Effective actions for SU(3) gauge theories

and mean-field solutions at finite density

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Jefferson Lab

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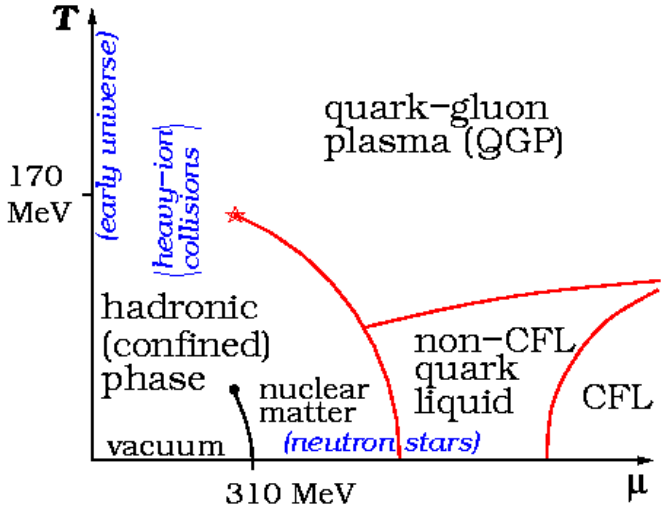
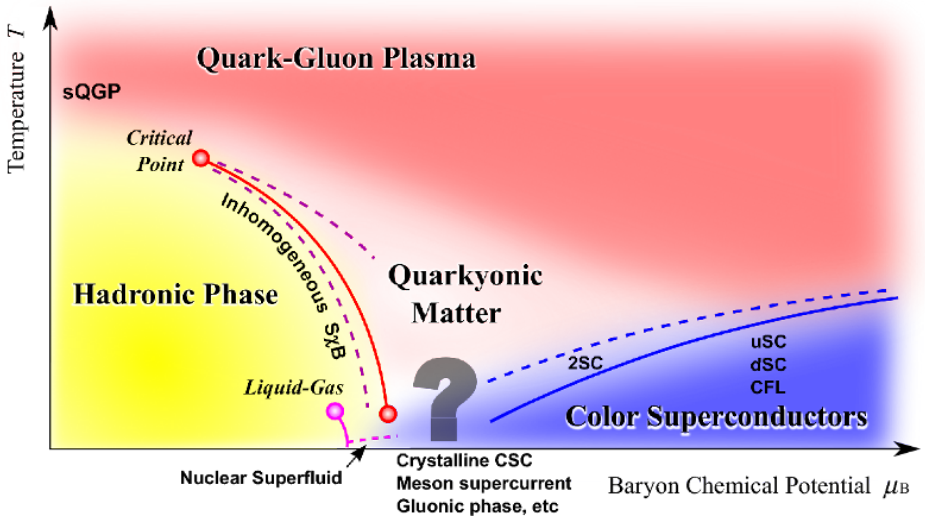
work in collaboration with Kurt Langfeld, Univ. of Plymouth
arXiv:1404.5844

The Sign Problem

Sketches of the QCD phase diagram (two of many):





Except near $\mu = 0$, it is mostly conjecture. The location and nature of phase transitions in the QCD phase diagram involves non-perturbative physics, but our most reliable tool, lattice Monte Carlo simulations, doesn't seem applicable at large finite densities.

sign problem 2

In order to vary particle density, we need to introduce a chemical potential μ into the fermion action.

When the quark fields are integrated out, each flavor contributes a factor

$$\det(\not{D} + m + \mu\gamma_0)$$

But

$$\gamma_5(\not{D} + m + \mu\gamma_0)\gamma_5 = \not{D}^\dagger + m - \mu\gamma_0 = (\not{D} + m - \mu^*\gamma_0)^\dagger$$

Take the determinant of both sides

$$\det(\not{D} + m + \mu\gamma_0) = \det{}^*(\not{D} + m - \mu^*\gamma_0)$$

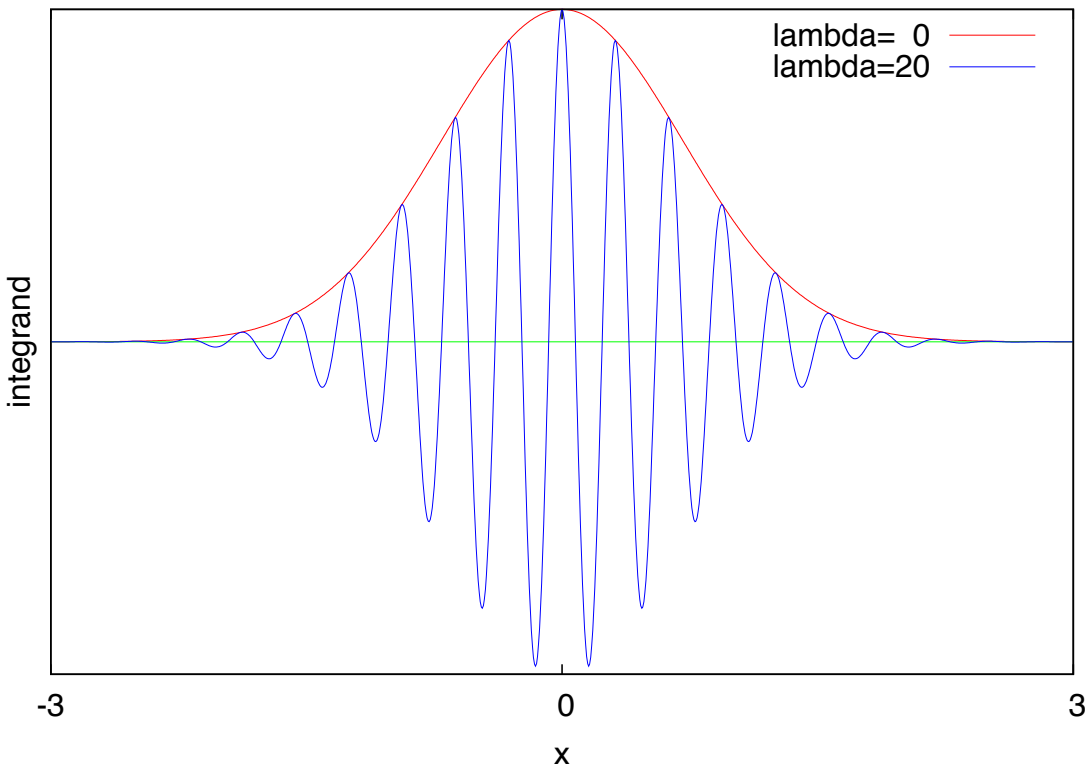
So the determinant factors are only real if $\mu = 0$, or purely imaginary, or if there are two flavors with μ of opposite sign (isospin chemical potential).

sign problem 3

Using $\det(M) = \exp \mathrm{Tr} \log(M)$, the fermion determinant becomes part of the action (the starting point for hybrid Monte Carlo), but for $\mu \neq 0$ the action is complex.

a toy 0-dimensional example:

$$Z(\lambda) = \int_{-\infty}^{\infty} dx \; e^{-x^2 + i\lambda x}$$



Straightforward importance sampling is impossible!

What to do?

There are several "direct" approaches to the sign problem in QCD, which are under development:

- **Reweighting + cumulant expansion** *(WHOT collaboration)*

  Treat the imaginary part of the action as an observable, rather than as part of the Boltzman weight.

- **Stochastic quantization** *(Aarts, Seiler, Sexty, Stamatescu...)*

  Complexify the field variables and apply the Langevin equation.

- **Lefschetz thimbles** *(Cristoforetti, Di Renzo, Mukherjee, Scorzato)*

  Shift functional integration contours into the complex plane.

In this talk I will discuss an "indirect" approach: Map the gauge-matter theory onto a much simpler theory – a **Polyakov line action (or "SU(3) spin") model** – where the sign problem has already been solved by a number of different techniques.

Effective Polyakov Line Action

Start with lattice gauge theory and integrate out all d.o.f. subject to the constraint that the Polyakov line holonomies are held fixed. In temporal gauge

$$e^{S_P[U_{\mathbf{x}}]} = \int DU_0(\mathbf{x},0)DU_k D\phi \left\{ \prod_{\mathbf{x}} \delta[U_{\mathbf{x}} - U_0(\mathbf{x},0)] \right\} e^{S_L}$$

At leading order in the strong coupling/hopping parameter expansion S_P has the form of an SU(3) spin model

$$\begin{aligned} S_{spin} &= J \sum_x \sum_{k=1}^{3} \Big(\mathrm{Tr}[U_x]\mathrm{Tr}[U^\dagger_{x+\hat{k}}] + \mathrm{c.c.} \Big) \\ &\quad + h \sum_x \Big(e^{\mu/T}\mathrm{Tr}[U_x] + e^{-\mu/T}\mathrm{Tr}[U_x] \Big) \end{aligned}$$

The SU(3) spin model has been solved successfully, for a wide range of parameters J, h, μ, in several different ways:

Methods

1. flux representation *(Gattringer and Mercado)*
2. stochastic quantization *(Aarts and James)*
3. reweighting *(Fromm, Langelage, Lottini and Philipsen)*
4. mean field *(Splittorff and JG)*

Since these methods work for the simple SU(3) spin model S_{spin}, perhaps they also work for the more complicated effective action S_P.

The problem is to find the effective action S_P, corresponding to lattice gauge theory at weaker couplings, finite μ, and light quark masses.

Avoid dynamical fermion simulations for now, work instead with an SU(3) gauge-Higgs model

$$S_L = \frac{\beta}{3}\sum_p \mathrm{ReTr}[U(p)] + \frac{\kappa}{3}\sum_x \sum_{\mu=1}^{4} \mathrm{ReTr}\left[\Omega^\dagger(x) U_\mu(x) \Omega(x+\hat{\mu})\right]$$

If we can derive S_P at $\mu = 0$, then we also have S_P at $\mu > 0$ by the following identity:

$$S_P^\mu[U_{\mathbf{x}}, U_{\mathbf{x}}^\dagger] = S_P^{\mu=0}\left[e^{N_t\mu} U_{\mathbf{x}}, e^{-N_t\mu} U_{\mathbf{x}}^\dagger\right]$$

which is true to all orders in the strong coupling/hopping parameter expansion.

How to compute S_P at $\mu = 0$?

- strong-coupling expansions *(Philipsen et al.)*
- inverse Monte Carlo *(Heinzl et al.)*
- relative weights *(this talk)*

And how do we know that we have derived S_P correctly?

One test: compare Polyakov line correlators

$$G(R) = \frac{1}{N_c^2}\left\langle \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger]\right\rangle \quad , \quad R = |\mathbf{x} - \mathbf{y}|$$

computed for the effective action, and in the underlying lattice gauge theory.

Agreement has not been demonstrated in other approaches to deriving S_P beyond $R = 2$ or 3 lattice spacings (see, e.g., *Bergner et al., arXiv:1311.6745*)

The Gold Standard - SU(2) via Relative Weights

In previous papers we worked out S_P for pure SU(2) gauge theory:

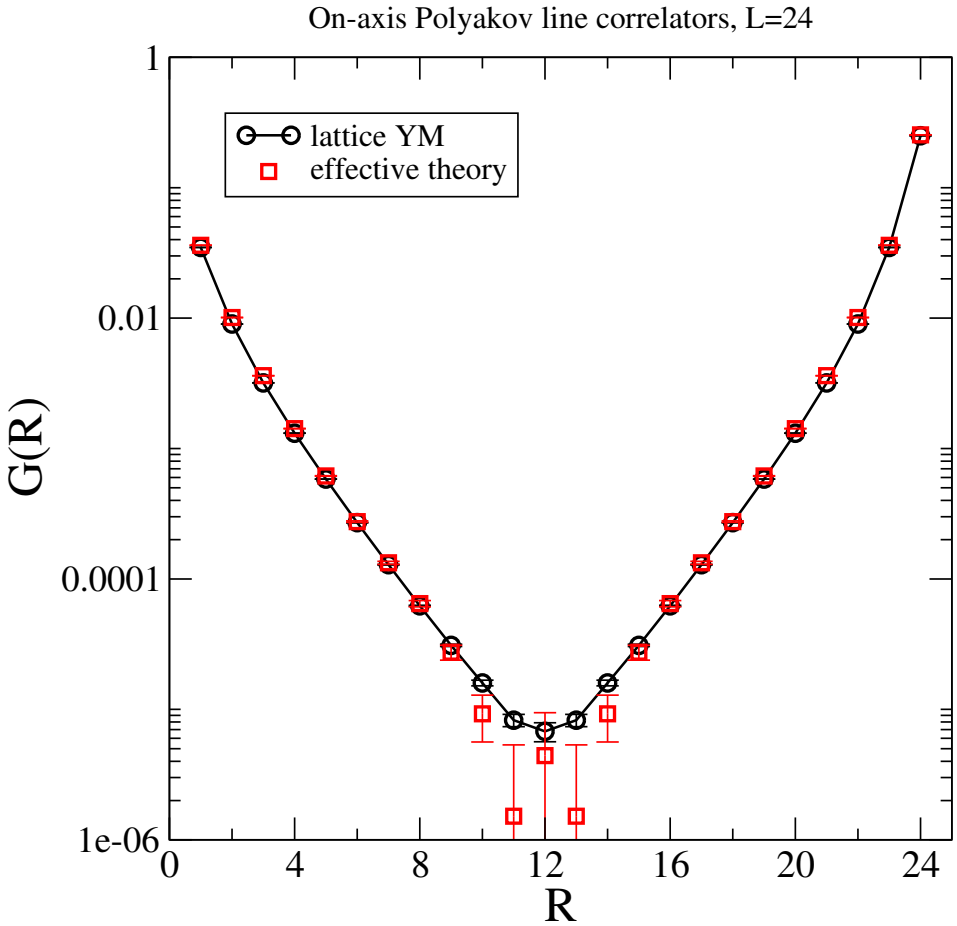
$$S_P = \sum_{\mathbf{x},\mathbf{y}} P_{\mathbf{x}} K(\mathbf{x}-\mathbf{y}) P_{\mathbf{y}}$$

where

$$P_{\mathbf{x}} = \tfrac{1}{2}\mathrm{Tr}\, U_{\mathbf{x}}$$

Here is the correlator comparison for

$$G(R) = \langle P_{\mathbf{x}} P_{\mathbf{y}} \rangle$$



The underlying lattice gauge theory is at $\beta = 2.2$ on a $24^3 \times 4$ lattice.

The Relative Weights Method

Let S'_L be the lattice action in temporal gauge with $U_0(\mathbf{x},0)$ fixed to $U'_{\mathbf{x}}$. It is not so easy to compute

$$\exp\Big[S_P[U'_{\mathbf{x}}]\Big] = \int DU_k D\phi \; e^{S'_L}$$

directly. But the ratio ("relative weights")

$$e^{\Delta S_P} = \frac{\exp[S_P[U'_{\mathbf{x}}]]}{\exp[S_P[U''_{\mathbf{x}}]]}$$

is easily computed as an expectation value

$$\begin{aligned}
\exp[\Delta S_P] &= \frac{\int DU_k D\phi \; e^{S'_L}}{\int DU_k D\phi \; e^{S''_L}} \\
&= \frac{\int DU_k D\phi \; \exp[S'_L - S''_L] e^{S''_L}}{\int DU_k D\phi \; e^{S''_L}} \\
&= \Big\langle \exp[S'_L - S''_L] \Big\rangle''
\end{aligned}$$

where $\langle ... \rangle''$ means the VEV in the Boltzman weight $\propto e^{S''_L}$.

Suppose $U_{\mathbf{x}}(\lambda)$ is some path through configuration space parametrized by λ, and suppose $U'_{\mathbf{x}}$ and $U''_{\mathbf{x}}$ differ by a small change in that parameter, i.e.

$$U'_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 + \tfrac{1}{2}\Delta\lambda) \quad , \quad U''_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 - \tfrac{1}{2}\Delta\lambda)$$

Then the relative weights method gives us the derivative of the true effective action S_P along the path:

$$\left(\frac{dS_P}{d\lambda}\right)_{\lambda=\lambda_0} \approx \frac{\Delta S}{\Delta\lambda}$$

The question is: which derivatives will help us to determine S_P itself?

Fourier components of P_x

$$P_{\mathbf{x}} \equiv \frac{1}{N_c}\mathrm{Tr}\,U_{\mathbf{x}} = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

We first set a particular momentum mode $a_{\mathbf{k}}$ to zero. Call the resulting configuration $\widetilde{P}_x$. Then define ($f \approx 1$)

$$\begin{aligned} P''_{\mathbf{x}} &= \left(\alpha - \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \\ P'_{\mathbf{x}} &= \left(\alpha + \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \end{aligned}$$

which uniquely determine (in SU(2) and SU(3)) the eigenvalues of the corresponding holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$.

S_P has a remnant local symmetry $U_{\mathbf{x}} \to g_{\mathbf{x}} U_{\mathbf{x}} g_{\mathbf{x}}^\dagger$, so the holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$ can be taken to be diagonal. We then compute

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a_{\mathbf{k}}^R}\right)_{a_{\mathbf{k}}=\alpha}$$

by the relative weights simulation ($a_{\mathbf{k}}^R$ is the real part of $a_{\mathbf{k}}$).

For a pure gauge theory, the part of S_P bilinear in P_x is constrained to have the form

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P_{\mathbf{y}}^\dagger K(\mathbf{x}-\mathbf{y})$$

Then, going over to Fourier modes

$$\frac{1}{\alpha}\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a_{\mathbf{k}}^R}\right)_{a_{\mathbf{k}}=\alpha} = 2\widetilde{K}(\mathbf{k})$$

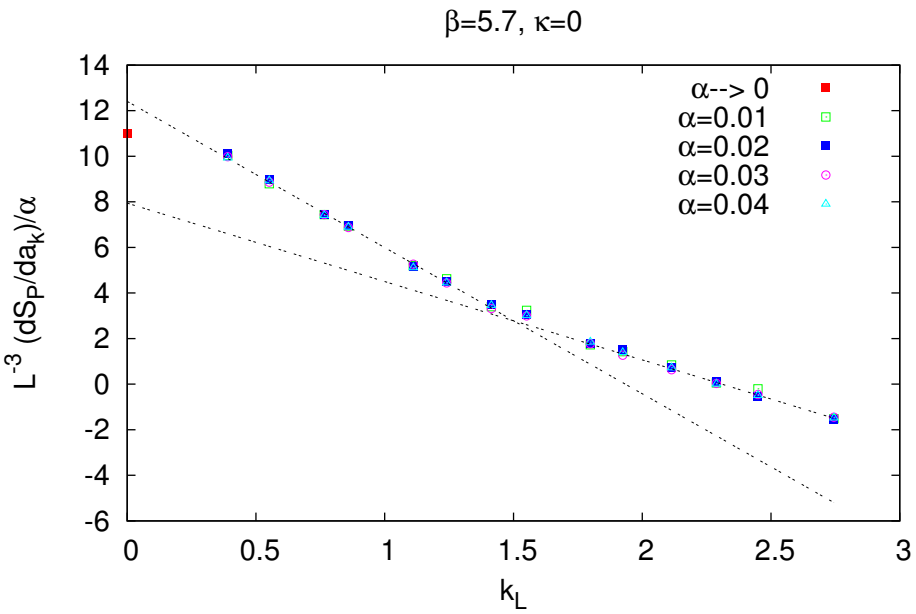
SU(3) Pure Gauge Theory

We work on a $16^3 \times 6$ lattice volume; there is a deconfinement transition at $\beta = 5.89$, but we are interested in the confinement (or, with matter, the "confinement-like") regime. Here are the relative weights results at $\beta = 5.7$:

1. Rotation invariance: data points only depend on **k** through k_L

$$k_L = 2\sqrt{\sum_{i=1}^{3} \sin^2(k_i/2)}$$

2. Except at $k_L = 0$, the data points fall on one of two straight lines, with different slopes.
3. Deviation at $k_L = 0$ is handled by a long range cutoff in the kernel $K(\mathbf{x} - \mathbf{y})$, which would otherwise be proportional to $\sqrt{-\nabla^2}$.



To compute $K(\mathbf{x}-\mathbf{y})$

1. Fit the $\widetilde{K}(k_L)$ data to

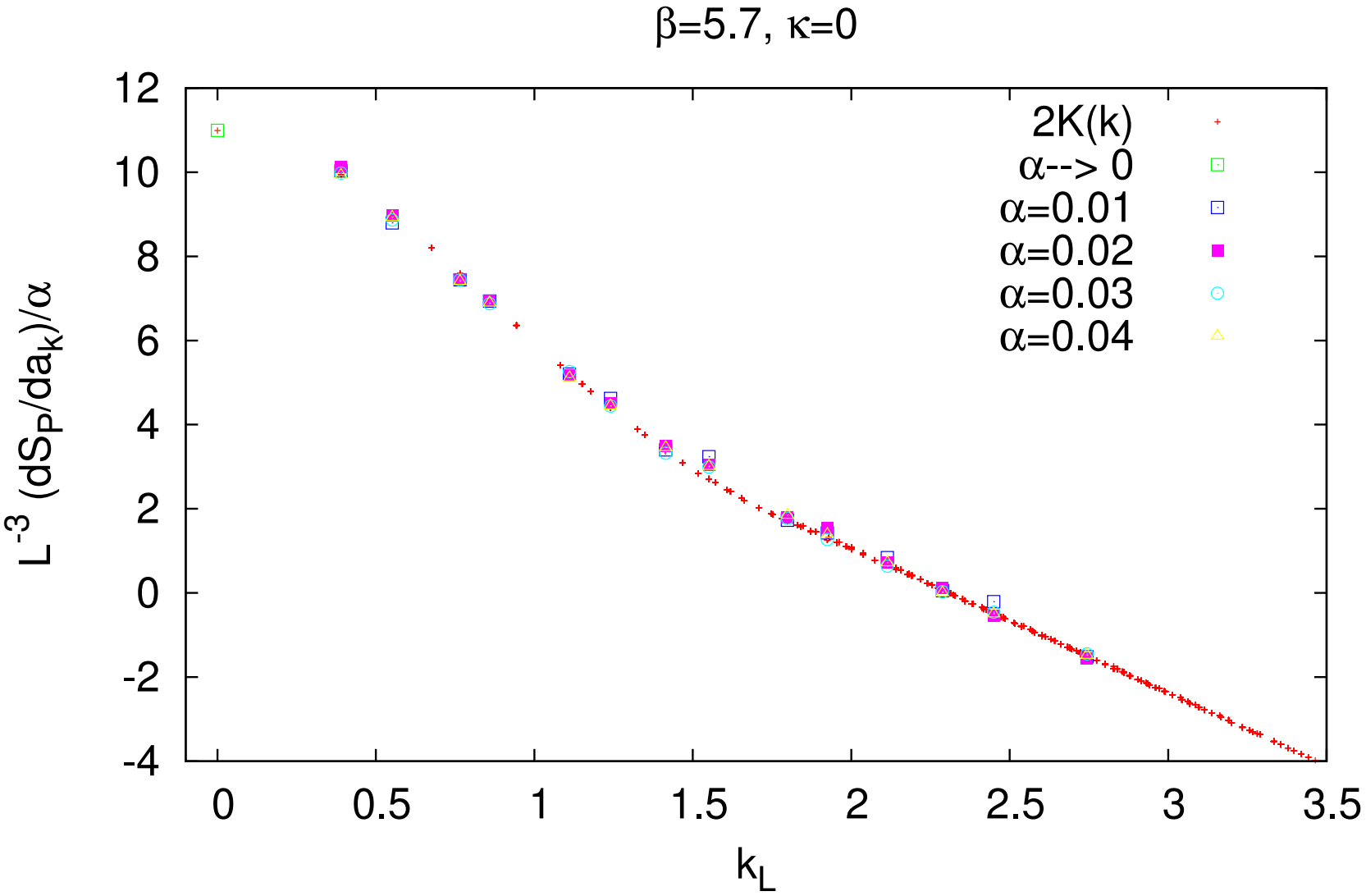
$$\widetilde{K}^{fit}(k_L) = \begin{cases} c_1 - c_2 k_L & k_L \leq k_0 \\ b_1 - b_2 k_L & k_L > k_0 \end{cases}$$

2. Introduce a long-range cutoff r_{max}

$$K(\mathbf{x}-\mathbf{y}) = \begin{cases} \frac{1}{L^3}\sum_{\mathbf{k}} \widetilde{K}^{fit}(k_L) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} & |\mathbf{x}-\mathbf{y}| \leq r_{max} \\ 0 & |\mathbf{x}-\mathbf{y}| > r_{max} \end{cases}$$

3. Transform back to momentum space. Choose cutoff r_{max} so that $\widetilde{K}(0)$ matches the data point at $k_L = 0$.

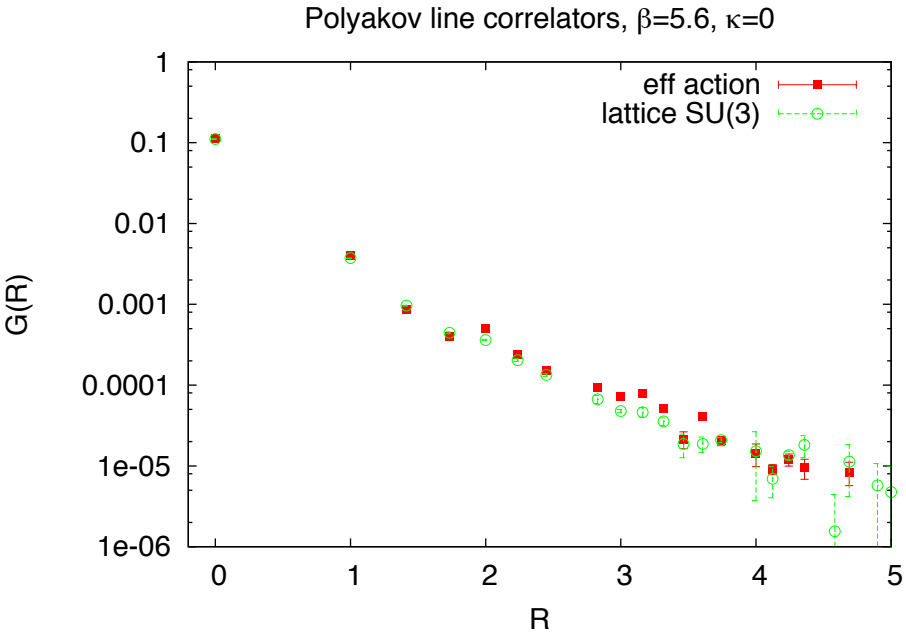
The red points are the Fourier transform of $K(\mathbf{x}-\mathbf{y})$, which gives us the effective action S_P



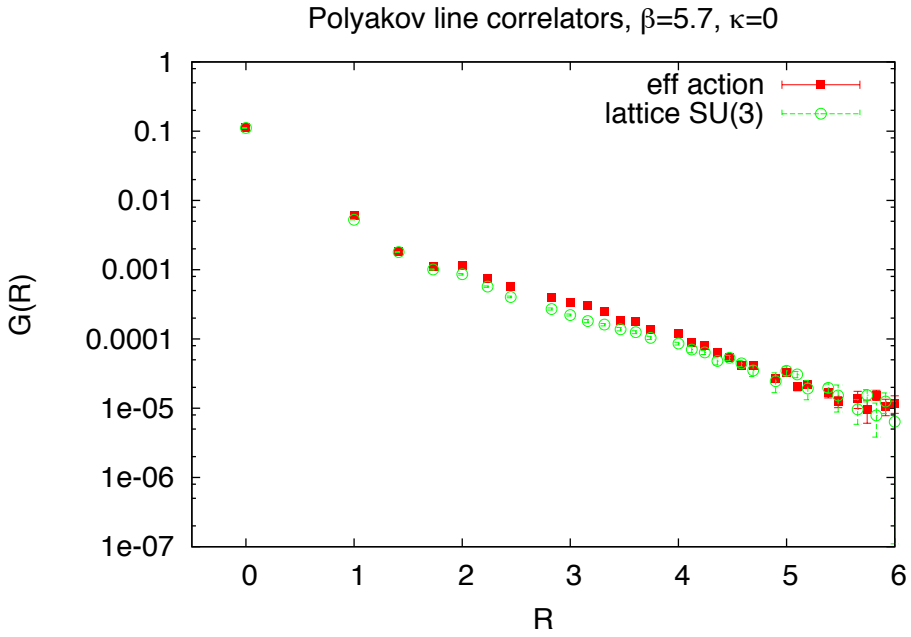
Correlator comparisons at $\beta = 5.6, 5.7$

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^{\dagger}_{\mathbf{y}} K(\mathbf{x} - \mathbf{y})$$

Simulate the effective theory in the usual way, and compare the Polyakov line correlators in the effective theory with the correlators in the underlying pure gauge theory



$\beta = 5.6$



$\beta = 5.7$

Gauge-Higgs model: General remarks

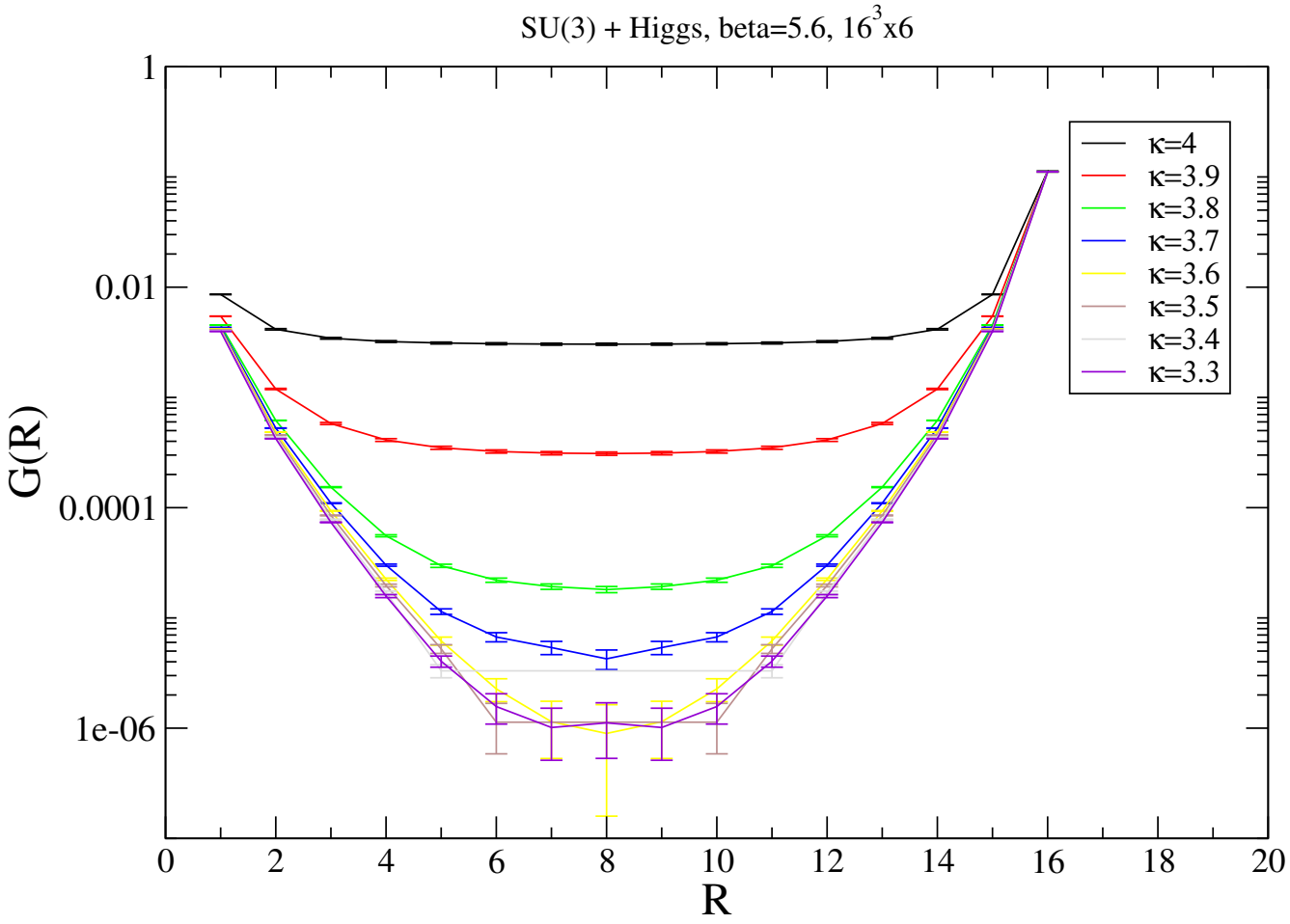
Fradkin-Shenker-Osterwalder-Seiler Theorem

In an SU(N) lattice gauge theory with matter in the fundamental representation, there is no absolute separation in coupling-constant space between a confining and a Higgs phase.

We are considering the SU(3) gauge-Higgs action

$$S_L = \frac{\beta}{3} \sum_p \mathrm{ReTr}[U(p)] + \frac{\kappa}{3} \sum_x \sum_{\mu=1}^{4} \mathrm{ReTr}\left[\Omega^\dagger(x) U_\mu(x) \Omega(x+\hat{\mu})\right]$$

In our case, keeping $\beta = 5.6$ fixed and varying κ, there is a rapid crossover from a "confinement-like" to a "Higgs-like" region at $\kappa \approx 4.0$.



This plot shows the Polyakov line correlator $G(R) = \langle P_{\mathbf{x}} P_{\mathbf{y}} \rangle$ vs. R for the SU(3) gauge-Higgs model, computed by standard lattice Monte Carlo (+ Lüscher-Weisz noise reduction), at $\beta = 5.6$ and various κ.

Finite density: General remarks

Introducing matter fields introduces a dependence on chemical potential in S_P:

$$S_P = \sum_s e^{s\mu/T} S_P^{(s)}[U_{\mathbf{x}}, U_{\mathbf{x}}^\dagger]$$

- Truncation is inevitable.
- But terms which are negligible at $\mu = 0$ can become significant at large enough μ.
- The hope is to calculate enough of S_P so that the approximation works in the region of interest in the $\mu - T$ plane.
- For now we will determine S_P up to 2nd order in fugacity, and 2nd order in products of Polyakov lines.

The starting point is to include, in the center symmetry-breaking terms, $\mathrm{Tr}U_{\mathbf{x}}, \mathrm{Tr}U_{\mathbf{x}}^2$ (+ complex conjugates), and products of no more than two of these terms.

S_P for the SU(3) gauge-Higgs model

1. Write down all possible terms in S_P involving $\mathrm{Tr}U_{\mathbf{x}}, \mathrm{Tr}U_{\mathbf{x}}^2, \mathrm{Tr}U_{\mathbf{x}}^\dagger, \mathrm{Tr}U_{\mathbf{x}}^{\dagger 2}$ and nonlocal products of any two of these terms.

2. Introduce a finite chemical potential via the transformation

$$U_{\mathbf{x}} \to e^{N_t \mu} U_{\mathbf{x}}, \;\; U_{\mathbf{x}}^\dagger \to e^{-N_t \mu} U_v^\dagger x$$

3. Make use of the SU(3) identities

$$\mathrm{Tr}[U_{\mathbf{x}}^2] = 9P_{\mathbf{x}}^2 - 6P_{\mathbf{x}}^\dagger \quad , \quad \mathrm{Tr}[U_{\mathbf{x}}^{\dagger 2}] = 9P_{\mathbf{x}}^{\dagger 2} - 6P_{\mathbf{x}}$$

   to express everything in terms of the $P_{\mathbf{x}}$ variables.

4. Discard terms involving a product of three or more $P_{\mathbf{x}}$'s.

We end up with the bilinear action

$$
\begin{aligned}
S_P &= \sum_{\mathbf{xy}} P_{\mathbf{x}} P_{\mathbf{y}}^\dagger K(\mathbf{x}-\mathbf{y}) + \sum_{\mathbf{xy}} (P_{\mathbf{x}} P_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y},\mu) + P_{\mathbf{x}}^\dagger P_{\mathbf{y}}^\dagger Q(\mathbf{x}-\mathbf{y};-\mu)) \\
&\quad + \sum_{\mathbf{x}} \left\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T}) P_{\mathbf{x}} + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T}) P_{\mathbf{x}}^\dagger \right\}
\end{aligned}
$$

where

$$
Q(\mathbf{x}-\mathbf{y};\mu) = Q^{(1)}(\mathbf{x}-\mathbf{y}) e^{-\mu/T} + Q^{(2)}(\mathbf{x}-\mathbf{y}) e^{2\mu/T} + Q^{(4)}(\mathbf{x}-\mathbf{y}) e^{-4\mu/T}
$$

The problem is to determine $K(\mathbf{x}-\mathbf{y}), d_1, d_2, Q(\mathbf{x}-\mathbf{y};\mu)$.

Use of the imaginary chemical potential $\mu/T = i\theta$

In terms of Fourier amplitudes

$$\frac{1}{L^3} S_P = \sum_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* \widetilde{K}(k_L) + a_0 \left(d_1 e^{i\theta} - d_2 e^{-2i\theta} \right) + a_0^* \left(d_1 e^{-i\theta} - d_2 e^{2i\theta} \right) + \sum_{\mathbf{k}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} \widetilde{Q}(k_L, \theta) + a_{\mathbf{k}}^* a_{-\mathbf{k}}^* \widetilde{Q}(k_L, \theta) \right)$$

Then

$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0 = \alpha} = 2\widetilde{K}(0)\alpha + 2d_1 \cos(\theta) - (2d_2 - 4\widetilde{Q}(0)\alpha) \cos(2\theta)$$

Fit to

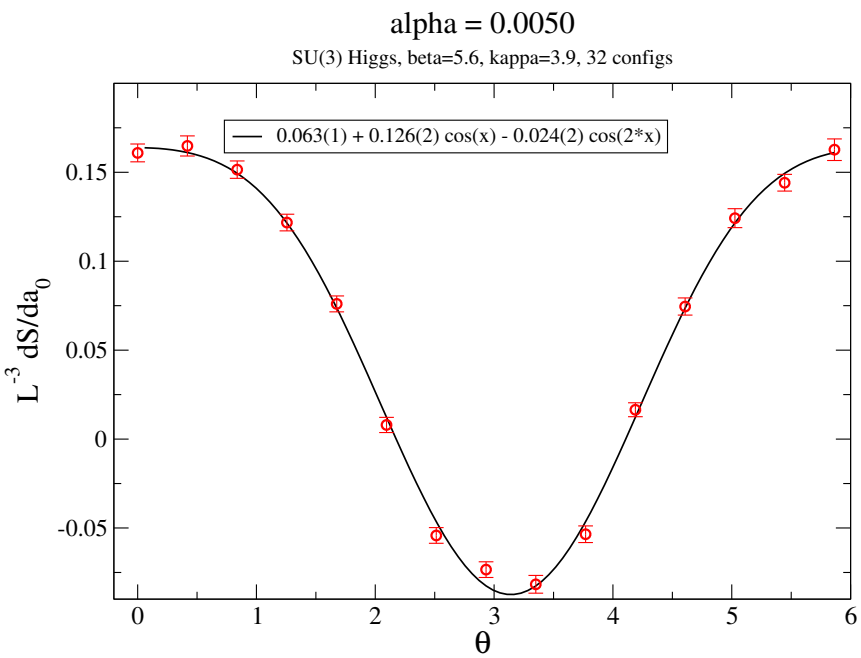
$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0^R = \alpha} = A(\alpha) + B(\alpha) \cos(\theta) - C(\alpha) \cos(2\theta)$$

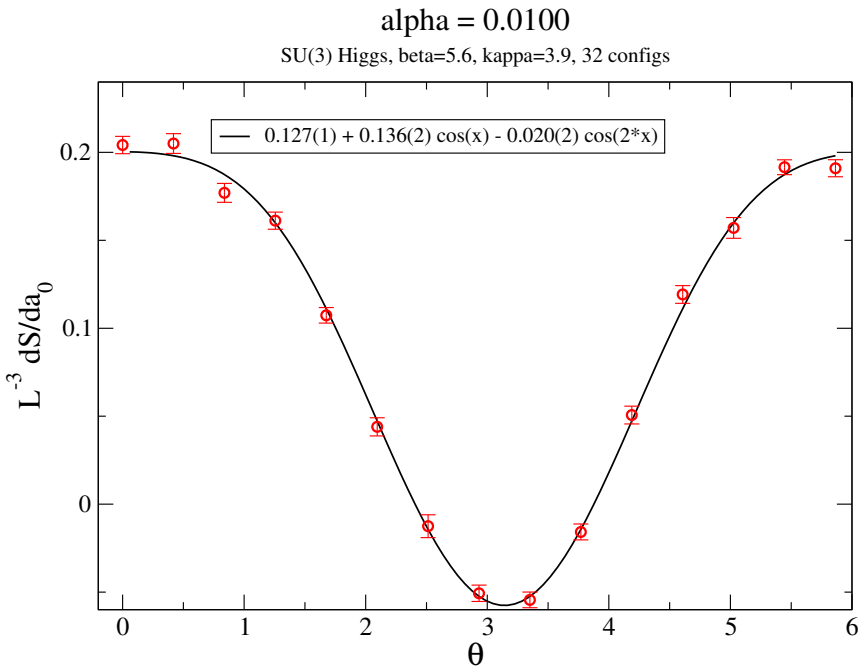
Compare the data to the fit, and we find $d_1, d_2, \widetilde{K}(0), \widetilde{Q}(0)$.

Gauge-Higgs theory at $\beta = 5.6, \kappa = 3.9$ on a $16^3 \times 6$ lattice. Calculate (lhs) and fit (rhs)

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a_0^R}\right)_{a_0^R=\alpha} = A(\alpha) + B(\alpha)\cos(\theta) - C(\alpha)\cos(2\theta)$$

at 15 values of θ and several α values:





We can then extract coefficients of center symmetry-breaking terms (in this case $d_1 = 0.0585, d_2 = 0.0115$), as well as $\widetilde{K}(0)$ and $\widetilde{Q}(0)$.

For $\mathbf{k} \neq 0$, the derivative wrt $a_{\mathbf{k}}$ has terms proportional to $a_{-\mathbf{k}}$. We set $a_{-\mathbf{k}}$ to some constant real value $a_{-\mathbf{k}} = \sigma$. Then

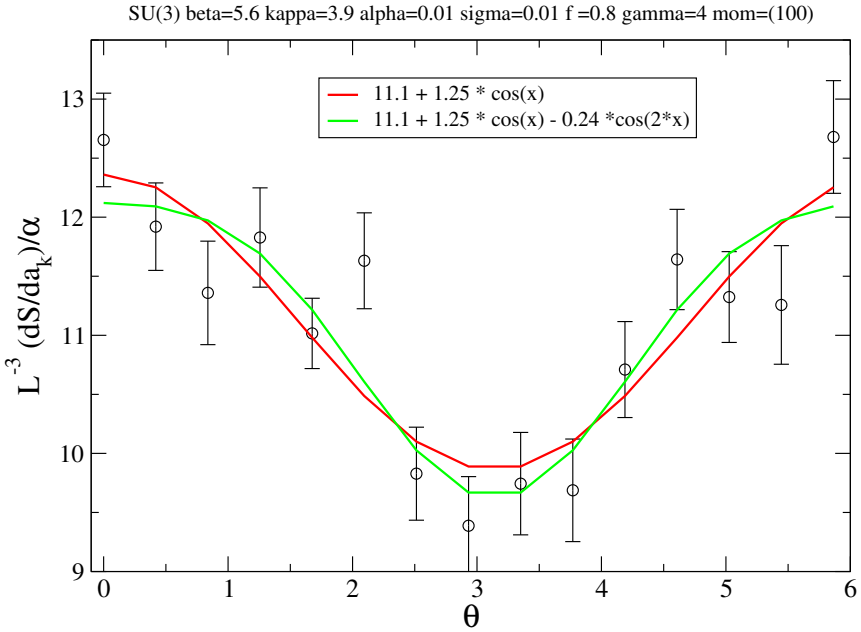
$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)^{a_{-\mathbf{k}}=\sigma}_{a_{\mathbf{k}}=\alpha} = 2\widetilde{K}(k_L)\alpha + 4\Big(\widetilde{Q}^{(1)}(k_L)\cos(\theta) + \widetilde{Q}^{(2)}(k_L)\cos(2\theta) + \widetilde{Q}^{(4)}(k_L)\cos(4\theta)\Big)\sigma$$

First, setting $\sigma = 0$, we have

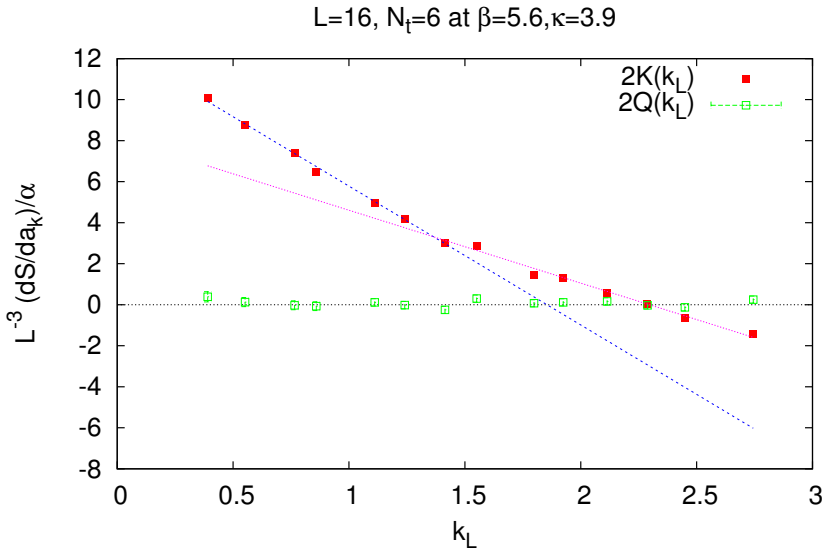
$$\widetilde{K}(k_L) = \frac{1}{2L^3}\frac{1}{\alpha}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)^{\alpha_{-\mathbf{k}}=0}_{a_{\mathbf{k}}=\alpha}$$

Then, at small but finite σ, we can determine the $\widetilde{Q}^{(n)}(k_L)$ from the θ-dependence of the data.

$\widetilde{Q}(k_L, \mu)$ seems calculable, but the magnitude is small and the errorbars are large:



(a) $a_{\mathbf{k}}$ derivative at smallest $k_L \neq 0$, vs. θ



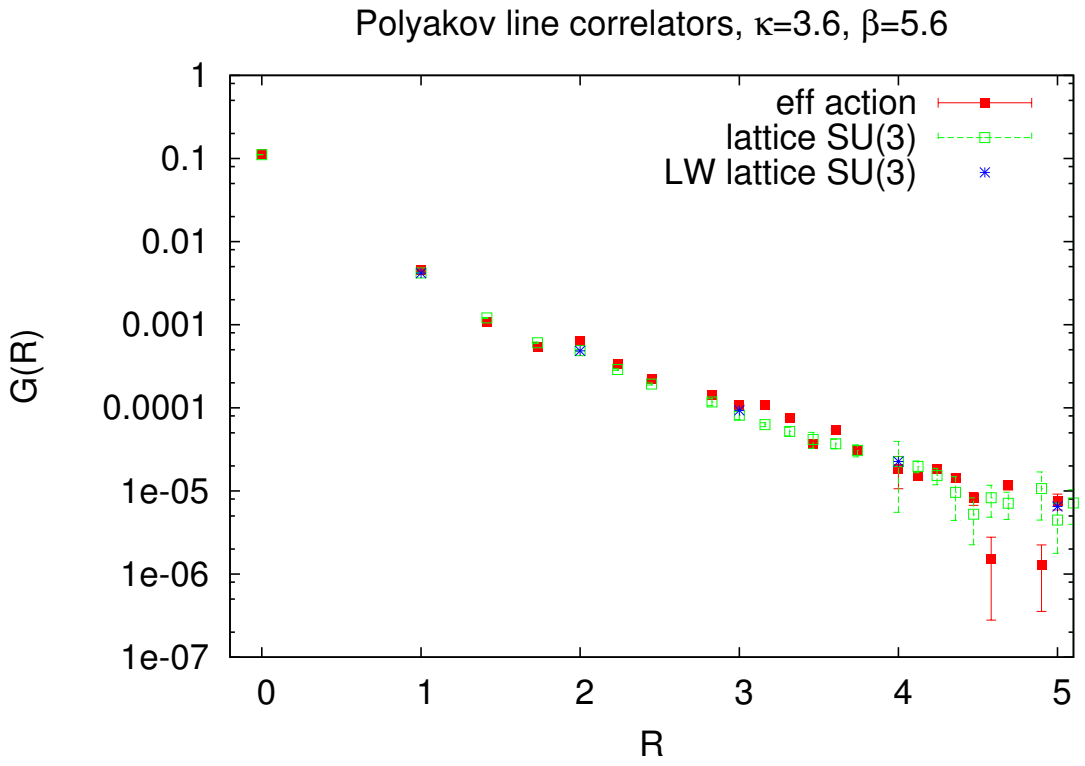
(b) $\widetilde{K}(k_L)$ and (estimate of) $\widetilde{Q}_1(k_L)$ vs. k_L

For now we will ignore the $Q(\mathbf{x} - \mathbf{y}; \mu)$ term in the action.

Gauge-Higgs Correlator Comparison

Effective action vs. lattice gauge theory

The underlying lattice gauge-Higgs theory is at $\beta = 5.6, \mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.



(a) $\kappa = 3.6$

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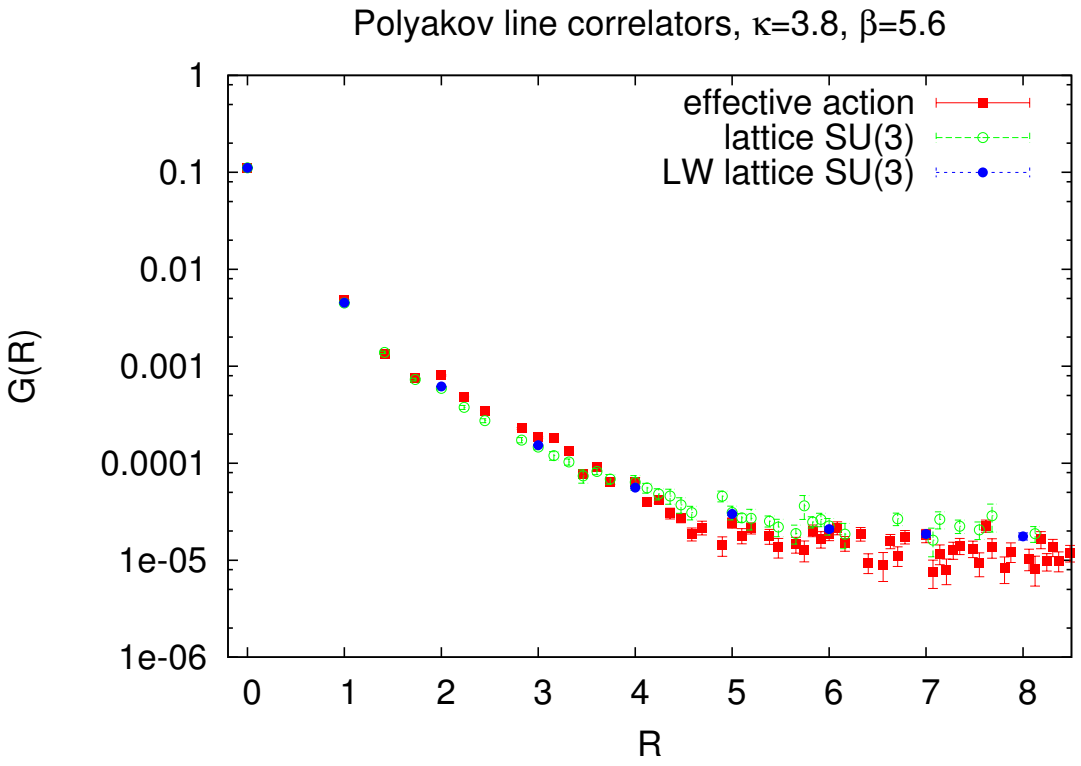

(b) $\kappa = 3.8$

Gauge-Higgs Correlator Comparison

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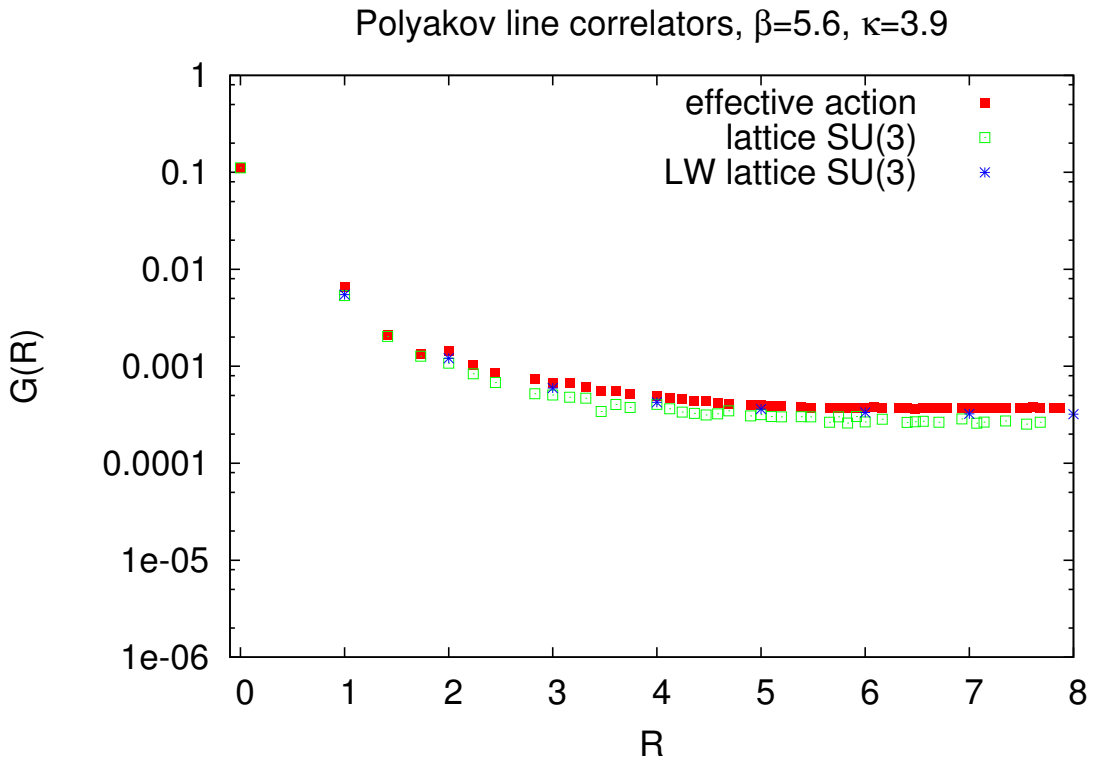

(c) $\kappa = 3.9$

Gauge-Higgs Correlator Comparison

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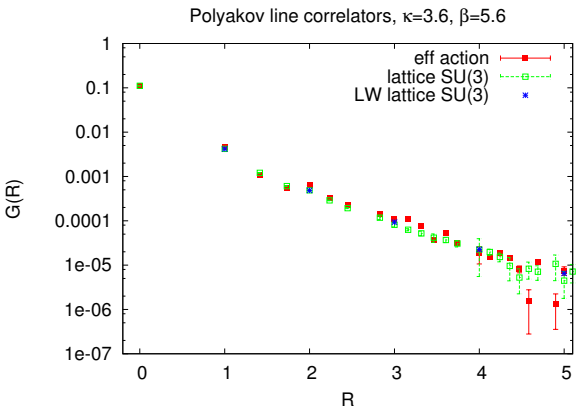
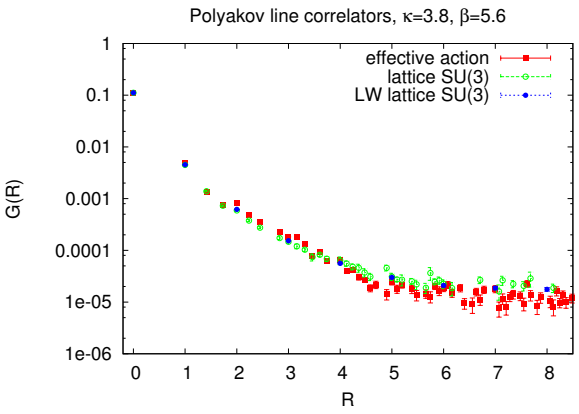
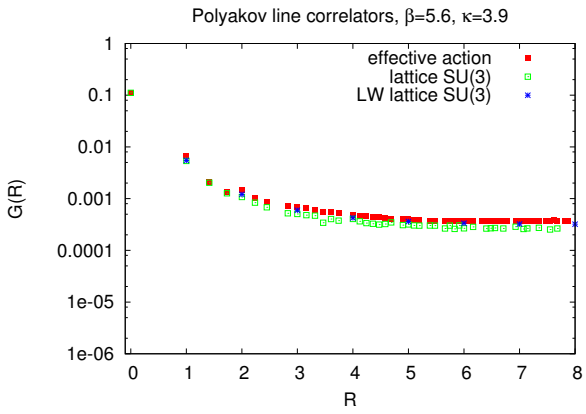


(a) $\kappa = 3.6$



(b) $\kappa = 3.8$

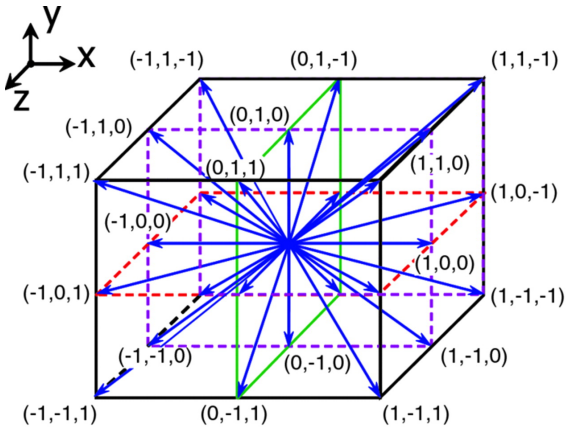


(c) $\kappa = 3.9$

Mean Field solution at $\mu > 0$

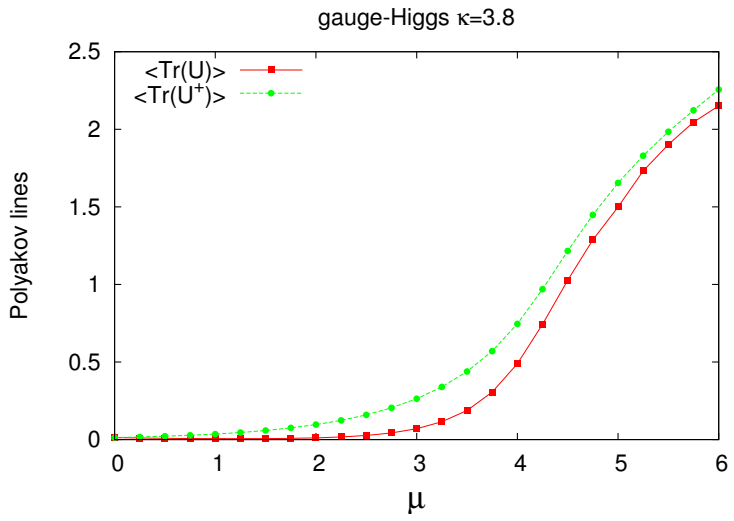
- S_P still has a sign problem.
- It can be addressed in various ways: flux representation, stochastic quantization, reweighting, and mean field.
- In general – mean field becomes more reliable the more spins are coupled to a given spin. But for S_P, *many* spins are coupled to any given spin, especially for light scalar masses, through the non-local kernel $K(\mathbf{x} - \mathbf{y})$.
- *Perhaps the mean field method is more reliable, when applied to S_P at finite μ, than one might expect.*

Whether or not that is true, we have applied mean field to S_P, following the treatment in *Splittorff and JG, arXiv:1206.1159* for the SU(3) spin model.

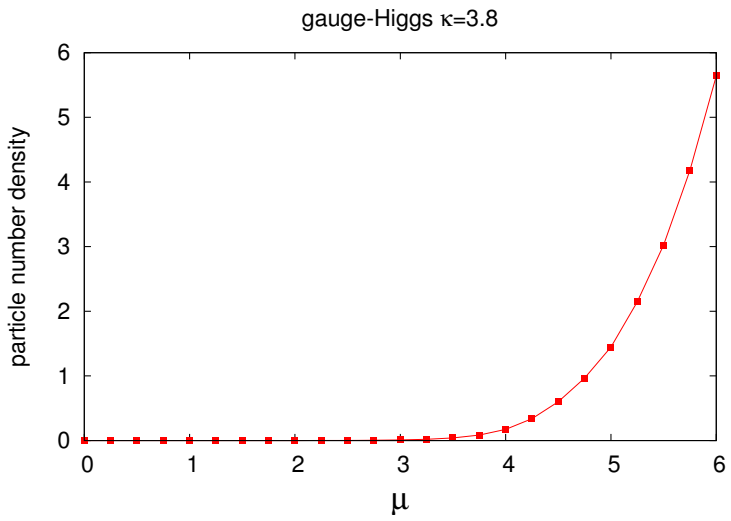


Gauge-Higgs model

Solution of S_P for $\langle \mathrm{Tr} U_{\mathbf{x}} \rangle$, $\langle \mathrm{Tr} U_{\mathbf{x}}^\dagger \rangle$ and particle number/site n, for an underlying lattice gauge-Higgs theory at $\beta = 5.6$ and $\kappa = 3.8$, $16^3 \times 6$ lattice volume, varying μ.



(a) $\langle \mathrm{Tr} U_{\mathbf{x}} \rangle$, $\langle \mathrm{Tr} U_{\mathbf{x}}^\dagger \rangle$



(b) particle number density

Heavy Quark model

In a certain limit where the inverse mass (staggered) or hopping parameter (Wilson) is very small, and the chemical potential μ is large, the fermion determinant simplifies. In temporal gauge, the lattice action is

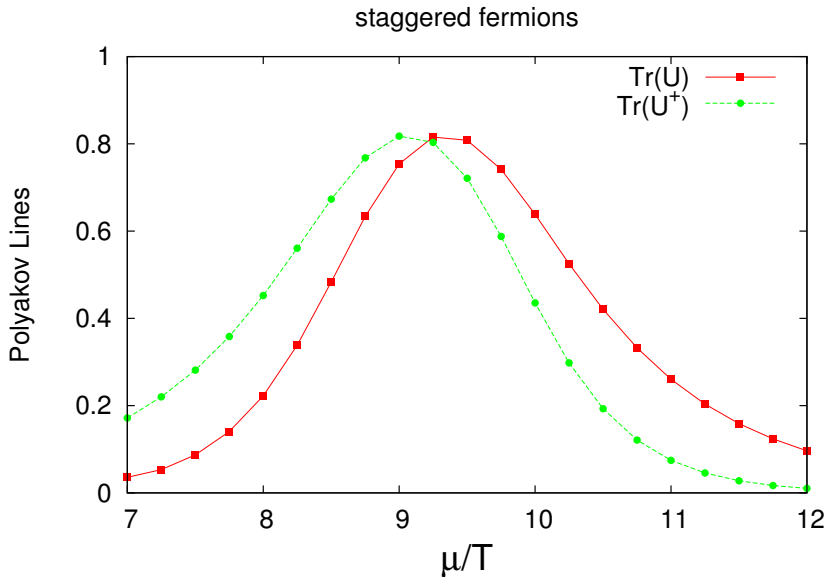
$$e^{S_L} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T} U_0(\mathbf{x}, 0)\right]^p \det\left[1 + he^{-\mu/T} U^\dagger(\mathbf{x}, 0)\right]^p e^{S_{plaq}}$$

where S_{plaq} is the plaquette action, $p = 1$ for staggered fermions, $p = 2N_f$ for Wilson fermions. If we know the Polyakov line action for the pure gauge theory S_P^{pg}, then the Polyakov line action in this heavy quark limit is obtained immediately:

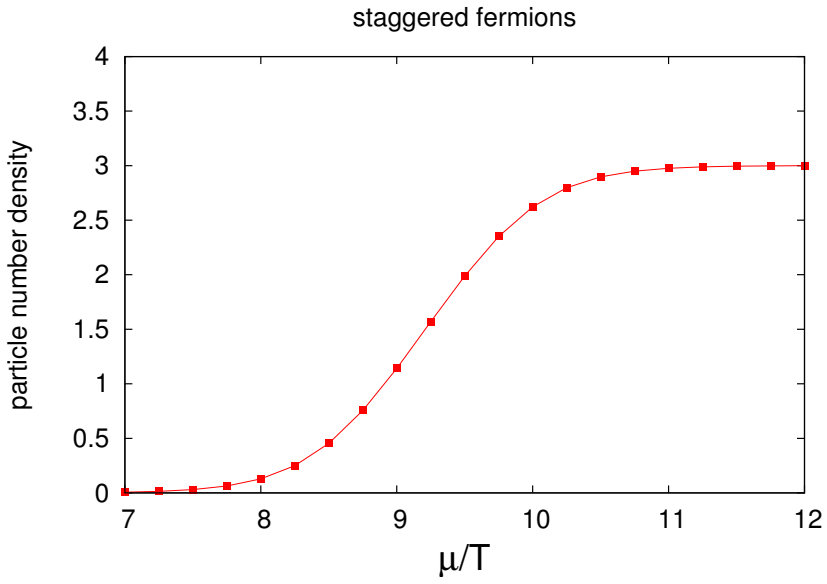
$$e^{S_P} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T} U_{\mathbf{x}}\right]^p \det\left[1 + he^{-\mu/T} U_{\mathbf{x}}^\dagger\right]^p e^{S_P^{pg}}$$

This action is also amenable to a mean field solution.

Here are some mean field results for staggered fermions at $\beta = 5.6$ and $h = 0.0001 \rightarrow m = 2.32/a$. Note the saturation in number density at large μ.



Polyakov lines



number density

CONCLUSIONS

We have determined the effective Polyakov line actions S_P, up to terms bilinear in $P_{\mathbf{x}}$, corresponding to SU(3) pure gauge theory, to SU(3) gauge-Higgs theory, and to SU(3) with heavy quarks, at finite chemical potential.

Next Steps:

1. Beyond bilinear: determine contributions to S_P involving products of three or four Polyakov line variables $P_{\mathbf{x}}$.
2. Beyond mean field: reweighting, stochastic quantization, flux representation...
3. Beyond scalars: relative weights for lighter dynamical fermions.