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AN INTRODUCTION TO SPIN DEPENDENT DEEP INELASTIC SCATTERING*

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Abstract

The main focus of these lectures is on those aspects of deep inelastic scattering that can be derived directly from QCD using quantum field theory, without recourse to phenomenological models. The emphasis is on spin dependent scattering, but the theory of spin averaged scattering is also discussed. A detailed analysis is given for the case of spin 1/2 targets, with a brief discussion of higher spin targets at the end. The QCD derivation of the Callan-Gross relation, the longitudinal structure function F_L, and the Bjorken and Ellis-Jaffe sum rules is presented. I also discuss the Wilczek-Wandzura contribution to g_2, and why the Gottfried sum rule does not hold in QCD.

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1. Introduction

These lectures provide an introductory account of the theory of deep inelastic scattering. Many aspects of deep inelastic scattering can be analysed starting directly from QCD, without having to introduce phenomenological models. It is precisely these aspects of deep inelastic scattering that are most important, and are the focus of these lectures. These are a set of introductory lectures for students, and are not intended to be a review. I have therefore not given any references in the text. A few references are given at the end which might be useful as an introduction to the vast literature on this subject.

The outline of these lectures is as follows. The kinematics of deep inelastic scattering is discussed in Sec. 2. Section 3 defines the leptonic tensor $\ell^{\mu\nu}$, the hadronic tensor $W^{\mu\nu}$, and discusses the symmetry properties of $W^{\mu\nu}$ and its decomposition into structure functions. The idea of scaling is covered in Sec. 4. The deep inelastic cross-section is derived in Sec. 5 for spin-1/2 targets in terms of structure functions. A useful inequality on the structure function g_1 that follows from unitarity is derived in Sec. 6. The physical interpretation of the structure functions in terms of virtual photon scattering is given in Sec. 7. This section also discusses the Compton amplitude $T_{\mu\nu}$, crossing symmetry, the optical theorem, and the analytic structure of the scattering amplitude. A brief discussion of deep inelastic scattering in the parton model can be found in Sec. 8. The operator product expansion and the derivation of the QCD sum rules for the moments of the structure functions are given in Sec. 9. This section is the most important part of these lectures, because it covers the properties of deep inelastic scattering that can be *derived* directly from QCD. Some applications of the results of Sec. 9, such as the Callan-Gross relation, the longitudinal structure function, the Ellis-Jaffe and Bjorken sum rules for g_1, and the Wilczek-Wandzura contribution to g_2 are discussed in Sec. 10. Anomalous dimensions, and scaling violation are discussed in Sec. 11. Section 12 concludes with a brief summary of the structure functions $b_1(x)$ and $\Delta(x)$ for spin one targets.

2. The Kinematics of Deep Inelastic Scattering

In a typical deep inelastic scattering experiment, an incoming beam of leptons with energy E scatters off a fixed hadronic target. The energy and direction of the scattered lepton are measured in the detector, but the final hadronic state (usually denoted by X) is not measured experimentally. The lepton interacts with the hadron target through the exchange of a virtual photon; the target hadron absorbs the virtual photon, to produce the final state X. If the target hadron remains intact, the process is elastic scattering. The deep inelastic region is where the target hadron is blown apart by the virtual photon, and fragments into many particles. I will only discuss the case of fixed-target deep inelastic scattering in detail. Deep inelastic scattering will soon be studied at HERA by colliding an electron beam with a proton beam. The kinematics for such colliding beam experiments is left as an exercise for the reader.

The basic diagram for deep inelastic scattering is show schematically in fig. 1. There are numerous kinematic variables which are used in the discussion of deep inelastic scattering. In the definitions given below, I will pick the $\hat{z}$ axis to be along the incident lepton beam direction. (Warning: in later sections, I will pick the $\hat{z}$ axis to be along the direction of the virtual photon.) The kinematic variables are:

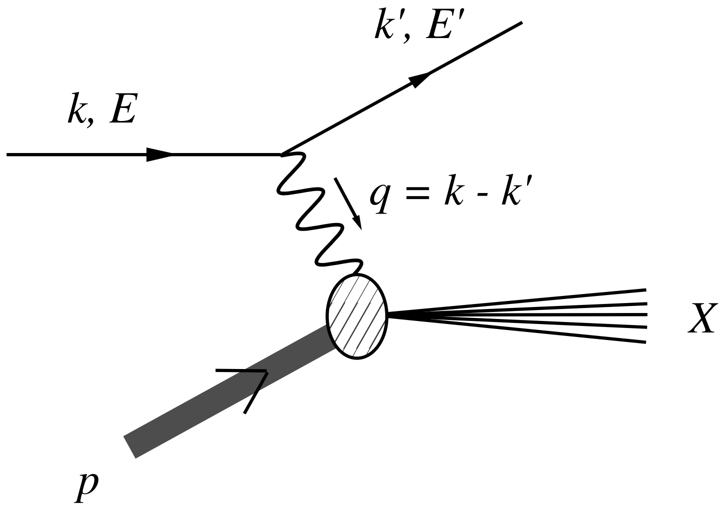

FIGURE 1.

The basic diagram for deep inelastic lepton hadron scattering. The virtual photon momentum is q. The final hadronic state is not measured, and is denoted by X.

Kinematic Variables

M The mass of the target hadron. The most important case is for a proton or neutron target, in which case M is the nucleon mass.

E The energy of the incident lepton.

k The momentum of the initial lepton. $k = (E, 0, 0, E)$, if the lepton mass is neglected.

Ω The solid angle into which the outgoing lepton is scattered.

E' The energy of the scattered lepton.

k' The momentum of the scattered lepton,
$k' = (E', E' \sin\theta \cos\phi, E' \sin\theta \sin\phi, E' \cos\theta)$.

p The momentum of the target, $p = (M, 0, 0, 0)$, for a fixed target experiment.

q $= k - k'$, the momentum transfer in the scattering process, *i.e.* the momentum of the virtual photon.

ν $= E - E' = p \cdot q/M$, the energy loss of the lepton.

y $= \nu/E = p \cdot q/p \cdot k$, the fractional energy loss of the lepton.

Q^2 $= -q^2 = 2EE'(1 - \cos\theta) = 4EE' \sin^2 \theta/2$.

x $= Q^2/2M\nu = Q^2/2p \cdot q = Q^2/2MEy$.

ω $= 1/x$.

The variable x was first introduced by Bjorken, and is crucial to understanding deep inelastic scattering because we will see that QCD predicts that structure functions are functions of x and independent of Q^2 to leading order, a property known as scaling. Higher order corrections in QCD produce a small logarithmic Q^2 dependence of the structure functions which is calculable for large Q^2 since QCD is asymptotically free. I will therefore use the following definition of deep inelastic scattering:

Deep inelastic scattering is the study of lepton-hadron scattering in the limit that x is fixed, and $Q^2 \to \infty$.

The invariant mass of the final hadronic system X is

$$M_X^2 = (p + q)^2 = M^2 + 2p \cdot q + q^2. \tag{2.1}$$

The invariant mass of X must be at least that of a nucleon, since baryon number is conserved in the scattering process. This gives the inequality

$$M_X^2 \geq M^2 \quad \Rightarrow \quad M^2 + 2p \cdot q - Q^2 \geq M^2 \quad \Rightarrow \quad x \leq 1. \tag{2.2}$$

Since Q^2 and ν are both positive, x must also be positive. The lepton energy loss $E - E'$ must be between zero and E, so the physically allowed kinematic region is

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1. \tag{2.3}$$

Eq. (2.2) can be written in the form

$$x=\frac{Q^2}{2p\cdot q}=1-\frac{M_X^2-M^2}{2p\cdot q}. \tag{2.4}$$

The value $x=1$ implies that $M_X^2=M^2$, and so $x=1$ corresponds to elastic scattering. Any fixed hadron state X with invariant mass M_X^2 contributes to the cross-section at the value of x obtained from Eq. (2.4),

$$x=\frac{1}{1+\left(M_X^2-M^2\right)/Q^2}. \tag{2.5}$$

In the deep inelastic limit $Q^2\to\infty$, so that any state X with fixed mass M_X gets driven to $x=1$. In particular, all nucleon resonances such as the N^* get pushed to $x=1$. The invariant mass of the hadronic state that contributes to the cross-section at a given value of Q^2 increases as $x\to 0$. For a fixed value of Q^2, there is a small region around $x=1$ of width $\Lambda_{\rm QCD}^2/Q^2$ which probes hadronic resonances with masses around that of the nucleon. This resonance region is present in any real experiment since Q^2 is finite, but is not present in the formal deep inelastic limit. Outside the resonance region, the invariant mass M_X^2 is of order Q^2.

The experimental measurements give the cross-section as a function of the final lepton energy and scattering angle, $d^2\sigma/dE'd\Omega$. The results are often presented instead by giving the differential cross-section as a function of $(x,\, Q^2,\, \phi)$ or $(x,\, y,\, \phi)$. The Jacobian for converting between these cases is easily worked out using the definitions of the kinematic variables,

$$\begin{aligned}
\frac{\partial(x,Q^2)}{\partial(x,y)}&=\begin{vmatrix} 1 & 0 \\ 2MEy & 2MEx \end{vmatrix}=2MEx=\frac{Q^2}{y},\\
\frac{\partial(x,y)}{\partial(E',\cos\theta)}&=\begin{vmatrix} ? & \frac{-2EE'}{2M\nu} \\ -\frac{1}{E} & 0 \end{vmatrix}=\frac{E'}{M\nu}.
\end{aligned} \tag{2.6}$$

Thus the cross-sections are related by

$$\frac{d^2\sigma}{dx\,dy\,d\phi}=\left(\frac{M\nu}{E'}\right)\frac{d^2\sigma}{dE'd\Omega}=\left(\frac{Q^2}{y}\right)\frac{d^2\sigma}{dx\,dQ^2d\phi}. \tag{2.7}$$

For most cases of interest, the cross-section will be independent of the azimuthal angle ϕ. In these lectures, I will give the differential cross-section in $(x,\, y,\, \phi)$. The variables x and y are Lorentz invariant, because they are defined using the inner products of four vectors. Thus the formula for the differential cross-section as a function of x, y, and ϕ is also valid for the kinematics at HERA, since cross-sections (and ϕ) are invariant under boosts along the collision axis. The only difference is that the transformation formula between $(x,\, y)$, and $(E',\, \theta)$ is modified.

There are three independent variables which describe the kinematics, E', θ and ϕ. The dependence on ϕ is trivial, so I will only discuss the dependence on the other two independent variables. It is convenient to plot the allowed kinematic region in the $(Q^2/2ME) - (\nu/E)(= y)$ plane, as shown in fig. 2. The boundary of the physical region is given by the requirements that

$$0 \le \theta \le \pi, \quad 0 \le \nu \le E, \quad 0 \le x \le 1. \tag{2.8}$$

Since $x = Q^2/2M\nu = (Q^2/2ME)/(\nu/E)$, the contours of constant x are straight lines through the origin with slope x. The relation between Q^2 and θ can be written as

$$Q^2 = 2EE'(1-\cos\theta) \quad \Rightarrow \quad \frac{Q^2}{2ME} = \frac{1}{M}(E-\nu)(1-\cos\theta). \tag{2.9}$$

Thus lines of constant θ are straight lines passing through the point $\nu/E = 1$, and intersecting the $Q^2/2ME$ axis at $Q^2/2ME = (E/M)(1-\cos\theta)$. Lines of fixed θ become steeper as the beam energy increases, whereas lines of fixed x remain constant.

The Q^2 dependence of the kinematic variables is crucial to understanding which terms are important in the deep inelastic limit. A generic point in the kinematic plane is given by some value of x and y. As $E \to \infty$ for a fixed value of x and y, the variables ν/E and $Q^2/2ME$ are fixed. This implies that in the deep inelastic limit, ν and E are each proportional to Q^2,

$$\nu \ \propto \ E \ \propto \ \frac{Q^2}{M}. \tag{2.10}$$

A generic point in the physical region has $(1-\cos\theta)$ of order M/E,

$$(1-\cos\theta) \ \propto \ \frac{M}{E} \ \propto \ \frac{M^2}{Q^2}, \tag{2.11}$$

so that the scattering angle θ is of order M/Q,

$$\theta \ \propto \ \frac{M}{Q}, \tag{2.12}$$

and goes to zero as $Q^2 \to \infty$.

The allowed region in fig. 2 also shows that for fixed beam energy E, there is a limit to the $Q^2 - x$ region which can be explored experimentally. The small x region is also the

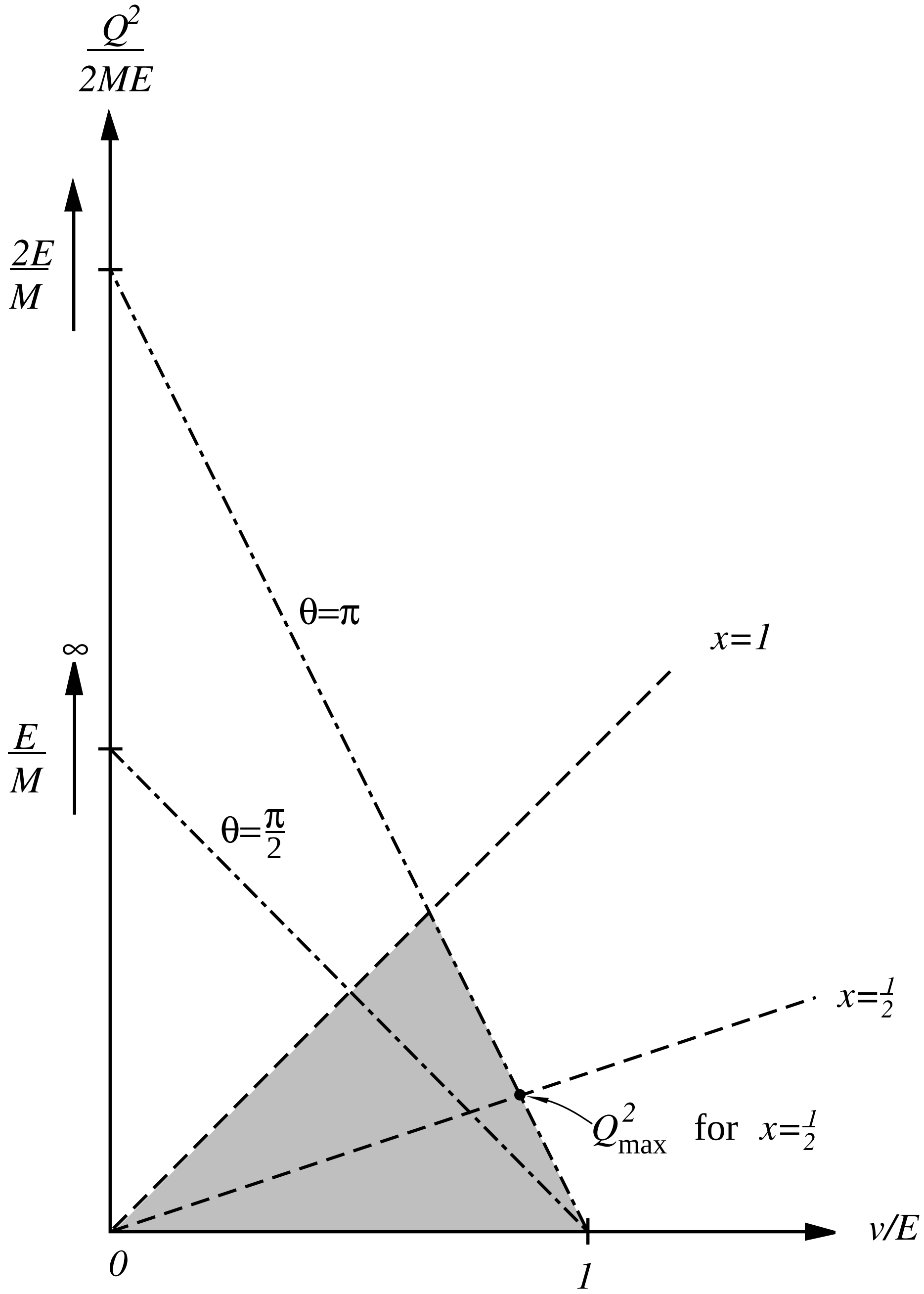

FIGURE 2.

The kinematically allowed region for deep inelastic scattering is shown in grey. The dot-dashed lines are lines of constant scattering angle θ, and the dashed lines are lines of constant x. In the deep inelastic limit, the intercept of the constant θ lines with the vertical axis becomes infinite.

small Q^2 region, because lines of constant x approach the horizontal axis for small x. For a fixed value of x, the maximum allowed value of Q^2 is at the intersection of the line $\theta=\pi$ with the line for fixed x. It is elementary to find the intersection point of the two lines,

$$Q^2_{\rm max} = 2MEx\left(\frac{2E}{2E+Mx}\right) \approx 2MEx, \qquad (E \gg Mx). \tag{2.13}$$

To be in the deep inelastic region, one needs Q^2 to be larger than a few $(\mathrm{GeV})^2$, so this places a limit on the smallest value of x accessible for a given beam energy. For example, with a 500 GeV lepton beam, and assuming $Q^2 \geq 10$ $(\mathrm{GeV})^2$ is large enough to be considered deep inelastic scattering, the smallest measurable value of x is 10^{-2}. The factor of $2ME$ in Eq. (2.13) is proportional to the centre of mass energy squared, *i.e.* to $s=(p+k)^2$. HERA will collide 28 GeV leptons on 820 GeV protons, so the value of s is 4(28)(820) $(\mathrm{GeV})^2$. This will allow the measurement of structure functions down to $x \approx 10^{-4}$. A similar measurement in a fixed target experiment would require a 50 TeV lepton beam.

Finally, it is useful to have formulæ for the different components of q as a function of x and y. The result is obtained by substituting for θ in terms of Q^2 into the definition of q,

$$q = k - k' = (E-E', E'\sin\theta\cos\phi, E'\sin\theta\sin\phi, E-E'\cos\theta), \tag{2.14}$$

$$\begin{aligned} q^0 &= \frac{Q^2}{2Mxy}, \qquad q^3 = \frac{Q^2}{2Mxy} + Mxy, \\ q^\perp &= Q\sqrt{1-y}\sqrt{1-\frac{M^2x^2y^2}{Q^2(1-y)}} \approx Q\sqrt{1-y}. \end{aligned} \tag{2.15}$$

Note that $q^\perp$ is of order Q, whereas q^0 and q^3 are both of order Q^2/M. Also, q^0 and q^3 are nearly equal, with q^0-q^3 being of order M. These properties of Q will be useful when we compute cross-sections in the deep inelastic limit.

3. Electroproduction, $\ell_{\mu\nu}$, and $W_{\mu\nu}$

The basic Feynman graph for deep inelastic scattering is fig. 1. The scattering amplitude $\mathcal{M}$ is given by

$$i\mathcal{M} = (-ie)^2\left(\frac{-ig_{\mu\nu}}{q^2}\right)\langle k'|\, j^\mu_\ell(0)\,|k, s_\ell\rangle\,\langle X|\, j^\nu_h(0)\,|p,\lambda\rangle\,, \tag{3.1}$$

where s_ℓ is the polarisation of the initial lepton, λ is the polarisation of the initial hadron, and j_ℓ^μ, j_h^μ are the leptonic and hadronic electromagnetic currents respectively. The coupling constant e is not included in the definition of the current, and hence has been included explicitly in $\mathcal{M}$. The quantity λ which describes the hadron polarisation can be chosen to be the value of the spin along an arbitrary quantisation axis (usually chosen to be the beam direction). For a spin 1/2 target, $\lambda = \pm 1/2$. We will discuss the hadron polarisation in greater detail later in this section.

The differential scattering cross-section is obtained from $\mathcal{M}$ by squaring, and multiplying by the phase space factors. The polarisation of the final lepton and hadron states are not measured, and have to be summed over. This gives the differential cross-section

$$\begin{aligned} d\sigma &= \sum_X \int \frac{d^3k'}{(2\pi)^3 2E'}\,(2\pi)^4\delta^4(k+p-k'-p_X)\,\frac{|\mathcal{M}|^2}{(2E)(2M)(v_{\rm rel}=1)}, \\ &= \sum_X \int \frac{d^3k'}{(2\pi)^3 2E'}\,\frac{(2\pi)^4\delta^4(k+p-k'-p_X)}{(2E)(2M)}\,\frac{e^4}{Q^4} \\ &\times \langle p,\lambda|\, j_h^\mu(0)\,|X\rangle \langle X|\, j_h^\nu(0)\,|p,\lambda\rangle \langle k,s_\ell|\, j_{\ell\mu}(0)\,|k'\rangle \langle k'|\, j_{\ell\nu}(0)\,|k,s_\ell\rangle , \end{aligned} \tag{3.2}$$

where I have used the relation

$$\langle\alpha|\, j^\mu\,|\beta\rangle^* = \langle\beta|\, j^\mu\,|\alpha\rangle , \tag{3.3}$$

which is true because the current is hermitian, $j_\mu^\dagger = j_\mu$. It is conventional to define the leptonic tensor $\ell_{\mu\nu}$ by

$$\ell^{\mu\nu} = \sum_{\text{final spin}} \langle k'|\, j_\ell^\nu(0)\,|k,s_\ell\rangle \langle k,s_\ell|\, j_\ell^\mu(0)\,|k'\rangle , \tag{3.4}$$

so that the leptonic current matrix elements in Eq. (3.2) can be replaced by $\ell_{\mu\nu}$. The definition of the hadronic tensor is slightly more complicated. The tensor $W_{\mu\nu}$ is defined by

$$W^{\mu\nu}(p,q)_{\lambda'\lambda} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\,\langle p,\lambda'|\,[j^\mu(x),j^\nu(0)]\,|p,\lambda\rangle \ , \tag{3.5}$$

where λ and λ' are the polarisations of the initial and final hadron. Inserting a complete set of states gives

$$\begin{aligned} W^{\mu\nu}(p,q)_{\lambda'\lambda} = \frac{1}{4\pi}\sum_X \int d^4x\, e^{iq\cdot x}\Big[&\langle p,\lambda'|\, j^\mu(x)\,|X\rangle \langle X|\, j^\nu(0)\,|p,\lambda\rangle \\ &- \langle p,\lambda'|\, j^\nu(0)\,|X\rangle \langle X|\, j^\mu(x)\,|p,\lambda\rangle\Big], \end{aligned} \tag{3.6}$$

where the sum on X is a sum over the allowed phase space for the final state X. Translation invariance implies that

$$\begin{aligned}\langle p,\lambda'|\, j^\mu(x)\,|X\rangle &= \langle p,\lambda'|\, j^\mu(0)\,|X\rangle\, e^{i(p-p_X)\cdot x},\\ \langle X|\, j^\mu(x)\,|p,\lambda\rangle &= \langle X|\, j^\mu(0)\,|p,\lambda\rangle\, e^{i(p_X-p)\cdot x}.\end{aligned} \tag{3.7}$$

Inserting Eq. (3.7) into Eq. (3.6) gives

$$\begin{aligned}W^{\mu\nu}(p,q)_{\lambda'\lambda} = \frac{1}{4\pi}\sum_X \Big[&(2\pi)^4\delta^4(q+p-p_X)\,\langle p,\lambda'|\, j^\mu(0)\,|X\rangle\,\langle X|\, j^\nu(0)\,|p,\lambda\rangle\\ &-(2\pi)^4\delta^4(q+p_X-p)\,\langle p,\lambda'|\, j^\nu(0)\,|X\rangle\,\langle X|\, j^\mu(0)\,|p,\lambda\rangle\Big].\end{aligned} \tag{3.8}$$

The only allowed final states are those with $p_X^0 \ge p^0$, since $M_X^2 \ge M^2$. Since $q^0 > 0$, only the first delta function in Eq. (3.8) can be satisfied, and the sum in $W_{\mu\nu}$ reduces to the expression in Eq. (3.2) involving the hadronic currents, and the energy-momentum conserving delta function (up to a factor of $1/4\pi$). Only the first term in Eq. (3.6) contributes, which is equivalent to the statement that one could have defined $W_{\mu\nu}$ in Eq. (3.5) simply as the matrix element of $j^\mu(x)j^\nu(0)$ without the commutator. The reason for using the commutator is that then $W_{\mu\nu}$ has a nicer analytic structure when continued away from the physical region which we will use extensively in Sec. 9. Substituting for $\ell_{\mu\nu}$ and $W^{\mu\nu}$ in Eq. (3.2) gives

$$d\sigma = \frac{e^4}{Q^4}\int \frac{d^3k'}{(2\pi)^3 2E'}\;\frac{4\pi\;\ell^{\mu\nu}\;W^{\mu\nu}(p,k-k')_{\lambda\lambda}}{(2E)(2M)(v_{\rm rel}=1)}, \tag{3.9}$$

so that

$$\begin{aligned}\frac{d^2\sigma}{dE'd\Omega} &= \frac{e^4}{16\pi^2 Q^4}\left(\frac{E'}{ME}\right)\ell_{\mu\nu}\;W^{\mu\nu}(p,q)_{\lambda\lambda},\\ \frac{d^2\sigma}{dx\,dy\,d\phi} &= \frac{e^4}{16\pi^2 Q^4}\;y\;\ell_{\mu\nu}\;W^{\mu\nu}(p,q)_{\lambda\lambda}.\end{aligned} \tag{3.10}$$

All the information about the deep inelastic scattering cross-section is contained in the leptonic and hadronic tensors $\ell_{\mu\nu}$ and $W_{\mu\nu}$.

The Leptonic Tensor $\ell^{\mu\nu}$

The leptonic tensor is trivial to compute since the leptons are pointlike fermions,

$$\ell^{\mu\nu} = \sum_{\text{final spin}} \overline{u}(k')\,\gamma^\nu\,u(k,s_\ell)\;\overline{u}(k,s_\ell)\,\gamma^\mu\,u(k'). \tag{3.11}$$

The sum over final state spinors uses the identity

$$\sum_{\text{final spin}} u(k')\,\overline{u}(k') = \not{k}' + m, \tag{3.12}$$

where the spinors are normalised to $2E$.

The polarisation of a spin $1/2$ particle can be described by a spin vector s_ℓ, defined in the rest frame of the particle by

$$2\vec{s}_\ell = \overline{u}(k,s_\ell)\,\vec{\sigma}\,u(k,s_\ell) = \overline{u}(k,s_\ell)\,\vec{\gamma}\,\gamma_5\,u(k,s_\ell), \tag{3.13}$$

where the states have the conventional relativistic norm of $2m$ in the rest frame. The vector s_ℓ^μ is defined in an arbitrary Lorentz frame by boosting $s_\ell = (0,\vec{s}_\ell)$ from the rest frame, or equivalently, by

$$2s_\ell^\mu = \overline{u}(k,s_\ell)\,\gamma^\mu\gamma_5\,u(k,s_\ell). \tag{3.14}$$

For a spin-$1/2$ particle at rest with spin up along the $\hat{z}$ axis, the spin vector is $\vec{s}_\ell = m\hat{z}$. This differs from the conventional normalisation of s_ℓ by a factor of the fermion mass m. It is extremely useful to use this normalisation for the spin to avoid unnecessary factors of m appearing in the spin dependent cross-section. In the extreme relativistic limit, all mass effects can be neglected in the calculation, and all formulæ should be written in terms of relativistic spinors normalised to $2E$, without any additional factors of m. This suggests that one define spin by Eq. (3.14) without any factors of m, and thus define a spin vector with the dimensions of mass, which is m times the conventional definition. With this normalisation for s_ℓ, longitudinally polarised fermions in the extreme relativistic limit have $s_\ell = \mathcal{H}_\ell k$, where k is the lepton momentum and $\mathcal{H}_\ell = \pm$ is the lepton helicity.

The initial state spinor product can be written in terms of the standard spin projection operator,

$$u(k,s_\ell)\,\overline{u}(k,s_\ell) = (\not{k} + m)\;\frac{1+\gamma_5\not{s}_\ell/m_\ell}{2}. \tag{3.15}$$

Substituting Eq. (3.12) and (3.15) into Eq. (3.11) gives

$$\begin{aligned}
\ell^{\mu\nu} &= \mathrm{Tr}\;(\not{k}' + m_\ell)\,\gamma^\nu\,(\not{k} + m_\ell)\;\frac{1+\gamma_5\not{s}_\ell/m_\ell}{2}\;\gamma^\mu, \\
&= 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k' - m_\ell^2) - i\epsilon^{\mu\nu\alpha\beta}q_\alpha s_{\ell\beta}\right), \\
&\approx 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}k\cdot k' - i\epsilon^{\mu\nu\alpha\beta}q_\alpha s_{\ell\beta}\right),
\end{aligned} \tag{3.16}$$

neglecting the lepton mass. There are no factors of m_ℓ in the spin term with our normalisation convention Eq. (3.14). Note that the spin-independent part of the leptonic tensor is symmetric in $\mu\nu$, and the spin-dependent part is antisymmetric in $\mu\nu$. Thus an unpolarised lepton beam probes only the symmetric part of $W_{\mu\nu}$, and the polarisation asymmetry in the cross-sections probes only the antisymmetric part of $W_{\mu\nu}$.

The Hadronic Tensor $W^{\mu\nu}$ for Spin-1/2 Targets

The structure of the hadron relevant for deep inelastic scattering can be completely characterised by the hadronic tensor $W_{\mu\nu}$. Unlike the leptonic tensor which is known, $W_{\mu\nu}$ cannot be computed directly from QCD because of non-perturbative effects in the strong interactions. The strong interactions are invariant under parity and time reversal, and these symmetries place restrictions on the form of $W_{\mu\nu}$. In this section, I will concentrate on hadronic targets with spin-1/2, which is the most important case experimentally. The tensor $W_{\mu\nu}$ depends on the polarisations λ, λ', which for a spin-1/2 target can each take the values $\pm 1/2$. We will need to discuss the case where $\lambda \neq \lambda'$ to describe an arbitrary target polarisation, even though it appears that only $\lambda = \lambda'$ occurs in Eq. (3.10).

The most general polarisation state of a spin J target in its rest frame can be written as a density matrix

$$\rho = \sum_{\lambda=-J}^{J} \sum_{\lambda'=-J}^{J} \left|\lambda\right\rangle \rho_{\lambda\lambda'} \left\langle\lambda'\right| . \tag{3.17}$$

The states $|\lambda\rangle$ and $|\lambda'\rangle$ transform as the spin-J representation under the rotation group. Thus the density matrix ρ transforms under the representation $J \otimes J = 0 \oplus 1 \oplus \ldots \oplus 2J$. The density matrix of a spin-J target can thus be described by $2J+1$ irreducible tensors. We will restrict ourselves in this section to $J = 1/2$, whose polarisation density matrix can be decomposed into irreducible tensors of spin 0 and 1, *i.e.* by a scalar and a vector. It is easy to see that one can define a scalar and a vector from ρ by

$$\rho_{\text{scalar}} \equiv \text{Tr}\,\rho = 1, \qquad \vec{\rho}_{\text{vector}} \equiv \text{Tr}\,\rho\,\vec{\sigma} = \vec{s}_h/M, \tag{3.18}$$

which defines $\vec{s}_h$. ρ can be expressed in terms of these irreducible tensors,

$$\rho = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\frac{\vec{s}_h}{M}\right). \tag{3.19}$$

The only complication for targets with spin greater than 1/2 is that the decompositions analogous to Eqs. (3.18) and (3.19) are more complicated, which is why I have restricted the analysis of $W_{\mu\nu}$ to spin-1/2 targets. A spin-1/2 polarised target can be described by an axial vector s_h, which has been defined with an additional factor of the target mass

M for the same reasons that we absorbed a factor of the leptonic mass into the lepton polarisation vector s_ℓ. The magnitude of $\vec{s}_h$ depends on the degree of polarisation of the target; for an unpolarised target, $\vec{s}_h = 0$, for a 100% polarised target, $|\vec{s}_h| = M$, and for a target with fractional polarisation f, $|\vec{s}_h| = fM$. The four vector s_h^μ is defined to be $(0, \vec{s}_h)$ in the rest frame of the target, and is defined in other reference frames by a Lorentz boost.

The tensor $W_{\mu\nu}(p, q, s_h)$ for a spin-1/2 target is defined by

$$W_{\mu\nu}(p, q, s_h) = \sum_{\lambda,\lambda'} \rho_{\lambda\lambda'}\, W_{\mu\nu}(p, q)_{\lambda'\lambda} = \mathrm{Tr}\, \rho\, W_{\mu\nu} \tag{3.20}$$

treating $W_{\mu\nu}$ as a matrix in spin space with labels λ, λ'. Thus the most general tensor $W_{\mu\nu}$ for a spin-1/2 target can be written as a function of the vectors p, q and s_h. Note that s_h contains the spin information of both the initial and final state spinors through the decomposition of the density matrix Eq. (3.19). Thus s_h can only occur *linearly* in $W_{\mu\nu}$; there are no terms with two factors of s_h.

The Decomposition of $W^{\mu\nu}$ into Structure Functions

The deep inelastic structure functions are obtained by writing down the most general tensor decomposition of $W_{\mu\nu}$. The strong interactions are invariant under parity. The parity transform of the electromagnetic current is

$$\mathcal{P} j^\mu(x) \mathcal{P}^{-1} = j^{\mu_P}(x_P), \tag{3.21}$$

with $\mu_P = \mu$ for $\mu = 0$, and $\mu_P = -\mu$ for $\mu = 1, 2, 3$. To avoid writing complicated formulæ, I will use the obvious convention $j^{-1} = -j^1$, *etc.*, for the transformation of the various components of a vector under parity. The parity transform of the point x is $x_P = (t, -\vec{x})$, and the parity transform of the polarisation density matrix is

$$\mathcal{P} \rho \mathcal{P}^{-1} = \rho_P, \qquad \mathrm{Tr}\, \rho_P = 1, \qquad M\, \mathrm{Tr}\, \rho_P \vec{\sigma} = \vec{s}_h, \tag{3.22}$$

since s_h is an axial vector. This implies

$$\begin{aligned} W^{\mu\nu}(p, q, s_h) &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x}\, \mathrm{Tr}\, \rho\, [j^\mu(x), j^\nu(0)] \\ &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x}\, \mathrm{Tr}\ \mathcal{P}\rho\mathcal{P}^{-1}\ \mathcal{P}[j^\mu(x), j^\nu(0)]\mathcal{P}^{-1} \\ &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x}\, \mathrm{Tr}\, \rho_P\, [j^{\mu_P}(x_P), j^{\nu_P}(0)]. \end{aligned} \tag{3.23}$$

Defining $q_P = (q^0, -\vec{q})$, $(s_h)_P = (-s_h^0, \vec{s}_h)$, and using $q_P \cdot x_P = q \cdot x$, one can rewrite the last equality in Eq. (3.23) as

$$W^{\mu\nu}(p, q, s_h) = W^{\mu_P \nu_P}(p_P, q_P, (s_h)_P), \tag{3.24}$$

which is the condition that $W_{\mu\nu}$ is parity invariant.

Invariance under time-reversal is only slightly more complicated, because time reversal is antiunitary. The time-reversal of momentum is defined by $p_T = (p^0, -\vec{p})$, spin by $(s_h)_T = (s_h^0, -\vec{s}_h)$, and coordinates by $x_T = (-t, \vec{x})$. Note that time reversal takes x^0 to $-x^0$, but still takes p^0 to $+p^0$, *i.e.* time reversal changes the sign of time, but not of energy. Time reversal invariance then implies that

$$\begin{aligned} W^{\mu\nu}(p, q, s_h) &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x}\, \mathrm{Tr}\, \rho\, [j^\mu(x), j^\nu(0)] \\ &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x} \left(\mathrm{Tr}\, \mathcal{T} \rho \mathcal{T}^{-1}\ \mathcal{T}[j^\mu(x), j^\nu(0)]\mathcal{T}^{-1}\right)^* \\ &= \frac{1}{4\pi} \int d^4x\, e^{iq\cdot x} \left(\mathrm{Tr}\, \rho_T\, [j^{\mu_T}(x_T), j^{\nu_T}(0)]\right)^*, \end{aligned} \tag{3.25}$$

where the complex conjugation in the second line is present because time reversal is antiunitary. Using the identity $q \cdot x = -q_T \cdot x_T$, we get

$$e^{iq\cdot x} = \left(e^{iq_T \cdot x_T}\right)^*, \tag{3.26}$$

so that time reversal invariance implies that

$$W^{\mu\nu}(p, q, s_h) = W^{\mu_T \nu_T}(p_T, q_T, (s_h)_T)^*. \tag{3.27}$$

The electromagnetic currents and density matrix are hermitian, which implies

$$\begin{aligned} W^{\mu\nu}(p, q, s_h)^* &= \frac{1}{4\pi} \int d^4x\, e^{-iq\cdot x}\, \mathrm{Tr}\, \rho^\dagger\, [j^\mu(x), j^\nu(0)]^\dagger \\ &= \frac{1}{4\pi} \int d^4x\, e^{-iq\cdot x}\, \mathrm{Tr}\, \rho\, [j^\nu(0), j^\mu(x)] \\ &= \frac{1}{4\pi} \int d^4x\, e^{-iq\cdot x}\, \mathrm{Tr}\, \rho\, [j^\nu(-x), j^\mu(0)] = W^{\nu\mu}(p, q, s_h), \end{aligned} \tag{3.28}$$

where the first line follows from the identity $(\mathrm{Tr}\, AB)^* = \mathrm{Tr}(AB)^\dagger$ and the last line follows from translation invariance. Note that all three identities, Eqs. (3.24), (3.27), and (3.28) relate $W^{\mu\nu}$ for q^0 to a transformed $W^{\mu\nu}$ at the same value of q^0.

There is a final identity that relates $W_{\mu\nu}$ for $q^0 > 0$ to $W_{\mu\nu}$ for $q^0 < 0$ which follows from the antisymmetry of the commutator and translation invariance:

$$\begin{aligned} W^{\mu\nu}(p,q,s_h) &= -\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\,\mathrm{Tr}\,\rho\,[j^\nu(0),j^\mu(x)] \\ &= -\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\,\mathrm{Tr}\,\rho\,[j^\nu(-x),j^\mu(0)] \\ &= -\frac{1}{4\pi}\int d^4x\, e^{-iq\cdot(-x)}\,\mathrm{Tr}\,\rho\,[j^\nu(-x),j^\mu(0)] = -W^{\nu\mu}(p,-q,s_h). \end{aligned} \tag{3.29}$$

This identity is the photon crossing symmetry identity for $W_{\mu\nu}$, since exchanging $\mu\nu$ and letting $q \to -q$ is equivalent to exchanging the outgoing and incoming photons. Finally current conservation, $\partial_\mu j^\mu(x) = 0$, implies that

$$q_\mu W^{\mu\nu}(p,q,s_h) = q_\nu W^{\mu\nu}(p,q,s_h) = 0, \tag{3.30}$$

which greatly restricts the possible tensor structures allowed for $W_{\mu\nu}$.

The most general hadronic tensor for polarised deep inelastic scattering from spin-1/2 targets can be written in terms of p, q, s_h and the invariant tensors $g_{\alpha\beta}$ and $\epsilon_{\alpha\beta\lambda\sigma}$ with the constraints Eqs. (3.24)–(3.30). The algebra is straightforward but tedious. It is conventional to write $W_{\mu\nu}$ in the form

$$\begin{aligned} W_{\mu\nu} = {}& F_1\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{F_2}{p\cdot q}\left(p_\mu - \frac{p\cdot q\; q_\mu}{q^2}\right)\left(p_\nu - \frac{p\cdot q\; q_\nu}{q^2}\right) \\ &+ \frac{ig_1}{p\cdot q}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s^\sigma + \frac{ig_2}{(p\cdot q)^2}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda\left(p\cdot q\, s^\sigma - s\cdot q\, p^\sigma\right)\ , \end{aligned} \tag{3.31}$$

where the coefficients of the different tensor structures are called structure functions. The leptonic current is also conserved, so that $q^\mu \ell_{\mu\nu} = q^\nu \ell_{\mu\nu} = 0$. Thus it is convenient to simplify the expression for $W_{\mu\nu}$ by omitting all q_μ and q_ν terms before contracting with $\ell_{\mu\nu}$,

$$\begin{aligned} W_{\mu\nu} = {}& -F_1\, g_{\mu\nu} + \frac{F_2}{p\cdot q}\, p_\mu p_\nu + \frac{ig_1}{\nu}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s_h^\sigma \\ &+ \frac{ig_2}{\nu^2}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda\left(p\cdot q\, s_h^\sigma - s_h\cdot q\, p^\sigma\right)\ , \end{aligned} \tag{3.32}$$

(Alternatively, one can drop the q_μ and q_ν terms in $\ell_{\mu\nu}$. However, it is incorrect to drop the q_μ and q_ν terms in both $\ell_{\mu\nu}$ and $W_{\mu\nu}$!) There are also structure functions W_1, W_2, G_1 and G_2 proportional to F_1, F_2, g_1 and g_2 which are sometimes used in the literature,

$$\begin{aligned} MW_1 &= F_1, & \nu W_2 &= F_2, \\ M^2\nu G_1 &= g_1, & M\nu^2 G_2 &= g_2, \end{aligned} \tag{3.33}$$

but I will never refer to these, because they do not scale in the deep inelastic limit.

For a spin-1/2 target, the symmetric part of $W_{\mu\nu}$ is independent of the hadron spin, and the spin dependent part of $W_{\mu\nu}$ is antisymmetric in $\mu\nu$. We have already seen that the symmetric part of $\ell_{\mu\nu}$ is independent of the lepton spin, and the spin dependent part of $\ell_{\mu\nu}$ is antisymmetric in $\mu\nu$. Thus the combination $W^{\mu\nu}\ell_{\mu\nu}$ has no terms which have only the hadron spin, or only the lepton spin; all terms contain either both or none. Thus the structure functions F_1 and F_2 can be measured using an unpolarised beam and target, but to measure the structure functions g_1 and g_2 requires both a polarised beam and a polarised target. There is no advantage to doing an experiment with only a polarised beam or only a polarised target. This is not true for target spins greater than 1/2. For example, for a spin one target, there is target polarisation dependence in the symmetric part of $W_{\mu\nu}$, which can be measured using a polarised target and an unpolarised beam.

4. Scaling

I have referred to scaling at several points so far. We are now in a position to understand what scaling means. The tensor $W_{\mu\nu}$ defined in Eq. (3.5) is dimensionless, as are the structure functions defined in Eq. (3.32). The structure functions are dimensionless functions of the Lorentz invariant variables $p^2 = M^2$, $p \cdot q$, and q^2. It is conventional to write them as functions of $x = Q^2/2p \cdot q$ and $Q^2 = -q^2$, so they can be written as dimensionless functions of the dimensionless variables x and Q^2/M^2. In elastic scattering there is a strong dependence on Q^2/M^2, and the elastic form factors fall off like a power of Q^2/M^2. It was thought that the same behaviour would persist for the deep inelastic structure functions. The scale M is a typical hadronic scale, at which confinement (and other non-perturbative) effects become important. Bjorken was the first to point out that if the constituents of the hadron were essentially free pointlike objects at high energies, then the hadronic scale M should be irrelevant, and the structure functions then only depend on x, and must be independent of Q^2. This is the famous prediction of scaling.

We now know that QCD is an asymptotically free theory, and the strong interaction coupling constant α_s becomes small at short distances. Thus at large Q, non-perturbative effects (such as the hadronic mass scale) are irrelevant, and QCD is described by a dimensionless coupling constant. Thus one recovers the prediction of scaling in QCD, since there is no dimensionful scale in the problem. As is well known, scale invariance in a quantum field theory is broken because of quantum corrections. This introduces scaling violations because of anomalous dimensions and the running of α_s. Since α_s is small at high energies, these scaling violations can be reliably computed in QCD perturbation theory. Thus QCD predicts scaling violations (*i.e.* Q^2 dependence) for the structure functions which are calculable. We will discuss this in more detail in Sec. 11.

5. The Cross-Section for Spin-1/2 Targets

The cross-section for spin-1/2 targets can now be obtained by combining the expressions Eq. (3.32) for $W_{\mu\nu}$, Eq. (3.16) for $\ell_{\mu\nu}$, Eq. (3.10) for the cross-section, and using the identity

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\lambda\sigma} = -2\left(g^\alpha_\lambda g^\beta_\sigma - g^\alpha_\sigma g^\beta_\lambda\right), \tag{5.1}$$

$$\begin{aligned}\frac{d^2\sigma}{dx\,dy\,d\phi} = \frac{e^4 ME}{4\pi^2 Q^4}\Bigg[\, & xy^2 F_1 + (1-y)F_2 + y^2 g_1\left(2x\,\frac{s_\ell\cdot s_h}{p\cdot q} + \frac{q\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q}\right) \\ & + 2xy^2 g_2\left(\frac{s_h\cdot s_\ell}{p\cdot q} - \frac{p\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q}\right)\Bigg].\end{aligned} \tag{5.2}$$

As mentioned earlier, the spin dependent terms involve both s_ℓ and s_h. Let us look at the cross-section for the two cases which are most interesting for current experiments, a longitudinally polarised lepton beam incident on a target which is either longitudinally or transversely polarised. For a longitudinally polarised lepton beam, the polarisation is $s_\ell = \mathcal{H}_\ell k$, where $\mathcal{H}_\ell = \pm$ is the lepton helicity. The lepton polarisation terms in Eq. (5.2) can be written as

$$\frac{q\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell\,\frac{q\cdot k}{p\cdot q} = -\mathcal{H}_\ell\,x, \qquad \frac{p\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell\,\frac{p\cdot k}{p\cdot q} = \frac{\mathcal{H}_\ell}{y}, \qquad \frac{s_h\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell\,\frac{s_h\cdot k}{p\cdot q}. \tag{5.3}$$

Longitudinally Polarised Target

A target polarised along the incident beam direction has $\vec{s}_h = M\mathcal{H}_h \hat{z}$, where $\mathcal{H}_h = \pm$ for a target polarised parallel or antiparallel to the beam. s_h is dotted into either q or k both of which have 0 and 3 components which are almost equal. Thus s_h can be replaced by $-\mathcal{H}_h\, p$ in the evaluation of the cross-section in the deep inelastic limit. (The additional minus sign occurs because of the relative minus sign between the space and time components in a dot product such as $k \cdot s$.) The result is

$$\frac{d^2\sigma}{dx\, dy} = \frac{e^4 ME}{2\pi Q^4}\left[xy^2 F_1 + (1-y)F_2 - \mathcal{H}_\ell \mathcal{H}_h y(2-y)xg_1 + \mathcal{O}\left(M^2/Q^2\right)\right], \tag{5.4}$$

where the azimuthal angle ϕ has been integrated over, since the cross-section is independent of ϕ. (The same expression can be used for HERA kinematics with the replacement $2ME \to s$.) Thus the polarisation asymmetry in the cross-section can be used to measure the structure function g_1. The effect of the structure function g_2 is suppressed relative to the leading terms in Eq. (5.4) by an additional factor of M^2/Q^2, and has been omitted.

Transversely Polarised Target

The polarisation vector of a transversely polarised target can be chosen to point along the $\hat{x}$ axis. In this case

$$\frac{d^2\sigma}{dx\, dy\, d\phi} = \frac{e^4 ME}{2\pi Q^4}\left[xy^2 F_1 + (1-y)F_2 + 2x^2\, \mathcal{H}_\ell \mathcal{H}_h\, \frac{M}{Q}\sqrt{1-y}\, \cos\phi\, \left(g_1 y + 2g_2\right)\right], \tag{5.5}$$

where I have used Eq. (2.15) for the magnitude of the transverse component of q. The structure functions g_1 and g_2 are equally important for a transversely polarised target, and so an experiment with a transversely polarised target can be used to determine g_2, once g_1 has been measured using a longitudinally polarised target. Note that the spin-dependent piece of the cross-section for a transversely polarised target is smaller than the spin-averaged piece by a factor of M/Q, unlike in Eq. (5.4) where both terms are comparable in magnitude. Thus a small misalignment of the target relative to the beam direction can lead to a large contamination of the cross-section asymmetry due to the fractional longitudinal polarisation. It is therefore best to measure the $\cos\phi$ dependence of the polarisation asymmetry to measure g_2 using a transversely polarised target.

6. Structure Function Inequalities

There are some simple inequalities among structure functions that follow from unitarity. The differential cross-section Eq. (5.4) must be positive over the entire kinematic region, and for any sign of $\mathcal{H}_h$ and $\mathcal{H}_\ell$. This requires that

$$xy^2 F_1 + (1-y)F_2 \geq xy(2-y)\,|g_1| \geq 0, \tag{6.1}$$

for all values of x and y. In particular, the inequality at $y=1$ implies that

$$F_1 \geq |g_1| \geq 0. \tag{6.2}$$

The n^{th} moment of a function f is defined by

$$M_n(f) \equiv \int_0^1 dx\ x^{n-1} f(x), \tag{6.3}$$

We will derive sum rules for the moments of F_1 and g_1 in Sec. 9. There are useful inequalities amongst the moments that must be satisfied because of the restriction, Eq. (6.2). For example,

$$\begin{aligned} |M_n(g_1)| = \left|\int_0^1 dx\ x^{n-1} g_1(x)\right| &\leq \int_0^1 dx\ x^{n-1}\,|g_1(x)| \\ &\leq \int_0^1 dx\ x^{n-1} F_1(x) = M_n(F_1), \end{aligned} \tag{6.4}$$

using the triangle inequality $|a+b| \leq |a| + |b|$. Thus any moment of g_1 is bounded by the corresponding moment of F_1. There is also the inequality

$$M_n(F_1) = \int_0^1 dx\ x^{n-1} F_1(x) \geq \int_0^1 dx\ x^{n-1} x F_1(x) = M_{n+1}(F_1), \tag{6.5}$$

since $F_1 \geq 0$ and $0 \leq x \leq 1$. Thus we find the sequence of inequalities

$$M_1(F_1) \geq \boxed{M_2(F_1)} \geq M_3(F_1) \geq \boxed{M_4(F_1)} \ldots \tag{6.6}$$

$$M_1(F_1) \geq \boxed{|M_1(g_1)|}, \quad \boxed{M_2(F_1)} \geq |M_2(g_1)|, \quad M_3(F_1) \geq \boxed{|M_3(g_1)|}, \quad \ldots \tag{6.7}$$

The boxed quantities are the moments for which we will derive QCD sum rules in Sec. 9. The second moment of F_1, $M_2(F_1)$ bounds all the higher moments of F_1, as well as all moments of g_1 for which there are sum rules, except the first. This has an important experimental consequence which will be studied in Sec. 10, since the sum rule for the first moment of g_1 is the Ellis-Jaffe sum rule.

7. Helicity Amplitudes: The Interpretation of Structure Functions

The structure functions F_1, F_2, g_1, and g_2 defined in Eq. (3.32) have simple physical interpretations. To understand them better, it is useful to look at the amplitude for forward Compton scattering off a hadron target (see fig. 3)

$$(T_{\mu\nu})_{\lambda'\lambda} = i \int d^4x\, e^{iq\cdot x} \langle p, \lambda' | \, T\left(j_\mu(x) j_\nu(x)\right) |p, \lambda\rangle \, . \tag{7.1}$$

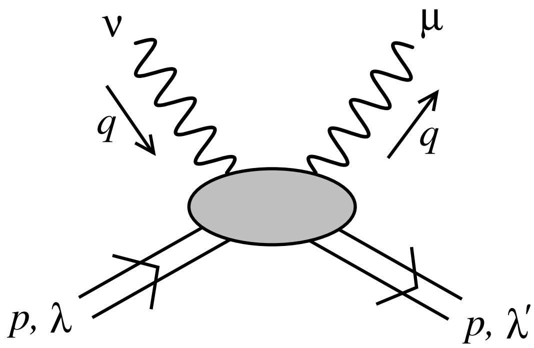

FIGURE 3.
The forward Compton scattering amplitude.

The tensor $T_{\mu\nu}(p, q, s_h)$ can be written in terms $(T_{\mu\nu})_{\lambda'\lambda}$ by analogy with Eq. (3.20),

$$T_{\mu\nu}(p, q, s_h) = \sum_{\lambda,\lambda'} \rho_{\lambda\lambda'}\, T_{\mu\nu}(p, q)_{\lambda'\lambda} = \mathrm{Tr}\, \rho\, T_{\mu\nu}. \tag{7.2}$$

The tensor $T_{\mu\nu}(p, q, s_h)$ has the same symmetry properties as $W_{\mu\nu}(p, q, s_h)$, Eqs. (3.24), (3.27), (3.28), and the current conservation constraint Eq. (3.30). Thus $T_{\mu\nu}(p, q, s_h)$ can be expanded in the same tensors as $W_{\mu\nu}$. The structure functions for $T_{\mu\nu}(p, q, s_h)$ will be denoted by a tilde to distinguish them for the corresponding structure functions for $W_{\mu\nu}$:

$$\begin{aligned} T_{\mu\nu} = {} & \widetilde{F}_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{\widetilde{F}_2}{p\cdot q} \left(p_\mu - \frac{p\cdot q\; q_\mu}{q^2} \right) \left(p_\nu - \frac{p\cdot q\; q_\nu}{q^2} \right) \\ & + \frac{i\,\widetilde{g}_1}{p\cdot q}\, \epsilon_{\mu\nu\lambda\sigma} q^\lambda s^\sigma + \frac{i\,\widetilde{g}_2}{(p\cdot q)^2}\, \epsilon_{\mu\nu\lambda\sigma} q^\lambda \left(p\cdot q\, s^\sigma - s\cdot q\, p^\sigma \right) \, . \end{aligned} \tag{7.3}$$

The photon crossing symmetry relation for $T_{\mu\nu}$

$$T_{\mu\nu}(p, q, s_h) = T_{\nu\mu}(p, -q, s_h), \tag{7.4}$$

differs by a sign from Eq. (3.29) for $W_{\mu\nu}$, because exchanging j^μ and j^ν inside a commutator produces a minus sign, whereas exchanging them inside a time-ordered product does not. Eq. (7.4) implies that $\widetilde{F}_1$ is an even function of ω, and $\widetilde{F}_2$, $\widetilde{g}_1$, and $\widetilde{g}_2$ are odd functions of ω,

$$\begin{aligned} \widetilde{F}_1(-\omega) &= \widetilde{F}_1(\omega), & \widetilde{F}_2(-\omega) &= -\widetilde{F}_2(\omega), \\ \widetilde{g}_1(-\omega) &= -\widetilde{g}_1(\omega), & \widetilde{g}_2(-\omega) &= -\widetilde{g}_2(\omega). \end{aligned} \tag{7.5}$$

The Compton amplitude for photon-antinucleon scattering can be obtained from the photon-nucleon scattering amplitude by fermion crossing symmetry, *i.e.* by replacing an outgoing hadron with momentum p and spin s_h by an incoming antihadron with momentum $-p$ and spin s_h. Thus the Compton amplitude for antinucleon scattering is $T_{\mu\nu}(-p, q, s_h)$. This implies

$$\begin{aligned} \widetilde{F}_1^{\rm antinucleon}(\omega) &= \widetilde{F}_1^{\rm nucleon}(-\omega), & \widetilde{F}_2^{\rm antinucleon}(\omega) &= -\widetilde{F}_2^{\rm nucleon}(-\omega), \\ \widetilde{g}_1^{\rm antinucleon}(\omega) &= -\widetilde{g}_1^{\rm nucleon}(-\omega), & \widetilde{g}_2^{\rm antinucleon}(\omega) &= -\widetilde{g}_2^{\rm nucleon}(-\omega). \end{aligned} \tag{7.6}$$

using Eq. (7.3) and noting that $\omega \to -\omega$ if $p \to -p$. Eqs. (7.5) and (7.6) imply that the antinucleon structure functions are equal to the corresponding nucleon structure functions,

$$\begin{aligned} \widetilde{F}_1^{\rm antinucleon}(\omega) &= \widetilde{F}_1^{\rm nucleon}(\omega), & \widetilde{F}_2^{\rm antinucleon}(\omega) &= \widetilde{F}_2^{\rm nucleon}(\omega), \\ \widetilde{g}_1^{\rm antinucleon}(\omega) &= \widetilde{g}_1^{\rm nucleon}(\omega), & \widetilde{g}_2^{\rm antinucleon}(\omega) &= \widetilde{g}_2^{\rm nucleon}(\omega). \end{aligned} \tag{7.7}$$

This relation could have been obtained trivially by applying charge conjugation to $T_{\mu\nu}$, since it converts the nucleon to an antinucleon. There are no additional signs because the product of two electromagnetic currents is even under charge conjugation.

The optical theorem implies that twice the imaginary part of the forward scattering amplitude is the total cross-section. There are factors of i in the expansion of $T_{\mu\nu}$ and $W_{\mu\nu}$, so what the optical theorem implies in our case is that twice the imaginary part of the $T_{\mu\nu}$ structure functions is equal to corresponding $W_{\mu\nu}$ structure functions times 4π. The extra factor of 4π is due to the extra factor of $1/4\pi$ in the definition of $W_{\mu\nu}$ relative to $T_{\mu\nu}$. The analytic structure of the amplitudes $\widetilde{F}_1$, *etc.* is shown in fig. 4 as a function of $\omega = 1/x$.

There are cuts in the physical region $1 \le |\omega| \le \infty$. The optical theorem implies that

$$\begin{aligned} {\rm Im}\, \widetilde{F}_1(\omega + i\epsilon) &= 2\pi F_1(\omega), & {\rm Im}\, \widetilde{F}_2(\omega + i\epsilon) &= 2\pi F_2(\omega), \\ {\rm Im}\, \widetilde{g}_1(\omega + i\epsilon) &= 2\pi g_1(\omega), & {\rm Im}\, \widetilde{g}_2(\omega + i\epsilon) &= 2\pi g_2(\omega). \end{aligned} \tag{7.8}$$

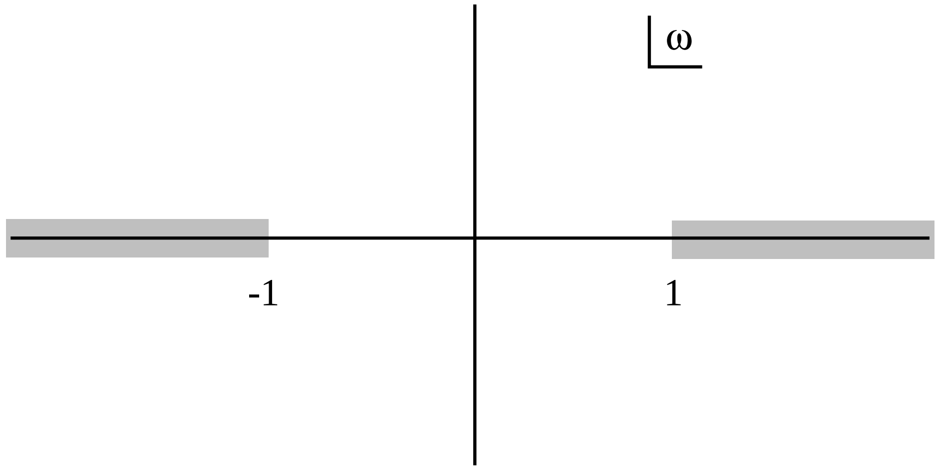

FIGURE 4.
The analytic structure of $T_{\mu\nu}$ in the complex ω plane. The discontinuity across the cuts $1\leq|\omega|\leq\infty$ is related to $W_{\mu\nu}$.

The discontinuity of $T_{\mu\nu}$ across the right hand cut gives $W_{\mu\nu}$ for nucleon targets. The antinucleon structure functions F_1, F_2, g_1 and g_2 are equal to the corresponding nucleon structure functions, because of Eq. (7.7), a result which also follows trivially from charge conjugation. We also see from Eq. (7.6) that the discontinuity across the left hand cut in the complex ω plane is the physical antinucleon structure function.

The forward Compton amplitude can be expanded in terms of helicity amplitudes. Pick the $\hat{z}$ axis to be along the direction of the virtual photon, and choose this axis to quantise the angular momentum of the target and photon. The initial and photon spin components along $\hat{z}$ will be called h and h', and the initial and final target spin components will be called H and H'. The helicity amplitude for forward Compton scattering (see fig. 5):

$$\gamma_h + \text{target}_H \to \gamma_{h'} + \text{target}_{H'} \tag{7.9}$$

will be denoted by $\mathcal{A}(h, H; h', H')$. Angular momentum conservation along the $\hat{z}$ axis implies that

$$h + H = h' + H'. \tag{7.10}$$

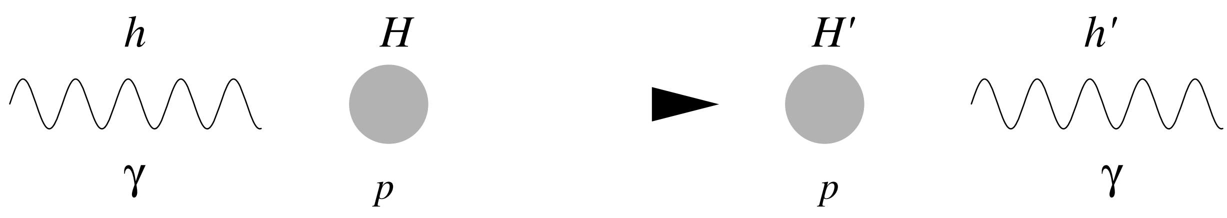

FIGURE 5.

The incoming photon scatters off the target proton which is at rest. The helicity amplitude for this process is denoted by $\mathcal{A}(h,H;h',H')$.

Reflecting in the plane of the scattering does not change any of the momenta, but reverses all the spins. Thus parity invariance implies that

$$\mathcal{A}(h,H;h',H') = \mathcal{A}(-h,-H;-h',-H'). \tag{7.11}$$

Time reversal exchanges initial and final states, and reverses the directions of momenta and spin. Since the helicity of the photons and the target is defined as the component of spin along the direction of motion of the photon, helicity does not change sign under time reversal. Thus time reversal invariance implies that

$$\mathcal{A}(h,H;h',H') = \mathcal{A}(h',H';h,H). \tag{7.12}$$

The virtual photon has momentum $q = (q^0, 0, 0, q^3)$, with three possible polarisations,

$$\begin{aligned}
\epsilon_+^\mu &= -\frac{1}{\sqrt{2}}(0,1,i,0), \quad h = +1, \\
\epsilon_-^\mu &= \frac{1}{\sqrt{2}}(0,1,-i,0), \quad h = -1, \\
\epsilon_0^\mu &= \frac{1}{Q}(q^3,0,0,q^0), \quad h = 0,
\end{aligned} \tag{7.13}$$

where $\epsilon_\pm$ are spacelike, and ϵ_0 is timelike. The fourth polarisation proportional to q^μ does not contribute since the electromagnetic current is conserved. Denoting $H = 1/2$ by $\uparrow$, and $H = -1/2$ by $\downarrow$ and using Eqs. (7.10)–(7.12), we see that for a spin-1/2 target there are only four independent helicity amplitudes which can be chosen to be

$$\mathcal{A}(+,\uparrow;+,\uparrow), \quad \mathcal{A}(+,\downarrow;+,\downarrow), \quad \mathcal{A}(0,\uparrow;0,\uparrow), \quad \mathcal{A}(+,\downarrow;0,\uparrow). \tag{7.14}$$

There are four independent structure functions for spin-1/2 targets because there are four independent helicity amplitudes. The helicity amplitudes can be computed in terms of $T_{\mu\nu}$ using

$$\mathcal{A}(h, H; h', H') = \epsilon_{h'}^{\mu *}\, \epsilon_h^{\nu}\; T_{\mu\nu}(s), \tag{7.15}$$

with

$$\vec{s} = M\; \overline{u}(H')\, \vec{\sigma}\, u(H). \tag{7.16}$$

For spin-1/2 targets, $u(\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u(\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The computation of Eq. (7.15) is simple for spin-1/2 targets in the deep inelastic limit. Using Eq. (7.3) and neglecting terms of order M/Q, we find

$$\begin{aligned} \mathcal{A}(+,\uparrow;+,\uparrow) &= \widetilde{F}_1 - \widetilde{g}_1, & \mathcal{A}(+,\downarrow;+,\downarrow) &= \widetilde{F}_1 + \widetilde{g}_1, \\ \mathcal{A}(0,\uparrow;0,\uparrow) &= \widetilde{F}_2/2x\widetilde{F}_1 \equiv \widetilde{F}_L, & \mathcal{A}(+,\downarrow;0,\uparrow) &= \big(2\sqrt{2}xM/Q\big)(\widetilde{g}_1 + \widetilde{g}_2). \end{aligned} \tag{7.17}$$

For example, the coefficient of $\widetilde{F}_1$ in Eq. (7.3) is $-g_{\mu\nu}$. Thus the coefficient of $\widetilde{F}_1$ in Eq. (7.17) is $+1$ for $\mathcal{A}(+,\uparrow;+,\uparrow)$ and $\mathcal{A}(+,\downarrow;+,\downarrow)$ because ϵ_+ is spacelike, is -1 for $\mathcal{A}(0,\uparrow;0,\uparrow)$ because ϵ_0 is timelike, and is zero for $\mathcal{A}(+,\downarrow;0,\uparrow)$ because ϵ_+ and ϵ_0 are orthogonal. Taking imaginary parts of the left hand side gives identical formulæ with $\widetilde{F} \to 2\pi F$, *etc.* Thus we see that

$$2\pi F_1 = \frac{1}{2}\,\mathrm{Im}\left[\mathcal{A}(+,\uparrow;+,\uparrow) + \mathcal{A}(+,\downarrow;+,\downarrow)\right], \tag{7.18}$$

is the cross-section for scattering a transverse photon off an unpolarised target,

$$2\pi g_1 = -\frac{1}{2}\,\mathrm{Im}\left[\mathcal{A}(+,\uparrow;+,\uparrow) - \mathcal{A}(+,\downarrow;+,\downarrow)\right] \tag{7.19}$$

is the spin asymmetry in the scattering cross-section for a transverse photon, and

$$2\pi F_L = \mathrm{Im}\,\mathcal{A}(0,\uparrow;0,\uparrow) = \mathrm{Im}\,\mathcal{A}(0,\downarrow;0,\downarrow) \tag{7.20}$$

is the cross-section for scattering longitudinally polarised photons off an unpolarised target. F_L is called the longitudinal structure function. The physical interpretation of g_2 is obscure, the combination $g_1 + g_2$ is proportional to the "single helicity flip" amplitude $\mathcal{A}(+,\downarrow;0,\uparrow)$. Note that the single helicity flip amplitude is suppressed by a factor of M/Q relative to the diagonal helicity amplitudes.

The helicity amplitude formalism is useful in discussing targets with spin greater than 1/2. It provides a simple way of counting the number of independent structure functions and understanding their physical significance.

8. The Parton Model

The parton model provides a simple physical picture which helps in interpreting the structure functions of deep inelastic scattering. The hadron target is considered to be made up of a number of free on-shell partons (quarks and gluons) in the deep inelastic limit. The scattering cross-section can then be computed in terms of the incoherent scattering of free quarks and gluons. The hadronic tensor $W_{\mu\nu}$ can be computed in terms of parton distribution functions. For example, the distribution function $u_{\uparrow}(\xi)$ is the probability to find an up quark in the hadron with polarisation parallel to that of the hadron, and momentum ξp. It is conventional to go to an infinite momentum frame when discussing the parton model, and to claim that partons are non-interacting in this infinite momentum frame. This is incorrect. Partons can be treated as free because of asymptotic freedom, not because of the choice of a particular Lorentz frame. At high energies, the QCD coupling constant becomes weak, and so one can treat parton interactions using perturbation theory. The main reason for going to the infinite momentum frame is to neglect target mass effects. A parton with momentum fraction ξ has invariant mass $\xi^2 M^2$, which is not zero, and depends on ξ. The infinite momentum frame hides this embarrassment, which is why it is used to discuss the parton model. Instead of using the infinite momentum frame, I will work in the rest frame of the target, and consistently neglect all target mass effects.

To lowest order in the strong interactions, only quarks scatter off the virtual photon, since gluons do not have electric charge. The contribution to $W_{\mu\nu}$ from a single polarised quark can be computed from the Feynman graph of fig. 6. The computation of the scattering amplitude can be obtained from

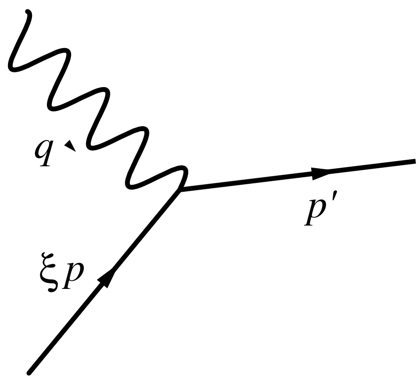

FIGURE 6.

The contribution to $W_{\mu\nu}$ from a single quark with momentum fraction ξ.

$\ell_{\mu\nu}$, Eq. (3.16), by the replacements $k \to \xi p$, $k' \to p'$, $s_\ell \to s$ and $q \to -q$, and multiplying by the square of the quark charge $\mathcal{Q}$. Comparing Eq. (3.16) and Eq. (3.8), we see that $W_{\mu\nu}$ has an additional factor of integration over the final particle phase space. Thus we find

$$\begin{aligned} W_{\mu\nu} = \frac{1}{4\pi}\mathcal{Q}^2 \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \, \frac{1}{\xi} \, (2\pi)^4 \delta^4(\xi p + q - p') \\ \times 2\left[\xi p^\mu p'^\nu + \xi p^\nu p'^\mu - g^{\mu\nu}\xi p \cdot p' + i\epsilon^{\mu\nu\alpha\beta} q_\alpha s_\beta\right]. \end{aligned} \tag{8.1}$$

The only subtlety is the factor of $1/\xi$. The proton states in $W_{\mu\nu}$ are normalised to $2p^0$, whereas the parton states in (8.1) are normalised to $2p^0\xi$ in the standard relativistic normalisation convention. This affects the particle flux in the computation of the cross-section. The extra factor of $1/\xi$ converts the parton flux to the correct normalisation for the proton flux. The integral in Eq. (8.1) is easily done using the identity

$$\int \frac{d^3p'}{(2\pi)^3 2E_{p'}} = \int \frac{d^4p'}{(2\pi)^4} \, (2\pi)\, \delta\left((\xi p + q - p')^2\right) = \int \frac{d^4p'}{(2\pi)^4} \, \frac{(2\pi)}{2p\cdot q} \, \delta\left(\xi + \frac{q^2}{2p\cdot q}\right). \tag{8.2}$$

Since the partons are massless, $s_\beta = h\xi p_\beta$, where h is the helicity of the parton, so that

$$\begin{aligned} W_{\mu\nu} &= \frac{\mathcal{Q}^2}{2\xi \; p\cdot q}\left[\xi p^\mu p'^\nu + \xi p^\nu p'^\mu - g^{\mu\nu}\xi p\cdot p' + ih\xi\,\epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta\right] \delta(\xi - x), \\ &= \frac{\mathcal{Q}^2}{2\xi \; p\cdot q}\left[2\xi^2 p^\mu p^\nu - g^{\mu\nu}\xi p\cdot q + ih\xi\,\epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta\right] \delta(\xi - x), \end{aligned} \tag{8.3}$$

where p' has been replaced by $\xi p + q$, and the q^μ and q^ν terms have been dropped. To compare with $W_{\mu\nu}$, we need Eq. (3.32) neglecting target mass effects. Thus the target spin s_h in Eq. (3.32) can be replaced by $\mathcal{H}p$, where $\mathcal{H}$ is the helicity of the target, and p is the target momentum. Comparing with Eq. (3.32), we see that the contribution of quarks to the structure functions is

$$F_1 = \frac{\mathcal{Q}^2}{2}\delta(\xi - x), \quad F_2 = \mathcal{Q}^2\xi\,\delta(\xi - x), \quad g_1 = \frac{\mathcal{Q}^2}{2}h\mathcal{H}\,\delta(\xi - x), \quad g_2 = 0. \tag{8.4}$$

The antiquark contribution can be computed similarly, and is identical to the quark contribution. The total structure function is obtained by integrating with the quark distribution functions, $q_+(\xi)$, the probability to find a quark in the nucleon with momentum fraction ξ and helicity equal to that of the nucleon ($h\mathcal{H} = +$), and $q_-(\xi)$, the probability to find a quark in the nucleon with momentum fraction ξ, and helicity opposite to that of the nucleon ($h\mathcal{H} = -$), and the antiquark distribution functions $\overline{q}_\pm(\xi)$:

$$
\begin{aligned}
F_1(x,Q^2) &= \sum_i \frac{Q_i^2}{2} \left(q_+(x) + q_-(x) + \overline{q}_+(x) + \overline{q}_-(x)\right), \\
F_2(x,Q^2) &= \sum_i Q_i^2 \; x \left(q_+(x) + q_-(x) + \overline{q}_+(x) + \overline{q}_-(x)\right), \\
g_1(x,Q^2) &= \sum_i \frac{Q_i^2}{2} \left(q_+(x) - q_-(x) + \overline{q}_+(x) - \overline{q}_-(x)\right), \\
g_2(x,Q^2) &= 0,
\end{aligned}
\tag{8.5}
$$

where the sum is over all quark flavours which can be considered light compared with Q^2. The most important feature of this result is that the structure functions depend only on x, and are independent of Q^2. We also see that the structure functions satisfy the Callan-Gross relation,

$$
F_2(x) = 2xF_1(x) \quad \Rightarrow \quad F_L(x) = 0. \tag{8.6}
$$

The Callan-Gross relation will get corrections at order α_s in QCD. The structure function g_2 vanishes identically; there is no simple interpretation for g_2 in the parton model.

We see from Eq. (8.5) that F_1 is the probability to find a quark in the hadron with momentum fraction x, and g_1 is the difference in probabilities to find a quark in the hadron with momentum fraction x with spin parallel and antiparallel to the hadron. Both quantities weight each quark flavour by the square of the electric charge. The combination $F_1(x) - g_1(x)$ depends only on $q_-(x)$. The reason is that $F_1(x) - g_1(x)$ is proportional to the imaginary part of the helicity amplitude $\mathcal{A}(+,\uparrow;+,\uparrow)$. There are two possible diagrams which contribute to the Compton scattering amplitude, one where the quark absorbs the initial photon, and then emits the final photon, and the crossed diagram where the final photon is emitted before the initial photon is absorbed. The crossed diagram has no imaginary part since the energy of the initial and final photons is positive, and $q^2 < 0$. The intermediate state when a quark with $J_z = m$ absorbs a helicity + photon is a virtual quark with $J_z = h + 1$. A virtual quark only has $J_z = \pm 1/2$, so only $J_z = -1/2$ quarks can absorb a helicity + photon. This implies that only $q_-(x)$ contributes to $F_1(x) - g_1(x)$, and similarly, only $q_+(x)$ contributes to $F_1(x) + g_1(x)$.

Isospin Symmetry and the Gottfried Sum Rule

It is conventional to use the symbol $q_\pm(x)$ to refer to the parton distributions in a proton target. The distributions in a neutron target are related to those in a proton target by isospin invariance. Making a 180° rotation in isospin space exchanges $p \leftrightarrow n$ and $u \leftrightarrow d$. Thus one finds

$$\begin{aligned}
u_{\pm,\ \rm proton}(x) &= d_{\pm,\ \rm neutron}(x), \qquad & \overline{u}_{\pm,\ \rm proton}(x) &= \overline{d}_{\pm,\ \rm neutron}(x),\\
d_{\pm,\ \rm proton}(x) &= u_{\pm,\ \rm neutron}(x), \qquad & \overline{d}_{\pm,\ \rm proton}(x) &= \overline{u}_{\pm,\ \rm neutron}(x),\\
s_{\pm,\ \rm proton}(x) &= s_{\pm,\ \rm neutron}(x), \qquad & \overline{s}_{\pm,\ \rm proton}(x) &= \overline{s}_{\pm,\ \rm neutron}(x),
\end{aligned} \tag{8.7}$$

Isospin invariance is broken by light quark masses, and by electromagnetism. The light quark mass effects are of order $m_q/\Lambda_{\rm QCD}$, and the electromagnetic corrections are of order $\alpha/4\pi$, so one expects isospin violation at the few percent level. There is no large isospin breaking effect proportional to m_u/m_d; the relevant quantity in QCD is the ratio of a light quark mass to the hadronic scale $\Lambda_{\rm QCD}$. There is no reason whatsoever for the "quark sea to be isospin symmetric," *i.e.* for

$$\overline{u}_{\rm proton}(x) = \overline{d}_{\rm proton}(x), \tag{8.8}$$

because the proton state is not isospin symmetric. One simple way to see this is to note that the production of $u\overline{u}$ pairs in the proton must be different from the production of $d\overline{d}$ pairs in the proton, since there are two "net" u quarks in the proton, and only one "net" d quarks, and any u quarks produced must be antisymmetric with respect to the existing quarks.

There is a (false) sum rule for the first moment of $F_1^{ep} - F_1^{en}$, the difference in F_1 between protons and neutrons. This is the Gottfried sum rule which can be derived as follows:

$$\begin{aligned}
\int_0^1 F_1^{ep}(x) - F_1^{en}(x) &= \int_0^1 \left[\tfrac{4}{9}\left[u(x)+\overline{u}(x)\right] + \tfrac{1}{9}\left[d(x)+\overline{d}(x)\right] + \tfrac{1}{9}\left[s(x)+\overline{s}(x)\right]\right]_{\rm proton}\\
&\quad - \left[\tfrac{4}{9}\left[u(x)+\overline{u}(x)\right] + \tfrac{1}{9}\left[d(x)+\overline{d}(x)\right] + \tfrac{1}{9}\left[s(x)+\overline{s}(x)\right]\right]_{\rm neutron}\\
&= \tfrac{1}{3}\int_0^1 \left[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)\right]_{\rm proton}\\
&= \tfrac{1}{3} \quad (\textit{Warning}:\ \textit{This is false}).
\end{aligned} \tag{8.9}$$

The second line follows from Eqs. (8.7), and the last line follows from using Eq. (8.8) to write

$$\tfrac{1}{3}\int_0^1 \left[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)\right]_{\rm proton} = \tfrac{1}{3}\int_0^1 \left[u(x) - d(x)\right] = \tfrac{1}{3}, \tag{8.10}$$

but since Eq. (8.8) is incorrect, so is the Gottfried sum rule. There is another reason why the Gottfried sum rule is incorrect; it is an attempt to derive a sum rule for the "wrong" moment of F_1 as we will discuss later. I repeat, *the Gottfried sum rule is false in QCD.*

9. The Operator Product Expansion and Sum Rules

The parton model provides an interpretation for the deep inelastic structure functions. One can instead derive sum rules for the structure functions directly from QCD using the operator product expansion. These sum rules are extremely important because they do not depend on any model of hadronic structure, and provide a direct test of QCD. They can be derived using only some very general results from quantum field theory. The starting point in the derivation is the time-ordered product of two currents

$$t_{\mu\nu} \equiv i \int d^4x\, e^{iq\cdot x} T\left(j_\mu(x) j_\nu(0)\right). \tag{9.1}$$

The nucleon matrix element of $t_{\mu\nu}$ gives the familiar Compton amplitude,

$$\left(T_{\mu\nu}\right)_{\lambda'\lambda} = \left\langle p, \lambda' \right| t_{\mu\nu} \left| p, \lambda \right\rangle. \tag{9.2}$$

The analytic structure of $T_{\mu\nu}$ and its relation to $W_{\mu\nu}$ was discussed in Sec. 7.

The key idea that permits the computation of $T_{\mu\nu}$ in QCD is the operator product expansion. Consider the product of two local operators,

$$\mathcal{O}_a(x)\ \mathcal{O}_b(0). \tag{9.3}$$

In the limit that $x \to 0$, the operators are at practically the same point. In this limit, the operator product can be written as an expansion in local operators,

$$\lim_{x\to 0} \mathcal{O}_a(x)\ \mathcal{O}_b(0) = \sum_k c_{abk}(x)\ \mathcal{O}_k(0). \tag{9.4}$$

The coefficient functions depend on the separation x. The left hand side is completely equivalent to the right hand side as long as one does not probe the operator product on distance scales which are small compared with the separation x. Thus one can replace the product $\mathcal{O}_a(x)\mathcal{O}_b(0)$ in the computation of matrix elements by the expansion Eq. (9.4) where the coefficients $c_{abk}(x)$ *are independent of the matrix elements, provided that the external states have momentum components which are small compared with the separation* x. In QCD, the coupling constant is small at short distances because of asymptotic freedom. Thus the coefficient functions can be computed in perturbation theory, since all non-perturbative effects occur at scales which are much larger than x, and thus do not affect the computation of the coefficient functions.

The momentum space version of the operator product expansion is for the product

$$\int d^4x\, e^{iq\cdot x}\ \mathcal{O}_a(x)\ \mathcal{O}_b(0). \tag{9.5}$$

In the limit that $q \to \infty$, the Fourier transform in Eq. (9.1) forces $x \to 0$, and again the operator product can be expanded in terms of local operators with coefficient functions that depend on q,

$$\lim_{q\to\infty} \int d^4x\, e^{iq\cdot x}\ \mathcal{O}_a(x)\ \mathcal{O}_b(0) = \sum_k c_{abk}(q)\ \mathcal{O}_k(0). \tag{9.6}$$

This expansion is valid for all matrix elements, provided q is much larger than the characteristic momentum in any of the external states.

We will use the Fourier transform version of the operator product expansion, Eq. (9.6). The product of two electromagnetic currents in Eq. (9.1) can be expanded in terms of a sum of local operators multiplied by coefficients which are functions of q. This expansion will be valid for matrix elements in the target, Eq. (9.2), provided that q is much larger than the typical hadronic mass scale $\Lambda_{\rm QCD}$. The local operators in the operator product expansion for QCD are quark and gluon operators with arbitrary dimension d and spin n. An operator with spin n and dimension d can be written as

$$\mathcal{O}_{d,n}^{\mu_1\cdots\mu_n}, \tag{9.7}$$

where $\mathcal{O}_{d,n}$ is symmetric and traceless in $\mu_1\ldots\mu_n$. The matrix element of $\mathcal{O}_{d,n}$ in the hadron target is proportional to

$$M^{d-n-2}\ \mathcal{S}\left[p^{\mu_1}\ldots p^{\mu_n}\right], \tag{9.8}$$

for a vector operator, and to

$$M^{d-n-2}\ \mathcal{S}\left[s^{\mu_1}p^{\mu_2}\ldots p^{\mu_n}\right], \tag{9.9}$$

for an axial operator. $\mathcal{S}$ acts on a tensor to project out the completely symmetric traceless component. The power of M follows from dimensional analysis, since a hadron state with the conventional relativistic normalisation has dimension minus one. The coefficient functions in the operator product expansion are functions only of q. Thus the free indices on the operator $\mathcal{O}$ must be either μ, ν, or be contracted with q^α. Every index on $\mathcal{O}$ contracted with q^α produces a factor of $p\cdot q$ (or $s\cdot q$) which is of order Q^2/M in the deep inelastic limit. An index μ or ν is contracted with $\ell_{\mu\nu}$, and produces a factor of $p \cdot k$ or $p \cdot k'$ (or $s \cdot k$ or $s \cdot k'$), both of which are also of order Q^2/M in the deep inelastic limit. In addition, since $t_{\mu\nu}$ has dimension two, the coefficient of $\mathcal{O}$ must have dimension Q^{2-d} in the operator product expansion. Thus the contribution of any operator $\mathcal{O}$ to $W_{\mu\nu}\ell^{\mu\nu}$ is of order

$$\begin{aligned} c_{\mu_1\ldots\mu_n}\mathcal{O}^{\mu_1\ldots\mu_n}_{d,n} &\to \frac{q_{\mu_1}}{Q}\cdots\frac{q_{\mu_n}}{Q}\ Q^{2-d}\left\langle \mathcal{O}^{\mu_1\ldots\mu_n}_{d}\right\rangle, \\ &\to \frac{q_{\mu_1}}{Q}\cdots\frac{q_{\mu_n}}{Q}\ Q^{2-d}M^{d-n-2}\ p^{\mu_1}\ldots p^{\mu_n}, \\ &\to \frac{(p\cdot q)^n}{Q^n}\ Q^{2-d}\ M^{d-n-2}, \\ &\to \omega^n\left(\frac{Q}{M}\right)^{2+n-d} = \omega^n\left(\frac{Q}{M}\right)^{2-t}, \end{aligned} \tag{9.10}$$

where the twist t is defined as

$$\text{twist} = t = d - n = \text{dimension} - \text{spin}. \tag{9.11}$$

The most important operators in the operator product expansion are those with the lowest possible twist. Twist two operators contribute a finite amount to the structure functions in the deep inelastic limit, twist three contributions are suppressed by M/Q, *etc.* The fundamental fields in QCD are quark and gluon fields, so the operators in the operator product expansion can be written in terms of quark fields ψ, the gluon field strength $G_{\mu\nu}$ and the covariant derivative D^μ. We can make a table listing the basic objects, with their dimension and twist:

	ψ	$G_{\mu\nu}$	D^μ
d	3/2	2	1
s	1/2	1	1
t	1	1	0

Any gauge invariant operator must contain at least two quark fields, or two gluon field strength tensors. Thus the lowest possible twist is two. A twist two operator has either two ψ's or two $G_{\mu\nu}$'s and an arbitrary number of covariant derivatives. The indices of the covariant derivatives are not contracted, because an operator such as D^2 has twist two, whereas $D^\alpha D^\beta$ has twist zero.

The first step in doing an operator product expansion is to determine all the linearly independent operators that can occur. We have just seen that the leading operators are twist two quark and gluon operators. The Lorentz structure of the quark operators must be either $\overline{\psi}\gamma^\mu\psi$ or $\overline{\psi}\gamma^\mu\gamma_5\psi$ in the limit that light quark masses can be neglected, because the operator $j^\mu j^\nu$ does not change chirality. The conventional basis for twist two quark operators is:

$$O_{V,a}^{\mu_1\ldots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\overline{\psi_a}\,\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_n}\psi_a\right\}\ , \tag{9.12}$$

$$O_{A,a}^{\mu_1\ldots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\overline{\psi_a}\,\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_n}\gamma_5\psi_a\right\}\ , \tag{9.13}$$

where the index $a = (u, d, s)$ runs over the light quark flavours. The twist two gluon operators are more complicated, and I will give one such operator tower:

$$O_{g,V}^{\mu_1\ldots\mu_n} = -\frac{1}{2}\left(\frac{i}{2}\right)^{n-2} S\left\{G_a^{\mu_1\alpha}\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_{n-1}}G_{a\alpha}{}^{\mu_n}\right\}\ . \tag{9.14}$$

In these lectures, I will only compute the operator product expansion to lowest order, so I do not need the gluon operators.

The second step in doing an operator product expansion is to determine the coefficient functions of the operators. The best way to do this to to evaluate enough on-shell matrix elements to determine all the coefficients. Since we have argued that the coefficients can be computed reliably in QCD perturbation theory, we will evaluate the coefficients by taking matrix elements in on-shell quark and gluon states. I will only illustrate the computation of the coefficients to lowest non-trivial order (*i.e.* $(\alpha_s)^0$) in this lecture. The generic term in the operator product expansion can be written as

$$jj \sim c_q\mathcal{O}_q + c_g\mathcal{O}_g, \tag{9.15}$$

where q and g refer to quark and gluon operators. Taking the matrix element of both sides in a free quark state, gives

$$\langle q|\, jj\,|q\rangle \sim c_q\,\langle q|\,\mathcal{O}_q\,|q\rangle + c_g\,\langle q|\,\mathcal{O}_g\,|q\rangle\,. \tag{9.16}$$

The electromagnetic current is a quark operator. Thus the left hand side is of order $(\alpha_s)^0$. The matrix element $\langle q|\,\mathcal{O}_q\,|q\rangle$ is also of order $(\alpha_s)^0$, whereas the matrix element $\langle q|\,\mathcal{O}_g\,|q\rangle$ is of order $(\alpha_s)^1$ since there are at least two gluons in $\mathcal{O}_g$, each of which contributes a factor of the QCD coupling constant g to the matrix element. Thus one can determine c_q to leading order by taking the quark matrix element of both sides of the operator product expansion, neglecting the gluon operators. I will work for simplicity in a theory with a

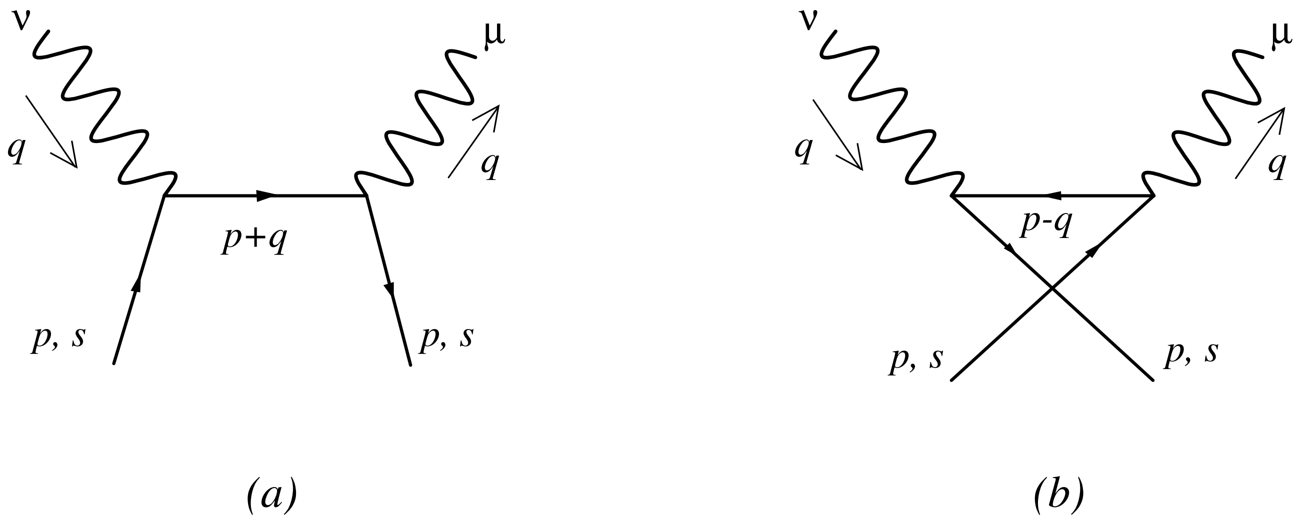

FIGURE 7.

The lowest order diagrams contributing to the quark matrix element of the product of two electromagnetic currents.

single quark flavour with charge one, and generalise the result to an arbitrary number of flavours with arbitrary charges at the end.

The quark matrix element of the left hand side of the operator product expansion, Eq. (9.15), is given by the Feynman graphs in fig. 7,

$$\mathcal{M}^{\mu\nu} = i\,\overline{u}(p,s)\,\gamma^\mu\, i\frac{\not p + \not q}{(p+q)^2}\,\gamma^\nu\, u(p,s) + i\,\overline{u}(p,s)\,\gamma^\nu\, i\frac{\not p - \not q}{(p-q)^2}\,\gamma^\mu\, u(p,s). \tag{9.17}$$

Note that there is an overall factor of i because we are computing i times the time ordered product in Eq. (9.1) and there are no factors of $-ie$ at the vertex. The crossed diagram (second term) can be obtained by the replacement $\mu \leftrightarrow \nu$, $q \to -q$ from the direct diagram (first term), so I will concentrate on simplifying the first term. Expanding the denominator gives

$$(p+q)^2 = 2p\cdot q + q^2 = q^2\left(1 + \frac{2p\cdot q}{q^2}\right) = q^2\,(1-\omega)\,, \tag{9.18}$$

since $p^2 = 0$ for an on-shell massless quark. The numerator can be simplified using the γ matrix identity

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu + g^{\nu\alpha}\gamma^\mu - g^{\mu\nu}\gamma^\alpha + i\epsilon^{\mu\nu\alpha\lambda}\gamma_\lambda\gamma_5, \tag{9.19}$$

$$\overline{u}(p,s)\gamma^\mu(\not p+\not q)\gamma^\nu u(p,s)=\overline{u}(p,s)\Big[(p+q)^\mu\gamma^\nu+(p+q)^\nu\gamma^\mu-g^{\mu\nu}(\not p+\not q)$$
$$+i\epsilon^{\mu\nu\alpha\lambda}(p+q)_\alpha\gamma_\lambda\gamma_5\Big]u(p,s). \tag{9.20}$$

For an on-shell massless quark,

$$\not p\, u(p,s)=0,\quad \overline{u}(p,s)\,\gamma_\lambda\, u(p,s)=2p_\lambda,\quad \overline{u}(p,s)\,\gamma_\lambda\gamma_5\, u(p,s)=2h\, p_\lambda, \tag{9.21}$$

where h is the quark helicity. Thus the $\not p$ and $\epsilon^{\mu\nu\alpha\lambda}p_\alpha\gamma_\lambda\gamma_5$ terms both give zero. (A note of caution: There have been many mistakes made by authors trying to distinguish $\overline{u}(p,s)\,\gamma_\lambda\gamma_5\, u(p,s)\, p_\sigma$ from $\overline{u}(p,s)\,\gamma_\sigma\gamma_5\, u(p,s)\, p_\lambda$. They are equal because of the equations of motion.)

Combining the various terms, and using

$$(1-\omega)^{-1}=\sum_{n=0}^{\infty}\omega^n, \tag{9.22}$$

gives

$$\mathcal{M}^{\mu\nu}_{\rm direct}=-\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[(p+q)^\mu p^\nu+(p+q)^\nu q^\mu-g^{\mu\nu}p\cdot q+ih\,\epsilon^{\mu\nu\alpha\lambda}\,q_\alpha p_\lambda\right]. \tag{9.23}$$

To complete the operator product expansion, we need the free quark matrix element of the right hand side of the operator product. The matrix element of the quark operators Eq. (9.12) and (9.13) in a free quark state is

$$\langle p,s|\,\mathcal{O}_V^{\mu_1\cdots\mu_n}\,|p,s\rangle=p^{\mu_1}\ldots p^{\mu_n}, \tag{9.24}$$

$$\langle p,s|\,\mathcal{O}_A^{\mu_1\cdots\mu_n}\,|p,s\rangle=h\ p^{\mu_1}\ldots p^{\mu_n}. \tag{9.25}$$

The factors of i and 2 in Eqs. (9.12) and (9.13) were chosen so that no such factors appear in the matrix elements.

I will determine the coefficient functions for the spin dependent term in the operator product expansion, and leave the computation of the spin independent terms as a homework problem. The spin dependent term on the left hand side of the operator product is

$$\mathcal{M}^{[\mu\nu]}=-\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\ ih\,\epsilon^{\mu\nu\alpha\lambda}\,q_\alpha p_\lambda+\ (\mu\leftrightarrow\nu,\ q\to-q,\ \omega\to-\omega), \tag{9.26}$$

since ω is odd in q. The crossed diagram causes half the terms to cancel, so that the matrix element is

$$
\begin{aligned}
\mathcal{M}^{[\mu\nu]} &= -\frac{4}{q^2} \sum_{n=0,2,4}^{\infty} \omega^n \; ih\, \epsilon^{\mu\nu\alpha\lambda}\, q_\alpha p_\lambda = -\frac{4}{q^2} \sum_{n=0,2,4}^{\infty} \left(\frac{-2p\cdot q}{q^2}\right)^n ih\, \epsilon^{\mu\nu\alpha\lambda}\, q_\alpha p_\lambda, \\
&= \sum_{n=0,2,4}^{\infty} -\frac{4}{q^2} \, \frac{2^n q^{\mu_1} \ldots q^{\mu_n}}{(-q^2)^n} \; ih\, \epsilon^{\mu\nu\alpha\lambda}\, q_\alpha p_\lambda \; p_{\mu_1} \cdots p_{\mu_n}, \\
&= \sum_{n=0,2,4}^{\infty} 2 \, \frac{2^{n+1} q^{\mu_1} \ldots q^{\mu_n}}{(-q^2)^{n+1}} \; ih\, \epsilon^{\mu\nu\alpha\mu_{n+1}}\, q_\alpha \; p_{\mu_1} \cdots p_{\mu_{n+1}},
\end{aligned}
\tag{9.27}
$$

where the dummy index λ has been replaced by μ_{n+1} in the last line. Replacing $n \to n+1$, and permuting the dummy indices $\mu_1 \ldots \mu_n$ gives

$$
\mathcal{M}^{[\mu\nu]} = \sum_{n=1,3,5}^{\infty} 2 \, \frac{2^n q^{\mu_2} \ldots q^{\mu_n}}{(-q^2)^n} \; ih\, \epsilon^{\mu\nu\alpha\mu_1}\, q_\alpha \; p_{\mu_1} \cdots p_{\mu_n}. \tag{9.28}
$$

The coefficient functions in the operator product depend only on q, and the matrix elements depend only on p. We have separated the operator product into pieces which depend only on q and only on p. By comparing with Eq. (9.25), we can write Eq. (9.28) as

$$
\mathcal{M}^{[\mu\nu]} = \sum_{n=1,3,5}^{\infty} 2 \, \frac{2^n q^{\mu_2} \ldots q^{\mu_n}}{(-q^2)^n} \; i\epsilon^{\mu\nu\alpha\mu_1}\, q_\alpha \, \langle O_A^{\mu_1 \cdots \mu_n} \rangle , \tag{9.29}
$$

so that

$$
t^{[\mu\nu]} = \sum_{n=1,3,5}^{\infty} 2 \, \frac{2^n q^{\mu_2} \ldots q^{\mu_n}}{(-q^2)^n} \; i\epsilon^{\mu\nu\alpha\mu_1}\, q_\alpha \, O_A^{\mu_1 \cdots \mu_n}. \tag{9.30}
$$

This is the operator product expansion for the spin dependent part of $t_{\mu\nu}$. We have omitted quark flavour types and quark charges in our analysis. The correct answer for Eq. (9.30) has $\mathcal{O}_A$ replaced by $\sum \mathcal{Q}_a^2 \mathcal{O}_{A,a}$, where $\mathcal{Q}_a$ is the charge of flavour a in units of e. Odd spin axial vector operators are charge conjugation even, and even spin axial vector operators are charge conjugation odd. Since electroproduction is a charge conjugation even process, only the odd spin axial vector operators appear in Eq. (9.30). I will leave the computation of the symmetric part of $t_{\mu\nu}$ from Eq. (9.23) as an exercise. The contribution can be written in terms of only even spin vector operators. (Even spin vector operators are charge conjugation even, and odd spin vector operators are charge conjugation odd.)

The most general operator product expansion at twist two can be written in the form

$$
\begin{aligned}
t_{\mu\nu} = & \sum_{n=2,4,\cdots}^{\infty} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \frac{2^n q_{\mu_1} \cdots q_{\mu_n}}{\left(-q^2\right)^n} \sum_j 2\, C^{(1)}_{j,n}\, O^{\mu_1 \cdots \mu_n}_{j,V} \\
& + \sum_{n=2,4,\ldots}^{\infty} \left(g_{\mu\mu_1} - \frac{q_\mu q_{\mu_1}}{q^2}\right) \left(g_{\nu\mu_2} - \frac{q_\nu q_{\mu_2}}{q^2}\right) \frac{2^n q_{\mu_3} \cdots q_{\mu_n}}{\left(-q^2\right)^{n-1}} \sum_j 2\, C^{(2)}_{j,n}\, O^{\mu_1 \cdots \mu_n}_{j,V} \\
& + \sum_{n=1,3,\ldots}^{\infty} i\epsilon_{\mu\nu\lambda\mu_1} q^\lambda \frac{2^n q_{\mu_2} \cdots q_{\mu_n}}{\left(-q^2\right)^n} \sum_j 2\, C^{(3)}_{j,n}\, O^{\mu_1 \cdots \mu_n}_{j,A} \\
& + \sum_{n=2,4,\cdots}^{\infty} \frac{2^n q_{\mu_1} \cdots q_{\mu_n}}{\left(-q^2\right)^n} \sum_j 2\, C^{(4)}_{j,n}\, O^{\mu\nu\mu_1\mu_2\mu_3 \cdots \mu_n}_{j,\Delta} \ .
\end{aligned}
\tag{9.31}
$$

The sum in the various terms is over twist two quark and gluon operators. What we have just shown is that $C^{(3)}_{j,n}$ at leading order is zero for the gluon operators, and is $\mathcal{Q}_a^2$ for the quark axial operator of flavour type a, as can be seen by comparing Eq. (9.31) with Eq. (9.30). A similar calculation for the spin-independent pieces shows that at leading order $C^{(1)}_n$ and $C^{(2)}_n$ are both equal to $\mathcal{Q}_a^2$ for quark operators of flavour type a, and all gluon operator coefficients are zero. (The terms of type $C^{(4)}_n$ have only gluon contributions, and so vanish to this order. They contribute to the structure function $\Delta(x)$ mentioned in Sec. 12.)

To compute $T_{\mu\nu}$, we need the hadronic matrix element of the operator product expansion, so we need to evaluate the hadronic matrix elements of the operators Eqs. (9.12) and (9.13). The matrix elements of these operators in the target are not known. We can parameterise the matrix elements in terms of a known tensor structure times an unknown normalisation coefficient. The matrix elements will be considered only in a spin-1/2 target. For arbitrary spin targets, the tensor structure of the matrix elements is much more complicated. The matrix elements of $\mathcal{O}_V$ and $\mathcal{O}_A$ can be written as

$$
\langle p, s|\, \mathcal{O}^{\mu_1 \cdots \mu_n}_V \,|p, s\rangle = V_n\, p^{\mu_1} \ldots p^{\mu_n}, \tag{9.32}
$$

$$
\langle p, s|\, \mathcal{O}^{\mu_1 \cdots \mu_n}_A \,|p, s\rangle = A_n\, \mathcal{S}\left[s^{\mu_1} \ldots p^{\mu_n}\right]. \tag{9.33}
$$

Using these matrix elements essentially undoes our free quark computation, except for factors of V_n and A_n, which were equal to one for the free quark matrix elements. Once again, let me concentrate on the term antisymmetric in $\mu\nu$. Substituting Eq. (9.33) into Eq. (9.31), one finds

$$
T^{[\mu\nu]} = \sum_{n=1,3,5}^{\infty} 2\, C^{(3)}_n\ i\epsilon^{\mu\nu\alpha\mu_1} q_\alpha\ \frac{2^n q^{\mu_2} \ldots q^{\mu_n}}{\left(-q^2\right)^n} A_n\, \mathcal{S}\left[s^{\mu_1} \ldots p^{\mu_n}\right]. \tag{9.34}
$$

Let us write

$$\mathcal{S}\left[s^{\mu_1}\dots p^{\mu_n}\right] = s^{\mu_1}\dots p^{\mu_n} + \mathcal{R}^{\mu_1\cdots\mu_n}, \tag{9.35}$$

where

$$\begin{aligned}\mathcal{R}^{\mu_1\cdots\mu_n} &= \mathcal{S}\left[s^{\mu_1}\dots p^{\mu_n}\right] - s^{\mu_1}\dots p^{\mu_n} \\ &= \frac{1}{n}\left[s^{\mu_1}p^{\mu_2}\dots p^{\mu_n} + p^{\mu_1}s^{\mu_2}\dots p^{\mu_n} + \dots p^{\mu_1}p^{\mu_2}\dots s^{\mu_n}\right] - s^{\mu_1}\dots p^{\mu_n} \\ &= -\frac{n-1}{n}s^{\mu_1}p^{\mu_2}\dots p^{\mu_n} + \frac{1}{n}p^{\mu_1}s^{\mu_2}\dots p^{\mu_n} + \dots + \frac{1}{n}p^{\mu_1}p^{\mu_2}\dots s^{\mu_n}.\end{aligned} \tag{9.36}$$

The tensor $\mathcal{R}^{\mu_1\cdots\mu_n}$ has no completely symmetric part, and so has spin $n-1$ rather than spin n. Thus the contribution of $\mathcal{R}$ to deep inelastic scattering is twist three, rather than twist two, *even though it came from the matrix element of a twist two operator.* This is the Wilczek-Wandzura contribution to g_2, as will be discussed in Sec. 10. Omitting the $\mathcal{R}$ term in Eq. (9.35) for the moment, and substituting into Eq. (9.34), we find

$$\begin{aligned}T^{[\mu\nu]} &= \sum_{n=1,3,5}^{\infty} -\frac{4}{q^2}\, C_n^{(3)} i\epsilon^{\mu\nu\alpha\mu_1} q_\alpha s_{\mu_1} A_n \omega^{n-1}, \\ &= i\epsilon^{\mu\nu\alpha\lambda} q_\alpha s_\lambda\, \frac{\widetilde{g}_1}{p\cdot q},\end{aligned} \tag{9.37}$$

using the definition of $\widetilde{g}_1$ from Eq. (7.3). This gives us the power series expansion

$$\begin{aligned}\widetilde{g}_1 &= \sum_{n=1,3,5}^{\infty} -\frac{4p\cdot q}{q^2}\, C_n^{(3)}\, A_n\, \omega^{n-1}, \\ \Rightarrow \quad \widetilde{g}_1 &= \sum_{n=1,3,5}^{\infty} 2\, C_n^{(3)}\, A_n \omega^n.\end{aligned} \tag{9.38}$$

The operator product expansion has allowed us to compute $\widetilde{g}_1$ as a power series in ω about $\omega = 0$ in QCD. The n^{th} term in the power series is due to an operator of twist two and spin n. The radius of convergence of the power series is $|\omega| = 1$, since that is the location of the first singularity in the complex ω plane. This is precisely where the physical region begins.

The structure functions can be computed near $\omega = 0$, but not in the physical region $1 \le |\omega| \le \infty$. To understand this better, it is useful to use light-cone coordinates,

$$q^{\pm} = \frac{1}{\sqrt{2}}\left(q^0 \pm q^3\right). \tag{9.39}$$

Then the deep inelastic limit, x fixed, $Q^2 \to \infty$, is the limit q^+ fixed, $q^- \to \infty$, because $Q^2 = 2q^+q^-$, and

$$x = \frac{1}{\omega} = \frac{q^+}{p^+}. \tag{9.40}$$

Thus deep inelastic scattering is not a short distance process; it probes the structure of the hadron along the light-cone. However, if we look at the unphysical region and let $\omega \to 0$, then $q^+ \to \infty$ (and we already had $q^- \to \infty$). The expansion of the scattering amplitude around $\omega = 0$ is a short distance expansion, which is why one can compute it in QCD.

We can relate $\widetilde{g}_1$ in the unphysical region to its value in the physical region using contour integration. The coefficient $2C_n^{(3)}A_n$ can be extracted by the contour integral

$$2C_n^{(3)}A_n = \frac{1}{2\pi i}\oint_{|\omega|=r<1} \widetilde{g}_1(\omega)\,\frac{d\omega}{\omega^{n+1}}, \tag{9.41}$$

over a circle in the complex ω plane around the origin, as shown in fig. 8. The contour of integration can then be deformed to the contour C. This gives

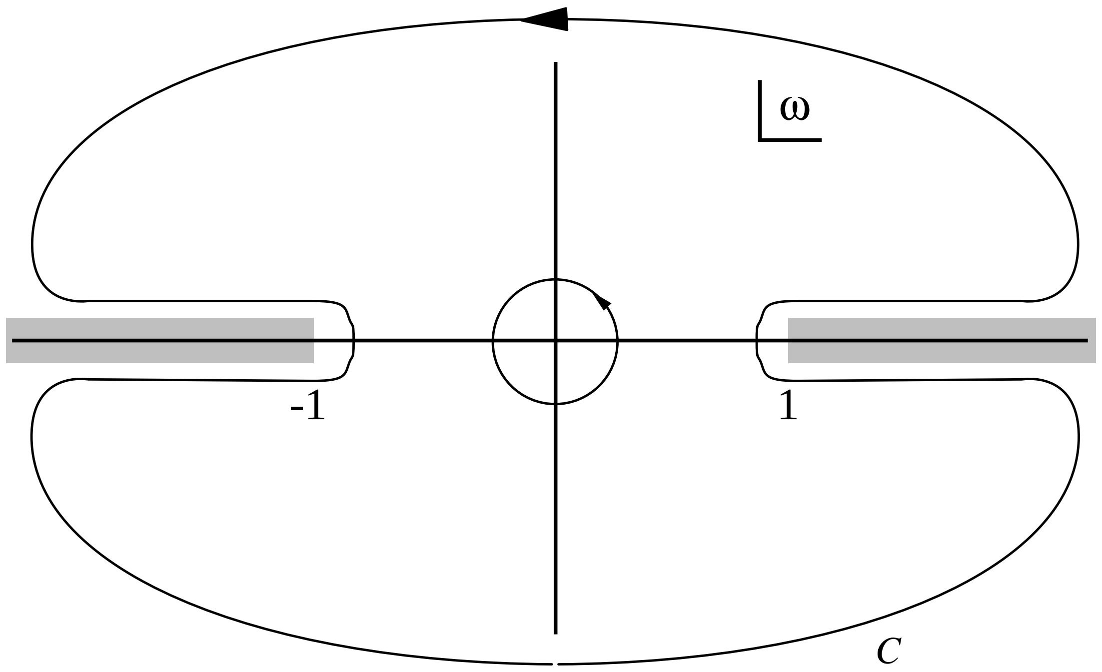

FIGURE 8.

The contours used to derive the moment sum rules.

$$2C_n^{(3)} A_n = \frac{1}{2\pi i} \oint_C \widetilde{g}_1(\omega) \, \frac{d\omega}{\omega^{n+1}}. \tag{9.42}$$

We now have to assume that the contour at infinity does not contribute to the integral, and can be neglected. In that case, only the discontinuities across the cuts contribute to the integral over contour C. Using the optical theorem, Eq. (7.8)

$$\widetilde{g}_1(\omega + i\epsilon) - \widetilde{g}_1(\omega - i\epsilon) = 4\pi i \, g_1(\omega), \tag{9.43}$$

where $g_1(\omega)$ is the experimentally measurable structure function, and using Eq. (7.5)

$$g_1(-\omega) = -g_1(\omega), \tag{9.44}$$

we get

$$2C_n^{(3)} A_n = 2\left[1 - (-1)^n\right] \int_1^\infty g_1(\omega) \, \frac{d\omega}{\omega^{n+1}}, \tag{9.45}$$

where the 1 term is from the right hand cut, and the $(-1)^n$ term is from the left hand cut. Substituting $\omega = 1/x$, and using the definition of a moment, Eq. (6.3), gives

$$2M_n(g_1) = 2\int_0^1 dx \ x^{n-1} g_1(x) = C_n^{(3)} A_n, \quad n \text{ odd}. \tag{9.46}$$

There is no sum rule for n even, because Eq. (9.45) reduces to the identity $0 = 0$. The sum rules for F_1 and F_2 are obtained in direct analogy to the derivation of the g_1 sum rule. The result is that

$$\begin{aligned} 2M_n(F_1) &= C_n^{(1)} V_n, \quad n \text{ even}, \\ 2M_n\left(\frac{F_2}{2x}\right) &= C_n^{(2)} V_n, \quad n \text{ even}. \end{aligned} \tag{9.47}$$

The moment sum rules relate a quantity defined at high energy such as F_1 to a low energy quantity, the zero momentum transfer matrix element of a local operator. One can derive sum rules in QCD only for the even moments of F_1, and the odd moments of g_1. There are no sum rules for the other moments. An attempt to derive a sum rule for such a "wrong moment" leads to the identity $0 = 0$, which is correct, but not very useful. This is due to the symmetry properties of the structure functions under charge conjugation (or crossing), Eq. (7.4). The Gottfried sum rule discussed in Sec. 8 is an attempt to derive a sum rule for the first moment of F_1, and is not true in QCD. It tries to relate a charge conjugation even structure function F_1 to a charge conjugation odd quantity, the number of u and d quark in the proton.

The only questionable assumption in the derivation of the sum rules is that the contour at infinity does not contribute to the integral. It should be possible to tell experimentally whether this is true. If the contour at infinity is important, then the sum rules will diverge near $x = 0$, which can be checked by measuring the small x behaviour of the structure functions. The sum rules become more convergent for the higher moments, so any problems with convergence will only occur for the lowest few moments. There is no evidence for any convergence problems for the sum rules in Eqs. (9.46) and (9.47).

10. Applications of the Moment Sum Rules

The previous section was a long and involved discussion of the derivation of the moment sum rules from QCD. In this section, I would like to discuss some of the consequences of the sum rules.

F_L in QCD

The sum rules for F_1 and F_2 imply that the moments of the longitudinal structure function $F_L = F_2/2x - F_1$ are given by

$$2M_n(F_L) = \left[C_n^{(2)} - C_n^{(1)}\right] V_n, \quad n \text{ even.} \tag{10.1}$$

We have already seen that at lowest order, $C_n^{(1)} = C_n^{(2)}$. This implies that $F_L = 0$ at leading order in QCD. This is the Callan-Gross relation that we obtained previously in Sec. 8 using the parton model. F_L is more interesting in QCD than in the parton model. At order α_s, $C_n^{(1)} \neq C_n^{(2)}$, so $F_L \neq 0$, but is suppressed relative to F_1 by $\alpha_s(Q)$. The same operator matrix elements V_n occur in F_L and F_2. The coefficients $C_n^{(1)}$ and $C_n^{(2)}$ can be computed in QCD perturbation theory, so one can predict F_L by using the measured value of F_1 to determine V_n. This prediction agrees with experiment.

The Ellis-Jaffe Sum Rule

Another interesting application is the sum rule for the first moment of g_1, the Ellis-Jaffe sum rule,

$$2\int_0^1 g_1(x)dx = C_1^{(3)} A_1. \tag{10.2}$$

At one loop the coefficient is,

$$C_1^{(3)} = 1 - \frac{\alpha_s(Q)}{\pi}. \tag{10.3}$$

The quantity A_1 is defined by the matrix element

$$2s^\mu A_1 = \langle p,s| \left[\frac{4}{9}\overline{u}\gamma^\mu\gamma_5 u + \frac{1}{9}\overline{d}\gamma^\mu\gamma_5 d + \frac{1}{9}\overline{s}\gamma^\mu\gamma_5 s \right] |p,s\rangle \,. \tag{10.4}$$

This is conventionally written as

$$A_1 = \frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s, \tag{10.5}$$

where Δq is defined by

$$2s^\mu \Delta q = \langle p,s|\, \overline{q}\gamma^\mu\gamma_5 q\, |p,s\rangle \,. \tag{10.6}$$

In terms of the parton distribution functions introduced in Sec. 8,

$$\Delta q = \int_0^1 \left[q_+(x) - q_-(x) + \overline{q}_+(x) - \overline{q}_-(x) \right] . \tag{10.7}$$

The notation is slightly confusing because $q_+(x) - q_-(x)$ is sometimes referred to as $\Delta q(x)$.

The Ellis-Jaffe sum rule relates the first moment of the g_1 structure function to the matrix elements of axial currents. This sum rule is special because there is no twist two spin-one gauge invariant gluon operator, so the right hand side of the sum rule only has the quark axial current operator, even at higher order in QCD. For ep scattering, the linear combination $\Delta u - \Delta d$ in the proton is equal to the axial decay constant g_A in neutron beta decay by isospin symmetry

$$\langle \text{proton}|\, \overline{u}\gamma^\mu\gamma_5 u - \overline{d}\gamma^\mu\gamma_5 d\, |\text{proton}\rangle = \langle \text{proton}|\, \overline{u}\gamma^\mu\gamma_5 d\, |\text{neutron}\rangle = 2g_A s^\mu. \tag{10.8}$$

The other non-singlet combination $\Delta u + \Delta d - 2\Delta s$ is determined by using $SU(3)$ symmetry and fitting to the F and D coefficients in hyperon semileptonic decay,

$$\Delta u + \Delta d - 2\Delta s = 3F - D, \quad g_A = F + D. \tag{10.9}$$

The Ellis-Jaffe sum rule can thus be written in the form

$$\int_0^1 g_1(x)dx = \frac{C_1^{(3)}}{18}\left[9F - D + 6\Delta s \right] = 0.126 \pm 0.010 \pm 0.015, \tag{10.10}$$

where the experimentally measured value for the sum rule is determined by the EMC collaboration (Ref. 6).

The values of F and D can be obtained by fitting to the hyperon semileptonic data. The standard results used in the literature are now out of date, because the experimental results have changed significantly. I will use the values

$$F = 0.47 \pm 0.04, \quad D = 0.81 \pm 0.03, \tag{10.11}$$

from the paper by Jaffe and Manohar in Ref. 8. The EMC measurement, the values of F and D, and the Ellis-Jaffe sum rule determine all three Δq's,

$$\Delta u = 0.74 \pm 0.10, \quad \Delta d = -0.54 \pm 0.10, \quad \Delta s = -0.20 \pm 0.11, \tag{10.12}$$

and

$$\Delta\Sigma \equiv \Delta u + \Delta d + \Delta s = 0.01 \pm 0.29. \tag{10.13}$$

Much of the recent interest in the spin structure of the proton is because of the non-zero value for Δs, and the small value for $\Delta\Sigma$.

The structure function inequality Eq. (6.2) implies that $s(x)+\overline{s}(x) \geq |\Delta s(x) + \Delta\overline{s}(x)|$. (This can be proved by noting that the inequality holds for arbitrary values of the electromagnetic charges of the quarks, and in particular holds for $\mathcal{Q}_u = \mathcal{Q}_d = 0$, $\mathcal{Q}_s \neq 0$.) The corresponding moment inequalities, Eqs. (6.6) and (6.7) imply that the first moment of $s(x) + \overline{s}(x)$, *i.e.* the momentum fraction carried by s quarks, bounds all other moments of $s(x)+\overline{s}(x)$ and $\Delta s(x)+\overline{\Delta}s(x)$, except the moment that appears in the Ellis-Jaffe sum rule. The strange quark momentum fraction has been measured experimentally to be 0.026, so all other strange quark matrix elements should be small. The value of Δs, Eq. (10.12), escapes this constraint.

The Bjorken sum rule is obtained by taking the first moment of the difference $g_1^{ep} - g_1^{en}$ for proton and neutron targets. Using isospin invariance, we see that

$$\begin{aligned}\int_0^1 \left(g_1^{ep}(x) - g_1^{en}(x)\right) dx &= \frac{1}{2}\, C_1^{(3)} \left[\frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s\right]_{\text{proton}-\text{neutron}} \\ &= \frac{1}{6}\, C_1^{(3)} \left(\Delta u - \Delta d\right)_{\text{proton}} \\ &= \frac{1}{6}\, C_1^{(3)} g_A.\end{aligned} \tag{10.14}$$

Any flavour singlet contribution such as Δs or gluons cancels out in the Bjorken sum rule. This sum rule will soon be tested experimentally.

The Wilczek-Wandzura contribution to g_2

Let us return to the tensor $\mathcal{R}$ that we left out in the computation of g_1 in Sec. 9. Inserting $\mathcal{R}$ into the formula for $T_{\mu\nu}$, Eq. (9.34),

$$T^{[\mu\nu]} = \sum_{n=1,3,5}^{\infty} -\frac{(n-1)}{n}\left[s^{\mu_1}\,\frac{2^n\,(p\cdot q)^{n-1}}{(-q^2)^n} - p^{\mu_1}\,\frac{2^n\,(p\cdot q)^{n-2}\,q\cdot s}{(-q^2)^n}\right] 2\,C_n^{(3)} A_n\; i\epsilon^{\mu\nu\alpha\mu_1} q_\alpha \tag{10.15}$$

since each term with a $\mu_2\ldots\mu_n$ index on s contributes to the second tensor structure. Comparing with the expansion Eq. (7.3) of $T^{\mu\nu}$, we see that Eq. (10.15) contributes to the structure function $\widetilde{g}_2$,

$$\begin{aligned}\widetilde{g}_2(\omega) &= \sum_{n=1,3,5}^{\infty} 2C_n^{(3)} A_n\left[\frac{1}{n}-1\right]\omega^n \\ &= -\widetilde{g}_1(\omega) + \int_0^\omega \widetilde{g}_1(\omega')\,\frac{d\omega'}{\omega'}\end{aligned} \tag{10.16}$$

using Eq. (9.38) for $\widetilde{g}_1$. Taking the imaginary part of both sides converts $\widetilde{g}_{1,2}$ into the experimentally measurable structure functions $g_{1,2}$. Since $\widetilde{g}_1$ has an imaginary part only in the physical region, the integral over ω' can be restricted to the region $1\le\omega'\le\omega$. Changing the integration variable to $x=1/\omega$ gives the Wilczek-Wandzura relation

$$g_2(x) = -g_1(x) + \int_x^1 g_1(x')\,\frac{dx'}{x'}. \tag{10.17}$$

This contribution to g_2 came from twist two operators, but we have already seen that it is really a twist three contribution because $\mathcal{R}$ has spin $n-1$ instead of n. There are also twist three operators in the operator product expansion that contribute to g_2, so the total contribution to g_2 can be written as

$$g_2(x) = \left[g_2(x)\right]_{\rm Wilczek-Wandzura} + \left[g_2(x)\right]_{\rm twist\ three}\,. \tag{10.18}$$

Both terms are equally important, and there is no reason why the Wilczek-Wandzura piece should be taken as a prediction for g_2.

The Wilczek-Wandzura contribution arises because the twist two operator Eq. (9.13) is symmetric in all its indices. This weights the $p\cdot s$ term in $T^{\mu\nu}$ by a factor of $(n-1)/n$ relative to the $q\cdot s$ term, where n is the spin of the operator. The standard decomposition of $T^{\mu\nu}$, Eq. (7.3) weights the $p\cdot s$ and $q\cdot s$ terms by coefficients which are independent of the power of ω in the structure function. It is this mismatch which leads to twist two operators producing an effectively twist three contribution to g_2.

11. Anomalous Dimensions and Scaling Violation

The Q^2 dependence of deep inelastic structure functions is an extremely important subject, but I do not have the time to analyse it in any detail in these lectures. I just want to give an outline of the QCD analysis. The starting point is the moment sum rules for the structure functions,

$$M_n(F(x)) \sim c_q\, \mathcal{O}_q + c_g\, \mathcal{O}_g, \tag{11.1}$$

in terms of quark and gluon operators, where F stands for a generic structure function. In deep inelastic scattering, the renormalisation scale μ is chosen to be Q, to avoid large logarithms in the computation of the coefficient functions. Thus Eq. (11.1) is more properly written as

$$M_n(F(x,Q^2)) \sim c_q(Q)\, \mathcal{O}_q(Q) + c_g(Q)\, \mathcal{O}_g(Q). \tag{11.2}$$

The Q^2 evolution of the structure functions is due to the Q^2 dependence in the coefficients and operators in Eq. (11.2). The coefficient $c_q(Q^2) = 1+\mathcal{O}(\alpha_s(Q))$ depends only weakly on Q^2. The largest Q^2 dependence comes from operator renormalisation. The renormalisation group equations for the operator can be written in the form

$$\mu \frac{d}{d\mu}\, \mathcal{O}^{\mu_1\cdots\mu_n} = -\frac{g^2}{16\pi^2}\, \gamma_n\, \mathcal{O}^{\mu_1\cdots\mu_n}, \tag{11.3}$$

at one loop, where γ_n is a calculable anomalous dimension. Dividing by the coupling constant evolution equation

$$\mu \frac{d}{d\mu}\, g = -b\, \frac{g^3}{16\pi^2}, \tag{11.4}$$

and integrating, gives

$$\mathcal{O}(Q) = \left[\frac{\alpha_s(Q)}{\alpha_s(Q_0)}\right]^{\gamma_n/2b} \mathcal{O}(Q_0), \tag{11.5}$$

so we can relate the moment of the structure function at Q_0 to the moment at Q. This leads to a calculable Q^2 dependence of the structure functions. As $Q^2 \to \infty$, $\mathcal{O}(Q) \to 0$, if $\gamma_n > 0$.

The actual situation is slightly more complicated. The quark operator can be divided into singlet and non-singlet pieces. The non-singlet pieces evolve as discussed above, but the singlet pieces can mix with gluon operators. In the singlet sector, one needs to solve a two dimensional matrix renormalisation group equation. The computation is similar to the one sketched above, and the moment evolution equation is a 2×2 matrix equation in terms of the eigenvectors and eigenvalues of the anomalous dimension matrix.

The Q^2 dependence of the Ellis-Jaffe sum rule is particularly simple. There is no gluon operator that can contribute to the sum rule, and thus there is no gluon operator which can mix with the singlet quark currents. The non-singlet currents are conserved, and have no anomalous dimension. The singlet current is not conserved because of the axial anomaly, and has an anomalous dimension at two loops,

$$\mu \frac{d}{d\mu}\, j^\mu_{\text{singlet}} = -N_f \left(\frac{g^2}{4\pi^2}\right)^2, \tag{11.6}$$

where N_f is the number of light flavours. Integrating the two loop equation by dividing by Eq. (11.4), gives

$$j^\mu_{\text{singlet}}(Q) = \exp\left[\frac{N_f}{8\pi}\left(\alpha(Q) - \alpha(Q_0)\right)\right] j^\mu_{\text{singlet}}(Q_0). \tag{11.7}$$

As $Q^2 \to \infty$, $j^\mu(Q)$ approaches a finite value in contrast to the case where the operator had a one loop anomalous dimension, Eq. (11.5). The Q^2 variation due to the two loop anomalous dimension Eq. (11.7) produces a very weak Q^2 dependence in Σ in Eq. (10.13).

There are some constraints on the possible anomalous dimensions for the structure functions in QCD. The unitarity inequality Eq. (6.2) must be true at all Q^2. This implies that the anomalous dimensions of the axial vector operators must be larger than the vector operators, so that for large Q^2, the g_1 structure functions falls off at least as fast as F_1. Left and right handed quarks have the same QCD couplings. This implies that non-singlet axial and vector operators have the same anomalous dimension, since they cannot mix with gluons. This is not true for the singlet operators, because the Dirac trace in a closed Fermi loop depends on whether or not there is an additional γ_5 at the vertex. However, at one loop, it is still possible to show that the qq element of the singlet anomalous dimension matrix for vector and axial operators is the same, since there are no closed Fermi loops in the Feynman graph for the anomalous dimension.

12. Outlook and Conclusions

There were several new experiments on spin dependent deep inelastic scattering discussed at this Winter Institute. The g_1 structure function of the proton will be measured more accurately, to check the EMC measurement. The g_1 structure function of the deuteron is also going to be measured. This, in combination with the EMC measurement, will give the first measurement of g_1 for the neutron, and should be able to test the Bjorken sum rule. There are experiments at HERA to measure two new structure functions which exist for spin one targets, $b_1(x)$ and $\Delta(x)$, which I did not have time to discuss here. The new generation of experiments are now sensitive enough to test the structure of the proton that is not "trivial," *i.e.* effects from pair production such as the strange quark distribution, Fermi statistics effects such as $\overline{u} \neq \overline{d}$, *etc.*

Let me briefly discuss the structure functions b_1 and Δ for spin one targets. In terms of helicity amplitudes, the structure function b_1 is proportion to

$$b_1 \; \propto \; \left[\mathcal{A}(+,1;+1) + \mathcal{A}(+,-1;+,-1) - 2\mathcal{A}(+,0;+0)\right], \tag{12.1}$$

and is the angular momentum two asymmetry in the cross-section for a transverse photon to scatter off a polarised spin one target. The structure function b_1 is a twist two structure function that can be measured with a polarised target and an unpolarised beam. (All even angular momentum structure functions can be measured this way; odd angular momentum structure functions such as g_1 $(J=1)$ need both a polarised beam and polarised target.) The structure function $\Delta(x)$ for a spin one target is proportional to

$$\Delta \; \propto \; \mathcal{A}(+,-1;-,+1), \tag{12.2}$$

and changes the helicity of the incident photon and target by two units. It is also a twist two structure function, but gets no quark contribution because the helicity of a quark cannot be changed by two units. Thus the structure function is leading twist but order α_s, because it arises from gluon operators in the terms of type $C_{j,n}^{(4)}$ in Eq. (9.31). $\Delta(x)$ also vanishes for higher spin targets that can be considered as a collection of non-interacting spin-1/2 nucleons, so it should provide some interesting information on the structure of higher spin targets.

I have tried to avoid any mention of models in this talk, except for a brief discussion of the parton model in Sec. 9. Instead, I concentrated on those aspects of deep inelastic scattering that can be derived from QCD using only quantum field theory. If a model calculation disagrees with experiment it is the model which is wrong (or needs to be "improved"); but if a QCD field theory calculation disagrees with experiment, it is the experimental result which is wrong, or it is evidence for a new fundamental interaction.

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