Poincaré Invariant Quantum Theory

J. Lab Lectures

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1 Background:

Local quantum field theory is the theory of choice for modeling reactions involving energy and distance scales where relativity and quantum theory both have to be considered. Quantum Chromodynamics (QCD) is generally accepted to be the quantum field theory that governs the strong interactions. Given this, it is natural to ask why one might consider an alternative theoretical approach.

The short answer is that we do not really know how to solve QCD, or any non-trivial local four-dimensional field theory. From a practical point of view it is not known how to compute ab-initio error bounds to any field-theoretic calculations. For example, no one has been tempted to dismiss QCD just because a “QCD prediction” does not agree with experiment. From a mathematical point of view, even the existence of QCD is a Millennium problem[1]. This is why non-relativistic potential models are still used for realistic computations.

One of the difficulties with all local field theories is due to the requirement that the theory be local. This condition means that “experiments” done in arbitrarily small space-like separated spacetime regions should be independent. This sounds like a sensible requirement, however formulating this condition requires a theory with an infinite number of degrees of freedom that is defined on all energy scales. This leads to many of the infinities that make the theory so difficult to define. This can be fixed by introducing cutoffs, but this leads to violations of locality on some scale, and it is known [2] that violations on one scale lead to violations on all scales. Establishing that the cutoffs can be removed in a controlled manner leaving a non-trivial theory with the expected properties is the difficult problem. In addition, even if the resulting theory is defined on all scales, we must come to grips with the realization that it is unlikely to be correct at the Planck scale.

The point is that the field theory does a lot more than is needed to realistically describe a class of reactions at some finite energy scale. The key properties of a field theory that should survive at finite energy scales are the quantum mechanical interpretation, Poincaré invariance, cluster properties, and a spectral condition. In addition, the dynamics should dominated by a finite number of degrees of freedom.

While it is a central goal of nuclear physics to resolve all of these issues with QCD, one still wants to develop a quantitative understanding of classes of phenomena that dominate the physics at a given energy scale. Effective field theory does this for low-energy reactions, but it is not generally applicable in the few GeV region. At higher energies quasipotential approaches, such as the Gross equation[3], provide a phenomenology that is motivated by quantum field theory. Poincaré invariant quantum theory is a framework for constructing quantum mechanical models of systems of a finite number of degrees for freedom consistent with Poincaré invariance, cluster properties, and a spectral condition. It appears to be the only known alternative to local field theory satisfying all of these properties [4], although so far cluster properties have only been established for models with a finite number of degrees of freedom. It is suitable for describing reactions in the few GeV region, which are still dominated by a relatively small number of degrees of freedom. There is an inverse scattering theorem [5], so there are interactions that can describe any reaction. Cluster properties provide relations between reactions involving different numbers of degrees of freedom. Poincaré invariant quantum mechanics is a minimal extension of the standard potential theory that is successfully being used to obtain a quantitative understanding of low-energy physics to energies in the few GeV range. While there is no direct connection to QCD, one expects that QCD could be used to provide insight into both the choice of relevant degrees of freedom and an understanding of the operator structures that appear in the interactions and current operators.

2 Quantum theory

Quantum mechanics is a linear theory. The mathematical setting for a quantum theory is a complete complex linear vector space, or Hilbert space. I represent vectors in the Hilbert space by "kets"

$$|\psi\rangle. \tag{1}$$

The Hilbert space inner product in Dirac's Bra-Ket notation has the familiar form:

$$\langle\psi|\phi\rangle. \tag{2}$$

All of the predictions of a quantum theory are expressed in terms of this inner product. In what follows I assume that the Hilbert space vectors are normalized to unity:

$$\langle\psi|\psi\rangle = 1. \tag{3}$$

The quantities that are measured in a quantum theory are probabilities, expectation values, and ensemble averages. Probabilities are given directly in terms of the Hilbert space inner product by

$$P_{\phi,\psi} = |\langle\phi|\psi\rangle|^2. \tag{4}$$

This represents the probability of measuring a system to be in the state represented by the vector $|\phi\rangle$ if it was initially prepared in the state represented by the vector $|\psi\rangle$.

A related quantum observable is the expectation value of a Hermitian operator A in the state $|\psi\rangle$. Hermitian operators have a complete set of eigenvectors, $|n\rangle$, with real eigenvalues a_n. The expectation value of A in the state $|\psi\rangle$,

$$\langle\psi|A|\psi\rangle = \sum_n |\langle\psi|n\rangle|^2 a_n = \sum_n P_{n,\psi} a_n, \tag{5}$$

is the weighted average of the eigenvalues of A by the probabilities of measuring the system to be in the n-th eigenstate of A if it is initially prepared in the state $|\psi\rangle$.

To measure probabilities or expectation values it is necessary to perform a statistically significant number of measurements on identically prepared initial states. In most experiments it is impossible to ensure the initial states are identically prepared. For example, a polarized beam of particles is never 100% polarized. Instead, it consists of a statistical distribution of polarizations. This situation is treated by using an ensemble average. In this case I assume that the initial state is in a statistical ensemble of quantum states $|\psi_m\rangle$ distributed with classical probability P_m. The expectation value of the Hermitian operator A in this ensemble is defined by

$$\langle A\rangle = \sum_m P_m \langle\psi_m|A|\psi_m\rangle$$

This expectation value can be expressed by the trace of the product of A multiplied by a density matrix ρ:

$$< A >:= \mathrm{Tr}(\rho A) \qquad \rho = \sum_m |\psi_m\rangle P_m \langle\psi_m| \qquad \sum_m P_m = 1$$

where here the different $|\psi_m\rangle$ do not have to be orthogonal.

An important observation is that Hilbert space scalar products and eigenvalues of Hermitian operators are unitarily invariant. Since quantum probabilities, expectation values, and ensemble averages are constructed out of

Hilbert space scalar products and eigenvalues of Hermitian operators, all quantum observables are unitarily invariant.

Abstract observables are useful in quantum field theories, but the observables that are relevant to scattering experiments are normally associated with isolated particles. A complete measurement of the state of an asymptotically free particle involves a determination of the particle's linear momentum, up to some finite experimental resolution in a particular coordinate system, and the projection of its spin along some axis in the same (or a different) coordinate system. A complete experiment would involve a measurement of the state (momentum and spin projection) of each of the asymptotic particles produced in a scattering reaction.

These measurements normally involve using a combination of conservation laws and observations of the trajectories of charged particles in classical electromagnetic fields.

The identification of a complete measurement provides a means to pass from an abstract formulation of the Hilbert space of quantum theory to an explicit representation of a model Hilbert space. Any complete measurement of a scattering process will give probabilities of measuring the projection of the particle's spin along a given axis and its linear momentum to be in a finite volume V of momentum space. Thus vectors in a single-particle Hilbert space are square integrable functions of the linear momentum and magnetic quantum number $\langle \mathbf{p}, \mu | \psi \rangle$, where

$$\int_V |\langle \mathbf{p}, \mu | \psi \rangle|^2 d^3p \tag{6}$$

represents the probability that a particle in state $|\psi\rangle$ will be measured to have linear momentum in the momentum volume V and magnetic quantum number μ. The normalization condition

$$\sum_\mu \int |\langle \mathbf{p}, \mu | \psi \rangle|^2 d^3p = 1 \tag{7}$$

means that the probability of finding the particle any of its allowed states is 1.

Thus, the Hilbert for a single particle can be chosen as the space of square integrable functions in the particle's momentum and spin projection. N-particle Hilbert spaces can be taken as N-fold tensor products of single particle spaces. Spaces describing states with variable particle number can be taken as orthogonal direct sums of N-particle spaces.

For example, a suitable Hilbert space to describe nucleon-nucleon scattering at an energy sufficient to produce no more than one pion can be taken to be

$$\mathcal{H} = (\mathcal{H}_n \otimes \mathcal{H}_n) \oplus (\mathcal{H}_n \otimes \mathcal{H}_n \otimes \mathcal{H}_\pi) \tag{8}$$

Wave functions have the form

$$\langle \cdot | \psi \rangle = \begin{pmatrix} \langle \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 | \psi_{nn} \rangle \\ \langle \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2, \mathbf{p}_\pi | \psi_{nn\pi} \rangle \end{pmatrix} \tag{9}$$

with normalization

$$1 = \langle \psi | \psi \rangle = \langle \psi_{nn} | \psi_{nn} \rangle + \langle \psi_{nn\pi} | \psi_{nn\pi} \rangle. \tag{10}$$

The quantity

$$\langle \psi_{nn} | \psi_{nn} \rangle \tag{11}$$

is the probability that the state $|\psi\rangle$ will be measured to have two nucleons and no pions and

$$\langle \psi_{nn\pi} | \psi_{nn\pi} \rangle \tag{12}$$

is the probability that the state $|\psi\rangle$ will be measured to have two-nucleons an one pion.

Thus, for any bounded energy range the observable experimental reaction products determine a representation of the Hilbert space with sufficient structure to describe all accessible experimental observables. In some applications, for example with models involving confined degrees of freedom, it is possible and useful to use different degrees of freedom. In any representation the experimental degrees of freedom (physical particles) must eventually appear in the formulation of the scattering asymptotic conditions that are used to define scattering probabilities.

3 Special Relativity

A fundamental assumption of special relativity is the existence of inertial coordinates systems. Inertial coordinate systems have the property that equivalent experiments done in different inertial coordinate systems lead to identical results.

Experimentally, the Michelson-Morley experiment established that different inertial coordinate systems are related by transformations that preserve the proper distance (time)

$$|\mathbf{x}-\mathbf{y}|^2 - c^2(t_x - t_y)^2 = |\mathbf{x}'-\mathbf{y}'|^2 - c^2(t'_x - t'_y)^2. \tag{13}$$

between the space-time coordinates of "events", which are labeled by their space and time coordinates $(t, \mathbf{x})$ and $(t', \mathbf{x}')$ in different inertial coordinate systems.

The Poincaré group is the group of transformations that preserves the quadratic form (13). I use 4-vectors, x^μ, to label events

$$x \to x^\mu = (x^0, x^1, x^2, x^2) = (ct, x^1, x^2, x^2) \tag{14}$$

and use the Minkowski metric

$$\eta_{\mu\nu} = \eta^{\mu\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

The most general point transformation, $x' = f(x)$, satisfying (14) is called a Poincaré transformation which has the general form

$$x^\mu \to x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{16}$$

where a^μ is constant and $\Lambda^\mu{}_\nu$ is a constant matrix, called a Lorentz transformation, satisfying

$$\eta^{\mu\nu} = \sum_{\alpha\beta} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta}. \tag{17}$$

In what follows I use Einstein's summation convention, which assumes that repeated lower case Greek letters are summed from 0 to 3.

The full Poincaré group is generated by space translations, time translations, rotations, and rotationless Lorentz boosts which depend continuously on a parameter, as well as the discrete transformations of space reflection, time reversal, and four dimensional reflections.

It is experimentally observed that the discrete Poincaré transformations are not symmetries of the weak interaction. In what follows the symmetry group of special relativity will be taken as the subgroup of the Poincaré group where the Lorentz transformations can be continuously deformed to the identity. I use the term Poincaré transformation to refer to this subgroup of the full Poincaré group.

Classically relativistic invariance is interpreted to mean that dynamical equations remain unchanged under changes of inertial coordinate system. This is because the solution of the dynamical equations is observable. This leads to the notion that the equations of a relativistically invariant theory should be "covariant". This is no longer necessary in a quantum theory.

In a quantum theory the results of experimental measurements are quantum probabilities, expectation values, and ensemble averages. These quantities are not solutions of dynamical equations. Special relativity requires that these quantum observables remain invariant under change of inertial coordinate system.

The invariance of quantum observables with respect to changes in inertial coordinate system means that the group of Poincaré transformations continuously connected to the identity is a symmetry of the quantum theory. In 1939 Wigner [6] showed that this requirement is equivalent to the existence of a unitary representation of the Poincaré group on the quantum mechanical Hilbert space. Below are some observations related to Wigner's theorem. Poincaré invariant quantum mechanics is simply a quantum theory with a unitary representation of the Poincaré group.

1. Wigner's results apply to any quantum theory including quantum field theory. Wigner's theorem motivated all serious attempts to axiomatize quantum field theory [2][7][8].

2. As stated above, discrete Poincaré transformations are not considered part of the Poincaré group when discussing special relativity.

3. Antiunitary transformations do not appear because any continuous Poincaré transformation can be written at the square of another Poincaré transformation.

4. Wigner's unitary representations of the Poincaré group are ray representations - Later Bargmann [9] showed that they could be replaced by single-valued representations of the covering group, which replaces Lorentz transformations by the group $SL(2,\mathbb{C})$.

5. Wigner's theorem says nothing about microscopic causality; however the Poincaré invariance is not compatible with the existence of a sensible position operator for particles [10] , which would be needed test locality in a theory of particles. Implementation of a test of microscopic causality requires additional degrees of freedom normally associated with fields. The difficulty in defining a suitable position operator for a particle has nothing to do with antiparticles.

6. The basic building blocks of unitary representations of the Poincaré group are the irreducible representations, which were also classified by Wigner in [6]. Vectors in the positive mass positive energy irreducible representations have exactly the same quantum numbers as a particle of mass m and spin s. This will be discussed in section 7.

4 Parameterization of the Poincaré group

A general Poincaré group element (Λ, a) is labeled by a Lorentz transformation Λ and a space-time translation four vector a. The group product is

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{18}$$

where $\Lambda_2\Lambda_1$ means $\Lambda^{\mu}_{2\alpha}\Lambda^{\alpha}_{1\nu}$ and $\Lambda_2 a_1$ means $\Lambda^{\mu}_{2\alpha}a^{\alpha}_1$. The identity is

$$I = (I, 0) \tag{19}$$

and inverse is

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a). \tag{20}$$

The Poincaré group is a 10 parameter group. There are 10 independent one-parameter subgroups associated with rotations about the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axes, rotationless Lorentz transformations in the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ direction, translations in the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions and time translations. These elementary transformation can be used to generate any Poincaré transformation.

The rotations can be parameterized by an axis and angle of rotation, $\boldsymbol{\theta}$:

$$\Lambda \to R(\boldsymbol{\theta}) = e^{i\mathbf{L}\cdot\boldsymbol{\theta}} \tag{21}$$

where

$$\mathbf{L} = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

while the rotationless Lorentz boost that transforms a particle of mass m at rest to linear momentum $\mathbf{p}$ is

$$\Lambda \to \mathbf{B}(\mathbf{p}/m) = \begin{pmatrix} h/m & \mathbf{p}/m \\ \mathbf{p}/m & I + \frac{\mathbf{p}\otimes\mathbf{p}}{m(m+h)} \end{pmatrix} \tag{23}$$

where $h = \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. This Lorentz transformation can be expressed in terms of the rapidity $\boldsymbol{\rho}$:

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{p}} \qquad \frac{|\mathbf{p}|}{m} = \sinh(|\rho|) \qquad \frac{|\omega(\mathbf{p})|}{m} = \cosh(|\rho|) \tag{24}$$

which plays a similar role as the angles in the rotation group. When expressed in terms of angles or rapidity, $\mathbf{B}(\mathbf{p}/m) \to \mathbf{B}(\boldsymbol{\rho})$, rotations about a specific axis or rotationless Lorentz transformations in given direction become one parameter groups

$$R(\theta_1\hat{\mathbf{n}})R(\theta_2\hat{\mathbf{n}}) = R((\theta_1+\theta_2)\hat{\mathbf{n}}) \qquad B(\rho_1\hat{\mathbf{n}})B(\rho_2\hat{\mathbf{n}}) = B((\rho_1+\rho_2)\hat{\mathbf{n}}) \tag{25}$$

Lorentz transformations can also be represented by 2×2 complex matrices with determinant $= 1$[2][11]. This representation is useful for computations and well as for establishing general properties of the Lorentz group. To motivate this representation note that any four vector can be expressed as a 2×2 Hermitian matrix as follows:

$$X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(X\sigma_\mu) \tag{26}$$

where $\sigma_0 = I$ and σ_i are the three Pauli matrices. The connection with the Lorentz group follows because

$$\det(X) = -\eta_{\mu\nu}x^\mu x^\nu = -x^2. \tag{27}$$

Real Lorentz transformations correspond to linear transformations that preserve this determinant and the Hermiticity of X. Up to irrelevant constant multiplicative factors, the most general transformation with these properties is

$$X' = AXA^\dagger \qquad \det(A) = 1. \tag{28}$$

It follows from (26) that

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\mu A^\dagger). \tag{29}$$

The most general A with this property has the form

$$A = \pm e^{\frac{1}{2}(\boldsymbol{\rho}+i\boldsymbol{\theta})\cdot\boldsymbol{\sigma}} \tag{30}$$

where when $\boldsymbol{\theta}$ is zero the transformation is a rotationless boost with rapidity vector $\boldsymbol{\rho}$ and when $\boldsymbol{\rho}$ is zero this transformation is a rotation with angle of rotation $\boldsymbol{\theta}$. Both A and $-A$ correspond the same Lorentz transformations. In general there is a 2 to 1 correspondence with all Lorentz transformations continuously connected to the identity. In this representation boosts are positive (negative) Hermitian matrices with determinant 1 and rotations are unitary matrices with determinant 1.

5 Unitary representations

Unitary representations of the Poincaré group are unitary operators $U(\Lambda, a)$ satisfying the group representation property

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{31}$$

$$U(I, 0) = I \tag{32}$$

$$U[(\Lambda, a)^{-1}] = U(\Lambda^{-1}, -\Lambda^{-1}a) = U^\dagger(\Lambda, a). \tag{33}$$

The Poincaré group has 10 independent one parameter subgroups (25) labeled by angle of rotation (3), rapidity of a Lorentz boost (3) , spatial displacement (3), and temporal displacement (1). The generators of these transformations are the Hermitian operators

$$\mathbf{J}\cdot\hat{\mathbf{x}} = -i\frac{d}{d\theta}U(R(\theta\hat{\mathbf{x}}, 0)_{|\theta=0} \tag{34}$$

$$\mathbf{K}\cdot\hat{\mathbf{x}} = -i\frac{d}{d\rho}U(B(\rho\hat{\mathbf{x}}, 0)_{|\rho=0} \tag{35}$$

$$\mathbf{P}\cdot\hat{\mathbf{x}} = -i\frac{d}{da}U(I,a\hat{x})_{|a=0} \tag{36}$$

$$H = i\frac{d}{dt}U(I,(t,\mathbf{0})_{|t=0} \tag{37}$$

The generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ have the familiar interpretations; $\mathbf{J}$ is the angular momentum operator, $\mathbf{K}$ is the generator of rotationless Lorentz transformations, $\mathbf{P}$ is the linear momentum operator, and H is the Hamiltonian.

The group property (18) implies that the generators can be grouped into operators that transform as tensors with respect to Lorentz transformations

$$P^\mu = (H, \mathbf{P}) \tag{38}$$

$$J^{\mu\nu} = \begin{pmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & J_z & -J_y \\ K_y & -J_z & 0 & J_x \\ K_z & J_y & -J_x & 0 \end{pmatrix} \tag{39}$$

which have the transformation properties

$$U^\dagger(\Lambda,a)P^\mu U(\Lambda,a) = \Lambda^\mu{}_\nu P^\nu \tag{40}$$

$$U^\dagger(\Lambda,a)J^{\mu\nu}U(\Lambda,a) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (J^{\alpha\beta} - a^\alpha P^\beta + a^\beta P^\alpha). \tag{41}$$

The group representation property can be used to show that these operators also satisfy the commutations relations

$$[P^i, P^j] = 0 \qquad [P^i, H] = 0 \tag{42}$$

$$[J^i, J^j] = i\epsilon_{ijk}J^k \qquad [J^i, P^j] = i\epsilon_{ijk}P^k \tag{43}$$

$$[J^i, K^j] = i\epsilon_{ijk}K^k \qquad [J^i, H] = 0 \tag{44}$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k \qquad [K^i, H] = iP^i \qquad [K^i, P^j] = i\delta_{ij}H. \tag{45}$$

The spin is related to the Pauli-Lubanski vector [12]

$$W^\mu = \frac{1}{2}\epsilon^\mu{}_{\nu\alpha\beta}P^\nu J^{\alpha\beta} \tag{46}$$

which satisfies

$$[P^\mu, W^\nu] = 0 \qquad [W^\mu, W^\mu] = i\epsilon^{\mu\nu}{}_{\alpha\beta}W^\alpha P^\beta \tag{47}$$

The Poincaré group has two polynomial invariants

$$M^2 = H^2 - \mathbf{P}\cdot\mathbf{P} \qquad \text{and} \qquad W^2 \tag{48}$$

When $M^2 \neq 0$ the invariant W^2 can be replaced by the spin:

$$j^2 := W^2/M^2. \tag{49}$$

$M^2 > 0$	$P^0 > 0$	$p_0 = (M, 0, 0, 0)$	$SU(2)$
$M^2 > 0$	$P^0 < 0$	$p_0 = (-M, 0, 0, 0)$	$SU(2)$
$M^2 = 0$	$P^0 > 0$	$p_0 = (1, 0, 0, 1)$	$E(2)$
$M^2 = 0$	$P^0 < 0$	$p_0 = (-1, 0, 0, 1)$	$E(2)$
$M^2 = 0$	$P^0 = 0$	$p_0 = (0, 0, 0, 0)$	$SL(2, \mathbf{C})$
$M^2 < 0$		$p_0 = (0, 0, 0, \mu)$	$SU(1, 1)$

Table 1:

6 Irreducible representations

Irreducible representations of the Poincaré group are important in Poincaré invariant quantum theory because they are the elementary building blocks of any unitary representation of the Poincaré group.

Irreducible representation of the Poincaré group can be classified by the eigenvalues of the invariant operators M^2 and W^2 as well as the sign of P^0. Wigner [6] identified six classes of irreducible representations associated with the joint spectrum of M^2 and P^0. For each class there is a representative vector that is invariant under a subgroup of the Lorentz group, called the little group for that representation. These are listed in Table 1.

Here $E(2)$ is the covering group for the two-dimensional Euclidean group. Irreducible representations of the Poincaré group are constructed by starting with irreducible representations of the little group that leaves the canonical vector p_0 invariant, followed by a parameterized set of Lorentz transformation that change the value of p_0

The most interesting representations for physics are the representations with $M^2 > 0 \qquad P^0 > 0$ (massive particles) and the representations $M^2 = 0 \qquad P^0 > 0$ (massless particles). In these lectures I only consider the case of massive particles.

Relativistic invariance of an isolated particle implies the existence of a one-body unitary representation of the Poincaré group. For particles the eigenvalues of the operators

$$M \qquad j^2 = W^2/M^2 \tag{50}$$

are the particle's mass m and spin $j^2 = j(j+1)$.

A basis for a one-particle representation can be constructed as the set of simultaneous eigenstates of a maximal set of commuting Hermitian functions of the single-particle Poincaré generators H, $\mathbf{P}$ $\mathbf{J}$, and $\mathbf{K}$.

One set of operators satisfying these conditions is:

$$M^2, W^2, \mathbf{P}, \hat{\mathbf{z}} \cdot \hat{\mathbf{W}} \tag{51}$$

It is useful to replace W^μ by the spin variables

$$j^2 = W^2/M^2 \tag{52}$$

and

$$(0, \mathbf{j}_c) := \frac{1}{m} B(-\mathbf{p}/m)^\mu{}_\nu W^\nu \tag{53}$$

where $B(-\mathbf{p}/m)^\mu_\nu = B^{-1}(\mathbf{p}/m)^\mu_\nu$ is a 4×4 matrix of operators obtained by replacing the parameter $\mathbf{p}$ by the momentum operator in (23):

$$(0, \mathbf{j}_c) := \frac{1}{m} B(-\mathbf{p}/m)^\mu{}_\nu W^\nu = \tag{54}$$

$$(0, \frac{\mathbf{W}}{m} + \mathbf{p}\frac{\mathbf{p} \cdot \mathbf{W} + W^0(m+h)}{m^2(m+h)}). \tag{55}$$

It follows from (47) that

$$[j_c^k, j_c^l] = i\epsilon^{kln} j_c^n \qquad \mathbf{j}_c \cdot \mathbf{j}_c = j^2 \qquad [\mathbf{j}, \mathbf{p}] = 0 \tag{56}$$

A suitable set of commuting observables for a single particle is the mass, spin, linear momentum and $\hat{\mathbf{z}}$ component of the canonical spin.

The $SU(2)$ commutation relations imply the spin can only have integral or half-integral eigenvalues j. The spectrum of the linear momentum operator is $\mathbb{R}^2$ because it linear momentum can be boosted to any frame.

Thus, the Hilbert space for a particle of mass m spin j is the space of square integrable functions of the linear momentum and spin,

$$\langle \mathbf{p}, \mu | \psi \rangle \qquad 1 = \int d\mathbf{p} \sum_{\mu=-j}^{j} |\langle \mathbf{p}, \mu | \psi \rangle|^2 = 1 \tag{57}$$

The action of $U(\Lambda, a)$ on these states is expressed in terms of

$$\langle \mathbf{p}, \mu | U(\Lambda, a) | \psi \rangle = \int \sum_{\mu'=-j}^{j} \langle \mathbf{p}, \mu | U(\Lambda, a) | \mathbf{p}', \mu' \rangle d\mathbf{p}' \langle \mathbf{p}', \mu' | \psi \rangle =$$

$$\int \sum_{\mu'=-j}^{j} D^{jm}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda,a) d\mathbf{p}' \langle \mathbf{p}',\mu'|\psi\rangle \tag{58}$$

where

$$D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda,a) := \langle \mathbf{p},\mu|U(\Lambda,a)|\mathbf{p}',\mu'\rangle \tag{59}$$

It is a consequence of the definitions that the matrices $D^{mj}_{\mu,\mathbf{p};\mu'',\mathbf{p}''}(\Lambda,a)$ are unitary representation of the Poincaré group.

$$\int \sum_{\mu''=-j}^{j} d\mathbf{p}'' D^{mj}_{\mu,\mathbf{p};\mu'',\mathbf{p}''}(\Lambda_2,a_2) d\mathbf{p}'' D^{mj}_{\mu'',\mathbf{p}'';\mu',\mathbf{p}'}(\Lambda_1,a_1) =$$

$$D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda_2\Lambda_1,\Lambda_2 a_1 + a_2) \tag{60}$$

These representations are irreducible. The matrices were derived for a single particle of mass m and spin j, but all positive mass positive energy irreducible representations of the Poincaré group have this form in the basis (57).

In the next section I show how to construct the matrices

$$D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda,a) \tag{61}$$

for any $m > 0$ and j.

7 Factorization theorem

The action of $U(\Lambda,a)$ on the one-particle Hilbert space is determined by the matrix $D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda,a)$. To compute $D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda,a)$ I use the following factorization theorem:

Factorization Theorem: Let (Λ,a) be any Poincaré transformation and p be any time-like four momenta, $p := (\omega_m(\mathbf{p}),\mathbf{p})$. Then (Λ,a) can be expressed as the product of (1) a rotationless Lorentz transformation to a frame where $\mathbf{p}$ is zero, (2) a rotation, (3) a translation of the system at rest and (4) a rotationless Lorentz transformation to the frame with the transformed momentum, $\mathbf{\Lambda}p$:

$$(\Lambda,a) =$$

$$(B(\mathbf{\Lambda}p/m),0)(I,B^{-1}(\mathbf{\Lambda}p/m)a)\times$$

$$(B^{-1}(\mathbf{\Lambda}p/m)\Lambda B(\mathbf{p}/m),0)(B^{-1}(\mathbf{p}/m),0) \tag{62}$$

The proof of this theorem follows by evaluating the above expression using the group multiplication property.

The factorization theorem implies

$$U(\Lambda, a)|\mathbf{p}', \mu\rangle =$$
$$U[B(\mathbf{\Lambda} p'/m), 0]U[I, B^{-1}(\mathbf{\Lambda} p'/m)a)]\times$$
$$U[B^{-1}(\mathbf{\Lambda} p'/m)\Lambda B(\mathbf{p}'/m)], 0]U[B^{-1}(\mathbf{p}'/m), 0]|\mathbf{p}', \mu\rangle \tag{63}$$

where I use a "prime" to indicate an eigenvalue.

The factorization theorem reduces the computation of $D^{mj}_{\mu,\mathbf{p};\mu',\mathbf{p}'}(\Lambda, a)$ to the following four steps:

1. **Inverse boost to a rest state:**
$$U[B^{-1}(\mathbf{p}/m), 0]|\mathbf{p}, \mu\rangle = |\mathbf{0}, \mu\rangle \sqrt{\frac{m}{\omega(\mathbf{p})}} \tag{64}$$

2. **Rotation of a rest state:**
$$U[B^{-1}(\mathbf{\Lambda} p'/m)\Lambda B(\mathbf{p}'/m)], 0]|\mathbf{0}, \mu\rangle =$$
$$\sum_{\nu=-j}^{j} |\mathbf{0}, \nu\rangle D^{j}_{\nu\mu}[B^{-1}(\mathbf{\Lambda} p/m)\Lambda B(\mathbf{p}/m)] \tag{65}$$

   where
$$R_w(\Lambda, \mathbf{p}/m) := B^{-1}(\mathbf{\Lambda} p/m)\Lambda B(\mathbf{p}/m) \tag{66}$$

   is a Wigner rotation and
$$D^{j}_{\nu\mu}[R] = \langle j, \nu|U(R, I)|j, \mu\rangle \tag{67}$$

   is the standard spin j irreducible representation of the rotation group.

3. **Translation of a rest state:**
$$U[I, B^{-1}(\mathbf{\Lambda} p/m)a)|\mathbf{0}, \nu\rangle = e^{ia\cdot\Lambda p}|\mathbf{0}, \nu\rangle \tag{68}$$

4. **Rotationless boost of a rest state:**

$$U[B(\mathbf{p}'/m),0]|\mathbf{0},\mu\rangle = |\mathbf{p}',\mu\rangle|\frac{\omega(\mathbf{p}')}{m})|^{1/2} \tag{69}$$

Combining these four elementary unitary transformations gives

$$U(\Lambda,a)|(m,j)\mathbf{p},\mu\rangle \tag{70}$$

$$\sum_{\nu=-j}^{j}|\boldsymbol{\Lambda}p,\nu\rangle e^{i\Lambda p\cdot a}D^{j}_{\nu\mu}[B^{-1}(\boldsymbol{\Lambda}p/m)\Lambda B(\mathbf{p}/m)]|\frac{\omega(\boldsymbol{\Lambda}p)}{\omega(\mathbf{p})}|^{1/2} \tag{71}$$

Comparing (58) and (71) gives

$$D^{mj}_{\mu',\mathbf{p}';\mu,\mathbf{p}}(\Lambda,a) =$$

$$\delta(\mathbf{p}'-\boldsymbol{\Lambda}p)e^{ip'\cdot a}D^{j}_{\nu\mu}[B^{-1}(\boldsymbol{\Lambda}p/m)\Lambda B(\mathbf{p}/m)]|\frac{\omega(\boldsymbol{\Lambda}p)}{\omega(\mathbf{p})}|^{1/2} \tag{72}$$

This is the mass m spin j irreducible representation of the Poincaré group in the $|\mathbf{p},\mu\rangle$ basis. All positive-mass positive-energy irreducible representations have this explicit form in the momentum canonical spin basis.

A second important observation is that this four step process can be used to construct irreducible representation from any rest state that transforms irreducibly with respect to rotations. This will be used to construct Clebsch-Gordan coefficients in the next section and dynamics in the following section.

In the remainder of this section I discuss the proof of the elementary relations (64), (65), (68) and (69).

The transformation properties of the linear momentum in equations (64) and (69) is a consequence of the transformation properties of the four momentum operator.

Invariance of the spin in (64) and (69) follows from the definition of the spin in terms of the generators and the transformation properties of the generators:

$$U(\Lambda,0)\mathbf{j}U^{\dagger}(\Lambda,0) =$$

$$U(\Lambda,0)\frac{1}{m}B^{-1}(\mathbf{p}/m)WU^{\dagger}(\Lambda,0) =$$

$$\frac{1}{m}B^{-1}(\boldsymbol{\Lambda}^{-1}p/m)\Lambda^{-1}W =$$

$$\frac{1}{m}B^{-1}(\mathbf{\Lambda}^{-1}p/m)\Lambda^{-1}B(\mathbf{p}/m)B^{-1}(\mathbf{p}/m)W =$$
$$B^{-1}(\mathbf{\Lambda}^{-1}p/m)\Lambda^{-1}B(\mathbf{p}/m)\mathbf{j} \tag{73}$$

The operator multiplying $\mathbf{j}$ becomes a Wigner rotation when it is applied to a momentum eigenstate. For $\Lambda = B^{-1}(\mathbf{p}/m)$ this Wigner rotation becomes the identity, which implies that μ is unchanged in (64) and (69).

The square root factors that appear in (64) and (69) are needed to ensure that $U[B^{-1}(\mathbf{p}'/m), 0]$ and $U[B(\mathbf{p}'/m), 0]$ are unitary for states with a delta function normalization:

$$\langle \mathbf{\Lambda}p|\mathbf{\Lambda}p'\rangle = \langle \mathbf{p}|\mathbf{p}'\rangle \delta(\mathbf{\Lambda}p - \mathbf{\Lambda}p') = \delta(\mathbf{p}-\mathbf{p}')|\frac{\partial \mathbf{p}}{\partial \mathbf{\Lambda}p}| = \langle \mathbf{p}|\mathbf{p}'\rangle |\frac{\partial \mathbf{p}}{\partial \mathbf{\Lambda}p}| \tag{74}$$

which leads to the identification

$$|\mathbf{\Lambda}p'\rangle = |\mathbf{p}'\rangle |\frac{\partial \mathbf{p}}{\partial \mathbf{\Lambda}p}|^{1/2} \tag{75}$$

The Jacobian

$$|\frac{\partial \mathbf{p}}{\partial \mathbf{\Lambda}p}| = \frac{\omega(\mathbf{p})}{\omega(\mathbf{\Lambda}p)} \tag{76}$$

can be read off of

$$\int \delta(p^2+m^2)d^4p = \int \frac{d\mathbf{p}}{\omega(\mathbf{p})} = \int \frac{d\mathbf{\Lambda}p}{\omega(\mathbf{\Lambda}p)} \tag{77}$$

Equation (65) follows because the transformation is a rotation in an irreducible basis for the rotation group, while (68) follows because the basis state is an eigenstate of the four momentum.

8 Clebsch-Gordan Coefficients

The one-particle representations of the Poincaré group constructed in the previous section are positive-mass positive-energy irreducible representations of the Poincaré group. In general any positive-mass positive-energy irreducible representation has the form (72).

In this section I show how to decompose a product of two irreducible representations of the Poincaré group into mutually orthogonal irreducible subspaces. The final result looks very much like the non-relativistic decomposition into center of mass and relative momentum variables. These non-interacting irreducible representations are used in the next section to construct dynamical irreducible representations.

The tool for performing this construction is the Clebsch-Gordan coefficients for the Poincaré group. I construct these coefficients in this section.

Consider of tensor product of two irreducible basis vectors:

$$|\mathbf{p}_1, \mu_1\rangle \otimes |\mathbf{p}_2, \mu_2\rangle \tag{78}$$

Define the kinematic variables

$$P^\mu = \mathbf{p}_1 + \mathbf{p}_2 \tag{79}$$

$$M_0^2 = -\eta_{\mu\nu} P^\mu P^\nu \tag{80}$$

where all of the single particle momenta are on their mass shells - i.e. $\mathbf{p}_i^0 = \sqrt{m_i^2 + \mathbf{p}_i^2}$

I define

$$k_i^\mu = B^{-1}(\mathbf{P}/M_0)^\mu{}_\nu p_i^\nu \tag{81}$$

and

$$\mathbf{k} := \mathbf{k}_1 = -\mathbf{k}_2 \tag{82}$$

Apply

$$U_0[B^{-1}(\mathbf{P}/M_0)] := U_1[B^{-1}(\mathbf{P}/M_0)] \otimes U_2[B^{-1}(\mathbf{P}/M_0)] \tag{83}$$

to the basis vector (78) to get

$$\begin{gathered} U_0[B^{-1}(\mathbf{P}/M_0)]|\mathbf{p}_1, \mu_1\rangle \otimes |\mathbf{p}_2, \mu_2\rangle = \\ |\mathbf{k}, \mu_1'\rangle \sqrt{\frac{\omega_{m_1}(\mathbf{k})}{\omega_{m_1}(\mathbf{p}_1)}} D^{j_1}_{\mu_1'\mu_1}[B^{-1}(\mathbf{k}/m_1)B^{-1}(\mathbf{P}/M_0)B^{-1}(\mathbf{p}_1/m_1)] \otimes \\ |-\mathbf{k}, \mu_2'\rangle \sqrt{\frac{\omega_{m_2}(\mathbf{k})}{\omega_{m_2}(\mathbf{p}_2)}} D^{j_2}_{\mu_2'\mu_2}[B^{-1}(-\mathbf{k}/m_2)B^{-1}(\mathbf{P}/M_0)B^{-1}(\mathbf{p}_2/m_2)] \end{gathered} \tag{84}$$

This defines a rest state of the two-particle system. If

$$|\mathbf{k}, \mu_1'\rangle \otimes |-\mathbf{k}, \mu_2'\rangle \tag{85}$$

can be decomposed into irreducible representations with respect to rotations then the factorization theorem can be used to construct two-particle irreducible representations.

In order to understand transformation properties $\mathbf{k}, \mathbf{j}_1$, and $\mathbf{j}_2$ under rotations R note

$$k' := B^{-1}(R\mathbf{P}/M_0)Rp_1 = B^{-1}(R\mathbf{P}/M_0)RB(\mathbf{P}/M_0)k \tag{86}$$

The transformation

$$B^{-1}(R\mathbf{P}/M_0)RB(\mathbf{P}/M_0) \tag{87}$$

is a Wigner rotation of the rotation R. The rotationless boosts have the property that

$$B^{-1}(R\mathbf{P}/M_0)RB(\mathbf{P}/M_0) = R \tag{88}$$

I will prove this at the end of this section using the 2 matrix representations. Using (88) in (8) gives

$$k' = Rk \tag{89}$$

The spins have a similar transformation property:

$$\mathbf{j}'_1 =$$

$$\frac{1}{m_1}B^{-1}(R\mathbf{p}_1/m_1)RW = B^{-1}(R\mathbf{p}_1/m_1)RB(\mathbf{p}_1/m_1)\frac{1}{m_1}B^{-1}(\mathbf{p}_1/m_1)W =$$

$$B^{-1}(R\mathbf{p}_1/m_1)RB(\mathbf{p}_1/m_1)\mathbf{j}_1 \tag{90}$$

This is a different Wigner rotation of R, but it also involves rotationless boosts, so I have

$$B^{-1}(R\mathbf{p}_1/m_1)RB(\mathbf{p}_1/m_1) = R \tag{91}$$

Similar results hold for $\mathbf{j}_2$.

Since the quantities $\mathbf{k}$, $\mathbf{j}_1$ and $\mathbf{j}_2$ all rotate together, they can be coupled using ordinary spherical harmonics and $SU(2)$ Clebsch-Gordan coefficients:

$$|\mathbf{0}, k, l, s, j, \mu\rangle :=$$

$$\int |\mathbf{k}, \mu_1\rangle \otimes |-\mathbf{k}, \mu_2\rangle \times d\Omega(\hat{\mathbf{k}})Y^l_{\mu_l}(\hat{\mathbf{k}})$$

$$\langle j_1, \mu_1, j_2, \mu_2|s, \mu_s\rangle\langle l\mu_l, j_2, s, \mu_s|j, \mu\rangle \tag{92}$$

This state is a zero momentum eigenstate of state the two particle systems that transforms under a $2j+1$-dimensional representation of the rotation group. It satisfies

$$U(R, 0)|\mathbf{0}, k, l, s, j, \mu\rangle := |\mathbf{0}, k, l, s, j, \mu'\rangle D^j_{\mu'\mu}(R) \tag{93}$$

I use the factorization theorem to construct irreducible representations. It follows from (69) that the irreducible state with linear momentum $\mathbf{P}$ is

$$|\mathbf{P}, k, l, s, j, \mu\rangle$$

$$U_0[B^{-1}(\mathbf{P}/M_0)]|\mathbf{0}, k, l, s, j, \mu\rangle \sqrt{\frac{M_0}{\sqrt{\mathbf{P}^2 + M_0^2}}} \tag{94}$$

where as in (69) the multiplicative factor is chosen to make $U_0[B^{-1}(\mathbf{P}/M_0)]$ unitary if $|\mathbf{P}, k, l, s, j, \mu\rangle$ has a $\delta(\mathbf{P} - \mathbf{P}')$ normalization.

Equation (92) can be inverted to give

$$|\mathbf{k}, \mu_1\rangle \otimes | - \mathbf{k}, \mu_2\rangle =$$

$$\sum |\mathbf{0}, k, l, s, j, \mu\rangle Y^{l*}_{\mu_l}(\hat{\mathbf{k}})\langle l\mu_l j_2 s\mu_s|j\mu\rangle\langle j_1\mu_1 j_2\mu_2|s\mu_s\rangle \tag{95}$$

Using (95) in (84) gives

$$U_0[B^{-1}(\mathbf{P}/M_0)]|\mathbf{p}_1, \mu_1\rangle \otimes |\mathbf{p}_2, \mu_2\rangle =$$

$$\sum |\mathbf{0}, k, l, s, j, \mu\rangle Y^{l*}_{\mu_l}(\hat{\mathbf{k}})\langle l, \mu_l, s, \mu_s|j, \mu\rangle\langle j_1, \mu_1', j_2, \mu_2'|s, \mu_s\rangle\times$$

$$\sqrt{\frac{\omega_{m_1}(\mathbf{k})}{\omega_{m_1}(\mathbf{p}_1)}}\sqrt{\frac{\omega_{m_2}(\mathbf{k})}{\omega_{m_2}(\mathbf{p}_2)}}\times$$

$$D^{j_1}_{\mu_1'\mu_1}[B^{-1}(\mathbf{k}/m_1)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_1/m_1)]\times$$

$$D^{j_2}_{\mu_2'\mu_2}[B^{-1}(-\mathbf{k}/m_2)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_2/m_2)] \tag{96}$$

Combining (94) and (96) gives the desired decomposition of tensor products of irreducible representations into

$$|\mathbf{p}_1, \mu_1\rangle \otimes |\mathbf{p}_2, \mu_2\rangle =$$

$$\sum |\mathbf{P}, k, l, s, j, \mu\rangle \sqrt{\frac{M_0}{\sqrt{\mathbf{P}^2 + M_0^2}}}\sqrt{\frac{\omega_{m_1}(\mathbf{k})}{\omega_{m_1}(\mathbf{p}_1)}}\sqrt{\frac{\omega_{m_2}(\mathbf{k})}{\omega_{m_2}(\mathbf{p}_2)}}\times$$

$$Y^{l*}_{\mu_l}(\hat{\mathbf{k}})\langle l, \mu_l, j_2, s, \mu_s|j, \mu\rangle\langle j_1, \mu_1', j_2, \mu_2'|s, \mu_s\rangle\times$$

$$D^{j_1}_{\mu_1'\mu_1}[B^{-1}(\mathbf{k}/m_1)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_1/m_1)]\times$$

$$D^{j_2}_{\mu_2'\mu_2}[B^{-1}(-\mathbf{k}/m_2)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_2/m_2)] \tag{97}$$

The Clebsch-Gordan coefficients can be read off by taking matrix elements with single particle states

$$\begin{aligned}
&\langle \mathbf{P}, k, l, s, j, \mu | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle = \\
&\int d\Omega(\hat{\mathbf{k}}) \delta(\mathbf{p}_1 - \mathbf{p}_1(\mathbf{P}, \mathbf{k})) \delta(\mathbf{p}_2 - \mathbf{p}_2(\mathbf{P}, \mathbf{k})) \times \\
&\sqrt{\frac{M_0}{\sqrt{\mathbf{P}^2 + M_0^2}}} \sqrt{\frac{\omega_{m_1}(\mathbf{k})}{\omega_{m_1}(\mathbf{p}_1)}} \sqrt{\frac{\omega_{m_2}(\mathbf{k})}{\omega_{m_2}(\mathbf{p}_2)}} \times \\
&Y^{l*}_{\mu_l}(\hat{\mathbf{k}}) \langle l, \mu_l, j_2, s, \mu_s | j, \mu \rangle \langle j_1, \mu_1', j_2, \mu_2' | s, \mu_s \rangle \times \\
&D^{j_1}_{\mu_1'\mu_1}[B^{-1}(\mathbf{k}/m_1)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_1/m_1)] \times \\
&D^{j_2}_{\mu_2'\mu_2}[B^{-1}(-\mathbf{k}/m_2)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_2/m_2)] =
\end{aligned} \tag{98}$$

$$\begin{aligned}
&\delta(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2)) \delta(\mathbf{k} - \mathbf{k}_1(\mathbf{p}_1, \mathbf{p}_2)) \times \\
&\sqrt{\frac{\sqrt{\mathbf{P}^2 + M_0^2}}{M_0}} \sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)}{\omega_{m_1}(\mathbf{k})}} \sqrt{\frac{\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_2}(\mathbf{k})}} \times \\
&Y^{l*}_{\mu_l}(\hat{\mathbf{k}}) \langle l, \mu_l, j_2, s, \mu_s | j, \mu \rangle \langle j_1, \mu_1', j_2, \mu_2' | s, \mu_s \rangle \times \\
&D^{j_1}_{\mu_1'\mu_1}[B^{-1}(\mathbf{k}/m_1)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_1/m_1)] \times \\
&D^{j_2}_{\mu_2'\mu_2}[B^{-1}(-\mathbf{k}/m_2)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_2/m_2)]
\end{aligned} \tag{99}$$

where in the second term I used the relation

$$\frac{\partial(\mathbf{P}\mathbf{k})}{\partial(\mathbf{p}_1\mathbf{p}_2)} = \frac{\sqrt{\mathbf{P}^2 + M_0^2}}{M_0} \frac{\omega_{m_1}(\mathbf{p}_1)}{\omega_{m_1}(\mathbf{k})} \frac{\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_2}(\mathbf{k})} \tag{100}$$

to change the variables that appear in the delta functions.

The new feature of the Poincaré Clebsch-Gordan coefficients that do not appear in the corresponding $SU(2)$ Clebsch-Gordan coefficients is the appearance of the quantum numbers l and s. Intuitively these correspond to the spin and orbital angular momentum. In the Clebsch-Gordan coefficient (98) or (99) they represent degeneracy parameters, indicating that in decomposing the tensor product of two irreducible representations into irreducible representations, the same values of M_0 and $\mathbf{j}$ appear more than once. In these expression we have replaced M_0 by the continuous variable $k := |\mathbf{k}|$; where k and M_0 are related by

$$M_0 = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2} \tag{101}$$

In general the structure of the Clebsch-Gordan coefficients depends on the choice of basis for the irreducible representations. In the above I chosen the irreducible basis to be simultaneous eigenstates of linear momentum and $\hat{\mathbf{z}}$ of the canonical spin. Different choices of basis are possible and used in applications.

The key property of the rotationless boost is the observation that the Wigner rotation of a rotation is the rotation. This justified the used of the partial wave analysis of the rest vector.

This property of the rotationless boost is elementary to prove using the $SL(2,\mathbb{C})$ representations. First note that (28) applied for $A = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}$ associated with a rotation R gives

$$e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} x^\mu \sigma_\mu e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = (Rx)^\mu \sigma_\mu = x^\mu (R^{-1}\sigma)_\mu \tag{102}$$

Using (102) in the $SL(2,\mathbb{C})$ representation of the Wigner rotation of the rotation R gives

$$e^{-\frac{1}{2}R\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{-\frac{1}{2}R\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} =$$

$$e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{-\frac{1}{2}R\boldsymbol{\rho}\cdot R\boldsymbol{\sigma}} e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \tag{103}$$

which is the $SL(2,\mathbb{C})$ representation of the original rotation.

9 Dynamical representations

The representation $U(\Lambda, a)$ is abstract. Specific realizations of Poincaré invariant quantum mechanics involve both a choice of Hilbert space representations on which $U(\Lambda, a)$ acts and a particular implementation of the dynamics. The problem of adding interactions to the Hamiltonian while retaining the Poincaré commutation relations is a non-linear problem. The problem is most easily understood by considering the commutator

$$[K^i, P^j] = i\delta_{ij} H \tag{104}$$

implies that one cannot add interactions to the Hamiltonian without also adding them to the left side of the commutator. Once interactions appear in the operators on the left side of the commutator one must still satisfy the commutation relations.

While this problem is non-linear, the resulting generators must satisfy the Poincaré Lie algebra, and the resulting unitary representation of the Poincaré group must have a decomposition into irreducible representations. A sensible strategy is to add interactions to a direct integral of irreducible representations that does not change the group structure.

I discuss the two-particle case [13], but the result can be extended to any number of particles [14][15][16]. The starting point is a description of two free particles. The Hilbert space is the tensor product of two one-particle irreducible representation spaces

$$|(m_1, j_1)\mathbf{p}_1, \mu_1\rangle \otimes |(m_2, j_2)\mathbf{p}_2, \mu_2\rangle \tag{105}$$

Using the Clebsh-Gordan coefficients derived in section 8 this can be replaced by the irreducible basis

$$|(k, j)\mathbf{P}, \mu, l, s\rangle \tag{106}$$

where

$$U_0(\Lambda, a)|(k, j)\mathbf{P}, \mu, l, s\rangle = |(k, j)\mathbf{\Lambda}P, \mu', l, s\rangle \times$$
$$\sqrt{\frac{\omega_{M_0}(\mathbf{\Lambda}P)}{\omega_{M_0}(\mathbf{P})}} D^j_{\mu'\mu}[B^{-1}(\mathbf{\Lambda}P/M_0)\Lambda B(\mathbf{P}/M_0)] e^{i\Lambda P\cdot a} \tag{107}$$

Here $U_0(\Lambda, a)$ indicates the non-interacting unitary representation of the Poincaré group.

Note that the only variables that are transformed in this basis are $\mathbf{P}$ and μ. The noninteracting mass operator in this representation is

$$M_0 = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2} \tag{108}$$

The basis vectors are simultaneous eigenstates of $\mathbf{M}_0$, j^2, $\mathbf{j} \cdot \hat{\mathbf{z}}$, and $\mathbf{P}$. All of these operators, in addition to the conjugate operators, commute with M_0.

Comparing to a non-relativistic model, M_0 is the relativistic analog of the center of mass Hamiltonian. Its eigenvalues represent the energy of the non-interacting two particle-system in the two-particle rest frame.

Following the non-relativistic procedure I add an interaction to M_0:

$$M = M_0 + V \tag{109}$$

to construct an interacting mass operator. It is desirable to do this is a manner that does not disrupt the Poincaré commutation relations. The simplest way to do this is to require that the interaction commutes with j^2, $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$ and the operators conjugate to $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$. It follows that if this interaction is evaluated in the non-interacting irreducible basis then it must have the form

$$\langle (k',j')\mathbf{P}',\mu',l',s'|V|(k,j)\mathbf{P},\mu,l,s\rangle =$$

$$\delta(\mathbf{P}'-\mathbf{P})\delta j'j\delta_{\mu'\mu}\langle k',l,' s'\|v^j\|k,l,s\rangle \tag{110}$$

The requirement that V commutes with the operators conjugate to $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$ means that the reduced kernel $\langle k',l,' s'\|v^j\|k,l,s\rangle$ does not depend on $\mathbf{P}$ or μ. It is encouraging to note that this interacting has the same number of degrees of freedom as a rotationally invariant non-relativistic interaction in a partial-wave basis.

Because M and M_0 satisfy the same commutation relations with with j^2, $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$ and the operators conjugate to $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$, it follows that it is possible to find simultaneous eigenstates of M j^2, $\mathbf{j}\cdot\hat{\mathbf{z}}$, and $\mathbf{P}$, and furthermore that these eigenstates transform just like the corresponding non-interacting irreducible states, with the eigenvalue of M_0 replaced by the eigenvalue of M. The desired eigenstates can be constructed by diagonalizing M in the free-particle irreducible basis.

$$\langle (k',j')\mathbf{P}',\mu',l',s'|(\lambda,j)\mathbf{P}',\mu'\rangle = \tag{111}$$

$$\delta_{j'j}\delta(\mathbf{P}'-\mathbf{P})\delta_{\mu'\mu}\langle j',k',l',s'|j,\lambda\rangle \tag{112}$$

$$(\sqrt{m_1^2+\mathbf{k}^2}+\sqrt{m_2^2+\mathbf{k}^2})\langle j,k,l,s|j,\lambda\rangle + \tag{113}$$

$$\sum_{s=|j_1-j_2|}^{|j_1+j_2|}\sum_{l=|j-s|}^{|j+s|}\int_0^\infty \langle k,l,s\|v^j\|k',l',s'\rangle k'^2 dk' \langle j,k',l',s'|j,\lambda\rangle \tag{114}$$

$$= \lambda\langle j,k',l',s'|j,\lambda\rangle \tag{115}$$

For a reasonable interaction these states will be complete. The commutation relations imply

$$U(\Lambda,a)|(\lambda,j)\mathbf{P},\mu\rangle =$$

$$|(\lambda, j)\mathbf{\Lambda}P, \mu'\rangle\sqrt{\frac{\omega_\lambda(\mathbf{\Lambda}P)}{\omega_\lambda(\mathbf{P})}}D^j_{\mu'\mu}[B^{-1}(\mathbf{\Lambda}P/\lambda)\Lambda B(\mathbf{P}/\lambda)]e^{i\Lambda P\cdot a} \tag{116}$$

This leads defines the dynamical representation of the Poincaré group.

The eigenfunctions and the Clebsch-Gordan coefficients can be used to express these relations in plane wave bases.

This basic construction was first done by Bakamjian and Thomas, [13]. They used the same basis for the irreducible representation the I used above, resulting in an "instant-form dynamics"[17], where the mass eigenvalue λ does not appear in the coefficients (116) when the Poincaré transformation is a rotation or spatial translation.

10 Fields

While Poincaré invariant dynamics is not a local field theory, it is possible to construct fields that transform covariantly with respect to the dynamical representation of the Poincaré group. An important example of a field in applications of this formalism is a current operator, however it is easy to construct fields of any spin.

The method of construction is based on the Wigner-Eckart theorem for the Poincaré group. In general a covariant field is a set of operators that depend on a space-time coordinate x:

$$\Psi_n(x) \tag{117}$$

and transform covariantly:

$$U(\Lambda, a)\Psi_n(x)U^\dagger(\Lambda, a) = \Psi_{n'}(\Lambda x + a)S(\Lambda)_{n'n} \tag{118}$$

where $S(\Lambda)_{n'n}$ is a finite-dimensional representation of the Lorentz group. These representations are well known; the irreducible building blocks are symmetrized tensor products of $SL(2,\mathbb{C})$ matrices [2].

There is a large class of operators that satisfy (118). A field operator is defined if all of its matrix elements in a given basis are known. I evaluate the matrix elements of field operators in the basis of irreducible eigenstates of $U(\Lambda, a)$. To specify the operator $\Psi_n(x)$ it is necessary to determine the matrix elements:

$$\langle(m', j')\mathbf{p}', \mu'|\Psi_n(x)|(m, j)\mathbf{p}, \mu\rangle \tag{119}$$

Inserting $U^\dagger(\Lambda,a)U(\Lambda,a)=I$ on both sides of the field operators in (119), using the covariance relation, implies the identity

$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(x)|(m,j)\mathbf{p},\mu\rangle =$$
$$\langle (m',j')\mathbf{\Lambda}p',\nu'|\Psi_{n'}(\Lambda x+a)|(m,j)\mathbf{\Lambda}p,\nu\rangle \times$$
$$e^{i\Lambda p\cdot a}D^j_{\nu\mu}[B^{-1}(\mathbf{\Lambda}p/m)\Lambda B(\mathbf{p}/m)]\sqrt{\frac{\omega(\mathbf{\Lambda}p)}{\omega(\mathbf{p})}}\times$$
$$e^{-i\Lambda p'\cdot a}D^{j*}_{\nu'\mu'}[B^{-1}(\mathbf{\Lambda}p'/m')\Lambda B(\mathbf{p}'/m')]\sqrt{\frac{\omega(\mathbf{\Lambda}p')}{\omega(\mathbf{p}')}}S(\Lambda)_{nm}. \tag{120}$$

This equation can be used to relate an arbitrary matrix element to a reduced set of invariant independent matrix elements.

For example, if I set $\Lambda=I$ and $a=-x$ in (120) I get

$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(x)|\langle (m,j)\mathbf{p},\mu\rangle =$$
$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(0)|(m,j)\mathbf{p},\mu\rangle e^{i(p'-p)\cdot x} \tag{121}$$

which shows that matrix elements of the field operator for any x can be expressed, using translational covariance, in terms of matrix elements with $x=0$.

Since the initial four momentum is time-like it is possible to use a rotationless Lorentz transformation $\Lambda=B^{-1}(\mathbf{p}/m)$, $a=0$ to transform the initial momentum to its rest value:

$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(0)|(m,j)\mathbf{p},\mu\rangle =$$
$$\langle (m',j')\mathbf{\Lambda}p',\nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\mu\rangle\sqrt{\frac{m}{\omega(\mathbf{p})}}\sqrt{\frac{\omega(\mathbf{\Lambda}p')}{\omega(\mathbf{p}')}}\times$$
$$D^{j*}_{\nu'\mu'}[B^{-1}(\mathbf{\Lambda}p'/m')\Lambda B(\mathbf{p}'/m')]S(\Lambda)_{n'n}. \tag{122}$$

Equation (122) , along with (121) implies that all matrix elements can be expressed in terms of the matrix elements

$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(0)|(m,j)\mathbf{0},\mu\rangle. \tag{123}$$

Finally I can use a rotation R about an axis parallel to $\hat{\mathbf{z}}\times\mathbf{p}'$ to orient $\mathbf{p}'$ in the $\hat{z}$ direction.

$$\langle (m',j')\mathbf{p}',\mu'|\Psi_n(0)|(m,j)\mathbf{0},\mu\rangle =$$

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\nu\rangle\times$$
$$D^{j}_{\nu\mu}[R]D^{j*}_{\nu'\mu'}[R]S(R)_{n'n}. \tag{124}$$

Combining (121) with (122) and (124) implies that every matrix element of $\Psi_n(x)$ can be expressed in terms of the matrix element

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\nu\rangle. \tag{125}$$

Finally I can still use rotations about the z axis to constrain the discrete indices

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\nu\rangle =$$
$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\nu\rangle e^{i(\mu'-\mu)\phi}S_{n'n}[R(\phi)]. \tag{126}$$

Differentiating with respect to ϕ and setting $\phi = 0$ gives

$$(\mu'-\mu)\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n}(0)|(m,j)\mathbf{0},\nu\rangle =$$

$$i\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|\Psi_{n'}(0)|(m,j)\mathbf{0},\nu\rangle \frac{\partial}{\partial\phi}S_{n'n}[R(\phi)]_{|\phi=0} \tag{127}$$

The last constraint has to be evaluated on a case by case basis. For a 4-vector field

$$\Psi_{n'}(0)\to J^{\alpha}(0)S_{n'n}[R(\phi)]\to \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos(\phi) & \sin(\phi) & 0\\ 0 & -\sin(\phi) & \cos(\phi) & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{128}$$

and

$$\frac{\partial}{\partial\phi}S_{n'n}[R(\phi)]_{|\phi=0}\to \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{129}$$

which gives the following constraints:

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^{0}(0)|(m,j)\mathbf{0},\nu\rangle =$$
$$\delta_{\nu\nu'}\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu|J^{0}(0)|(m,j)\mathbf{0},\nu\rangle \tag{130}$$

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^{3}(0)|(m,j)\mathbf{0},\nu\rangle =$$

$$\delta_{\nu\nu'}\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu|J^3(0)|(m,j)\mathbf{0},\nu\rangle \tag{131}$$

$$(\mu'-\mu)\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^x(0)|(m,j)\mathbf{0},\nu\rangle = \\ i\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^y(0)|(m,j)\mathbf{0},\nu\rangle \tag{132}$$

$$(\mu'-\mu)\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^y(0)|(m,j)\mathbf{0},\nu\rangle = \\ -i\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^x(0)|(m,j)\mathbf{0},\nu\rangle \tag{133}$$

The last two of these equation can combined to give

$$[(\mu'-\mu)^2-1]\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^x(0)|(m,j)\mathbf{0},\nu\rangle = 0 \tag{134}$$

$$[(\mu'-\mu)^2-1]\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^y(0)|(m,j)\mathbf{0},\nu\rangle = 0 \tag{135}$$

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^y(0)|(m,j)\mathbf{0},\nu\rangle = \\ \frac{i}{(\mu-\mu')}\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu'|J^x(0)|(m,j)\mathbf{0},\nu\rangle \tag{136}$$

This implies that the most general for vector field, $J^\mu(x)$, can be uniquely specified by defining the independent matrix elements and using covariance to generate the remaining matrix elements:

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu|J^0(0)|(m,j)\mathbf{0},\nu\rangle \tag{137}$$

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu|J^3(0)|(m,j)\mathbf{0},\nu\rangle \tag{138}$$

$$\langle (m',j')\hat{\mathbf{z}}|\mathbf{p}'|,\ \nu\pm 1|J^x(0)|(m,j)\mathbf{0},\nu\rangle \tag{139}$$

These independent matrix elements are Poincaré invariant functions - since one arrives that the same independent matrix elements from any starting frame.

These invariant matrix elements are the analog of the reduced matrix elements that appear in the standard $SU(2)$ Wigner Eckart theorem.

The form of the Wigner-Eckart theorem does not look exactly like the standard form because the operators are expressed as Lorentz covariant densities; however they could *equivalently* be expressed as Poincaré covariant operators [18]. This is not normally done because as the Fourier transform of x passes through the six classes of irreducible representation in Table

1, the transformation properties of the operator change. With Lorentz covariant densities transformation properties of the continuous parameter x is decoupled from the discrete field index.

If $J^\mu(x)$ is an electromagnetic current operator then current conservation and parity will further reduce the number of independent invariant matrix elements. The resulting number corresponds exactly to the number of invariant form factors.

The two main messages are from this section are (1) even though the representation $U(\Lambda, a)$ is not manifestly covariant, the theory has many fields that transform covariantly and (2) covariant field operators can be constructed using the Wigner-Eckart theorem for the Poincaré group.

11 Examples

11.1 Confined Quarks

Assume equal mass quarks and antiquarks.

1. Hilbert Space (identify the degrees of freedom - treat free quarks as massive spin 1/2 particles)

$$\mathcal{H} = \mathcal{H}_q \otimes \mathcal{H}_{\bar{q}} \tag{140}$$

2. Mass operator (include dynamics - confining interaction)

$$M^2 = 4(\mathbf{k}^2 + m^2) - \lambda \nabla_k^2 \tag{141}$$

3. Mass eigenfunctions (in the non-interacting irreducible basis)

$$\langle (k', j')\mathbf{P}', \mu'; l', s' | (M, j)\mathbf{P}, \mu \rangle = \delta(\mathbf{P}' - \mathbf{P})\delta_{j'j}\delta_{\mu'\mu}\phi^j_M(k', l', s') \tag{142}$$

4. Mass eigenvalue problem (solve in the non-interacting irreducible basis).

$$4(\mathbf{k}^2 + m^2)\phi^j(k, l, s) - \lambda \boldsymbol{\nabla}_k \phi^j(k, l, s) = M^2 \phi^j(k, l, s) \tag{143}$$

5. Relativistic Dynamics (simultaneous eigenstates of mass, spin, linear momentum and z-component of canonical spin are complete and transform irreducibly)

$$\langle (k,j)\mathbf{P},\mu;l,s|U(\Lambda,a)|(M,j)\mathbf{P},\mu\rangle =$$

$$\langle (k,j)\mathbf{P},\mu;l,s|U(\Lambda,a)|(M,j)\mathbf{\Lambda}P,\nu\rangle\times$$

$$\sqrt{\frac{\omega_M(\mathbf{\Lambda}P_M)}{\omega_M(\mathbf{P})}}D^j_{\nu\mu}[B^{-1}(\mathbf{\Lambda P}/M)\Lambda B(\mathbf{P}/M)]e^{i\Lambda P_M\cdot a} \tag{144}$$

where

$$P_M = (\sqrt{\mathbf{P}^2+M^2},\mathbf{P}) \tag{145}$$

5. Representation in terms of quark degrees of freedom (use Poincaré Clebsh-Gordan coefficients - needed to calculate electromagnetic observables)

$$\langle \mathbf{p}_q,\mu_q,\mathbf{p}_{\bar q},\mu_{\bar q}|(M,j)\mathbf{P},\nu\rangle =$$

$$\sqrt{\frac{\sqrt{\mathbf{P}^2+M_0^2}}{M_0}}\sqrt{\frac{\omega_{m_q}(\mathbf{p}_q)}{\omega_{m_q}(\mathbf{k})}}\sqrt{\frac{\omega_{m_{\bar q}}(\mathbf{p}_{\bar q})}{\omega_{m_{\bar q}}(\mathbf{k})}}\times$$

$$Y^l_{\mu_l}(\hat{\mathbf{k}})\langle l,\mu_l,s\mu_s|j\mu\rangle\langle j_q,\mu'_q j_{\bar q}\mu'_{\bar q}|s\mu_s\rangle\times$$

$$D^{j_q *}_{\mu'_q\mu_q}[B^{-1}(\mathbf{k}/m_q)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_q/m_q)]\times$$

$$D^{j_{\bar q} *}_{\mu'_{\bar q}\mu_{\bar q}}[B^{-1}(-\mathbf{k}/m_{\bar q})B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_{\bar q}/m_{\bar q})]\phi^j(k,l,s) \tag{146}$$

These steps show that this model is mathematically equivalent to a quantum mechanical harmonic oscillator. Because the oscillator is associated with the square of the mass, taking square roots, it is easy to show that the mass eigenvalues grow linearly with the mean separation of the partons. More complicated spin-flavor dependent interactions can be included in step 3.

11.2 Pion production near threshold

This is an example of a model that does not conserve particle number.

1. Hilbert Space (identify degrees of freedom - in this example I treat the pions as physical rather than bare pions.)

$$\mathcal{H} = (\mathcal{H}_N \otimes \mathcal{H}_N) \oplus (\mathcal{H}_N \otimes \mathcal{H}_N \otimes \mathcal{H}_\pi) \tag{147}$$

2. Mass operator (include dynamics)

$$M = M_0 + V =$$
$$\begin{pmatrix} M_{0NN} & 0 \\ 0 & M_{0NN\pi} \end{pmatrix} + \begin{pmatrix} V_{NN} & V_{NN;NN\pi} \\ V_{NN\pi;NN} & V_{NN} + V_{N\pi} + V_{N'\pi} \end{pmatrix} \tag{148}$$

3. Mass eigenfunction (in the non-interacting irreducible basis)

$$\begin{pmatrix} \langle (k',j')\mathbf{P}',\mu';l',s'|(M,j)\mathbf{P},\mu\rangle \\ \langle (k',q',j')\mathbf{P}',\mu';L',S',j_2',l',s'|(M,j)\mathbf{P},\mu\rangle \end{pmatrix} = \tag{149}$$

$$\delta(\mathbf{P}'-\mathbf{P})\delta_{j'j}\delta_{\mu'\mu}\begin{pmatrix} \phi^j_{M1}(k',l',s') \\ \phi^j_{M2}(k',q',L',S',j_2',l',s') \end{pmatrix} \tag{150}$$

4. Mass eigenvalue problem (in the non-interacting irreducible basis).

$$(M - 2\sqrt{\mathbf{k}^2 + m_N^2})\phi^j_{M1}(k',l',s') =$$
$$\int \sum V^j_{NN}(k,l,s;k',l',s')\phi^j_{M1}(k',l',s')k'^2dk' +$$
$$\int \sum V^j_{NN;NN\pi}(k,l,s;k',q',L',S',j_2',l',s') \times$$
$$\phi^j_{M2}(k',q',L'S'j_2',l',s')k'^2dk'q'^2dq' \tag{151}$$

$$(M - \sqrt{4\mathbf{k}^2 + 4m_N^2 + q^2} + \sqrt{\mathbf{q}^2 + m_\pi^2})\phi^j_{M2}(k',q',L'S'j_2';l',s') =$$

$$\int \sum V^j_{NN\pi;NN}(k,q,L,S,j_2,l,s;k',l',s')\phi^j_{M1}(k',l',s')k'^2dk' +$$
$$\int \sum V^j_{NN\pi;NN\pi}(k,q,L,S,j_2,l,s;k',q',L',S',j_2',l',s')\times$$
$$\phi^j_{M2}(k',q',L',S',j_2',l',s')k'^2dk'q'^2dq' \qquad (152)$$

where $V^j_{NN\pi;NN\pi}$ is a sum of two body interactions in (148). Different orders of coupling of the irreducible representation are natural for each pairwise interaction in the three-particle sector. The transformations that change the order of the coupling can be computed using four Clebsch-Gordan coefficients - they are the analog of "Racah coefficients" for the Poincaré group.

The solution to this problem has the complexity of a three-body problem. The scattering problem with all of the correct asymptotic conditions must be solved using Faddeev methods.

For the pion to be physical the off diagonal parts of the interaction should be short ranged "2-3" operators rather than elementary vertices.

5. Relativistic Dynamics (simultaneous eigenstates of mass, spin, linear momentum and z-component of canonical spin are complete and transform irreducibly):

$$\begin{pmatrix} \langle (k',j')\mathbf{P}',\mu';l',s'|U(\Lambda,a)|(M,j)\mathbf{P},\mu\rangle \\ \langle (k',q',j')\mathbf{P}',\mu';L',S',j_2',l',s'|U(\Lambda,a)|(M,j)\mathbf{P},\mu\rangle \end{pmatrix} =$$
$$\begin{pmatrix} \langle (k',j')\mathbf{P}',\mu';l',s'|(M,j)\mathbf{\Lambda}P,\mu'\rangle \\ \langle (k',q',j')\mathbf{P}',\mu';L',S',j_2',l',s'|(M,j)\mathbf{\Lambda}P,\mu'\rangle \end{pmatrix} \times$$
$$\sqrt{\frac{\omega_M(\mathbf{\Lambda}P_M)}{\omega_M(\mathbf{P})}} D^j_{\mu'\mu}[B^{-1}(\mathbf{\Lambda P}/M)\Lambda B(\mathbf{P}/M)]e^{i\Lambda P_M\cdot a} \qquad (153)$$

where

$$P_M = (\sqrt{\mathbf{P}^2+M^2},\mathbf{P}) \qquad (154)$$

5. Representation in terms of single particle degrees of freedom. The irreducible free-particle basis is constructed by first coupling the nucleon irreducible representations to two-nucleon irreducible representations and then coupling the resulting two-nucleon irreducible representations

to the pion irreducible representation. This involves using two sets of Poincaré Clebsch-Gordan coefficients. These can be inverted using the Poincare Clebsch-Gordan coefficients to express the irreducible three particle basis in terms of the tensor product of the single particle irreducible bases. The computation is straightforward, but the result is messy and not very illuminating.

This is a simple extension of the NN model to allow for the production of a single pion. In this model the deuteron will have a two-nucleon and two-nucleon one pion component.

11.3 Quark string model

The flexibility of Poincaré invariant quantum mechanics can be illustrated by considering a quark model motivated by strong-coupling lattice QCD. The basic building blocks of strong coupling Lattice QCD are quarks and links[19]. The physical degrees of freedom involve combinations of quarks, antiquarks and links that are connected to form a color singlet at each lattice site. In the absence of the interactions the energy of each configuration is the sum of the quark and antiquark masses and a quantity proportional to the total length of the links, and color singlets are confined. These degrees of freedom then interact in a manner that couples color singlets to color singlets.

To make a Poincaré invariant model based on these degrees of freedom I consider a model with quarks and antiquark degrees of freedom, where interactions are added in a two step process. First quarks and antiquarks connected by links are modeled by quarks and antiquarks interacting via a confining interaction. Because this represents a locally gauge invariant object the color degrees of freedom are assumed to be summed out. The mass operator can be diagonalized and the resulting eigenstates transform like particles. The interactions that couple singlets or multi-singlets are modeled by short-range interactions. For example, a string breaking interaction is one that couples one confined singlet quark antiquark pair to a pair of confined singlet quark-antiquark pairs. The structure of the interactions can be motivated by lattice degrees of freedom [19] or axiomatic models [20] of these degrees of freedom. The interactions between different singlets can be expressed in terms of internal quark degrees of freedom or the Poincaré irreducible labels of the confined states.

I consider an example of a model that has a meson spectrum, meson decay, and meson-meson scattering. I begin by constructing the singlet subspaces. For this model I include three such subspaces. The quarks and antiquarks in each subspace are assumed to couple to color singlets so the quarks in these subspaces do not have color quantum numbers. These are tensor products of two or four irreducible representations of the Poincaré group for each flavor combination:

$$\mathcal{H}_{q\bar{q}} \tag{155}$$

$$\mathcal{H}_{q\bar{q}*} \tag{156}$$

$$\mathcal{H}_{qq\bar{q}\bar{q}} \tag{157}$$

Each of these subspaces can be decomposed into free particle irreducible subspaces using the Poincaré Clebsh-Gordan coefficients. An invariant mass operator is defined by adding a confining interaction to each non-interacting mass operator:

$$M_{q\bar{q}} = M_{0q\bar{q}} + V_{q\bar{q}} \tag{158}$$

$$M_{q\bar{q}*} = M_{0q\bar{q}} + V_{q\bar{q}*} \tag{159}$$

$$M_{qq\bar{q}\bar{q}} = M_{qq\bar{q}\bar{q}} + V_{qq\bar{q}\bar{q}} \tag{160}$$

It is useful to think of the lattice counterpart of these three operators in a Born Oppenheimer type of approximation. For $V_{q\bar{q}}$ can be thought of as the lowest energy state of a quark anti-quark pair coupled to a single as a function of the distance between the quark and antiquark; $V_{q\bar{q}*}$ can be thought of as the energy of the first excited state of a quark anti-quark pair coupled to a singlet as a function of the distance between the quark and antiquark; $V_{qq\bar{q}\bar{q}}$ can be considered as the lowest energy state of a two quark two anti-quark singlet that cannot be decomposed into a pair of non-interacting singlets as a function of the quark and antiquark coordinates. In Poincaré invariant quantum mechanics these interactions can be modeled. The interaction in each of these singlet mass operators is assumed to have only discrete spectra.

A model Hilbert space is defined by:

$$\mathcal{H} =$$

$$\mathcal{H}_{q\bar{q}} \oplus \mathcal{H}_{q\bar{q}*} \oplus \mathcal{H}_{qq\bar{q}\bar{q}} \oplus (\mathcal{H}_{q\bar{q}} \otimes \mathcal{H}_{q\bar{q}}) \oplus (\mathcal{H}_{q\bar{q}} \otimes \mathcal{H}_{q\bar{q}*}) \oplus (\mathcal{H}_{q\bar{q}*} \otimes \mathcal{H}_{q\bar{q}*}) \tag{161}$$

On this space I define the a mass operator that only includes interactions between quarks and antiquarks in the same color singlet:

$$M_c = \begin{pmatrix} M_{q\bar{q}} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{q\bar{q}*} & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{qq\bar{q}\bar{q}} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{(q\bar{q})(q\bar{q})} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{(q\bar{q})(q\bar{q})*} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{(q\bar{q}*)(q\bar{q})*} \end{pmatrix} \tag{162}$$

where

$$M_{(q\bar{q})(q\bar{q})*} = \sqrt{M^2_{(q\bar{q})} + \mathbf{k}^2} + \sqrt{M^2_{(q\bar{q}*)} + \mathbf{k}^2} \tag{163}$$

$$M_{(q\bar{q})(q\bar{q})} = \sqrt{M^2_{(q\bar{q})_1} + \mathbf{k}^2} + \sqrt{M^2_{(q\bar{q}*)_2} + \mathbf{k}^2} \tag{164}$$

$$M_{(q\bar{q})(q\bar{q})*} = \sqrt{M^2_{(q\bar{q})_1} + \mathbf{k}^2} + \sqrt{M^2_{(q\bar{q}*)_2} + \mathbf{k}^2} \tag{165}$$

The relative momenta $\mathbf{k}$ in the two-singlet subspaces is obtained by using (81) with M_0 replaced by one of the above masses, which can also be expressed in terms of the individual singlet mass eigenvalues and momenta.

Additional interactions allow the quarks in different singlets to interact. These interaction have the matrix form

$$\begin{pmatrix} 0 & V_a & V_b & V_c & V_d & V_e \\ V_a^\dagger & 0 & V_f & V_g & V_h & V_i \\ V_b^\dagger & V_f^\dagger & 0 & V_j & V_k & V_l \\ V_c^\dagger & V_g^\dagger & V_j^\dagger & V_{s1} & V_m & V_n \\ V_d^\dagger & V_h^\dagger & V_k^\dagger & V_m^\dagger & V_{s2} & V+_o \\ V_e^\dagger & V_i^\dagger & V_l^\dagger & V_n^\dagger & V_o^\dagger & V_{s3} \end{pmatrix} \tag{166}$$

This construction requires that each of interactions commutes with and is independent of the total $\mathbf{P}$, $\mathbf{j}^2$, $\mathbf{j}\cdot\hat{\mathbf{z}}$. In this case the spins are not the kinematic spins - they are the spins obtained by treating the confined bound states as particles. These interactions are assumed to be short range interactions when they couple different confined irreducible singlets, however the existence of infinite towers of confined singlet states in each sector puts additional constraints on a model if one wants of have non-trivial meson-meson scattering theory.

Thus, the dynamics of this model is constructed in two steps. First the confining mass operator $M_c = M_0 + V_c$ is diagonalized in the non-interacting irreducible basis, to construct a complete set of confined irreducible mass eigenstates. Then the $M = M_c + V$ is diagonalized in the confined irreducible basis to get a dynamical mass operator, which along with linear momentum and the spin of the confined system can be used to construct a dynamical irreducible representation.

This model is a fully relativistic quantum mechanical model. While this model it is not complete, it illustrates some of the problems that need to be addressed in complex systems. For suitable interactions this mass operator will support bound states, unstable resonances, and scattering states. The bound states correspond to physical mesons in this model. These in general will have a different mass than the corresponding bare mesons that only include the confining interaction. The continuous spectrum is associated with scattering of the bare mesons. Finally bare mesons with mass in the scattering continuum should to be unstable. There is the potential for an interesting interplay between the resonances and the scattering states. The scattering problem involves the sums over an infinite number of short range interactions - it is not automatic that the sum of an infinite number of short ranged interactions results in a short ranged interaction. This is also related to the decay widths of the high lying bare mesons. There will clearly be an interplay between the lifetimes of high lying states and the existence of a scattering theory that needs to be investigated in such a model. Some of these questions are addressed in [21]

12 Position in Poincaré Invariant Quantum Mechanics

One principle that is given up in Poincaré invariant quantum mechanics is microscopic locality. The other thing that is given up is the use of local field operators, which are replaced by particle degrees of freedom. Locality cannot be tested in these theories because there are no suitable position operators for particles in Poincaré invariant quantum theory. Thus, while the theory gives up microscopic locality, it also eliminates the degrees of freedom that are needed to test microscopic locality.

In order to understand the difficulties associated with finding a suitable position operator in Poincaré invariant quantum mechanics I begin by considering the wave function of a spinless particle at the origin at time $t = 0$. I denote the wave function of this particle by $\langle \mathbf{p} | \mathbf{x} = 0; t = 0 \rangle$. If I make the naively sensible assumption that such a state is invariant under homogeneous Lorentz transformations, then

$$\langle \mathbf{p} | \mathbf{x} = 0; t = 0 \rangle = \langle \mathbf{p} | U(\Lambda, 0) | \mathbf{x} = 0; t = 0 \rangle =$$

$$\sqrt{\frac{\omega_m(\mathbf{\Lambda}^{-1} p)}{\omega_m(\mathbf{p})}} \langle \mathbf{\Lambda}^{-1} p | \mathbf{x} = 0; t = 0 \rangle. \tag{167}$$

Comparing the left and right sides of (167) it follows that wave function $\langle \mathbf{p} | \mathbf{x} = 0; t = 0 \rangle$ must have the form

$$\langle \mathbf{p} | \mathbf{x} = 0; t = 0 \rangle = \frac{1}{\sqrt{\omega_m(\mathbf{p})}} f(p^2) = \frac{1}{\sqrt{\omega_m(\mathbf{p})}} f(m^2) = \frac{C}{\sqrt{\omega_m(\mathbf{p})}}, \tag{168}$$

where C is constant. I can now translate this eigenstate to construct an eigenstate corresponding to a particle localized at $\mathbf{x}$:

$$\langle \mathbf{p} | \mathbf{x}; t = 0 \rangle = \langle \mathbf{p} | U(I, \mathbf{x}) | \mathbf{x} = 0; t = 0 \rangle = e^{i \mathbf{p} \cdot \mathbf{x}} \frac{C}{\sqrt{\omega_m(\mathbf{p})}}. \tag{169}$$

If I take the overlap between the state at $(\mathbf{x} = 0; t = 0)$ with a state at $(\mathbf{x} \neq 0; t = 0)$, the result is ([22]:

$$\langle \mathbf{0} | \mathbf{x} \rangle = |C|^2 \int \frac{d^3 p}{\omega_m(\mathbf{p})} e^{i \mathbf{p} \cdot \mathbf{x}}$$

$$-(2\pi)^3 |C|^2 \frac{i}{2} D_+^*(0, \mathbf{x}) =$$

$$(2\pi)^3 |C|^2 \frac{i}{2} \left[\lim_{t \to 0} \frac{1}{4\pi} \epsilon(t) \delta(\mathbf{x}^2) - \frac{mi}{4\pi^2 |\mathbf{x}|} K_1(m|\mathbf{x}|) \right], \tag{170}$$

where $D_+(x)$ is the positive frequency part of the Pauli-Jordan commutator function. For $\mathbf{x} \neq 0$, this expression is non-zero, but falls off like $K_1(m|\mathbf{x}|)$, vanishing as $|m\mathbf{x}|^{-1/2} e^{-|m\mathbf{x}|}$ as $|\mathbf{x}| \to \infty$. Thus, these two states have an overlap which falls off exponentially when the coordinates are separated by more than a Compton wavelength. The assumption that a particle localized at the origin can be described in an invariant way implies that it is not orthogonal to a state at a different point at the same time. The Compton wavelength of the particle again sets the scale for the violation of orthogonality.

It is possible to obtain additional insight by noting that the position operator discussed above canonically conjugate to the linear momentum in an irreducible representation. If the representation has a spin, this operator is also required to commute with canonical spin (the resulting operator is the so called Newton-Wigner position operator). This means that it is essentially $-i$ multiplied by the partial derivative of the linear momentum holding the canonical spin constant. Because the spins undergo momentum dependent Wigner rotations, this is a non-trivial requirement that depends on which boost is used to define the spin.

Changing spin observable involves momentum-dependent rotations, which do not commute with the momentum derivatives. To understand this, consider two spin-1/2 wave functions in a canonical spin and helicity spin basis, respectively:

$$ {}_c\langle m\, j; \mathbf{p}\, \mu|\phi\rangle = f_\mu(\mathbf{p}); \tag{171} $$

$$ {}_h\langle m\, j; \mathbf{p}\, \mu|\psi\rangle = f_\mu(\mathbf{p}). \tag{172} $$

The wave function $f_\mu(\mathbf{p})$ is chosen to be the same in each case, however the states are different. In both expressions, $\mathbf{p}$ is the three-momentum. The Fourier transforms of each of these wave functions are clearly the same. On the other hand, if we take the wave function ${}_c\langle m\, s; \mathbf{p}\, \mu|\phi\rangle$, and perform the unitary transformation that puts it into the same representation as the wave function ${}_h\langle m\, s; \mathbf{p}\, \mu|\psi\rangle$, then the new wave function is

$$ f_\mu(\mathbf{p}) \to f'_\mu(\mathbf{p}) = \sum_{\bar{\mu}} D^{\frac{1}{2}}_{\mu\bar{\mu}}[R_{hc}(\mathbf{p}/m)] f_{\bar{\mu}}(\mathbf{p}). \tag{173} $$

where

$$ R_{hc}(\mathbf{p}/m) = B_h^{-1}(\mathbf{p}/m) B_c(\mathbf{p}/m) \tag{174} $$

is the momentum dependent rotation constructed using a canonical boost to from the rest frame to a frame with linear momentum $\mathbf{p}$ followed by a helicity boost back to the rest frame. The resulting transformation is a momentum dependent rotation, called a generalized Melosh rotation [23][24]. Obviously the partial derivative with respect to the linear momentum holding the canonical spin constant differs from the partial derivative with respect to the linear momentum holding the helicity spin constant.

The conclusion is that although configuration space wave functions can be used as well as momentum space wave functions, one should never attempt to interpret the coordinates as observable quantities, especially on distance scales on the order of a Compton wavelength of a particle. We note that the concept of position gets even more complicated in models with interaction dependent spins.

13 Two-component spinor conventions:

The group $SL(2,\mathbb{C})$, which is a double cover of the Lorentz group, is useful for both computational and proving simple results [2][11][24].

Let σ_μ denote the 2×2 Pauli spin matrices and the identity. Let

$$X := x^\mu \sigma_\mu \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu X) \tag{175}$$

Note that

$$\det(X) = (x^0)^2 - (\mathbf{x})^2 = -x^2 \qquad X = X^\dagger \tag{176}$$

Any linear transformation that preserves the determinant and Hermiticity of X defines a real Lorentz transformation. If A is an arbitrary complex matrix with $\det(A) = 1$ and

$$X \to X' = AXA^\dagger \tag{177}$$

then

$$\det(X') = \det(X) \qquad \text{and} \qquad (X')^\dagger = X' \tag{178}$$

The corresponding Lorentz transformation is

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu A^\dagger) \tag{179}$$

It is obvious from (179) that

$$\Lambda(A)^\mu{}_\nu = \Lambda(-A)^\mu{}_\nu \tag{180}$$

It is not difficult to show that there is a 2 to 1 correspondence between the $SL(2,\mathbb{C})$ matrices A and Lorentz transformations connected to the identity.

A general element in $SL(2,\mathbb{C})$ has the form

$$A = \pm e^{\frac{1}{2}(\boldsymbol{\rho}+i\boldsymbol{\theta})\cdot\boldsymbol{\sigma}}. \tag{181}$$

$$A = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} \tag{182}$$

corresponds to a rotationless boost with rapidity $\boldsymbol{\rho}$ and

$$A = e^{i\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \tag{183}$$

corresponds to a rotation through an angle $\boldsymbol{\theta}$.

Canonical Boosts:

$SL(2,C)$ representatives of canonical boosts are given by:

$$\sinh(\omega) = \frac{|\mathbf{p}|}{m} = |\mathbf{v}| \tag{184}$$

$$\cosh(\omega) = \frac{p^0}{m} = v^0 \tag{185}$$

$$\sinh(\frac{\omega}{2}) = \sqrt{\frac{p^0 - m}{2m}} = \sqrt{\frac{v^0 - 1}{2}} \tag{186}$$

$$\cosh(\frac{\omega}{2}) = \sqrt{\frac{p^0 + m}{2m}} = \sqrt{\frac{v^0 + 1}{2}} \tag{187}$$

$$\Lambda_c(v) := \cosh(\omega/2)\sigma_0 + \sinh(\omega/2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} =$$

$$\sqrt{\frac{v^0+1}{2}}\sigma_0 + \sqrt{\frac{v^0-1}{2}}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} = \tag{188}$$

$$\frac{1}{\sqrt{2(v^0+1)}}\big((v^0+1)\sigma_0 + \mathbf{v}\cdot\boldsymbol{\sigma}\big) = \tag{189}$$

$$\frac{1}{\sqrt{2m(p^0+m)}}\big((p^0+m)\sigma_0 + \mathbf{p}\cdot\boldsymbol{\sigma}\big) \tag{190}$$

$$\Lambda_c^\dagger(v) = \Lambda_c(v) \tag{191}$$

$$\Lambda_c^{-1}(v) = \tilde{\Lambda}_c(v) = \cosh(\omega/2)\sigma_0 - \sinh(\omega_2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} = \tag{192}$$

$$\sqrt{\frac{v^0+1}{2}}\sigma_0 - \sqrt{\frac{v^0-1}{2}}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} = \tag{193}$$

$$\frac{1}{\sqrt{2(v^0+1)}}\big((v^0+1)\sigma_0 - \mathbf{v}\cdot\boldsymbol{\sigma}\big) = \tag{194}$$

$$\frac{1}{\sqrt{2m(p^0+m)}}\big((p^0+m)\sigma_0 - \mathbf{p}\cdot\boldsymbol{\sigma}\big) \tag{195}$$

Note that in all of the above expressions for the boosts v^0 or p^0 represent on-shell quantities.

14 Scattering theory in Poincaré invariant quantum mechanics

S and T operators

The formulation of scattering problems in Poincaré invariant quantum mechanics is based on standard time-dependent multichannel scattering. Scattering channels α are associated with asymptotically separated clusters, where the particles in each cluster are either in a bound state or the cluster consists of a single particle. Each distinct partition of the particles into clusters may correspond to more than one scattering channel or it may support no scattering channels. The scattering matrix is the inner product of incoming and outgoing wave scattering states

$$S_{\alpha\beta} = \langle \Psi^+_\alpha(0) | \Psi^-_\beta(0) \rangle \tag{196}$$

where α and β are channel labels and the initial and final scattering states are solutions of the time-dependent Schrödinger equation

$$i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle \tag{197}$$

satisfying the incoming and outgoing wave asymptotic conditions

$$\lim_{t\to\pm\infty} \| e^{-iHt}|\Psi^\pm_\alpha(0)\rangle - \Pi_\alpha e^{-iH_\alpha t}|\Phi^\pm_\alpha(0)\rangle \| = 0. \tag{198}$$

In this paper the $\pm$ on the scattering states and wave operators indicate the direction of the time limit ($-$ =past/$+$ =future), which is opposite to the sign of the $i\epsilon$ that appears in the resolvents used in time independent scattering. The operator Π_α is a channel projection operator,

$$\Pi_\alpha = \otimes_{i\in\alpha} \sum\!\!\!\!\!\!\int_{\nu_i} |(m_i, j_i)\mathbf{p}_i, \nu_i\rangle d\mathbf{p}_i \langle (m_i, j_i)\mathbf{p}_i, \nu_i| \tag{199}$$

which projects on the subspace associated with mutually non-interacting bound subsystems. The factors $|(m_i, j_i)\mathbf{p}_i, \nu_i\rangle$ are basis functions for the irreducible representation of the Poincaré group associated with the i-th cluster of the channel α with discrete mass eigenvalue, m_i.

The quantity $|S_{\alpha\beta}|^2$ represents the probability of a system prepared in a state that in the distant past looks like a system of asymptotically separated particles in channel β to be measured to be in a state that looks in the asymptotic future like a system of asymptotically separated particles in channel α.

It follows from (198) that the interacting and limiting non-interacting asymptotic states are related by the multichannel wave operators

$$|\Psi^{\pm}_{\alpha}(0)\rangle = \Omega_{\alpha\pm}(H, H_{\alpha})|\Phi^{\pm}_{\alpha}(0)\rangle \tag{200}$$

where the multichannel wave operators are defined by the strong limits

$$\Omega_{\alpha\pm} = \lim_{t\to\pm\infty} e^{iHt}\Pi_{\alpha}e^{-iH_{\alpha}t}. \tag{201}$$

In these equations H_{α} is the Hamiltonian with the interactions between particles in different asymptotic clusters set to zero. The multichannel scattering operator can then be expressed in terms of the wave operators as

$$S_{\alpha\beta} = \Omega^{\dagger}_{\alpha+}(H, H_{\alpha})\Omega_{\beta-}(H, H_{\beta}). \tag{202}$$

The wave operators can be expressed directly in terms of the mass operators. The Kato-Birman invariance principle [25][5] implies that H and H_{α} in the channel wave operators can be replaced by $f(H)$ and $f(H_{\alpha})$ where f is any piecewise differentiable function of bounded variation with positive derivative; specifically

$$M = \sqrt{H^2 - \mathbf{P}^2} \tag{203}$$

is a function with these properties. It follows that

$$\Omega_{\alpha\pm} = \lim_{t\to\pm\infty} e^{iMt}\Pi_{\alpha}e^{-iM_{\alpha}t} = \lim_{t\to\pm\infty} e^{iHt}\Pi_{\alpha}e^{-iH_{\alpha}t} \tag{204}$$

which leads to the equivalent expression for the multichannel scattering operator [16]:

$$S_{\alpha\beta} = \lim_{\tau,\tau'\to\infty} e^{iM_{\alpha}\tau}\Pi_{\alpha}e^{-iM(\tau+\tau')}\Pi_{\beta}e^{iM_{\beta}\tau'}. \tag{205}$$

To relate this to the time-independent formulation of scattering these limits are computed in eigenstates $|\alpha\rangle$ and $|\beta\rangle$ of M_{α} and M_{β} respectively. I prove that

$$\langle\alpha|S|\beta\rangle = \langle\alpha|\beta\rangle - 2\pi i\delta(\mathsf{W}_{\alpha} - \mathsf{W}_{\beta})\langle\alpha|T^{\alpha\beta}(\mathsf{W}_{\alpha} + i0^{+})|\beta\rangle \tag{206}$$

where

$$T^{\alpha\beta}(z) = V^{\beta} + V^{\alpha}(z - M)^{-1}V^{\beta}, \tag{207}$$

and

$$V^{\alpha} = M - M_{\alpha} \tag{208}$$

and

$$V^{\alpha} = M - M_{0} \tag{209}$$

for the breakup channel.

Here W_{α} and W_{β} are the eigenvalues of M_{α} and M_{β} in the channel eigenstates $|\alpha\rangle$ and $|\beta\rangle$. The first term in Eq. (206) is identically zero if the states $|\alpha\rangle$ and $|\beta\rangle$ correspond to different scattering channels.

To prove (206) I evaluate the S-matrix elements in the channel mass eigenstates:

$$\begin{aligned}
\langle\beta|S_{ba}|\alpha\rangle &= \lim_{\tau\to\infty}\langle\beta|e^{iM_{\beta}\tau}e^{-2iM\tau}e^{iM_{\alpha}\tau}|\alpha\rangle \\
&= \langle\beta|\alpha\rangle + \lim_{\tau\to\infty}\int_{0}^{\tau} d\tau'\,\frac{d}{d\tau'}\langle\beta|e^{i(\mathsf{W}_{\beta}+\mathsf{W}_{\alpha}-2M)\tau'}|\alpha\rangle \\
&= \langle\beta|\alpha\rangle + \lim_{\epsilon\to 0^{+}} i\int_{0}^{\infty} d\tau'\,\langle\beta| \\
&\times \left[(\mathsf{W}_{\beta}-M)e^{i(\mathsf{W}_{\beta}+\mathsf{W}_{\alpha}-2M+i\epsilon)\tau'} + e^{i(\mathsf{W}_{\beta}+\mathsf{W}_{\alpha}-2M+i\epsilon)\tau'}(\mathsf{W}_{\alpha}-M)\right]|\alpha\rangle \\
&= \langle\beta|\alpha\rangle + \lim_{\epsilon\to 0^{+}}\frac{1}{2}\langle\beta| \\
&\times \left[(M-\mathsf{W}_{\beta})\frac{1}{\bar{\mathsf{W}}-M+i\epsilon} + \frac{1}{\bar{\mathsf{W}}-M+i\epsilon}(M-\mathsf{W}_{\alpha})\right]|\alpha\rangle, \qquad (210)
\end{aligned}$$

where $M_{\alpha}|\alpha\rangle = \mathsf{W}_{\alpha}|\alpha\rangle$ and $M_{\beta}|\beta\rangle = \mathsf{W}_{\beta}|\alpha\rangle$ and $\bar{\mathsf{W}} := \frac{1}{2}(\mathsf{W}_{\alpha}+\mathsf{M}_{\beta})$ is the average invariant mass eigenvalues of the initial and final asymptotic states. The $i\epsilon$ are introduced because formally the sharp eigenstates should be first integrated against wave packets in the cluster momenta before the time limit is computed. Adding the $i\epsilon$ has no effect if these integrals are done first, however when $i\epsilon$ is included it is possible to change the order of the time limit and the integration over the wave packets. I will reinsert the wave packets when I compute the cross section. In deriving (211) the two strong limits in (206) are replaced a single weak limit. Equation (206) is interpreted as the kernel of an integral operator. S-matrix elements are obtained by integrating the sharp eigenstates in Eq. (211) over normalizable functions of the energy and other continuous variables.

To simply this expression I define the residual interactions V^{α} and V^{β} by:

$$V^{\alpha} := M - M_{\alpha}; \qquad V^{\beta} = M - M_{\beta}, \tag{211}$$

where
$$V^\alpha|\alpha\rangle = (M - \mathsf{W}_\alpha)|\alpha\rangle; \qquad V^\beta|\beta\rangle = (M - \mathsf{W}_\beta)|\beta\rangle. \tag{212}$$
The resolvent operators of the mass operator and the channel mass operator,
$$G(z) := \frac{1}{z - M} \qquad G_\alpha(z) := \frac{1}{z - M_\alpha}, \tag{213}$$
are related by the second resolvent relations [26]:
$$G(z) - G_\alpha(z) = G_\alpha(z)V^\alpha G(z) = G(z)V^\alpha G_\alpha(z). \tag{214}$$
which when used in Eq. (206) gives
$$\begin{aligned}
\langle\beta|S|\alpha\rangle &= \langle\beta|\alpha\rangle \\
&+ \lim_{\epsilon\to 0^+} \frac{1}{2}\langle\beta| \left[V^\beta \left(1 + G(\bar{\mathsf{W}} + i\epsilon)V^\alpha\right) G_\alpha(\bar{\mathsf{W}} + i\epsilon)\right. \\
&+ \left. G_\beta(\bar{\mathsf{W}} + i\epsilon)\left(1 + V^\beta G(\bar{\mathsf{W}} + i\epsilon)\right) V^\alpha\right] |\alpha\rangle \\
&= \langle\beta|\alpha\rangle \left[1 - \lim_{\epsilon\to 0^+} \frac{\mathsf{W}_\beta - \mathsf{W}_\alpha}{\mathsf{W}_\beta - \mathsf{W}_\alpha + 2i\epsilon}\right] \\
&\quad + \lim_{\epsilon\to 0^+}\left[\frac{1}{\mathsf{W}_\beta - \mathsf{W}_\alpha + 2i\epsilon} + \frac{1}{\mathsf{W}_\alpha - \mathsf{W}_\beta + 2i\epsilon}\right] \\
&\times \langle\beta| \left(V^\alpha + V^\beta G(\bar{\mathsf{W}} + i\epsilon)V^\alpha\right)|\alpha\rangle \\
&= \langle\beta|\alpha\rangle \lim_{\epsilon\to 0^+}\left[\frac{2i\epsilon}{\mathsf{W}_\beta - \mathsf{W}_\alpha + 2i\epsilon}\right] \\
&+ \lim_{\epsilon\to 0^+}\left[\frac{-4i\epsilon}{(\mathsf{W}_\beta - \mathsf{W}_\alpha)^2 + 4\epsilon^2}\right]\langle\beta| \left(V^\alpha + V^\beta G(\bar{\mathsf{W}} + i\epsilon)V^\alpha\right)|\alpha\rangle
\end{aligned} \tag{215}$$
It is now possible to evaluate the limit as $\epsilon \to 0$. It is important to remember that this is the kernel of an integral operator.

The first term in square brackets is unity when the initial and final mass eigenvalues are identical, and zero otherwise; however, the limit in the bracket is a Kronecker delta and *not* a Dirac delta function. For $\alpha \neq \beta$, $\langle\beta(\mathsf{W}')|\alpha(\mathsf{W})\rangle$ are Lebesgue measurable in W' for fixed W, so there is no contribution from the first term in Eq. (215). For the case that $\mathsf{W}_\alpha = \mathsf{W}_\beta$, we have $\langle\beta(\mathsf{W}')|\alpha(\mathsf{W})\rangle \propto \delta(\mathsf{W}' - \mathsf{W})$. The matrix element vanishes by orthogonality unless $\mathsf{W}_\beta = \mathsf{W}_\alpha$, but then the coefficient is unity. Thus, the first term in (215) is $\langle\beta|\alpha\rangle$ if the initial and final channels are the same, but zero otherwise. The matrix elements also vanish by orthogonality for two different channels governed by the same asymptotic mass operator with the same invariant mass. The first term in (215) therefore includes a *channel* delta function.

For the second term, the quantity in square brackets becomes $-2\pi i\delta(\mathsf{W}_\beta - \mathsf{W}_\alpha)$, which leads to the relation

$$\langle\beta|S|\alpha\rangle = \langle\alpha|\beta\rangle - 2\pi i\delta(\mathsf{W}_\beta - \mathsf{W}_\alpha)\langle\beta|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)|\alpha\rangle, \tag{216}$$

where

$$T^{\beta\alpha}(z) = V^\alpha + V^\beta G(z) V^\beta. \tag{217}$$

and $\langle\alpha|\beta\rangle$ is zero if the initial and final channels are different and is the overlap of the initial and final states if the initial and final channels are the same. Equation (216) is exactly eq. (206).

The channel projection operators Π_α are absorbed in the channel states, $|\alpha\rangle$. The translational invariance of the interaction (211) requires that

$$\langle\mathbf{P}, \cdots|T^{\alpha\beta}(z)|\cdots, \mathbf{P}\,'\rangle = \delta(\mathbf{P} - \mathbf{P}\,')\langle\cdots\|T^{\alpha\beta}(z)\|\cdots\rangle. \tag{218}$$

With our choice of irreducible basis the residual interactions and the resolvent commute with the total linear momentum operator, and if the sharp channel states $|\alpha\rangle$ and $|\beta\rangle$ are simultaneous eigenstates of the appropriate partition mass operator and the linear momentum, then a three-momentum conserving delta function can be factored out of the T-matrix element:

$$\langle\beta|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)|\alpha\rangle = \delta^3(\mathbf{P}_\beta - \mathbf{P}_\alpha)\langle\beta\|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)\|\alpha\rangle. \tag{219}$$

When combined with the three-momentum conserving delta function the invariant mass delta function can be replaced an energy conserving delta function

$$\delta(\mathsf{W}_\beta - \mathsf{W}_\alpha) = \left|\frac{d\mathsf{W}}{d\mathsf{E}}\right| \delta(\mathsf{E}_\beta - \mathsf{E}_\alpha) \qquad \left|\frac{d\mathsf{W}}{d\mathsf{E}}\right| = \frac{\mathsf{W}}{\mathsf{E}}. \tag{220}$$

The S-matrix elements can be expressed in terms of the *reduced channel transition operators* as follows:

$$\langle\beta|S|\alpha\rangle = \langle\alpha|\beta\rangle\delta_{\beta\alpha} - i(2\pi)\delta^4(P_\beta - P_\alpha)\frac{\mathsf{W}_\alpha}{\mathsf{E}_\alpha}\langle\beta\|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)\|\alpha\rangle \tag{221}$$

In this expression the S operator is invariant while the single particle asymptotic states have a non-covariant normalization.

Scattering cross sections

The representation of the scattering matrix (221) is used to calculate the cross section. I derive the cross section following standard methods used by Brenig and Haag [27]. An initial state consisting of a target t in a state $|\varphi_t\rangle$ and beam b in a state $|\varphi_b\rangle$ leads to the asymptotic differential probability amplitude for a n-particle final state in channel α:

$$\langle \mathbf{p}_1, \cdots, \mathbf{p}_n|\varphi\rangle := \int \langle \mathbf{p}_1, \cdots, \mathbf{p}_n|S_{\alpha\beta}|\mathbf{p}_b, \mathbf{p}_t\rangle d\mathbf{p}_b d\mathbf{p}_t \langle \mathbf{p}_b|\varphi_b\rangle\langle \mathbf{p}_t|\varphi_t\rangle. \quad (222)$$

where the spin degrees of freedom are suppressed. The differential probability for observing each final particle to be within $d\mathbf{p}_i$ of $\mathbf{p}_i$ is

$$dP = |\langle \mathbf{p}_1, \cdots, \mathbf{p}_n|\varphi\rangle|^2 d\mathbf{p}_1 \cdots d\mathbf{p}_n. \quad (223)$$

Inserting the expression (221) for S in terms of the wave packets in (222), assuming either different initial channels or non-forward scattering, so there is no contribution from the identity part of the S matrix, gives

$$\begin{aligned} dP &= d\mathbf{p}_1 \cdots d\mathbf{p}_n \int (2\pi)^2 \langle \mathbf{p}_1, \cdots, \mathbf{p}_n \| T^{\alpha\beta} \| \mathbf{p}_b', \mathbf{p}_t'\rangle \langle \mathbf{p}_1, \cdots, \mathbf{p}_n \| T^{\alpha\beta} \| \mathbf{p}_b'', \mathbf{p}_t''\rangle^* \\ &\times \ \delta\left(\sum_i \mathbf{p}_i - \mathbf{p}_b' - \mathbf{p}_t'\right) \delta\left(\sum_j \mathbf{p}_j - \mathbf{p}_b'' - \mathbf{p}_t''\right) \\ &\times \ \delta(\mathsf{W}_\alpha - \mathsf{W}_{bt}')\ \delta(\mathsf{W}_\alpha - \mathsf{W}_{bt}'') d\mathbf{p}_b' d\mathbf{p}_t' d\mathbf{p}_b'' d\mathbf{p}_t'' \\ &\times \ \langle \mathbf{p}_b'|\varphi_b\rangle \langle \mathbf{p}_b''|\varphi_b\rangle^* \langle \mathbf{p}_t'|\varphi_t\rangle \langle \mathbf{p}_t''|\varphi_t\rangle^*. \end{aligned} \quad (224)$$

The delta function for the conservation of linear momentum means that

$$\delta(\mathsf{W}_\alpha - \mathsf{W}_{bt}') = \delta(\mathsf{E}_\alpha - \mathsf{E}_{bt'}) \left|\frac{dE}{dM}\right| = \delta(\mathsf{E}_\alpha - \mathsf{E}_{bt}') \left|\frac{\mathsf{W}_\alpha}{\mathsf{E}_\alpha}\right| \quad (225)$$

where $\mathsf{E}_\alpha = \sqrt{\mathsf{W}_\alpha^2 + \mathbf{P}^2}$. With this replacement the delta functions in (225) can be replaced by products of four momentum conserving delta functions:

$$\delta^4(\sum_i p_i - p_b' - p_t')\ \delta^4(\sum_j p_j - p_b'' - p_t'') =$$

$$\delta^4(\sum_i p_i - p_b' - p_t')\ \delta^4(p_b' + p_t' - p_b'' - p_t''). \quad (226)$$

If the initial wave packets are sharply peaked about the target and beam momenta and the transition operator varies slowly on the support of these wave packets, then the transition operators can be factored out of the integral, replacing the momenta with the mean target and beam momenta, $\bar{\mathbf{p}}_b, \bar{\mathbf{p}}_t$. This approximation must be valid for the cross section to be independent of the shape of the wave packets. The result, after expressing the second four momentum conserving delta function in (226) by a Fourier integral representation

$$\frac{1}{2\pi^4}\int e^{ix\cdot(p'_b+p'_t-p''_b-p''_t)}d^4x \tag{227}$$

is

$$\begin{aligned} dP &= (2\pi)^4 d\mathbf{p}_1\cdots d\mathbf{p}_n \int \left|\langle \mathbf{p}_1,\cdots,\mathbf{p}_n\|T^{\alpha\beta}\|\bar{\mathbf{p}}_b,\bar{\mathbf{p}}_t\rangle\right|^2 |\frac{\mathsf{W}_\alpha}{\mathsf{E}_\alpha}|^2 \\ &\times |\langle \mathbf{x},t|\varphi_b\rangle|^2|\langle \mathbf{x},t|\varphi_t\rangle|^2 d\mathbf{x}\,dt \\ &\times \delta\left(\sum_i \mathbf{p}_i-\bar{\mathbf{p}}_b-\bar{\mathbf{p}}_t\right)\delta\left(\sum_i \mathsf{E}_{k_i}-\bar{\mathsf{E}}_b-\bar{\mathsf{E}}_t\right). \end{aligned} \tag{228}$$

This is the differential probability for a single scattering event. The space-time integral picks up a contribution whenever the beam and target are in the same place at the same time.

In a real experiment there is a statistical ensemble of N_t target particles with number density

$$\rho_t(\mathbf{x},t) = N_t\sum_l p_{lt}|\langle \mathbf{x},t|\varphi_{lt}\rangle|^2 \tag{229}$$

where N_t is the total number of target particles and p_{lt} are probabilities in the target density matrix. A normal beam current density with a flux of N_b beam particles per unit time per unit cross sectional area can be expressed as

$$j_b(\mathbf{x},t) = v_{bt}\rho_b(\mathbf{x},t) = v_{bt}N_b\sum_m p'_{mb}|\langle \mathbf{x},t|\varphi_{mt}\rangle|^2 \tag{230}$$

where p'_{mb} are probabilities in the beam density matrix and v_{bt} is the relative speed of the beam with respect to the target.

Using this in the expression for the scattering probability, assuming that there is no more than one scattering event per incident particle, gives the number of particles scattered per unit time per unit volume as

$$\frac{dN}{d^4x} = N_b N_t \frac{dP}{d\mathbf{x}dt} = \frac{(2\pi)^4}{v_{bt}} \left| \langle \mathbf{p}_1, \cdots, \mathbf{p}_n \| T^{\alpha\beta} \| \bar{\mathbf{p}}_{1b}, \bar{\mathbf{p}}_{1t} \rangle \right|^2 \frac{\mathsf{W}_\alpha^2}{\mathsf{E}_\alpha^2} \times \delta^4(p_1 + \cdots + p_n - \bar{p}_b - \bar{p}_t)\, d\mathbf{p}_1 \cdots d\mathbf{p}_n\, \rho_t(\mathbf{x},t) j_b(\mathbf{x},t). \quad (231)$$

This quantity is proportional to the beam current times the target density. The proportionality factor is the differential cross section, $d\sigma$. Comparing

$$\frac{dN}{d^4x} = d\sigma \rho_t(\mathbf{x},t) j_b(\mathbf{x},t) \quad (232)$$

with (231) gives the following expression for the differential cross section:

$$d\sigma = \frac{(2\pi)^4}{v_{bt}} \left| \langle \mathbf{p}_1, \cdots, \mathbf{p}_n \| T^{\alpha\beta} \| \bar{\mathbf{p}}_b, \bar{\mathbf{p}}_t \rangle \right|^2 \times \frac{\mathsf{W}_\alpha^2}{\mathsf{E}_\alpha^2} \delta^4 \left(\sum_{i=1}^n p_i - \bar{p}_b - \bar{p}_t \right) d\mathbf{p}_1 \cdots d\mathbf{p}_n. \quad (233)$$

For identical particles this must be multiplied by a statistical factor $\frac{1}{s}$, where s is the number of permutations of the identical particles in the final state (i.e. $s = n_1! n_2! \cdots$ if there n_1 identical particle of type 1, n_2 identical particle of type 2, etc. in the final state.)

For polarized beams or targets it is useful to introduce matrices S_{ia} and S_{fa} in the initial and final spin variables with the normalization

$$\mathrm{Tr}(S_{ia} S_{ib}) = \delta_{ab} \qquad \mathrm{Tr}(S_{fa} S_{fb}) = \delta_{ab} \quad (234)$$

Spin operators are linear combinations of these operators with constant coefficients s_{ia} and s_{fa}

$$S_f = \sum_a s_{fa} S_{fa} \qquad S_i = \sum_a s_{ia} S_{ia} \quad (235)$$

the resulting observable is

$$\langle O \rangle = \frac{\mathrm{Tr}(T^\dagger S_f T S_i)}{\mathrm{Tr}(T^\dagger T)} \quad (236)$$

where the traces are over the spins. All of the spin independent factors cancel in the ratio. In general the choice of spin-basis matrices S depend on the particle content of the initial and or final states.

Except for the factor $\mathsf{W}_\alpha^2/\mathsf{E}_\alpha^2$ Eq. (233) is identical to the corresponding non-relativistic expression. The additional factor of $\mathsf{W}_\alpha^2/\mathsf{E}_\alpha^2$ arises because we have chosen to calculate the transition operator using the mass operator instead of the Hamiltonian. The only difference in these formulas from standard formulas is that the transition operator is constructed from the difference of the mass operators with and without interactions and the appearance of the additional factor of $\mathsf{W}_\alpha^2/\mathsf{E}_\alpha^2$ which corrects for the modified transition operator. This factor becomes 1 when $\mathbf{P} = 0$.

The expression of Eq. (223) can be expressed in a manifestly invariant form. The relation to the standard expression of the invariant cross section using conventions of the particle data book [28] is derived below.

15 Invariance of S and relation to T

The expression (233) for the differential cross section can be rewritten in a manifestly invariant form. We write it as a product of an invariant phase space factor, an invariant factor that includes the relative speed, and an invariant scattering amplitude.

To establish the invariance of the invariant scattering amplitude note that the scattering operator S is Poincaré invariant:

$$\begin{aligned} U_f(\Lambda,a)S &= U_f(\Lambda,a)\Omega_+^\dagger(H,H_0)\Omega_-(H,H_0) \\ &= = \Omega_+^\dagger(H,H_0)U(\Lambda,a)\Omega_-(H,H_0) \\ &= \Omega_+^\dagger(H,H_0)\Omega_-(H,H_0)U_f(\Lambda,a) = SU_f(\Lambda,a). \end{aligned} \tag{237}$$

where U_f is the product of the cluster irreducible representations of the Poincaré group that act on the channel states.

The proof of the Poincaré invariance of the S operator above is a consequence of the intertwining relations for the wave operators

$$U(\Lambda,a)\Omega_\pm(H,H_0) = \Omega_\pm(H,H_0)U_0(\Lambda,a) \tag{238}$$

To show the intertwining property of the wave operators first note that the invariance principle gives the identity

$$\Omega_\pm(H,H_0) = \Omega_\pm(M,M_0). \tag{239}$$

The mass operator intertwines by the standard intertwining properties of wave operators. For our choice of irreducible basis the intertwining of the full Poincaré group follows because all of the generators can be expressed as functions of the mass operator and a common set of kinematic operators, $\{\mathbf{P}, j_z, j_x, j^2, -i\boldsymbol{\nabla}_P\}$, that commute with the wave operators.

The covariance of the S matrix elements follows from the Poincaré invariance of the S operator if the matrix elements of S are computed in a basis with a covariant normalization.

To extract the standard expression for the invariant amplitude the single particle states are replaced by states with the covariant normalization used in the particle data book [28]:

$$|\mathbf{p}, \mu\rangle \longrightarrow |p, \mu\rangle_{cov} = |\mathbf{p}, \mu\rangle \sqrt{2E_{k_m}}(2\pi)^{3/2}. \tag{240}$$

The resulting expression

$$-i(2\pi)\delta^4(P_\beta - P_\alpha)\frac{\mathsf{W}_\alpha}{\mathsf{E}_\alpha}{}_{cov}\langle\beta\|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)\|\alpha\rangle_{cov} \tag{241}$$

is invariant (up to spin transformation properties). Since the four dimensional delta function is invariant, the factor multiplying the delta function is also invariant (up to spin transformation properties). This means that

$${}_{cov}\langle\alpha\|M^{\alpha\beta}\|\beta\rangle_{cov} := \frac{1}{(2\pi)^3}\frac{\mathsf{W}_\alpha}{\mathsf{E}_\alpha}{}_{cov}\langle\beta\|T^{\beta\alpha}(\mathsf{W}_\alpha + i0^+)\|\alpha\rangle_{cov} \tag{242}$$

is a Lorentz covariant amplitude. The factor of $1/(2\pi)^3$ is chosen to agree with the normalization convention used in the particle data book [28].

The differential cross section becomes

$$\begin{aligned} d\sigma &= \frac{(2\pi)^4}{4E_{m_t}(\mathbf{p}_t)E_{m_b}(\mathbf{p}_b)v_{bt}} \left|{}_{cov}\langle p_1, \cdots, p_n, \|M^{\alpha\beta}\|\bar{p}_b, \bar{p}_t\rangle_{cov}\right|^2 \\ &\times \delta^4\left(\sum_i p_i - \bar{p}_b - \bar{p}_t\right) \frac{d\mathbf{p}_1}{2E_{m_1}(2\pi)^3}\cdots\frac{d\mathbf{p}_n}{2E_{m_n}(2\pi)^3}. \end{aligned} \tag{243}$$

The identity

$$v_{bt} = \frac{\sqrt{p_t \cdot p_b)^2 - m_b^2 m_t^2}}{E_{m_b}E_{m_t}} \tag{244}$$

can be used to get an invariant expression for the relative speed between the projectile and target and

$$d\Phi_n(p_b + p_t; \mathbf{p}_1, \cdots, \mathbf{p}_n) = \delta^4\left(\sum_i p_i - \bar{p}_b - \bar{p}_t\right)$$

$$\times \frac{d\mathbf{p}_1}{2E_{m_1}(2\pi)^3} \cdots \frac{d\mathbf{p}_n}{2E_{m_n}(2\pi)^3} \tag{245}$$

is the standard Lorentz invariant phase space factor. Inserting these covariant expressions in the definition of the differential cross section gives the standard formula for the invariant cross section

$$d\sigma = \frac{(2\pi)^4}{4\sqrt{(p_t \cdot p_b)^2 - m_b^2 m_t^2}} \left| {}_{cov}\langle p_1, \cdots, p_n, \| M^{\alpha\beta} \| \bar{p}_b, \bar{p}_t \rangle_{cov} \right|^2$$

$$\times d\Phi_n(p_b + p_t; \mathbf{p}_1, \cdots, \mathbf{p}_n). \tag{246}$$

Because of the unitarity of the Wigner rotations and the covariance of

$$\left| {}_{cov}\langle p_1, \cdots, p_n, \| M^{\alpha\beta} \| \bar{p}_b, \bar{p}_t \rangle_{cov} \right|^2 \tag{247}$$

this becomes an invariant if the initial spins are averaged and the final spins are summed. In our model with spinless nucleon the total cross section is invariant.

This manifestly invariant formula for the cross section is identical to (233); in this form the invariant cross section can be evaluated in any frame. The index t refers to the target, which is in our case the deuteron.

References

[1] http://www.claymath.org/millennium/

[2] R. F. Streater and A. S. Wightman, *PCT, Spin And Statistics, And All That*, Benjamin/Cummings, Reading, Mass. (1964), Theorem 4.1.

[3] Franz L. Gross, "Relativistic Quantum Mechanics and Field Theory" (1999) - Science.

[4] Bert Schroer, arXiv:hep-th/0405105v4, 2004; arXiv:0711.4600v2 [hep-th], 2007

[5] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Birkhauser, 1983.

[6] E. P. Wigner, *Ann. Math.* **40**,149(1939).

[7] R. Haag, "Local Quantum Physics", Springer, 1982.

[8] K. Osterwalter and R. S. Schrader, *Comm. Math. Phys.* **42**,281(1975)

[9] V. Bargmann, *Ann. Math.* **59**,1 (1954).

[10] T.D.Newton and E.P.Wigner, *Rev. Mod. Phys.* **21**,400(1949).

[11] A. S. Wightman in *Relations de Dispersion et Particules Élémentaries*, ed. C. De Witt and R. Omnés, Hermann, Paris (1960).

[12] L.Lubanski, *Physica (Utrecht)* **9**,310(1942).

[13] B. Bakamjian and L. H. Thomas, *Phys. Rev.* **92**,1300 (1953).

[14] F. Coester, *Helv. Phys. Acta*, **38**,7(1965).

[15] S. N. Sokolov, *Sov. Phys. Dokl.* **233**,575(1977).

[16] F. Coester and W. N. Polyzou, *Phys. Rev. D* **26**,1348(1982).

[17] P. A. M. Dirac, *Rev. Mod. Phys*, **21**,392(1949).

[18] W. Polyzou and W. Klink, *Ann. Phys.* **185**,369(1988).

[19] K. G. Wilson, Erice Lectures, 1975, edited by A. Zichichi, Plenum Press, New York.

[20] E. Seiler, "Gauge theories as a problem in constructive quantum field theory and statistical mechanics", Springer 1982.

[21] R. F. Dashen, J. B. Healy, I. J. Muzinich, Ann. Physics, 102,1,(1976).

[22] N. N. Bogoliubov and D. V. Shirkov, "Introduction to the Theory of Quantized Fields" (Interscience Monographs in Physics and Astronomy) 1959.

[23] H. J. Melosh, *Phys. Rev. D* **9**,1095(1974).

[24] B.D. Keister, W. N. Polyzou, Phys. Rev. C **73**, 014005 (2006).

[25] C. Chandler and A. Gibson, *Indiana J. Math.* **25**,443(1976).

[26] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, American Mathematical Society Colloquium Publications, Vol 31, Providence, Rhode Island (1957), sec 4.8

[27] W. Brening and R. Haag, *Fort. der. Physik*, **7**,183(1959).
Mathematical Scattering Theory, Birkhauser, 1983.

[28] Particle Data Group - W.-M. Yao et al., Journal of Physics, G 33, 1 (2006).