Conformal

Slope vs. rigid cut-off regularization

$$\langle K_{\rm NLO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid} = \frac{d}{d\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid} - \langle K_{\rm LO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid}$$
$$\equiv \left[\frac{d}{\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle\right]^{\rm rigid}_{\text{"}+\text{"}}$$

Slope vs. rigid cut-off regularization

$$\langle K_{\rm NLO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid} = \frac{d}{d\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid} - \langle K_{\rm LO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm rigid}$$
$$\equiv \left[\frac{d}{\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle\right]^{\rm rigid}_{"+"}$$

But

$$\langle K_{\rm NLO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm slope} = \frac{d}{d\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm slope} - \langle K_{\rm LO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm slope}$$
$$\neq \left[\frac{d}{\eta}\langle {\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle\right]^{\rm slope}_{"+"}$$

$$\langle K_{\rm LO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}^{\rm rigid} - K_{\rm LO}{\rm Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\rm slope} \neq 0$$

The regularization by slope adds an extra term to the K_{NLO} calculated using rigid cut-off regularization. This extra term is not conformal invariant.

NLO kernel I. Balitsky, G.A.C. (2007)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \\
&= \frac{\alpha_s}{2\pi^2}\int d^2z\,\frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f \\
&\qquad - 2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}] \\
&+ \frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\Bigg[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} \\
&\qquad + \frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
&\qquad\qquad \times [\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\} - \mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\hat{U}_{z'}U_y^\dagger\hat{U}_z\hat{U}_{z'}^\dagger\} - (z'\to z)] \\
&+ \Big\{\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] - \frac{(x-y)^4}{X^2Y'^2X'^2Y^2}\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\,\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\} \\
&+ 4n_f\Big\{\frac{4}{(z-z')^4} - 2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\mathrm{Tr}\{t^a\hat{U}_xt^b\hat{U}_y^\dagger\}[\mathrm{Tr}\{t^a\hat{U}_zt^b\hat{U}_{z'}^\dagger\} - (z'\to z)]\Bigg]
\end{aligned}
$$

Gluon-loop in the shock wave

From NLO BK kernl

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

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$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

From NLO BFKL kernel

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

Gluon-loop in the shock wave

From NLO BK kernl

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

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$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
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$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

The coincidence of terms with the nontrivial γ dependence proves that there is no additional $O(\alpha_s)$ correction to the vertex of the gluon-shock wave interaction coming from the small loop inside the shock wave

About the discrepancy

From BK NLO side

The extra term $\frac{\alpha_s^2 N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\mathrm{U_x}\mathrm{U_y^\dagger}$ would contradict the requirement $\frac{d}{d\eta}U_x U_Y^\dagger = 0$ at $x = y$

From BFKL NLO side

The coefficient $6\zeta(3)$ agrees with the $j \to 1$ asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P} e^{ig \int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \mathrm{P} e^{ig \int_{-\infty}^{+\infty} d\frac{x^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

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$\Rightarrow$ The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\cancel{z}^4}\frac{(x-y)^2 \cancel{z}^4}{(x-z)^2(y-z)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \mathrm{P}e^{ig\int_{-\infty}^{+\infty} d\frac{x^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

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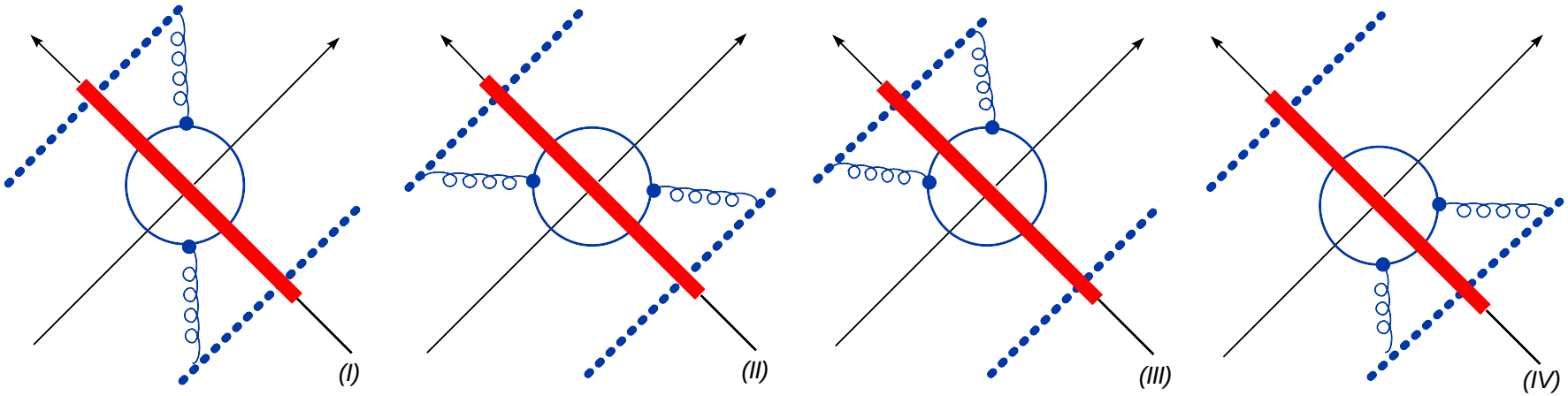
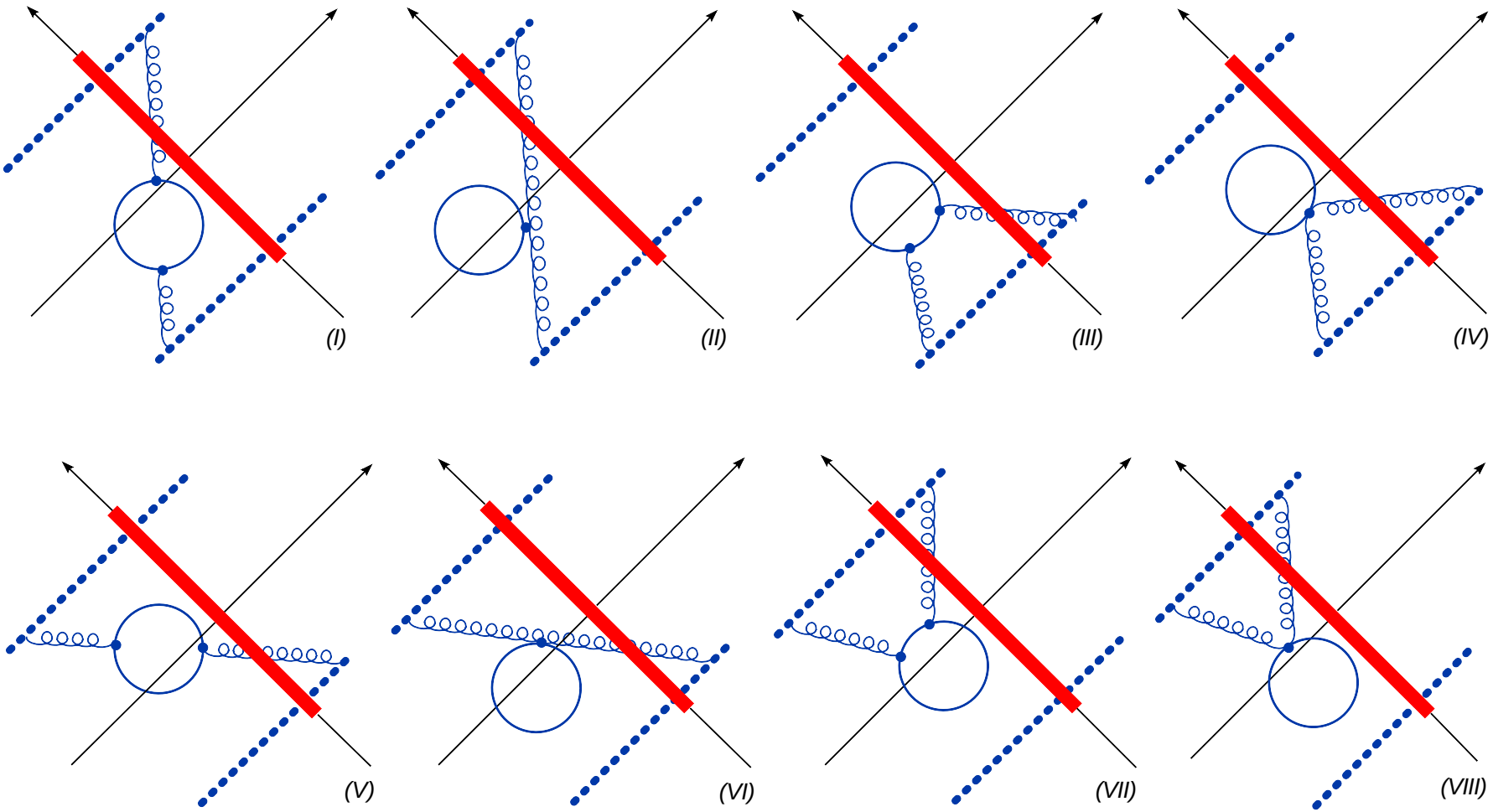
$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\cancel{z^4}} \frac{(x-y)^2 \cancel{z^4}}{(x-z)^2(y-z)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

In the leading order we have conformal invariance. What about the next-to-leading order?

$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c\pi}{12}\Big] - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{16\pi^2}\int d^2z'\;\;[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& \qquad -\mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c\pi}{12}\Big] - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{16\pi^2}\int d^2z'\;\;[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& \qquad -\mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Conformal scheme-dependent part

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1-\frac{\alpha_s N_c\pi}{12}\Big] - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{16\pi^2}\int d^2z'\;\;[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& \qquad -\mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Conformal scheme-dependent part $+$ Non-conformal part

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c\pi}{12}\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\;[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$
$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$

Conformal scheme-dependent part $+$ Non-conformal part
$+$ Conformal analitic part

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c\pi}{12}\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\;[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$
$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\} =$$

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL in $\mathcal{N}=4$
Lipatov and Kotikov, 2004

(Comparing the forward kernel)

Comparison of the dipole evolution and NLO BFKL

$$\int đ^2 p K^{\rm evol}(q,p) f(p) = 4\alpha_s N_c \int đ^2 p \Big\{ \frac{1}{(p-q)^2}\Big(1 - \frac{\alpha_s N_c \pi}{12}\Big)[f(p) - \frac{q^2}{2p^2} f(q)]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{2\ln q^2/p^2 \ln \frac{(q-p)^2}{p^2}}{(q-p)^2} + F(q,p)\Big)\Big\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(p)$$

Comparison of the dipole evolution and NLO BFKL

$$\int đ^2 p K^{\rm evol}(q,p) f(p) = 4\alpha_s N_c \int đ^2 p \Big\{ \frac{1}{(p-q)^2}\Big(1 - \frac{\alpha_s N_c \pi}{12}\Big)[f(p) - \frac{q^2}{2p^2} f(q)]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{2\ln q^2/p^2 \ln \frac{(q-p)^2}{p^2}}{(q-p)^2} + F(q,p)\Big)\Big\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(p)$$

In coordinate space

$$\int đ^2 p K^{\rm evol}(x,z) f(z) = 4\alpha_s N_c \int đ^2 z \Big\{ \frac{1}{(x-z)^2}\Big(1 + \frac{\alpha_s N_c}{4\pi}\Big)[f(z) - \frac{q^2}{2p^2} f(x)]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{\pi^2}{3} - \frac{2\ln z^2/x^2 \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q,p)\Big)\Big\} f(z) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(x)$$

Comparison of the dipole evolution and NLO BFKL

$$\int đ^2 p K^{\rm evol}(q,p) f(p) = 4\alpha_s N_c \int đ^2 p \Big\{ \frac{1}{(p-q)^2}\Big(1 - \frac{\alpha_s N_c \pi}{12}\Big)[f(p) - \frac{q^2}{2p^2} f(q)]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{2\ln q^2/p^2 \ln \frac{(q-p)^2}{p^2}}{(q-p)^2} + F(q,p)\Big)\Big\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(p)$$

In coordinate space

$$\int đ^2 p K^{\rm evol}(x,z) f(z) = 4\alpha_s N_c \int đ^2 z \Big\{ \frac{1}{(x-z)^2}\Big(1 + \frac{\alpha_s N_c}{4\pi}\Big)[f(z) - \frac{q^2}{2p^2} f(x)]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{\pi^2}{3} - \frac{2\ln z^2/x^2 \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q,p)\Big)\Big\} f(z) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(x)$$

While we have

$$\frac{d}{d\eta}\langle \mathcal{V}(x)\rangle = 4\alpha_s N_c \int đ^2 z \Big\{ \frac{1}{(x-z)^2}\Big(1 + \frac{\alpha_s N_c}{4\pi}\Big)[\langle \mathcal{V}(z)\rangle - \frac{q^2}{2p^2}\langle \mathcal{V}(x)\rangle]$$

$$+ \frac{\alpha_s N_c}{4\pi}\Big(- \frac{\pi^2}{3} - \frac{2\ln z^2/x^2 \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q,p)\Big)\Big\} \langle \mathcal{V}(z)\rangle$$

Comparison of the dipole evolution and NLO BFKL

$$\int đ^2pK^{\rm evol}(q,p)f(p) = 4\alpha_s N_c \int đ^2p\Big\{\frac{1}{(p-q)^2}\Big(1-\frac{\alpha_s N_c\pi}{12}\Big)[f(p)-\frac{q^2}{2p^2}f(q)]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big(-\frac{2\ln q^2/p^2\ln\frac{(q-p)^2}{p^2}}{(q-p)^2}+F(q,p)\Big)\Big\}f(p)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(p)$$

In coordinate space

$$\int đ^2pK^{\rm evol}(x,z)f(z) = 4\alpha_s N_c \int đ^2z\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big)[f(z)-\frac{q^2}{2p^2}f(x)]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big(-\frac{\pi^2}{3}-\frac{2\ln z^2/x^2\ln\frac{(x-z)^2}{x^2}}{(q-p)^2}+F(q,p)\Big)\Big\}f(z)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(x)$$

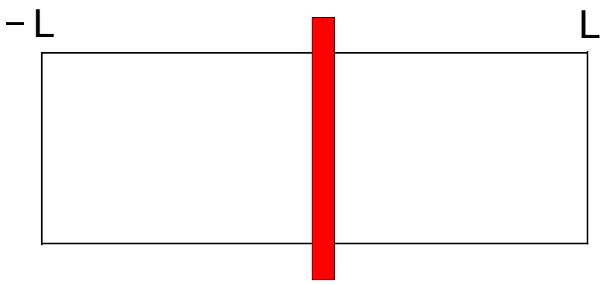
While we have

$$\frac{d}{d\eta}\langle\mathcal{V}(x)\rangle = 4\alpha_s N_c \int đ^2z\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big)[\langle\mathcal{V}(z)\rangle-\frac{q^2}{2p^2}\langle\mathcal{V}(x)\rangle]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big(-\frac{\pi^2}{3}-\frac{2\ln z^2/x^2\ln\frac{(x-z)^2}{x^2}}{(q-p)^2}+F(q,p)\Big)\Big\}\langle\mathcal{V}(z)\rangle$$

$\Rightarrow$Same kernel up to a term proportional to $\frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)$

Checking for the missing term

We have checked the possible missing term using gauge/scalar links at infinity



Gauge/scalar links

$$\mathrm{Tr}\{U_x U_y^\dagger\} = \lim_{L\to\infty} \mathrm{Tr}\{[Lp_1 + x_\perp, -Lp_1 + x_\perp][-Lp_1 + x_\perp, -Lp_1 + y_\perp] \times [-Lp_1 + y_\perp, Lp_1 + y_\perp][Lp_1 + y_\perp, Lp_1 + x_\perp]\}$$

Conclusions

- The NLO kernel for the evolution of the color dipole has been calculated. It consists of three parts: the running-coupling part proportional to β-function, the conformal part describing $1 \to 3$ dipoles transition and the non-conformal term.
- The result agrees with the forward NLO BFKL kernel up to a term proportional $\alpha_s^2 \zeta(3)$ times the original dipole.
- For the creation of dipoles in the small-x evolution, the coupling constant is determined by the size of the smallest dipole.
- The NLO evolution kernel depends on the precise definition of the cutoff in the longitudinal momenta.
- With $|\alpha| < \sigma$ cutoff, the NLO-BK and the NLO-BFKL for $\mathcal{N} = 4$ is (almost) conformally invariant in the transverse plane.

Outlook

- Although Wilson lines are formally conformal invariant, they are divergent. The rigid cutoff does not preserve conformal invariance.
- It is not known how to regularize the Wilson line in such a way that conformal invariance is preserved.
- Checking the conformal invariance for the NLO amplitude of the SYM $\mathcal{N} = 4$, and therefore the calculation of the NLO scalar impact factor.