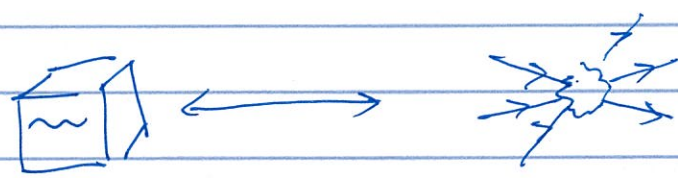
Three Body
Reft. Hansen & Sharpe 1408:5933

- Motivation
  - Gluex
  - $B \to K^* \ell\ell$, $B \to \rho \ell\ell$ [3.7σ tension]
  - nnn-force & neutron rich environments
  - Roper puzzle

- ~~Get~~ Goal: find an equation that relations the spectrum in a box to three particle scattering.



- Why the spectrum?

LQCD correlation functions:

$$C(t) = \langle \mathcal{O}(t)\,\mathcal{O}^\dagger(0)\rangle = \sum_n |\langle n|\mathcal{O}^\dagger|0\rangle|^2 e^{-E_n}$$

E_n = n^{th} interacting energy
$|n\rangle$ = nth eigenstate

Assume $\mathcal{O} \sim \pi(\vec{p})\,\pi(-\vec{p})$, where $\vec{p} = \frac{2\pi\vec{n}}{L}$ = free then unless the theory is <u>free</u> of interactions then

$$E_n \neq 2\sqrt{p^2 + m_\pi^2} \qquad \text{(off-shell)}$$

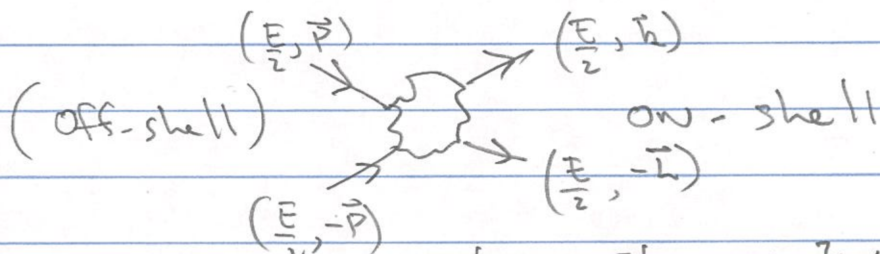
but there is a k_n such that

$$E_n = 2\sqrt{k_n^2 + m_\pi^2} \qquad \text{(on-shell)}$$

one can show

K-matrix

$$\left|\langle n|\,\pi^+(p)\,\pi^-(-p)\,|0\rangle\right| \sim \left|\mathbb{K}(E, p, k_n)\right|$$



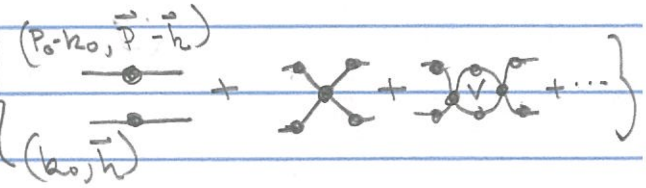
$$\mathcal{M}^{-1} = \mathbb{K}^{-1} - i\,\mathbb{P}^2/2$$

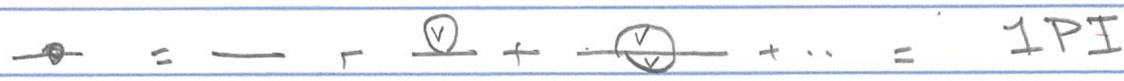
<u>So</u>, unless the spectrum depends on on-shell amplitudes ... we are at a loss.

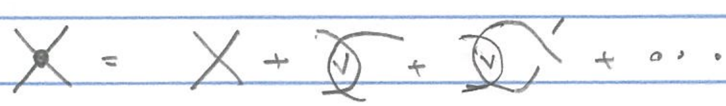
- Philosophy:
  Write down the finite volume correlation function & look at the poles.

- WARM UP: two-body problem
  Assume scalars & $m_1 = m_2$

$$C(t, \vec{P}) = \int \frac{dP_0}{2\pi} \frac{dk_0}{2\pi} L^6 e^{iP_0 t}$$



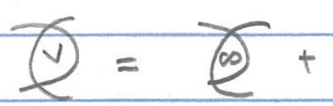




Using the Poisson:

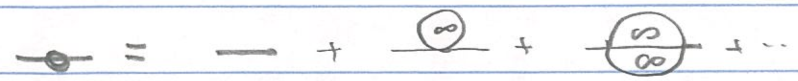
$$\left[\frac{1}{L^3}\sum_{\vec{q}} - \int \frac{d^3q}{(2\pi)^3}\right] f(\vec{q}) = \sum_{\vec{n}\neq\vec{0}} \int \frac{d^3q}{(2\pi)^3} f(\vec{q})\, e^{iL\vec{n}\cdot\vec{q}}$$

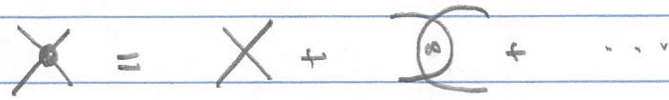
one can show that if intermediate particles cannot go on-shell then



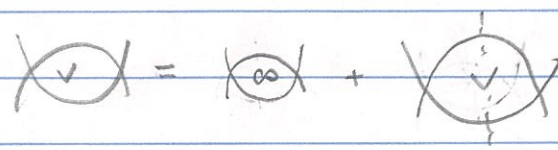
$$+ \mathcal{O}\left(e^{-m_\pi L}\right)$$

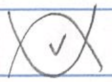
pick $\boxed{m_\pi L \gtrsim 4}$

$\Rightarrow$  $\mathcal{O}\left(e^{-m_\pi L}\right)$

 $\mathcal{O}\left(e^{-m_\pi L}\right)$

Note: the following diagram can have intermediate on-shell states



 $= \frac{1}{2}\frac{i^4}{L^3}\sum_{\vec{k}}\int\frac{dk_0}{2\pi}\,\underbrace{\frac{Z(k)}{k^2-m^2}}_{\text{Residues}=1\text{ @ pole}}\,\frac{Z(P-k)}{(P-k)^2-m^2}\,\underbrace{K_i(k,P-k)\,K_f(k,P-k)}_{\text{Kernels}}$

$$= \frac{-i}{2}\frac{1}{L^3}\sum_{\vec{k}}\frac{Z(E-\omega_k,\vec{P}-\vec{k})}{\left[(E-\omega_k)^2-\omega_{Pk}^2\right]2\omega_k}\,K_i(E,\vec{k},\vec{P})\,K_f(E,\vec{k},\vec{P})$$

$+$ [contribution from $k = E+\omega_{Pk}$ pole does not go on-shell!]

$\Rightarrow$ to evaluate this sum we use a simple trick. First, let $\vec{q}^{\,*}$ be the on-shell momentum

$$K_i(E,\vec{q},\vec{P}) = K_i^*(E^*,\vec{q}^{\,*})$$

$$E^2 = E^{*2} + P^2 \qquad E^{*} = 2\sqrt{m^2+\vec{q}^{\,*2}}$$

$$\vec{q} = \hat{k}\,q$$

direction is not ! magnitude is fixed by the onshell condition

$$K_i^*(E^*, \vec{q}^*) = \sqrt{4\pi}\, Y^*_{\ell m}(\hat{k})\, K^*_{i,\ell m}(q^*)$$

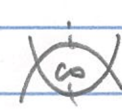
$$K_f^*(E^*, \vec{q}^*) = \sqrt{4\pi}\, Y_{\ell m}(\hat{k})\, K^*_{f\,\ell m}(q^*)$$

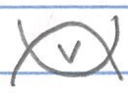
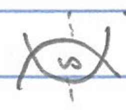
$\Rightarrow$  $-$  $= -\frac{i}{2}\frac{1}{L^3}\sum_{\vec{k}} \frac{1}{\left[(E-\omega_k)^2-\omega_{pk}^2\right]}\frac{1}{2\omega_k} \otimes$

$$\left[Z(E-\omega_k, \vec{P}-\vec{k})\, K_i(E,\vec{k},\vec{P})\, K_f(E,\vec{k},\vec{P}) - K_i^*(E^*,\vec{q}^*)\, K_f^*(E^*,\vec{q}^*)\right]$$

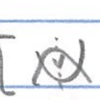
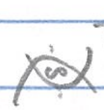
$$= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\, PV\left[\frac{1}{(E-\omega_k)^2-\omega_{pk}^2}\right]\left[Z\otimes K_i \otimes K_f - K_i^* K_f^*\right] + \mathcal{O}(e^{-m_\pi L})$$

this follows from the fact that the integrand is smooth

$=$ PV  $-$ PV 

$\Rightarrow$  $=$ PV  $+$ [ $-$ PV ]

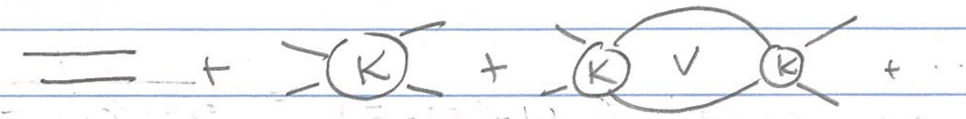
UV finite

$\Rightarrow$ [ $-$ PV ] $= -i\, K_{f,\ell' m'}\left[\frac{1}{2}\left[\frac{1}{L^3}\sum_k - \int\frac{d^3k}{(2\pi)^3}\right]\frac{Y_{\ell' m'}(\hat{k})\, Y^*_{\ell m}(\hat{k})}{\left[(E-\omega_k)^2-\omega_{pk}^2\right]}\right]\frac{K_{i,\ell m}}{2\omega_k}$

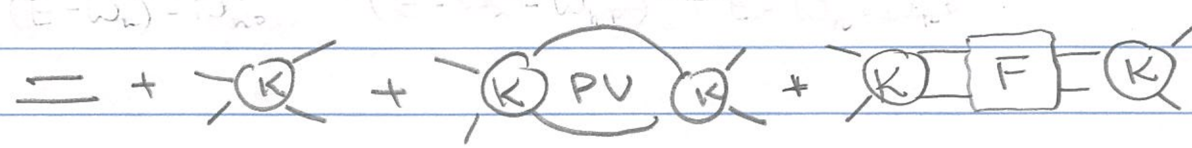
$$= -i\, K_{f\,\ell' m'}\left[\frac{1}{2}\left[\frac{1}{L^3}\sum_k - \int\frac{d^3k}{(2\pi)^3}\right]\frac{Y_{\ell' m'}(\hat{k})\, Y^*_{\ell m}(\hat{k})}{(2\omega_k)(2\omega_{pk})(E-\omega_k-\omega_{pk})}\right] K_{i,\ell m} + \mathcal{O}(e^{-m_\pi L})$$

$$\equiv -i\,\overrightarrow{K}\,\overleftrightarrow{F}\,\overleftarrow{K} \Rightarrow iK\; iF\; iK$$

$\Rightarrow C^V =$

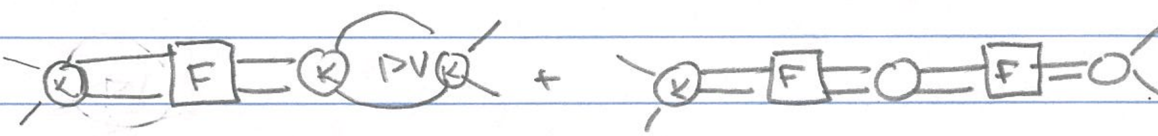


$=$



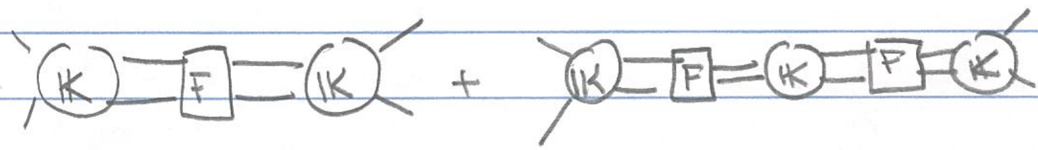
$+$

$+$



$+ \ldots$

$= C^\infty +$



$+ \ldots$

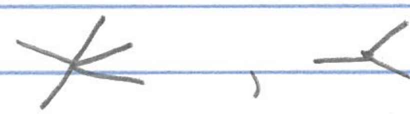
$$= C^\infty + iK\; iF\,\underbrace{\left(1 + KF + (-KF)^2 + (-KF)^3 + \cdots\right)}_{\frac{1}{1+KF}} K$$

$$\Rightarrow \text{pole @ } \boxed{\mathrm{Det}\left[1 + KF\right] = 0}$$

Onto three particles

- Assume identical, scalar bosons
- interactions

Not allowed



- current constraint: the two-particle K-matrix must be finite for all ℓ

$$\Rightarrow \quad K_\ell \sim \frac{1}{\cot \delta_\ell} \quad \Rightarrow \quad |\delta_\ell| \leq \pi/2$$

- Notation "All on-shell":

$(\vec{k}, \omega_k) \longrightarrow$ = Spectator

$\left.\begin{array}{l}(\vec{a}, \omega_a) \longrightarrow \\ (\vec{P}-\vec{k}-\vec{a}, \omega_{ka}) \longrightarrow\end{array}\right\}$ = pair with $P_2 = (E-\omega_k, \vec{P}-\vec{k})$

$\vec{b}_{ka} = \vec{P}-\vec{k}-\vec{a}$

this is true if All particles are on-shell...

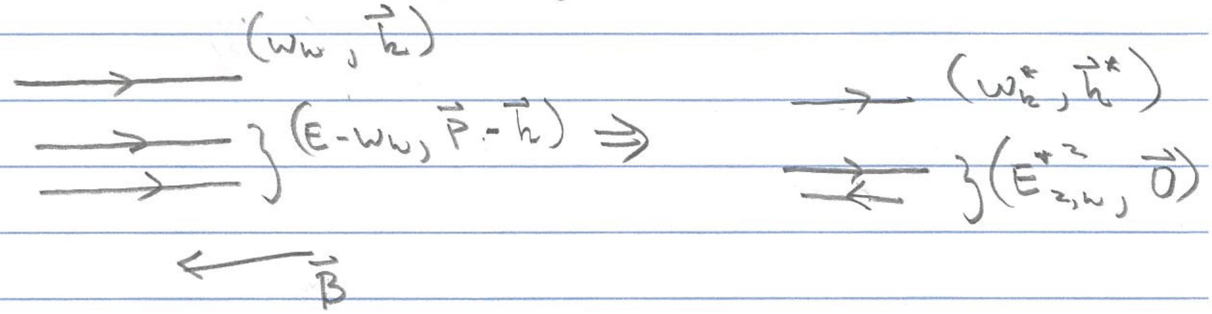
Energy of $(\vec{a}, \vec{b}_{ka})$ pair in its CM frame is labeled

holds for a small region of $\vec{k}$

$$E^{*2}_{2,k} = (P_2)^2 = (E-\omega_k)^2 - (\vec{P}-\vec{k})^2$$

boosting to this frame can be done by

$$\vec{\beta}_k = -\frac{(\vec{P} - \vec{k})}{E - \omega_k}$$



in this frame: $\omega_a^* = \omega_{ka}^* = \dfrac{E^*_{2,k}}{2}$ & $\vec{a}^* = -\vec{b}^*_{ka}$

& the magnitude is fixed:

$$a^* = b^*_{ka} = q^*_k = \sqrt{\frac{E^{*2}_{2k}}{4} - m^2}$$

$\Rightarrow$ if $(E, \vec{P})$ & $\vec{k}$ is fixed, the only remaining degree of freedom is $\hat{a}^*$.

<u>Note</u> q^*_k = on-shell momentum, & "k" denotes the spectator

- **Two on-shell & one off-shell:**

$\vec{k}, \vec{a}$ are on-shell, but $\vec{b}_{ka}$ is not

$\Rightarrow$ third particle has $(E-\omega_k-\omega_a, \vec{b}_{ka})$

boosting to two-particle CM frame

$(\omega_a, \vec{a}) \rightarrow (\omega_a^*, \vec{a}^*)$ & $(E-\omega_k-\omega_a, \vec{b}_{ka}) \rightarrow (E_{2k}^*-\omega_a^*, -\vec{a}^*)$

in this case $\boxed{a^* \neq q_k^*}$

$\Rightarrow$ degrees of freedom $E, \vec{P}, \vec{k}, \vec{a}^*$

- Scattering:

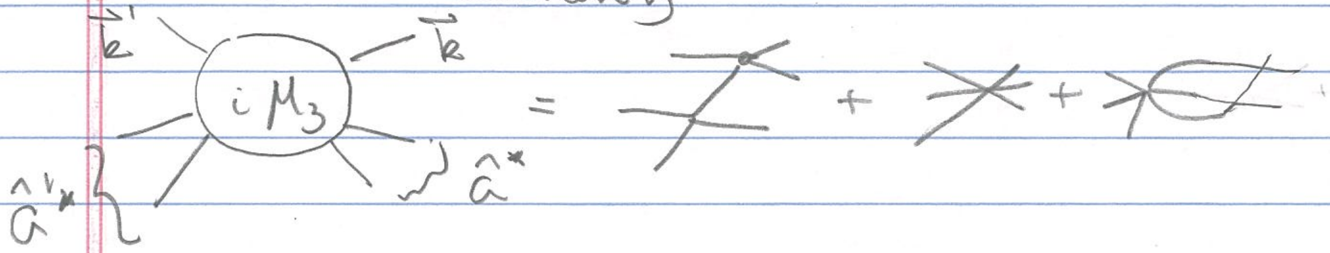
$$\mathbb{K}_2(\vec{k}, \hat{a}'^*, \hat{a}^*) = \text{two body K-matrix}$$

$\vec{k}$ label tells you total energy & momentum of subsystem $[E-\omega_k, \vec{P}-\vec{k}]$
Also, boost vector to C.M. frame, $\vec{\beta}_k$

$\hat{a}^*(\hat{a}^{*\prime})$ = initial (final) direction of one scattered particle

$$\Rightarrow \mathbb{K}_2(\vec{k}, \hat{a}'^*, \hat{a}^*) = 4\pi\, Y_{\ell'm'}^*(\hat{a}'^*)\, \mathbb{K}_{2\,\ell'm';\ell m}(\vec{k})\, Y_{\ell m}(\hat{a}^*)$$
$$\propto Y_{\ell'm'}^*(\hat{a}'^*)\, \delta_{\ell\ell'}\delta_{mm'}\, Y_{\ell m}(\hat{a}^*)$$

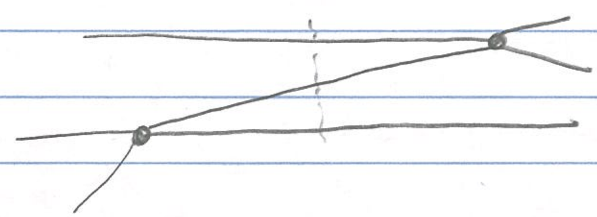
- three particle scattering



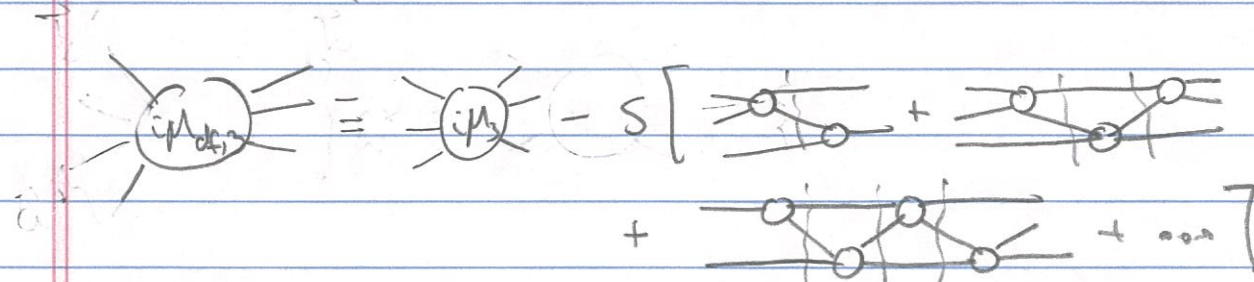
the on-shell scattering amplitude $\equiv \mathcal{M}_3(\vec{k}', \hat{a}'^*, \vec{k}, \hat{a}^*)$

$\mathcal{M}_3$ can be decomposed into spherical harmonics, but it CANNOT be reliably truncated.

this is because $\mathcal{M}_3$ has physical singularities that are NOT due to bound states.



Arbitrarily non-local, leads to finite contribution from higher partial waves.



"S" = symmetrize

$\Rightarrow$ $\mathcal{M}_{df,3}$ is smooth

$$\Rightarrow \mathcal{M}_{df,3}(\vec{k}', \hat{a}'^*; \vec{k}, \hat{a}^*) = 4\pi\, Y^*_{\ell'm'}(\hat{a}'^*)\, \mathcal{M}_{df,3\,\ell'm'\ell m}(\vec{k}', \vec{k})\, Y_{\ell m}(\hat{a})$$

<u>similarly</u>: (to be properly defined later)

$$\mathbb{K}_{df\,3}(\vec{k}', \hat{a}'^{*}, \vec{k}, \hat{a}^{*}) = 4\pi\, Y^{*}_{\ell' m'}(\hat{a}'^{*})\, \mathcal{K}_{df\,3,\ell' m', \ell m}(\vec{k}', \vec{k})\, Y_{\ell m}(\hat{a}^{*})$$

<u>Not</u> diagonal in <u>ℓ</u> or <u>m</u>

<u>let</u>:

$$K_{2, k'\ell' m', k\ell m} \equiv \delta_{k' k}\, \mathcal{K}_{2,\ell' m', \ell m}(\vec{k}) \quad ; \vec{k} = 2\pi \vec{n}/L$$

$$\mathcal{K}_{df,3, k'\ell' m', k\ell m} \equiv \mathcal{K}_{df,3\, \ell' m', \ell m}(\vec{k}', \vec{k}) \;;\; \vec{k}', \vec{k} = \frac{2\pi \vec{n}', \vec{n}}{L}$$

$\Rightarrow$ everything is a matrix in

$$\begin{pmatrix} \text{finite volume} \\ \text{mom } 2\pi\vec{n}/L \end{pmatrix} \times \begin{pmatrix} \text{two particle} \\ \text{angular momentum} \end{pmatrix}$$

- <u>QC</u>

$$\det\left[1 + F_3 \mathcal{K}_{df,3}\right] = 0$$

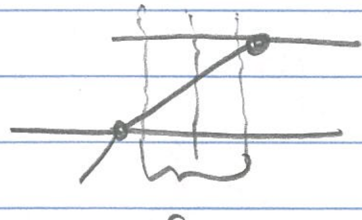
determinant in inner product

$$F_3 \equiv \frac{F}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + [1 + \mathcal{K}_2 G]^{-1} \mathcal{K}_2 F}\right]$$

$$G_{p\ell'm', k\ell m} = \left(\frac{k^*}{q_p^*}\right) 4\pi \frac{Y_{\ell' m'}(\hat{k}^*) H(\vec{k}) H(\vec{p}) Y^*_{\ell m}(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp})}$$

on-shell

$$\otimes \left(\frac{p^*}{q_k^*}\right)^{\ell} \frac{1}{2\omega_k L^3}$$



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$$F_{k'\ell'm', k\ell m} = \delta_{k'k} F_{\ell'm', \ell m}(\vec{k})$$

$$F_{\ell'm', \ell m}(\vec{k}) = F^{i\epsilon}_{\ell'm', \ell m}(\vec{k}) + \rho_{\ell'm', \ell m}(\vec{k})$$

$$F^{i\epsilon}_{\ell'm', \ell m}(\vec{k}) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{a}} - \int \frac{d\vec{a}}{(2\pi)^3}\right] \frac{4\pi Y_{\ell'm'}(\hat{a}^*) Y^*_{\ell m}(\hat{a}^*)}{2\omega_a 2\omega_{ka}}$$

$$\otimes \frac{H(\vec{k}) H(\vec{a}) H(\vec{b}_{ka})}{(E - \omega_k - \omega_a - \omega_{ka} + i\epsilon)} \left(\frac{a^*}{q_k^*}\right)^{\ell+\ell'}$$

$$\rho_{\ell'm', \ell m}(\vec{k}) = \delta_{\ell\ell'} \delta_{mm'} H(\vec{k}) \tilde{\rho}(P_2)$$

$$\tilde{\rho}_2(P_2) \equiv \frac{1}{16\pi\sqrt{P_2^2}} \begin{cases} -i\sqrt{P_2^2/4 - m^2} & (2m)^2 < E_{2k}^{*2} \\ \left|\sqrt{\frac{P_2^2}{4} - m^2}\right| & 0 < E_{2k}^{*2} \le (2m)^2 \end{cases}$$

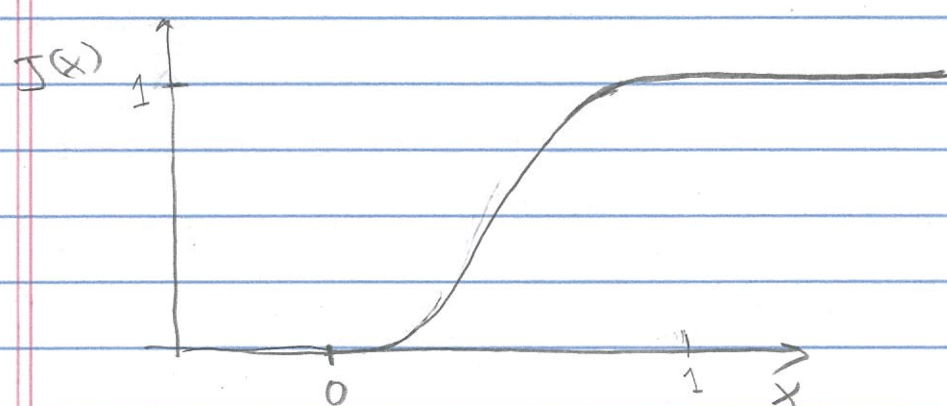
this assures that F is a smooth

$H(\vec{k})$ is a cutoff function

$$H(\vec{k}) = \begin{cases} 0 & E^{*2}_{2,k} \leq 0 \\ 1 & (2m)^2 < E^{*2}_{2,k} \end{cases}$$

$$H(\vec{k}) \equiv J\left(E^{*2}_{2,k} / 4m^2\right)$$

$$J(x) = \begin{cases} 0 & x \leq 0 \\ \exp\left[-\frac{1}{x}\exp\left(-\frac{1}{1-x}\right)\right] & 0 < x \leq 1 \\ 1 & 1 < x \end{cases}$$



- why do we need a cutoff function?

In the determinant we must consider scenarios when spectator has large momentum
⇒ pair will be off-shell. "H" damps these arbitrarily off-shell effects.