# Exploring the Schwinger-Dyson Equations of the gauge sector of QCD

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Jefferson Lab Cake Seminar

Work in collaboration with M.R. Pennington

# Hadronic Physics

- The physics of hadrons widely thought to be described by QCD at strong coupling.
- However, calculating anything from the fundamental Lagrangian in this limit is difficult.
- Many puzzles still remain,
  - The mechanism of confinement.
  - Dynamical Symmetry Breaking.
  - The particle composition of some resonances.
- Important for checking the standard model,
  - Flavour Physics observables used to calculate CKM elements.
  - Hadronic effects currently the largest theoretical unknown in $g_\mu - 2$.

# Talk Plan

1. Introduction to SDEs of QCD
2. Historical stuff – previous attempts at solving the SDEs.
3. Attempts at improving the truncation and the issues encountered.
4. Compare with Lattice
5. Outlook

# Strongly-coupled QCD

$$\mathcal{L} = \bar{\psi}^a \left( i\not{D}_{ab} - m \right) \psi^b - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{2\xi} \left( \partial_\mu A^\mu_a \right)^2 + \left( \partial^\mu \bar{c}_a \right) D^{ab}_\mu c_b$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$D^{ab}_\mu = \delta^{ab} \partial_\mu - g f^{abc} A^c_\mu$$

## Confinement and Hadronisation

- How is confinement realised in nature?
- How do the fields of the lagrangian combine to form hadrons?

## Nonperturbative Mass corrections

- What is the mechanism behind dynamical chiral symmetry breaking? $m(p^2) = m_0 \left( 1 + \alpha_s(\mu^2) c_1 \mathrm{Log}\left( p^2/\mu^2 \right) + \mathcal{O}(\alpha_s^2) \right)$
- Is there mass generation in QCD even if $m_0 \to 0$?

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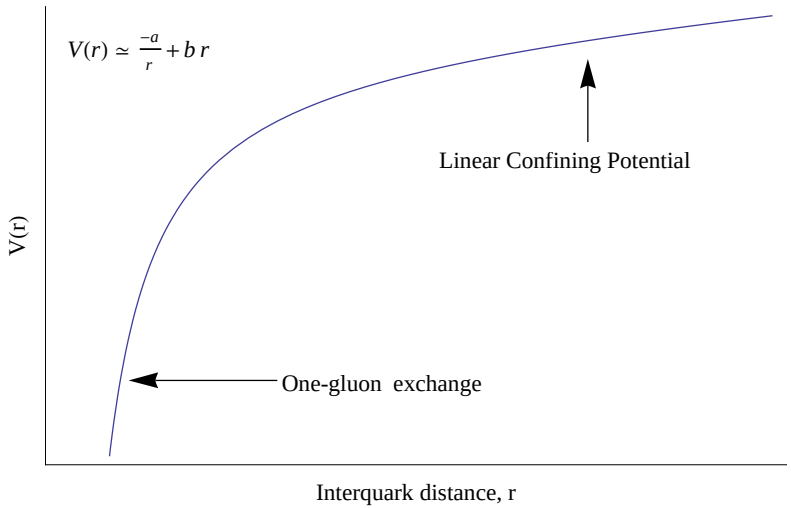
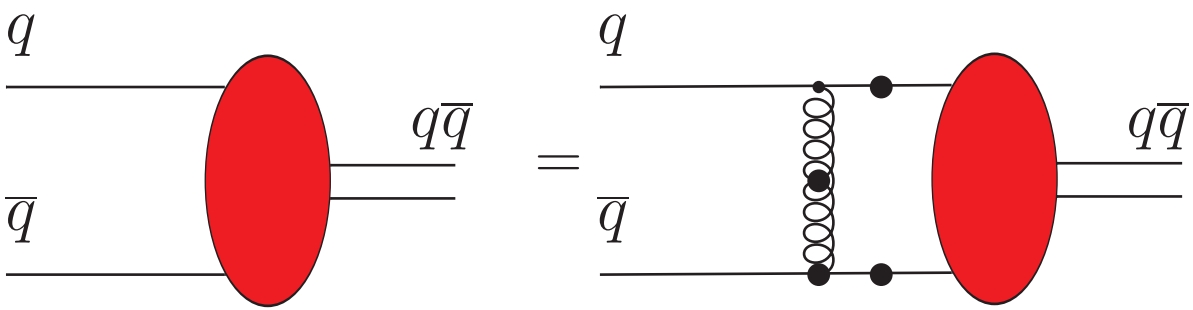
# Bound state studies

- SDEs are a necessary input for the Bethe–Salpeter Equation of mesons.
- Can in principle obtain all of the dressing functions of a quantum field theory from SDEs.
- The dressed Gluon and Quark propagators and the dressed Quark-Gluon vertex are important inputs in BSE studies.



# Schwinger-Dyson Equations

- Field equations of a QFT – can expand in $g$ to give perturbation theory.
- Valid non-perturbatively. Need to understand NP physics to understand how the particles of the Lagrangian become hadrons.
- Infinite tower of equations - 2 point Greens function depends on 3 and 4 point Greens functions. The 3 and 4 point Greens functions also depend on higher Greens functions, and so on...

- Always need to truncate the tower somewhere.
- Essential that the truncation respects gauge invariance to be physically meaningful.
- Neglect higher contributions or model them in a sensible way.
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- Multiplicative Renormalisability gives additional constraints.

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# Dressed propagators

Renormalisation provides an all-orders constraint on the structure of radiative corrections, an additional 'dressing function' multiplies the bare propagators,

$$D_{\mu\nu}(p) = \frac{G\ell(p^2)}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^4}$$

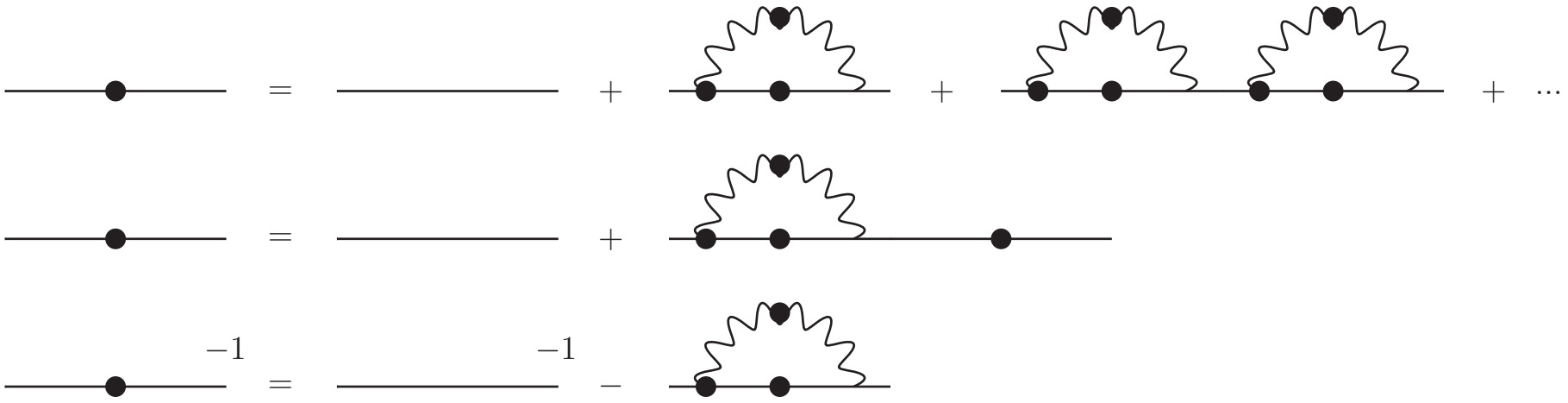
$$D^{-1}_{\mu\nu}(p) = \frac{p^2}{G\ell(p^2)}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{p_\mu p_\nu}{\xi}$$

$$D(p) = -\frac{Gh(p^2)}{p^2}$$

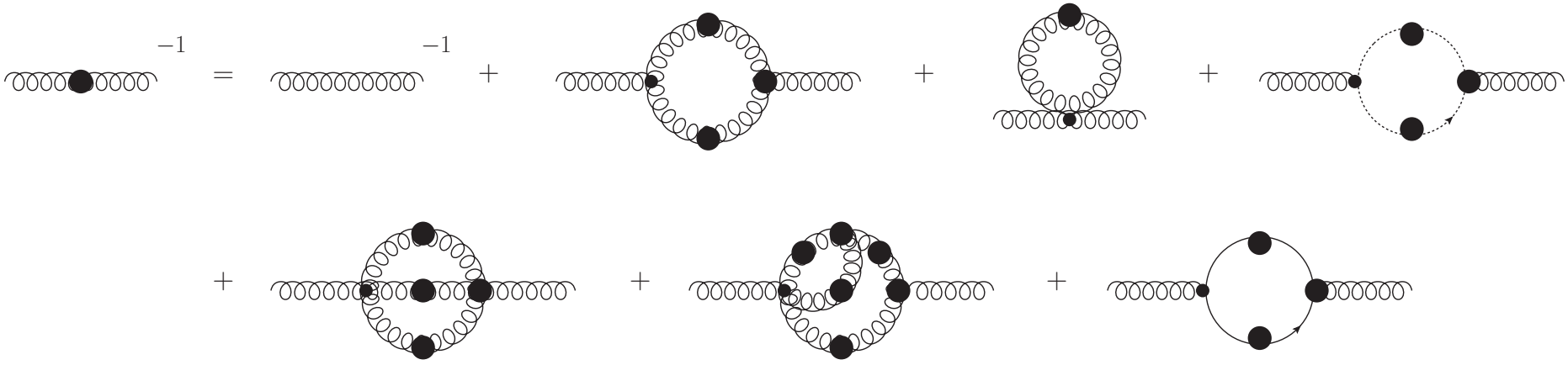
The new functions $G\ell(p^2)$ and $Gh(p^2)$ contain the non-perturbative effects. Setting these to 1 recovers the usual propagators. This simple form is a result of applying the Ward-Slavnov-Taylor Identities

# Graphical origin of the SDEs

For a fermion propagator in QED



# The Gluon SDE




Heavy dots correspond to fully dressed vertices and propagators, including all loop effects.
Light dots and normal propagators are those that we use in perturbative QCD.

- Couples gluons to gluons, ghosts and fermions simultaneously.
- Contains (superficially) quadratically divergent terms in individual diagrams.
- The dressed two-loop integrals are very tricky numerically – have never been solved exactly, only in certain limits.

# Results neglecting Ghosts

## First Numerical Studies

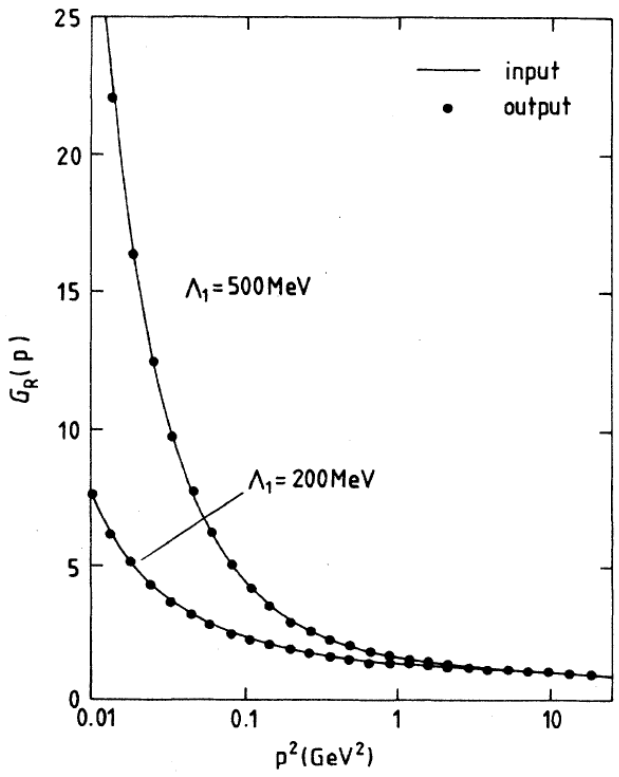
- Gluon dressing is singular in the IR
- Dressing function $G\ell(p^2) \sim p^{-2}$ in the IR.

→ Dressed propagator singular as $p^{-4}$.

→ A fourier transform of this propagator leads to a linearly rising confining potential for large distances.

- Similar to those found to work in potential model studies of heavy mesons.

Caveat: In Landau gauge, the Gluon propagator is neither gauge invariant nor an experimental observable.

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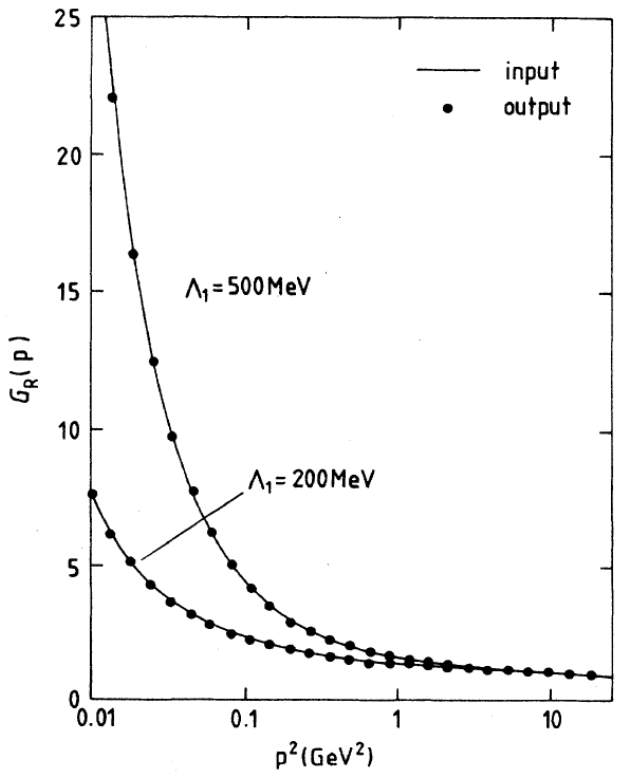


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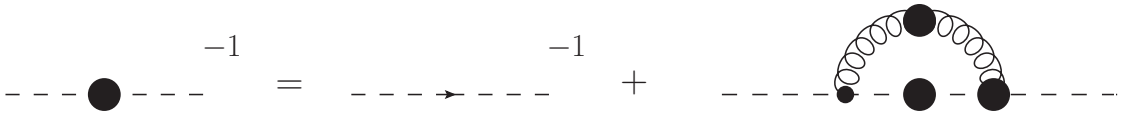
# A Typical Truncation

Refs: Alkofer, Fischer, von Smekal, *et al*

# The Ghost SDE




$$\frac{p^2}{Gh(p^2)} = p^2 + \frac{g^2 N_c}{(2\pi)^4} \int d^4\ell \, K(p^2, \ell^2, \theta) \frac{Gh(\ell_+^2)}{\ell_+^2} \frac{G\ell(\ell_-^2)}{\ell_-^2}$$

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- All recent SDE studies use covariant gauges – Ghosts are present.
- One vertex is always bare in these types of diagrams to avoid double counting.

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→ Not known at present – must introduce a truncation to approximate the vertex.

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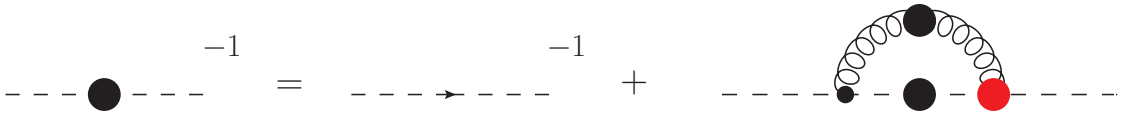


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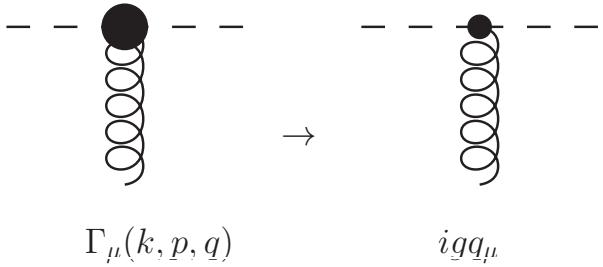
# Truncating the Ghost Eq.

- Replace full ghost-gluon vertex with bare ghost-gluon vertex

→ Use $\Gamma_\mu(k, p, q) = igq_\mu$




The most general form for this vertex is $\Gamma_\mu(k, p, q) = iq_\mu\alpha(k, p, q) + ik_\mu\beta(k, p, q)$. Using the bare vertex is equivalent to setting $\alpha = 1$ and $\beta = 0$ everywhere.

## Justification

→ Taylor's Non–Renormalisation Theroem: $\tilde{Z}_1 = 1$.

→ In Landau gauge this vertex reduces to the bare form for vanishing incoming ghost momentum.

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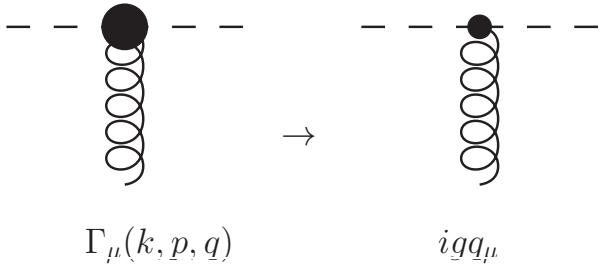


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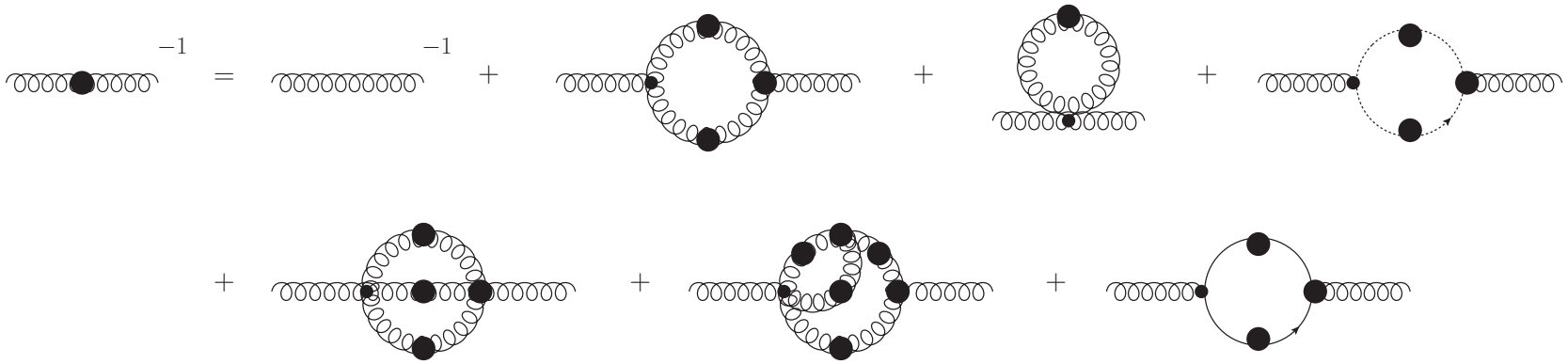


1. Drop quark interaction for now
   - Set $N_f = 0$, still have SU(3) YM theory.
   - Useful to compare with lattice QCD.
   - (Hopefully) Straightforward to add quarks later.
2. Full solutions need three– and four–gluon vertices.
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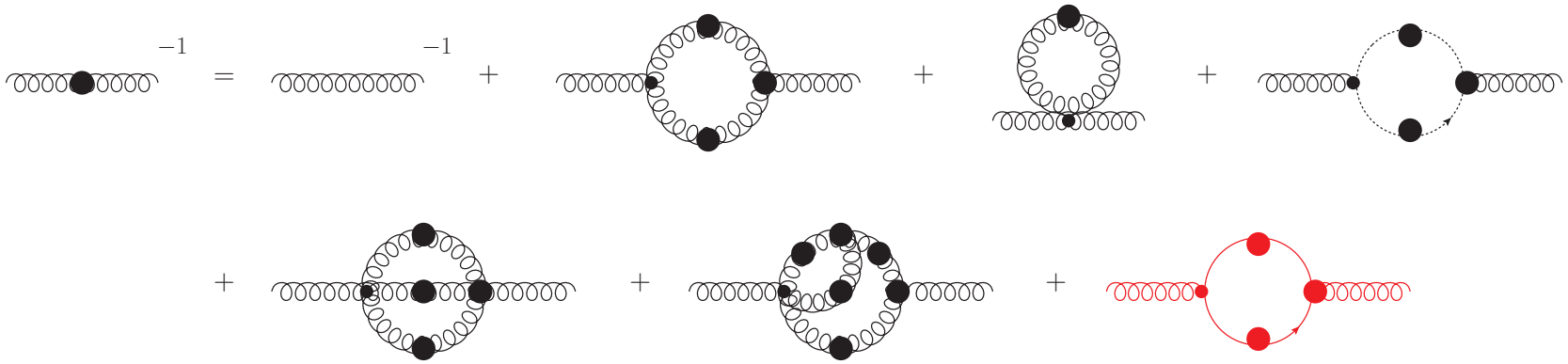


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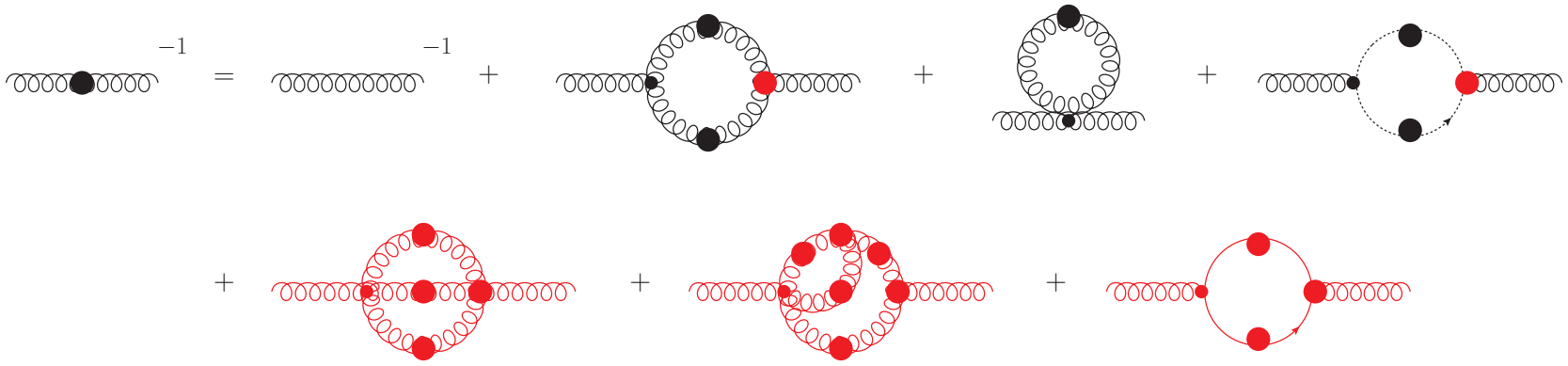


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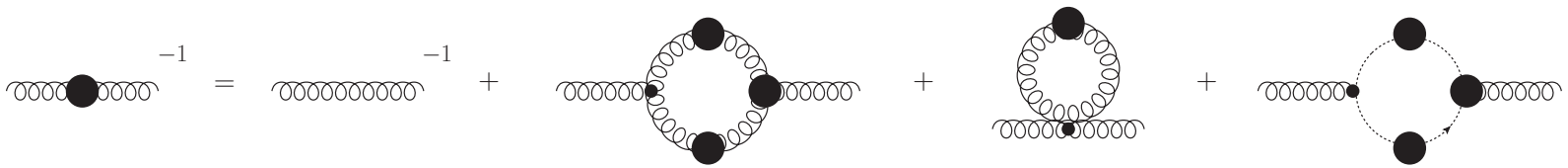


## Several sources of information are available

1. WSTIs suggest ratios of dressings of the form $\mathcal{G}h/\mathcal{G}\ell$.
2. The vertex should be Bose-symmetric.

→ Symmetric WSTI forms typically violate the propagtor one-loop running by some small amount.

3. Vertex should preserve the one-loop propagator results.

→ The form,

$$\frac{\mathcal{G}h(k^2)\mathcal{G}h(q^2)}{\mathcal{G}\ell(k^2)\mathcal{G}\ell(q^2)} \times (\text{bare vertex})$$

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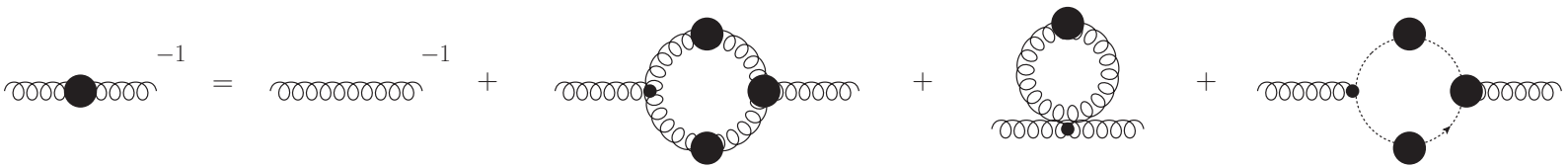


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# Confinement Scenarios $\rightarrow$ *IR Ghost enhancement?*

## Kugo-Ojima Confinement Criterion

- Unbroken BRST symmetry $\rightarrow$ Ghost more singular than a simple pole.
- Confinement restricts the physically allowed state space of QCD. Must separate physical and unphysical states.
- They consider BRST symmetry and the global charge and show that part of the state space contains only colour singlets.

Leads to the requirement on $D(p^2) = -\frac{Gh(p^2)}{p^2} = -\frac{1}{p^2}\frac{1}{1+u(p^2)}$ that $u(0) = -1$. Suggests the ghost propagator should be more singular than the bare propagator.

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Theoretical suggestion from the confinement scenarios:

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# Ghost equation IR solutions

## Analytic IR Power law solutions

Using a power law assumption for the deep infrared region, we can see if any solutions exist.

$$G\ell(x) = ax^{\kappa_1} \text{ and } Gh(x) = bx^{\kappa_2}$$

$$\frac{1}{bp^{2\kappa_2}} = -\frac{g^2 N_c}{(2\pi)^4} \int d^4\ell \, K(p,\ell) \frac{a\ell_+^{2\kappa_1}}{\ell_+^2} \frac{b\ell_-^{2\kappa_2}}{\ell_-^2}$$

## Self–Consistent IR power law conditions

- Matching powers on both sides leads to two solution types,

1. $G\ell_{IR}(x) = ax^{2\kappa}$, $Gh_{IR}(x) = bx^{-\kappa}$,
2. $G\ell_{IR}(x) = ax^{2\kappa}$, $Gh_{IR}(x) = b$, with $\kappa > 0$.

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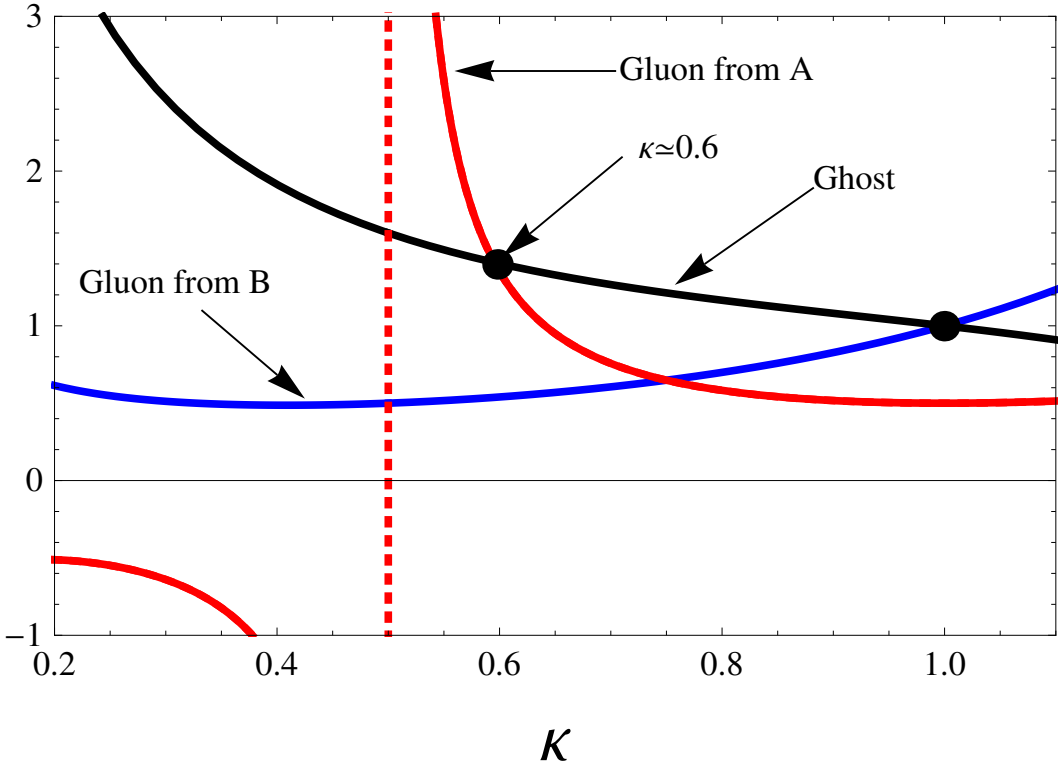
# Gluon IR Power–Law analysis

Similarly to the ghost equation, the deep IR region may be solved using a power–law. Value(s) of κ that simultaneously solve the equations in this limit may then be identified.

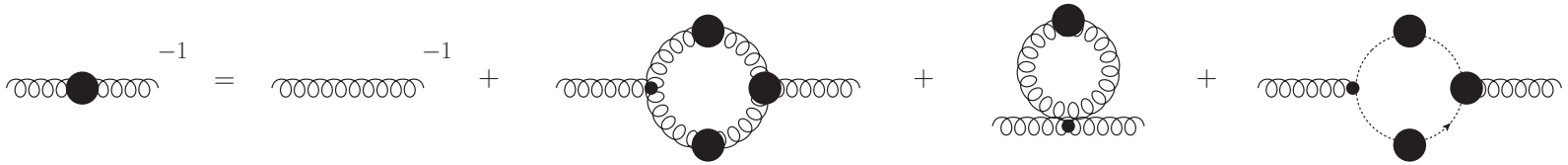



The intersections with the ghost curve (black) correspond to consistent values of κ in the IR.

refs: Alkofer *et al*, Bloch

# Solving the Gluon equation Numerically

We want to solve,

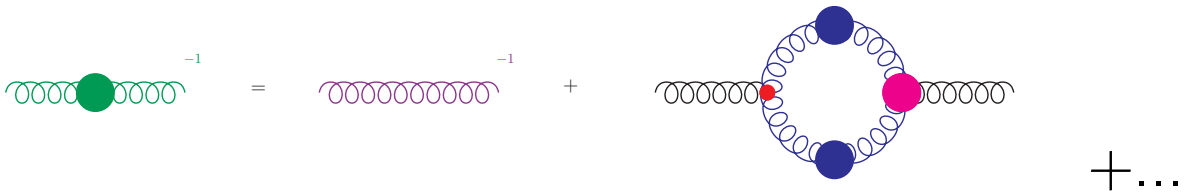


$$D^{-1}_{\mu\nu}(p) = D^{(0)\,-1}_{\mu\nu}(p) + \Pi^{(2g)}_{\mu\nu}(p) + \Pi^{(1g)}_{\mu\nu}(p) + \Pi^{(2c)}_{\mu\nu}(p).$$

- Contract tensor structure and remove quadratic divergences
  - Apply $\mathcal{P}_{\mu\nu}(p) = g_{\mu\nu} - \zeta \frac{p_\mu p_\nu}{p^2}$
  - This automatically removes quadratic divergences for $\zeta = d$
  - Retains a transverse gluon.
  - For $\zeta = 1$ has been frequently used by other authors.
- Each $\Pi_{\mu\nu}$ contains a loop integration
  - Tadpole term never contributes – no dependence on external momentum.
  - The ghost loop and gluon loop can be performed numerically, using a cutoff.
- Renormalise
- Solve for dressing functions self–consistently.

# Solving the Gluon equation Numerically (2)

Following these steps we obtain



$$D^{-1}_{\mu\nu}(p) = D^{(0),-1}_{\mu\nu}(p) + \int \frac{d^4k}{(2\pi)^4} \Gamma^{(0)}_{\mu\rho\sigma}\, D^{\rho\eta}(k_+)\, \Gamma_{\eta\phi\nu}\, D^{\phi\sigma}(k_-) + \ldots$$

$$G\ell(p^2)^{-1} = Z_3 + \frac{g^2 N_c Z_1}{3(2\pi)^3} \int_0^{\kappa^2} k^2 dk^2 \int_0^{\pi} d\theta \sin^2\theta\, Q(p,k)\Gamma(p,k) \frac{G\ell(k_+^2)}{k_+^2} \frac{G\ell(k_-^2)}{k_-^2} + \ldots$$

- The ghost–loop diagram in this equation is included in exactly the same way.
- May now solve for the dressing functions using an iterative procedure until input and outputs are numerically consistent.

# Fixing Renormalisation Point and Subtraction

Fix renormalisation constants $Z_3$ and $\tilde{Z}_3$ using a Momentum Subtraction (MOM) scheme,

**Eliminate Z's by subtracting equation from itself at some point.**

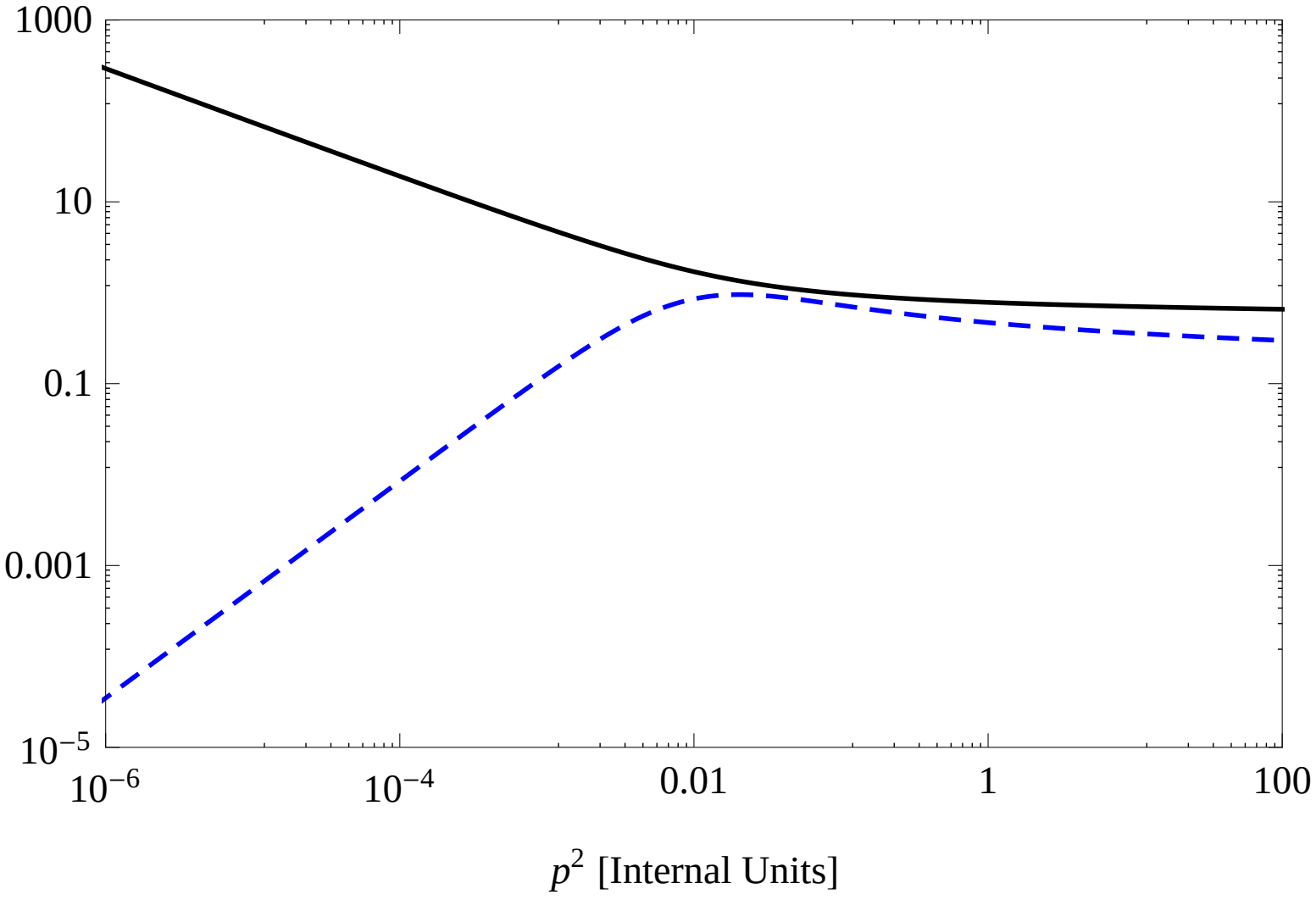
$$\begin{aligned} G\ell^{-1}(p^2) &= Z_3 + \Pi(p^2) \\ &= G\ell^{-1}(\mu^2) + \Pi(p^2) - \Pi(\mu^2) \end{aligned}$$

**Specify the value of the dressing function at that point.**

$$Gh(\mu^2) = 1 \text{ and } G\ell(\mu^2) = 1.$$

- If we renormalise at a perturbative point, eg $\mu = M_Z$, the renormalisation factors should take their perturbative values.
- Subtraction point is not necessarily fixed to the renormalisation point.
- For an IR divergent dressing function it is necessary to subtract at zero momentum.

# Infinite IR solutions




Ref: Alkofer and Fischer, hep-ph/0301094.

# Extending the Truncation

- So far this has been largely historical.
- Recent studies by Kondo and a reanalysis by Zwanziger has cast doubt on the singular solutions.
- It appears that both types may be possible.

$\rightarrow$ A finite ghost solution has been suggested with a similar vanishing gluon.

# A inconsistency in the truncation

$$D_{\mu\nu}(p) = \frac{G\ell(p^2)}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

## Transversality

The inverse gluon propagator has two tensor structures, however in Landau gauge the gluon must be transverse so they are simply related,

$$\begin{aligned}\Pi_{\mu\nu}(p) = D^{-1}_{\mu\nu}(p) &= A' p^2 g_{\mu\nu} + B' p_\mu p_\nu \\ &= A'\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) + \left(B' + A'\right) p_\mu p_\nu \\ &= \left(A p^2 + C\lambda^2\right) g_{\mu\nu} + \left(B + 1/\xi\right) p_\mu p_\nu\end{aligned}$$

- The type 1. solutions were produced **assuming** $B' + A' = 0$.

→ But self-consistent solutions don't exist with this constraint.

- Since the $C\lambda^2$ isn't known *a priori* for all momenta then it is more sensible to determine $B$.
- Result is only transverse if $A + B = 0$. This is guaranteed when solving for $B$.

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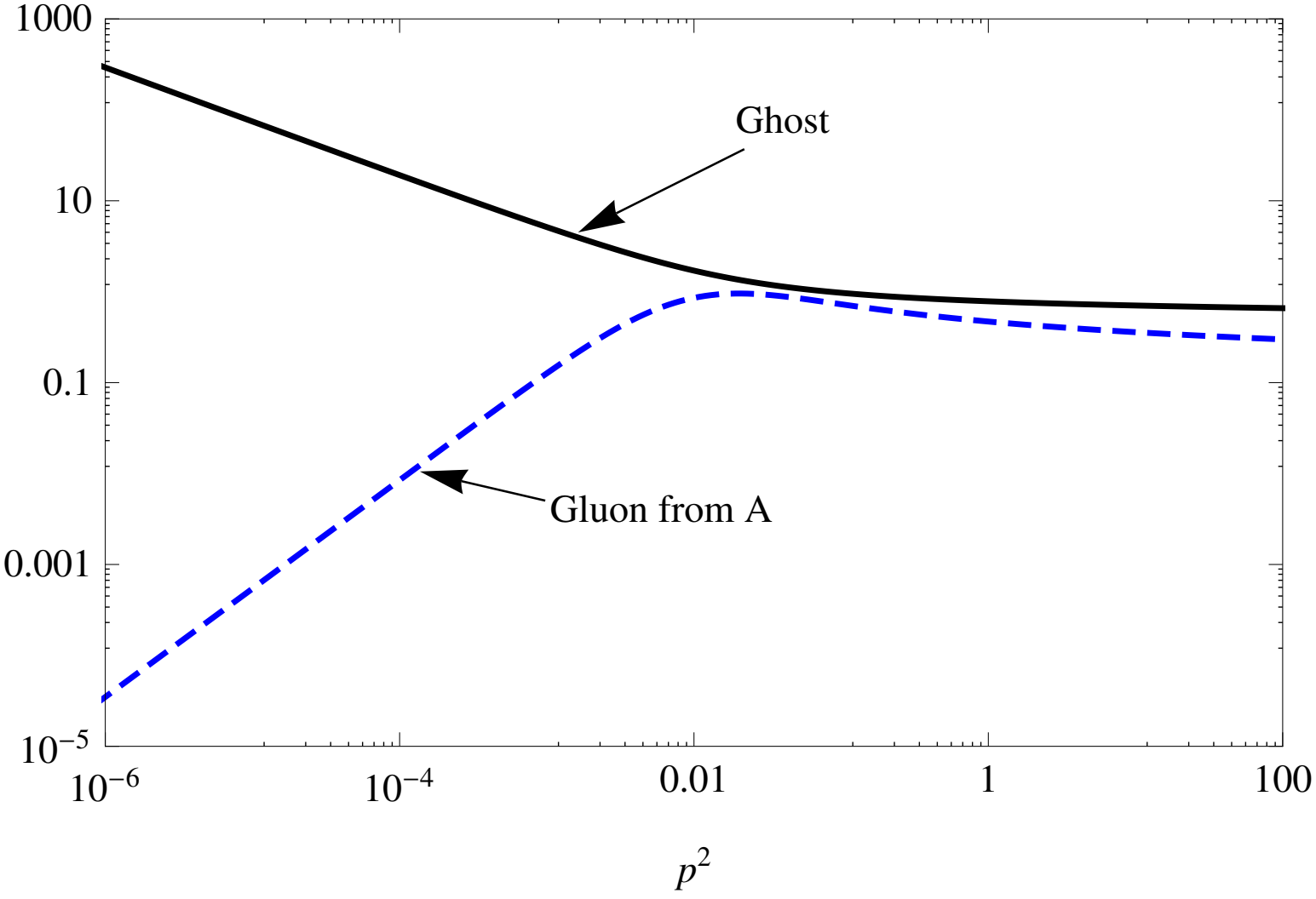


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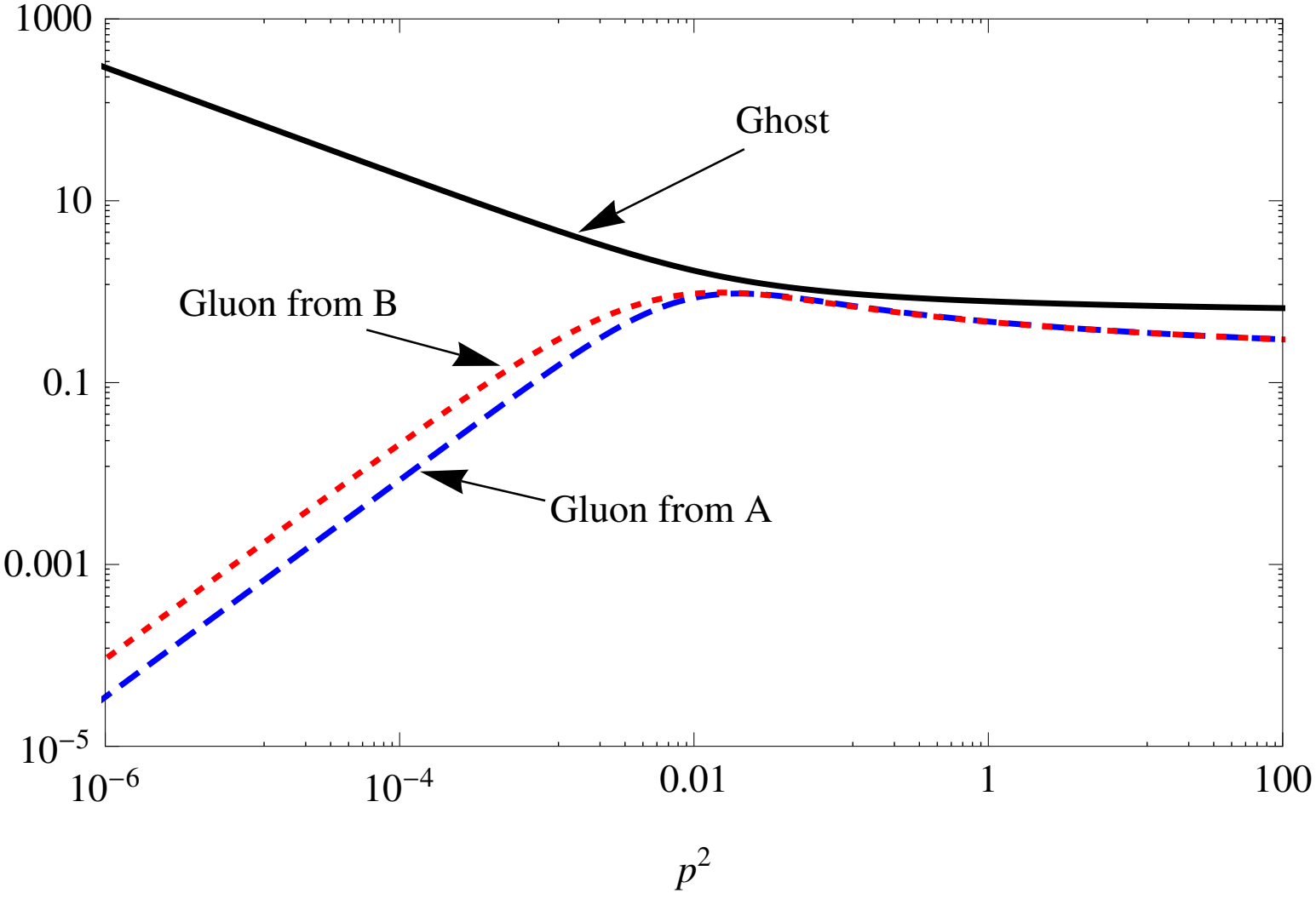



Ref: Alkofer and Fischer, hep-ph/0301094.

# Two types of Ghost solution

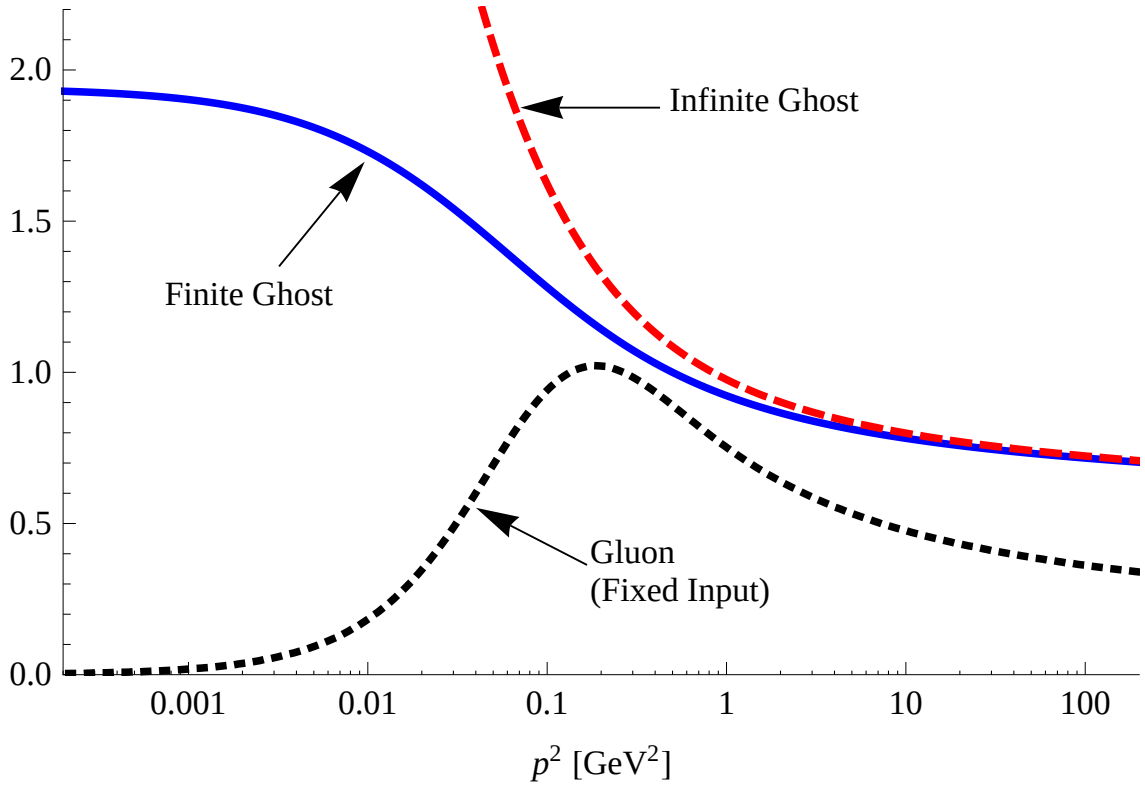
It turns out that the full ghost equation with a very wide range of vertices is compatible with a finite IR solution. This form is somewhat nicer since it is compatible with the perturbative renormalisation (and subtraction). Whereas the infinite solution requires the ghost equation to have a value specified in the IR ($\mathfrak{Gh} = \infty$ at $p^2 = 0$), such that the values in the perturbative region are implicit and no longer free.

The gluon equation however needs more work. Can fix the gluon and solve the ghost alone to see this.

# Two types of Ghost solution

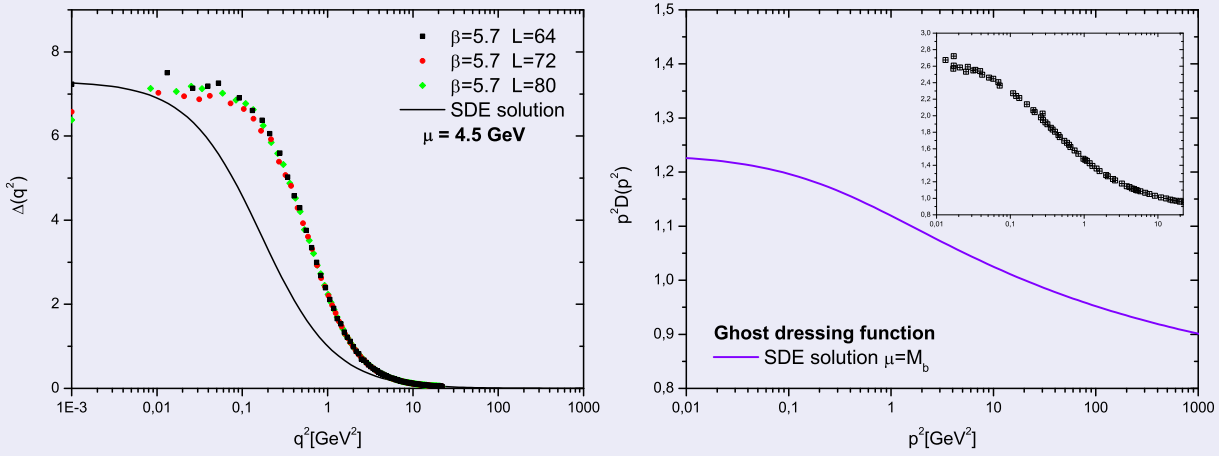
...from matching powers LHS and RHS in the ghost equation

1. $G\ell_{IR}(p^2) = a(p^2)^{2\kappa}$, $Gh_{IR}(p^2) = b(p^2)^{-\kappa}$,
2. $G\ell_{IR}(p^2) = ap^2$, $Gh_{IR}(p^2) = b$, (have set $\kappa = 0.5$).



# Why can't we solve the Gluon Equation for a finite ghost?

Has been done by Binosi & Papavassiliou in PT-BFM truncation.



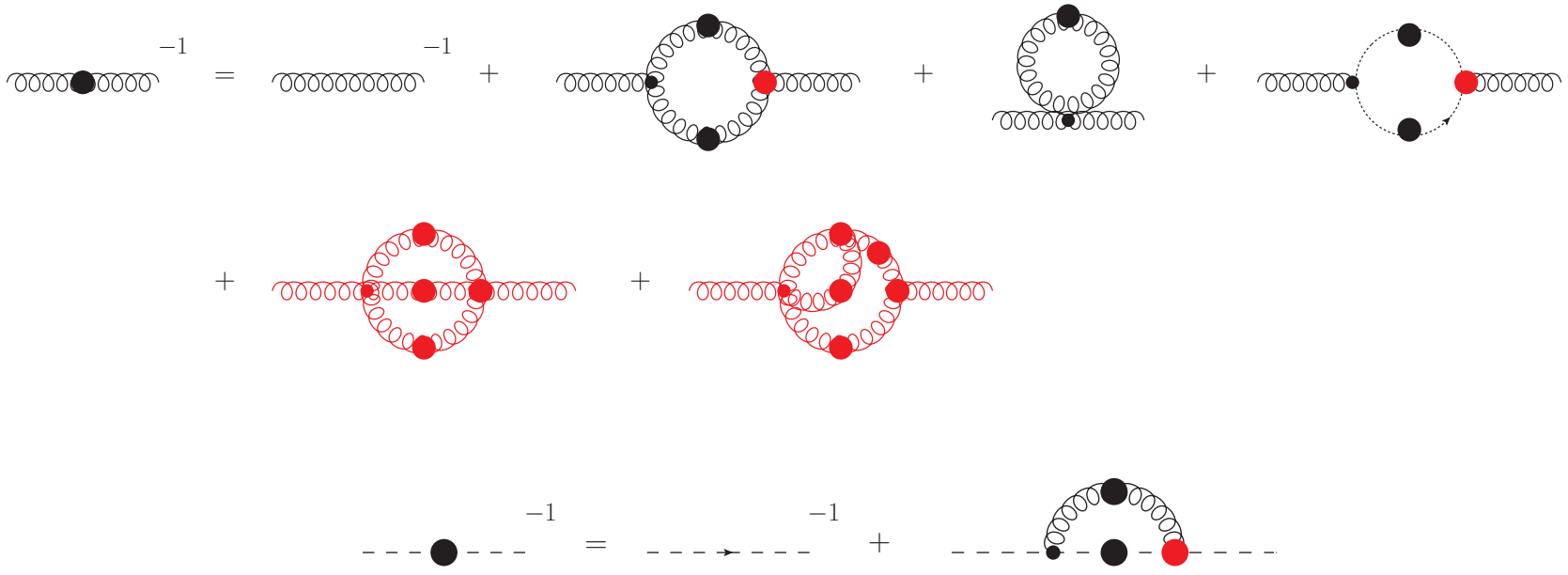

Left: $G\ell(p^2)/p^2$. Right: $Gh(p^2)$

- What about in standard Landau gauge QCD?

→ Practically, using bare vertices the Gluon dressing function wants to turn negative.

Fig Ref: Binosi and Papavassiliou arXiv:0909:2536

# Improving the Truncation




## Potential problems

- Neglects the **dressed two-loop diagrams** – it is thought that these have only have a quantitative effect in the perturbative region.
- Doesn't satisfy Ward–Slavnov-Taylor identities for either vertex.
- Uses the **bare ghost–gluon vertex**.

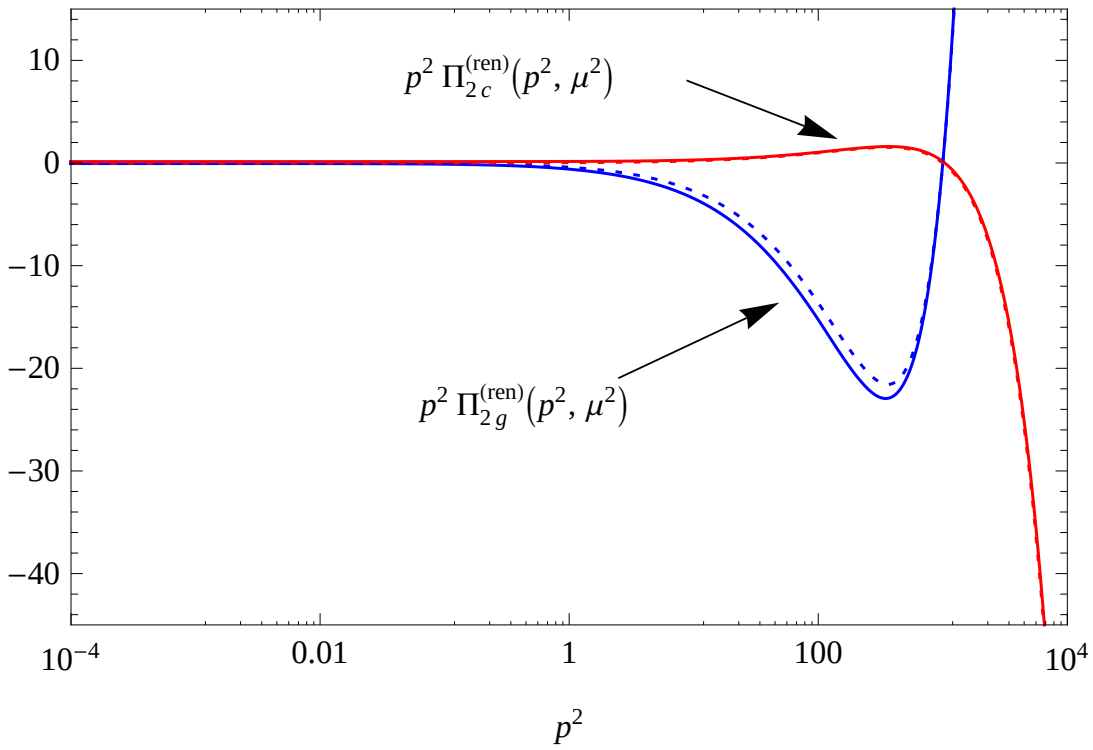
# Gluon SDE structure

**In the one–loop only truncation**

$$G\ell^{-1}(p^2) = G\ell^{-1}(\mu^2) + \Pi_{2c}(p^2, \mu^2) + \Pi_{2g}(p^2, \mu^2)$$

The failure of the numerical procedures to find self–consistent solutions is because $Gl(p^2)$ must remain positive for all $p^2$, however, the gluon loop term $\Pi_{2g}(p^2, \mu^2)$ is typically negative in the IR which must be more-than-cancelled by $G\ell^{-1}(\mu^2) + \Pi_{2c}(p^2, \mu^2)$. For the simple tree-level vertices this is not the case.

# Looking at the components of the Gluon Eq.




- Must satisfy $G\ell(p^2)^{-1} = G\ell(\mu^2)^{-1} + \Pi_{2c}(p^2,\mu^2) + \Pi_{2g}(p^2,\mu^2) > 0$
- Broken in the IR by the bare ghost-gluon vertex in the gluon equation.
- In one-loop-only truncations usually: $\lim_{p^2\to 0}\Pi_{2g}(p^2,\mu^2) < 0$.
- Must require:

$$\lim_{p^2\to 0}\Pi_{2c}(p^2,\mu^2) > 0\,.$$

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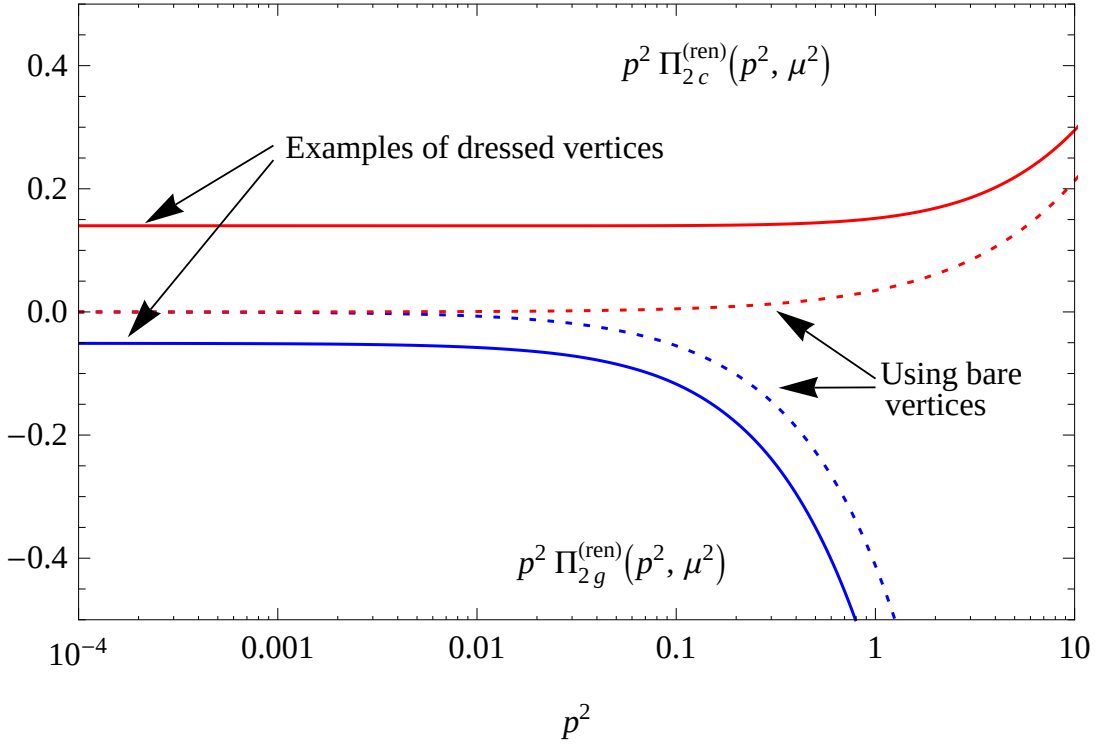


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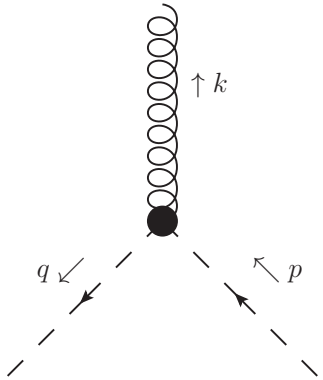
# Ghost–Gluon vertex structure

In Landau gauge the Ghost–Gluon vertex is remarkably simple,

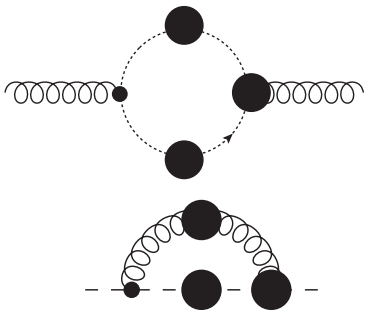



- Non-renormalisation: $\tilde{Z}_1 = 1$
- Reduces to tree-level form for vanishing incoming ghost momenta $\Gamma_\mu(-q, 0, q) = iq_\mu$

The most general vertex is,

$$\Gamma_\mu(k^2, p^2, q^2) = iq_\mu \alpha(k^2, p^2, q^2) + ik_\mu \beta(k^2, p^2, q^2)$$

- At one–loop order, and higher, in perturbation theory $\beta \neq 0$.
- $\beta$-term only appears in the ghost loop of the gluon equation.
- Drops out of loop in the ghost equation.



# Extended ghost-gluon vertices

$$\Gamma_\mu(k^2, p^2, q^2) = iq_\mu \alpha(k^2, p^2, q^2) + ik_\mu \beta(k^2, p^2, q^2)$$

## Taylor's Theorem

- Taylor's statement is actually: $\alpha - \beta = 1$ when $p = 0$.
- Can have significant structures that cancel between the terms, and these can lead to self-consistent solutions.

## A transverse vertex?

A transverse form has been suggested that does provide the desired contributions from the ghost loop term in the gluon equation by adding the second term $\beta$. This violates Taylor's theorem in the IR however we can demand that it is satisfied by adding compensating terms into $\alpha$. This works and self-consistent solutions may be found.

Refs: Boucaud *et al*, Fischer *et al*.

# Extended ghost-gluon vertices

The form of the vertex in existence already known to work in that it gives the correct contribution in the IR is derived by insisting that the vertex be transverse when all momenta are small,

$$k^{\mu}\Gamma_{\mu}^{(IR)}(k,p,q) = 0$$

where $k$ is the gluon momentum.

Using $\alpha = 1$ this leads to,

$$\Gamma_{\mu}(k,p,q) = q_{\mu} - \frac{k.q}{k^2}\mathcal{F}_{IR}k_{\mu}$$

There ghost-loop alone doesn't need to be transverse, only the combination of graphs.

In order to satisfy Taylor's theorem we use,

$$\Gamma_{\mu}(k,p,q) = q_{\mu} - \frac{k.q}{k^2}\mathcal{F}_{IR}(k+q)_{\mu}$$

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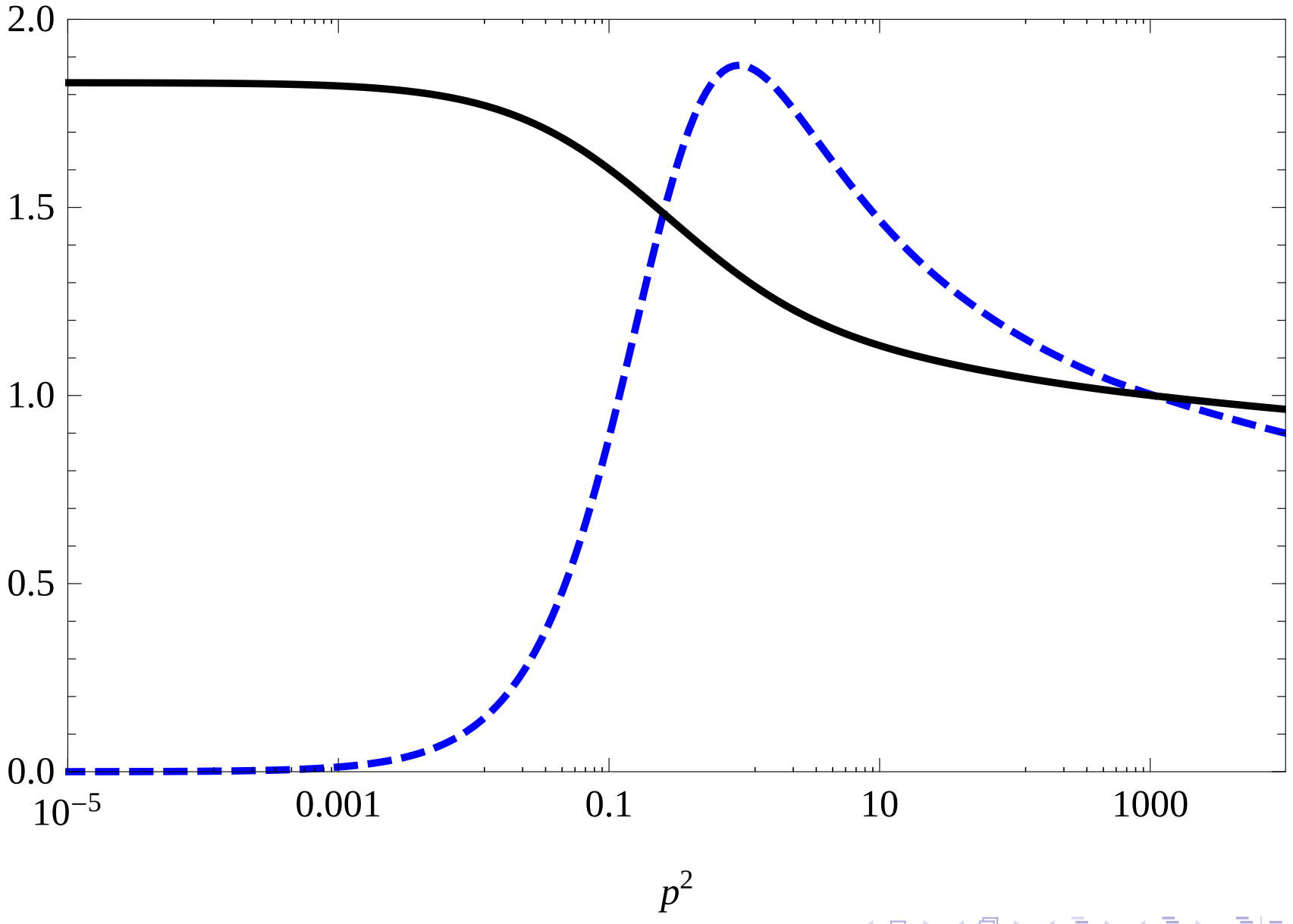
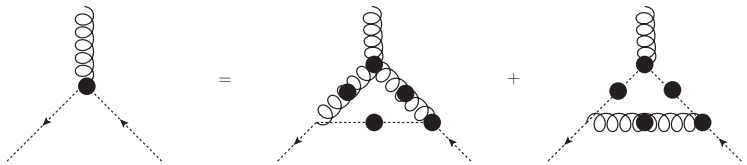
# Self-consistent solutions



# Ghost–Gluon vertex WSTI constraint and Vertex SDE

- *Bottom-up* type analysis is sufficient to obtain self-consistent solutions.
- Desirable to have a vertex derived *top-down* from field theory.

1. Can use WSTI.
2. Alternatively can solve the vertex schwinger–dyson equation.



## Vertex SDE

- Interesting structure when projecting out $\alpha$ and $\beta$ components,

$\rightarrow$ $\alpha$ term does not depend on $\beta$ term.

$\rightarrow$ Can solve $\alpha$ self-consistently given a propagator input only.

$\rightarrow$ $\beta$-term can then be found with a fixed $\alpha$

$\rightarrow$ Considerably simpler than solving for both functions simultaneously.

This work is still ongoing...

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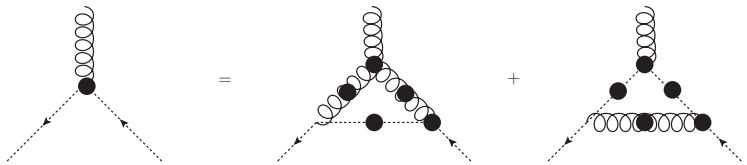

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# Ghost–Gluon vertex WSTI constraint and Vertex SDE

## WSTI

$$\frac{p_\mu \Gamma^\mu(p,k,q)}{Gh(p^2)} - \frac{k_\mu \Gamma^\mu(k,p,q)}{Gh(k^2)} = \frac{q^2}{Gh(q^2)}$$

WSTI constrains only the longitudinal part ($\beta$-term) of the vertex

$$\Gamma_\mu(k,p,q) = iq_\mu \alpha(k,p,q) + ik_\mu \left( \gamma(k,p,q) - \frac{k.q}{k^2} \alpha(k,p,q) \right)$$

then only the $\beta$-term contributes. Some very simple solutions exist,

$$\gamma(k,p,q) = \frac{k.q}{k^2} \frac{Gh(k^2)}{Gh(q^2)}$$

however these don't give self–consistent solutions in this truncation.

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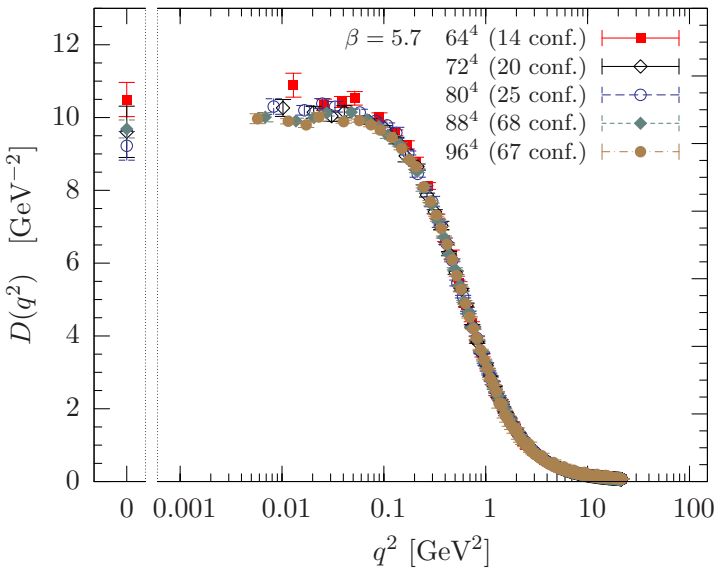
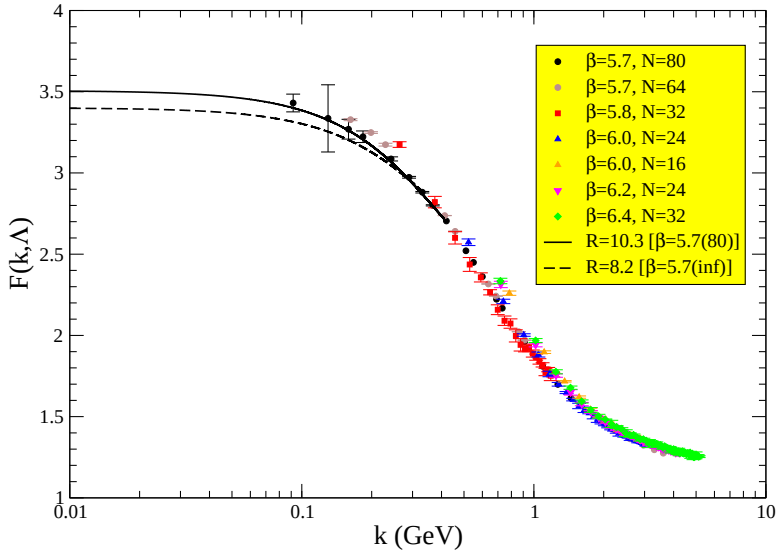
# Current Lattice status



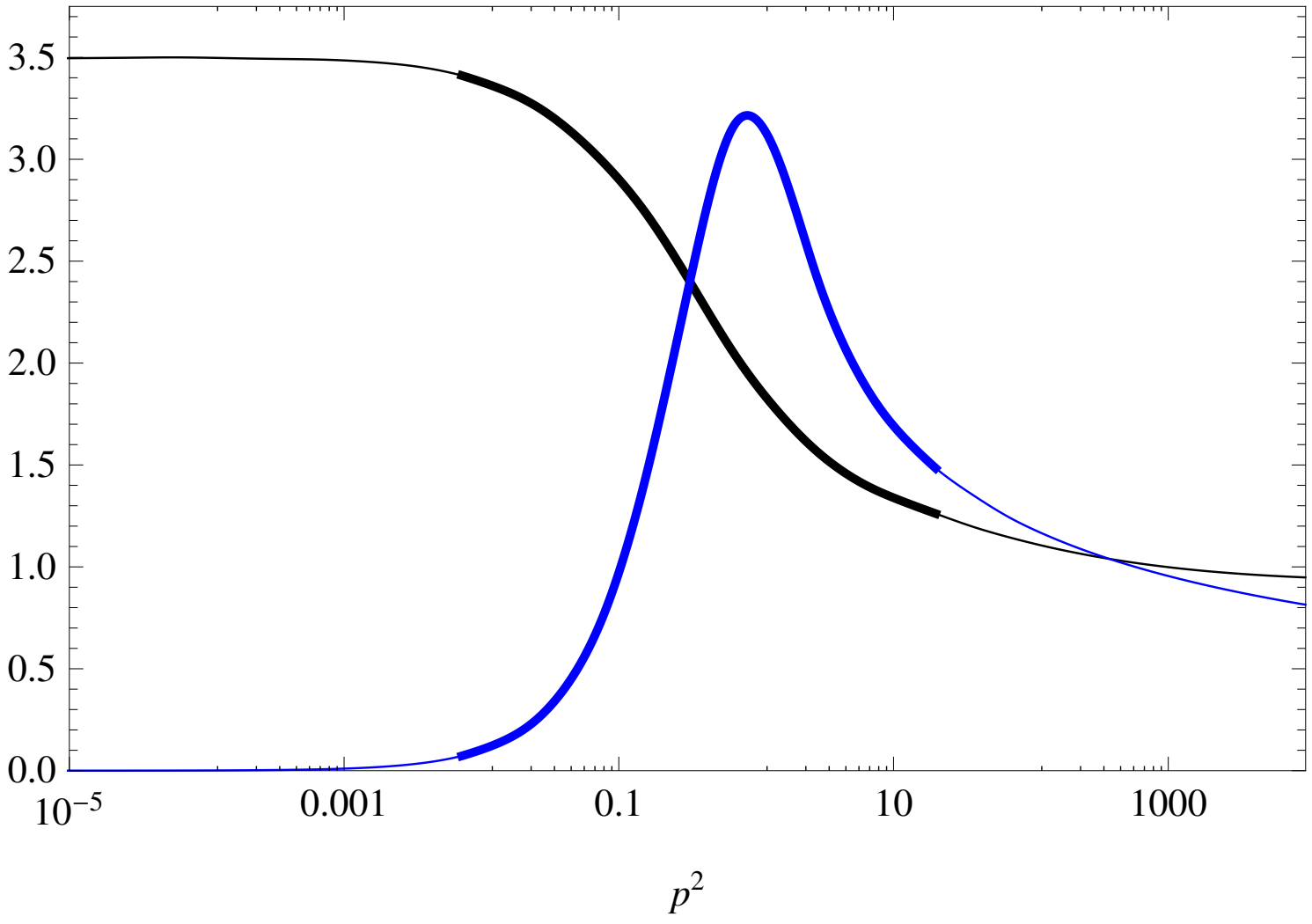

Left: The Gluon $D(p^2) = G\ell(p^2)/p^2$ appears to go to a constant.
Dressing is $G\ell(p^2) = \text{const} \times p^2$ in the IR.

Right: $F(p) = Gh(p^2)$. Ghost dressing function weakly increasing - weaker than $(p^2)^{-0.6}$.

- These solutions should be directly comparable. They are performed using the quenched approximation and in Landau gauge.
- Lattice solutions may suffer finite volume effects and finite lattice spacing effects (Always have $V < \infty$ and $a > 0$).
- Lattice QCD may be more susceptible to effects due to Gribov copies.

Refs: Bogolubsky *et al*, arXiv:0901.0736, Pene *et al*, arXiv:0911.0468.

# Direct Comparison with Lattice



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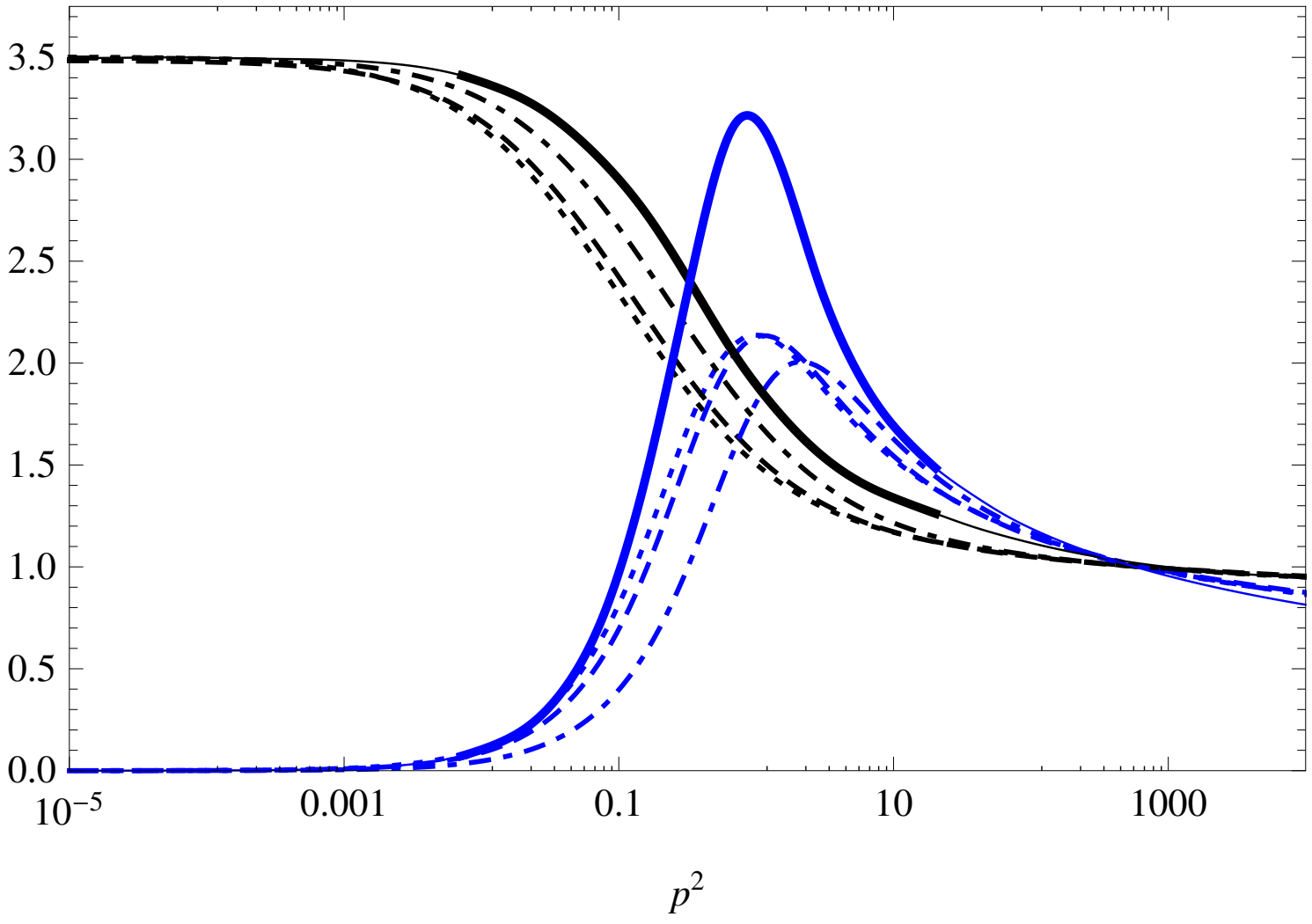


# Conclusions

- The strongly coupled gauge sector of QCD has been studied via its Schwinger-Dyson equations.
- In one–loop–only truncations we have found that a bare ghost–gluon vertex is insufficient.
- Simple vertex dressings have been found to reproduce results similar to those found in lattice QCD.
- We would like to find a ghost-gluon vertex derived from fundamental principles.
- Precise knowledge of the vertex is critical to give precise predictions for the gluon propagator.
- For full QCD, either the ghost-gluon vertex or the dressed two-loop graphs give vital contributions to the gluon propagator equation, which have often previously been overlooked.
- A precise gluon propagator is required for solving the quark equation and for Bethe-Salpeter studies.

# Extra Slide - The equations used

$$\mathrm{Gh}^{-1}(p^2) = Z_3 - \frac{g^2 N_c}{(2\pi)^3}\int_0^{\kappa^2} d\ell^2 \ell^2 \int_0^{\pi} d\theta \sin^2\theta \; K(p^2,\ell^2,\theta)\,\mathrm{Gh}\left(\ell_+^2\right)\,\mathrm{G}\ell\left(\ell_-^2\right)$$

$$\mathrm{G}\ell^{-1}(p^2) = Z_3 + \frac{g^2 N_c}{3(2\pi)^3}\int_0^{\kappa^2} d\ell^2 \ell^2 \int_0^{\pi} d\theta \sin^2\theta \; M(p^2,\ell^2,\theta)\,\mathrm{Gh}\left(\ell_+^2\right)\,\mathrm{Gh}\left(\ell_-^2\right)$$
$$+\frac{g^2 N_c}{3(2\pi)^3}\int_0^{\kappa^2} d\ell^2 \ell^2 \int_0^{\pi} d\theta \sin^2\theta \; Q(p^2,\ell^2,\theta)\,\frac{\mathrm{Gh}^{1-a/\delta-2a}\left(\ell_+^2\right)\,\mathrm{Gh}^{1-b/\delta-2b}\left(\ell_-^2\right)}{\mathrm{G}\ell^{a}\left(\ell_+^2\right)\,\mathrm{G}\ell^{b}\left(\ell_-^2\right)}$$

where $\ell_+ = \ell + \eta p$ and $\ell_- = \ell - (1-\eta)p$
The kernels for the symmetric momenta $\eta = \frac{1}{2}$ configuration, using $x = p^2$, $y = \ell^2$ and $\theta$ defined via $p^\mu k_\mu = (p^2 k^2)^{1/2}\cos\theta$, are given below. We use the $\zeta \to 4$ projector throughout such that we reproduce the expected logarithms.

$$K(x,y,\theta) = \frac{64y\sin^2\theta}{\left(4y + x - 4\sqrt{yx}\cos(\theta)\right)^2\left(4y + x + 4\sqrt{yx}\cos(\theta)\right)}$$

$$M(x,y,\theta) = \frac{4(-8y\cos(2\theta) - 4y + 3x)}{x\left(16y^2 - 8yx\cos(2\theta) + x^2\right)}$$

$$Q(x,y,\theta) = \frac{32y\left(2\left(48y^2 + 8yx + 9x^2\right)\cos(2\theta) + 48y^2 - 16yx\cos(4\theta) - 72yx - 9x^2\right)}{x\left(16y^2 - 8yx\cos(2\theta) + x^2\right)^2}$$

## Extra Slide - More numerical details (1)

### Subtraction

We subtract the equations at $x_0 = \mu^2$, and impose the MR condition $G\ell(x_0) = 1$

$$G\ell^{-1}(x) = Z_3 + \Pi(x) = G\ell^{-1}(x_0) + \Pi(x) - \Pi(x_0).$$

### Numerical represention

We divide the function up into 3 regions, $[0, \epsilon^2]$, $[\epsilon^2, \kappa^2]$, $[\kappa^2, \infty]$ For $x < \epsilon^2$ ($\epsilon$ usually $10^{-4}$),

$$G\ell_{IR}(x) = Ax^{\kappa_g}$$

$\kappa_g$ can be adjusted depending on the IR behaviour we are interested in. For $\epsilon^2 < x < \kappa^2$ then,

$$G\ell(x) = \left(\frac{x}{x+\lambda}\right)^{\kappa_g} \sum_i^n c_i T_i(s(x))$$

where $\kappa_g$ gives the required IR behaviour. $T_i$ are chebychev polynomials and $s(x)$ is the mapping from a log scale in $x \to s \in [-1, 1]$ as required by the $T_i$'s. Chebychev coefficients store the numerical information typically use $n \simeq 30$. For $x \gtrsim \kappa^2$,

$$G\ell_{UV}(x) = G\ell(\ell)\left(1 + \frac{11}{12\pi} N_c \alpha_s(\ell) \log \frac{x}{\ell}\right)^{\gamma}$$

we take $\ell$ to be some perturbative scale where the coupling is small, $\mu^2 (\sim 10^2)$ can be used or $\kappa^2 (\sim 10^3)$.

# Extra Slide - More numerical details (2)

## Iterative procedure – Generalised Newton–Raphson

Define,

$$F(x_j, c_i^n) = G\ell^{-1}(x_j) - G\ell^{-1}(x_0) - \Pi(x_j) + \Pi(x_0)$$

where $F = 0$ for the solved system. Do Taylor expansion in $c$,

$$F(x_j, c_i^{n+1}) = F(x_j, c_i^n) + \sum_i \frac{\partial F(x_j, c_i^n)}{\partial c_i^n} \delta c_i^n + \mathcal{O}(\delta c_i^n)^2$$

$$F(x_j, c_i^n) = -\sum_i \frac{\partial F(x_j, c_i^n)}{\partial c_i^n} \delta c_i^n$$

Solve linear system for $\delta c_i$'s, then correction can be found by, $c_i^{n+1} = c_i^n + \delta c_i^n$.

- Gives quadratic convergence when close to a solution.
- The step $\delta c_i$ is always in the direction of the minimum of $F$.
- Natural iterative method never has any guarantee that it will work whether solutions exist or not.

# Extra Slide - Gribov Copies

After gauge fixing there still exist configurations where,

$$A_\mu = U A_\mu U^\dagger + \frac{1}{g} U \partial_\mu U^\dagger \qquad U \in SU(N_c).$$

- → Gauge fixing (eg. Landau gauge $\partial^\mu A_\mu = 0$) isn't perfect, some configurations may still exist that are connected by a gauge transformation.
- → Restrict allowed configurations to the *First Gribov Horizon*. Contains $A_\mu = 0$.

## Gribov-Zwanziger

Gauge configurations close to horizon dominate the IR properties. In Coulomb gauge it can be shown these configurations lead to an almost linear confining potential. Zwanziger derives conditions that imply an IR vanishing gluon and IR enhanced ghost.

# Numerical Method

- Use Landau gauge with $\partial_\mu A^\mu = 0$ and $\xi \to 0$.
- We Wick rotate to Euclidean space – this allows use of squared momenta.
- Loop integrals may then be performed without approximation via,

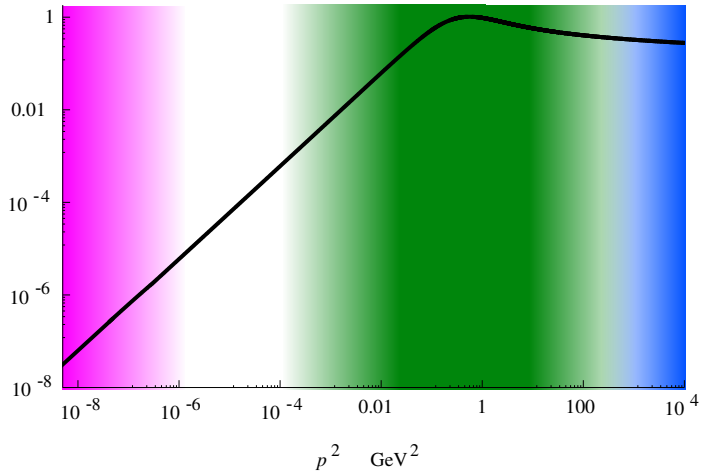
$$\int \frac{d^4k}{(2\pi)^4}\mathcal{F}(p^2,k^2,\psi) = \frac{1}{(2\pi)^3}\int_0^{\kappa^2} k^2 dk^2 \int_0^{\pi} d\psi \ \sin^2\psi \ \mathcal{F}(p^2,k^2,\psi)$$

- We must also contract the tensor structure in the Gluon equation, this is usually done with a "projector" of the form

$$\mathcal{P}_{\mu\nu}(p,\zeta) = g_{\mu\nu} - \zeta\frac{p_\mu p_\nu}{p^2}$$

where $p$ is the external momenta and $\zeta$ is a free parameter that can select different tensor structures.

To represent the dressing functions numerically, we divide $p^2$ into three distinct regions,




1. $\left[0, \epsilon^2\right]$ where $\epsilon \sim 10^{-4}$GeV.
2. $\left[\epsilon^2, \kappa^2\right]$ where $\kappa \sim 10^2$GeV.
3. $\left[\kappa^2, \infty\right]$

The dressing functions $\mathcal{G}h$ and $\mathcal{G}\ell$ are represented differently in each,

1. Use a power law, $\mathcal{G}\ell(x) = Ax^p$; choose $A$ and $p$ carefully to ensure smooth matching at $\epsilon^2$.
2. Use an interpolation routine; tabulate the function at a number of points.
   - Linear interpolation is very poor – discontinuous first derivative.
   - Chebychev Polynomials and Cubic Splines work well.
3. Use the 1–loop resummed perturbative running
   - Smooth matching is automatic for physical solutions – useful to check.

refs: Bloch, Williams, Fischer *et al*

# Iterative procedure

$$G\ell^{-1}(p^2) = G\ell^{-1}(\mu^2) + \Pi\left[G\ell\right](p^2) - \Pi\left[G\ell\right](\mu^2)$$

**An iterative procedure is applied:**

1. Choosing a sensible starting function for $Gh(p^2)$ and $G\ell(p^2)$.
2. Using this, calculate the result from the equations.
3. Use this to improve on the functions $Gh(p^2)$ and $G\ell(p^2)$ and repeat.

**Simplest method – 'Natural' Iterative Procedure**

- $G\ell_{i+1}^{-1}(p^2) = G\ell_i^{-1}(\mu^2) + \Pi\left[G\ell_i\right](p^2) - \Pi\left[G\ell_i\right](\mu^2)$
- This has some common problems:
  - No guarantees of convergence.
  - Can only iterate one equation at a time – can give stability issues.

Better method – Newton-Raphson type procedure

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# Newton–Raphson Iterative procedure

- In Region 2. $\left[\epsilon^2, \kappa^2\right]$, represent the functions using Chebychev polynomials.

$$G\ell(p^2, \vec{c}_i) = \sum_{j=0}^{n-1} c_{i,j} T_j(s(p^2))$$

where $s$ is some mapping $\left[\epsilon^2, \kappa^2\right] \to [-1, 1]$.

- Rewrite the equations we wish to solve as,

$$F(p^2, \vec{c}_i) = G\ell^{-1}(p^2, \vec{c}_i) - G\ell^{-1}(\mu^2, \vec{c}_i) - \Pi(p^2, \vec{c}_i) + \Pi(\mu^2, \vec{c}_i) = 0$$

- For a good starting function this won't be exact but we can Taylor expand,

$$F(p^2, \vec{c}_i + \delta\vec{c}_i) = F(p^2, \vec{c}_i) + \sum_{j=0}^{n-1} \frac{\partial F(p^2, \vec{c}_i)}{\partial c_{i,j}} \delta c_{i,j} + \ldots = 0$$

- We then identify $\vec{c}_{i+1} = \vec{c}_i + \delta\vec{c}_i$ defines our iterative procedure.