Lecture #3

χPT

@

NLO

NLO Chiral Lagrangian

Low energy power counting:

$\partial_\mu = \mathcal{O}(p)$, a_μ and $v_\mu = \mathcal{O}(p)$

$\chi = \mathcal{O}(p^2)$, $B_0 = \mathcal{O}(1) \Rightarrow \mathcal{M}_q = \mathcal{O}(p^2)$

NLO Lagrangian:

- four D , two χ , two D and one χ

- use LO Equations of motion to eliminate some terms.

- use that: $\langle U D_\mu U^\dagger \rangle = \langle U^\dagger D_\mu U \rangle = 0$

- use identity valid for traceless 3x3 matrices A and B:
$\langle ABAB \rangle = -2\langle A^2 B^2 \rangle + \frac{1}{2}\langle A^2 \rangle \langle B^2 \rangle + \langle AB \rangle^2$

The $\mathcal{O}(p^4)$ Lagrangian
[Gasser & Leutwyler]

$$\mathcal{L}_0^{(4)} = L_1 \langle D_\mu U^\dagger D^\mu U \rangle + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle$$

$$+ L_3 \langle D_\mu U^\dagger D^\mu U \, D_\nu U^\dagger D^\nu U \rangle$$

$$+ L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle$$

$$+ L_5 \langle D_\mu U^\dagger D^\mu U \, (\chi^\dagger U + \chi U^\dagger) \rangle$$

$$+ L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2$$

$$+ L_8 \langle \chi^\dagger U \chi^\dagger U + h.c. \rangle$$

$$- i L_9 \langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U \rangle$$

$$+ L_{10} \langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu} \rangle$$

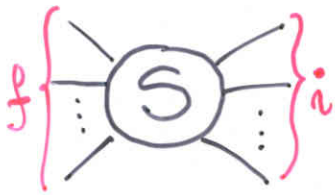
$$+ H_1 \langle F^R_{\mu\nu} F^{R\mu\nu} - R \to L \rangle$$

$$+ H_2 \langle \chi^\dagger \chi \rangle$$

$$F^R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i [r_\mu, r_\nu] \, , \ldots$$

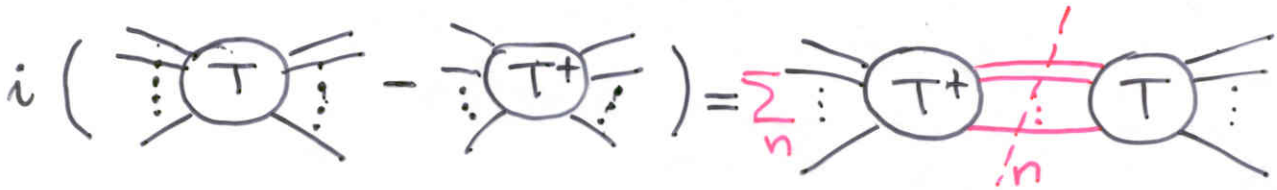
12 low energy constants (LECs)!

Unitarity Corrections (Loops)



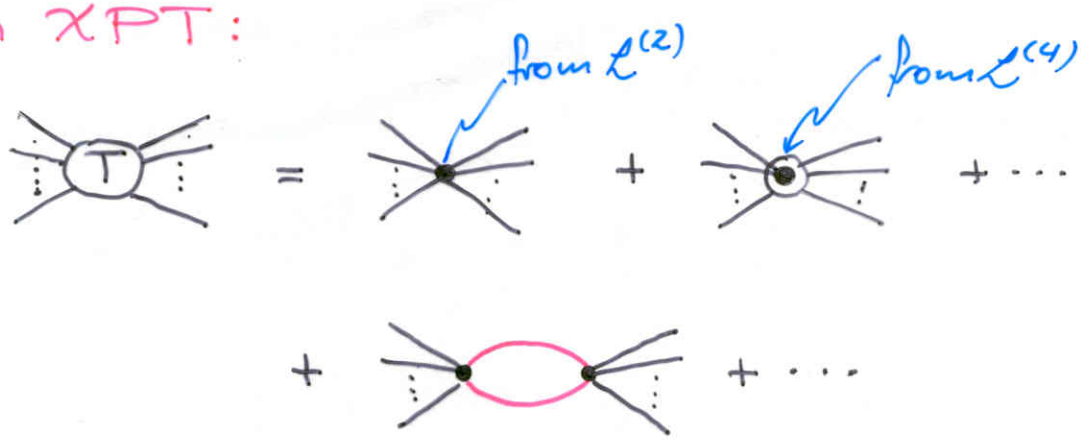
$$= \delta_{fi} + i\, T_{fi}$$

$$S^\dagger S = \mathbb{1} \quad \Rightarrow \quad i(T - T^\dagger) = -T^\dagger T$$



(Optical Theorem)

In χPT:



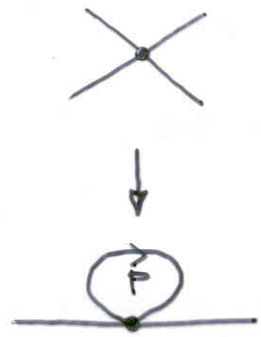
Loops are required by unitarity.

In χPT loop contributions are ordered according to χ expansion

[Weinberg]

Ultraviolet Divergencies (UV)

Example: tadpole diagram



from $\mathcal{L}^{(2)}$

$$\propto \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M_\pi^2}$$

integral diverges quadratically! . Regularization needed.

Requirements on regularization

- Regularization preserves all symmetries.
- Regularization provides a natural low energy counting for loop diagrams.

Dimensional Regularization

$$\int \frac{d^4p}{(2\pi)^4} f(p) \longrightarrow \int \frac{d^dp}{(2\pi)^d} f(p)$$

All one-loop integrals can be brought to combinations of integrals such as:

$(p_0 \to i p_0)$:

$$\int \frac{d^dp}{(2\pi)^d} (p^2)^\alpha = 0$$

$$\int \frac{d^dp}{(2\pi)^d} \frac{(p^2)^{\alpha \geq 0}}{(p^2+\mu^2)^\beta} = \frac{1}{(4\pi)^{d/2}} \mu^{d+2(\alpha-\beta)} \cdot \frac{\Gamma(\alpha+\frac{d}{2})\,\Gamma(\beta-\alpha-\frac{d}{2})}{\Gamma(\frac{d}{2})\,\Gamma(\beta)}$$

$\Gamma(\beta-\alpha-\frac{d}{2})$ has poles at $d = 2(n+\beta-\alpha)$

all U.V. divergencies contained in this factor.

(details about DR in any modern QFT book)

back to tadpole:

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^2-M_\pi^2)} \xrightarrow[4\to d]{p_0 \to i p_0} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+M_\pi^2}$$

$$= \frac{1}{(4\pi)^{d/2}} (M_\pi^2)^{\frac{d}{2}-1} \Gamma\left(1-\frac{d}{2}\right)$$

Expanding around $d=4$ and using $\varepsilon = \frac{4-d}{2}$

$$= \frac{M_\pi^2}{16\pi^2} \cdot \left(-\frac{1}{\varepsilon} + \gamma - 1 - \log 4\pi + \log M_\pi^2\right)$$

$\gamma = 0.577216\ldots$ (Euler's Constant)

Note that result is $\mathcal{O}(p^2)$ because integral is quadratically divergent.

General result: all 1-loop diagrams involving any number of insertions of $\mathcal{L}^{(2)}$ are $\mathcal{O}(p^4)$.

Renormalization:

It must be provided by $\mathcal{L}^{(4)}$.

Implementation [Gasser and Leutwyler]

$$L_i \longrightarrow L_i^r(\mu) + \Gamma_i \, \lambda(\mu)$$

$$\lambda(\mu) \equiv -\frac{1}{32\pi^2} \mu^{-2\varepsilon} \left(\frac{1}{\varepsilon} + \gamma - 1 - \log 4\pi \right)$$

Γ_i : β-functions

$\Gamma_1 = 3/32$; $\Gamma_2 = 3/16$; $\Gamma_3 = 0$; $\Gamma_4 = 1/8$; $\Gamma_5 = 3/8$

$\Gamma_6 = 11/144$; $\Gamma_7 = 0$; $\Gamma_8 = 5/48$; $\Gamma_9 = 1/4$; $\Gamma_{10} = -1/4$

(for $H_{1,2}$) $\Delta_1 = -1/8$, $\Delta_2 = 5/24$

Renormalized LECs:

$$L_i^r(\mu)$$

Rules of the chiral game

- @ $\mathcal{O}(p^2)$: tree level with $\mathcal{L}^{(2)}$
- @ $\mathcal{O}(p^4)$: 1-loop with $\mathcal{L}^{(2)}$ $\oplus$ tree level with $\mathcal{L}^{(4)}$
- @ $\mathcal{O}(p^6)$: 2-loops with $\mathcal{L}^{(2)}$ $\oplus$ 1-loop with $\mathcal{L}^{(2)}$ and a single insertion of $\mathcal{L}^{(4)}$ $\oplus$ tree level with $\mathcal{L}^{(6)}$. (Only for experienced players!)

Let's play @ $\mathcal{O}(p^4)$!

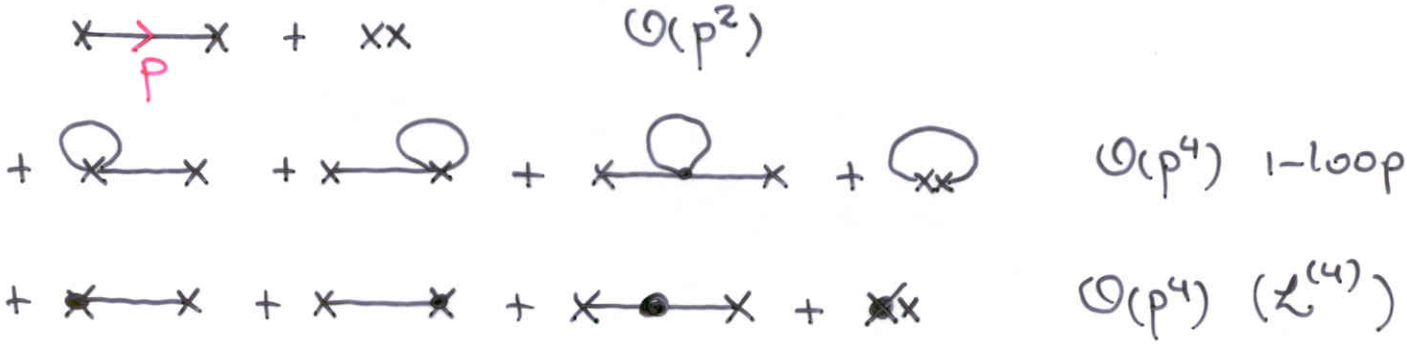
Masses & Decay Constants @ $\mathcal{O}(p^4)$

Study for this the 2-point function:

$$A^{ab}_{\mu\nu}(x) = \langle 0 | T \, A^a_\mu(x) \, A^b_\nu(0) | 0 \rangle$$

Neglect isospin breaking $\Rightarrow A^{ab}_{\mu\nu} \propto \delta^{ab}$

Diagrams



$$A^{ab}_{\mu\nu}(p) = \delta^{ab} \, p_\mu p_\nu \, F_a(p^2) \, \frac{i}{p^2 - M_a^2} + \text{contact terms}$$

Extract F_a and M_a^2 from 2-point function

$$\mu_M \equiv \frac{1}{32\pi^2}\frac{M^2}{F_0^2}\log\frac{M^2}{\mu^2}$$

$$F_\pi = F_0\left(1 - 2\mu_\pi - \mu_K + 2\hat{m}K_6 + K_7\right)$$

$$F_K = F_0\left(1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_\eta + (\hat{m}+m_s)K_6 + K_7\right)$$

$$\rightarrow F_\eta = F_\pi\left(\frac{F_K}{F_\pi}\right)^{4/3}\left(1 + \frac{1}{96\pi^2F_0^2}\left(3M_\eta^2\log\frac{M_\eta^2}{M_K^2} - M_\pi^2\log\frac{M_K^2}{M_\pi^2}\right)\right)$$

$$K_6 = \frac{4B_0}{F_0^2}L_5^r(\mu) \qquad K_7 = \frac{8B_0}{F_0^2}(2\hat{m}+m_s)L_4^r(\mu)$$

F_K/F_π does not contain K_7

Use F_K/F_π to determine L_5^r

$$\left.\frac{F_K}{F_\pi}\right|_{Exp} = 1.22 \pm 0.01 \Rightarrow L_5^r(\mu = M_\eta) = (2.2 \pm 0.5)\,10^{-3}$$

needs V_{us}

from lattice: (preliminary) $\left.\frac{F_K}{F_\pi}\right|_{Latt.} = 1.20 \pm 0.01 \Rightarrow L_5^r(\mu = M_\eta) = (2.0 \pm 0.5) \times 10^{-3}$

$$\mathring{M}_\pi^2 = 2\hat{m} B_0 \left(1+\mu_\pi - \tfrac{1}{3}\mu_\eta + 2\hat{m} K_3 + K_4\right)$$

$$\mathring{M}_K^2 = (\hat{m}+m_s) B_0 \left(1+\tfrac{2}{3}\mu_\eta + (\hat{m}+m_s) K_3 + K_4\right)$$

$$\mathring{M}_\eta^2 = \tfrac{2}{3}(\hat{m}+2m_s) B_0 \left(1+2\mu_K - \tfrac{4}{3}\mu_\eta + \tfrac{2}{3}(\hat{m}+2m_s) K_3 + K_4\right)$$

$$+ 2\hat{m} B_0 \left(-\mu_\pi + \tfrac{2}{3}\mu_K + \tfrac{1}{3}\mu_\eta\right) + K_5$$

$$K_3 = \frac{8B_0}{F_0^2}\left(2L_8^r(\mu) - L_5^r(\mu)\right)$$

$$K_4 = \frac{16B_0}{F_0^2}(2\hat{m}+m_s)\left(2L_6^r(\mu) - L_4^r(\mu)\right)$$

$$K_5 = \frac{128}{9}\frac{B_0^2}{F_0^2}(m_s-\hat{m})^2\left(2L_7 + L_8^r(\mu)\right)$$

- Deviation from GM-O

$$2L_7 + L_8^r(\mu = M_\eta) = (0.3 \pm 0.4)\times 10^{-3}$$

- $$\frac{\mathring{M}_K^2}{\mathring{M}_\pi^2} = \frac{m_s+\hat{m}}{2\hat{m}}(1+\Delta_M)$$

- If one includes $m_u \neq m_d$

$$\frac{\mathring{M}_{K^0}^2 - \mathring{M}_{K^+}^2}{\mathring{M}_K^2 - \mathring{M}_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}(1+\Delta_M)$$

$$\Delta_M = -\mu_\pi + \mu_\eta + \frac{8}{F_0^2}(M_K^2 - M_\pi^2)(2L_8^r(\mu) - L_5^r(\mu))$$

- Parameter free relation

$$\frac{\overset{\circ}{M}{}^2_{K^0} - \overset{\circ}{M}{}^2_{K^+}}{M_K^2 - M_\pi^2}\,\frac{M_\pi^2}{M_K^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2} = 1/Q^2$$

- Estimate of Δ_M

need $R \equiv \dfrac{m_s - \hat{m}}{m_d - m_u}$ from other source than masses of GB's.

$R = 40 \pm 5$ from various sources

and use Dashen's Theorem (with corrections $\mathcal{O}(p^2 e^2)$) for $(\overset{\circ}{M}{}^2_{K^0} - \overset{\circ}{M}{}^2_{K^+})$

$$\Delta_M = \pm 0.15$$

$$\Rightarrow \quad L_8^r(\mu = M_\eta) \sim \frac{1}{2} L_5^r(\mu = M_\eta)$$

$$\Rightarrow \quad L_7 \sim -0.4 \times 10^{-3}$$

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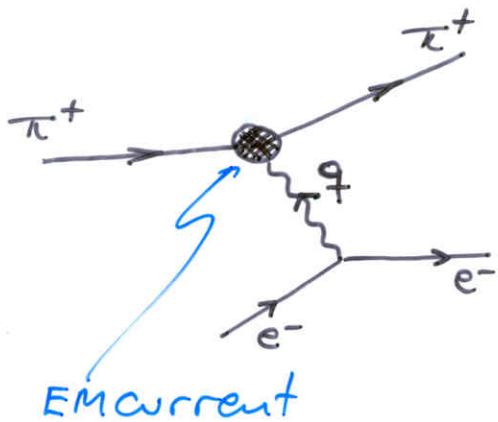
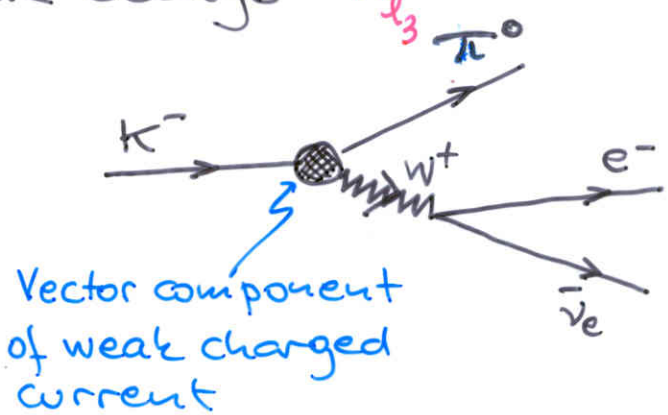
Form Factors @ $O(p^4)$

EM



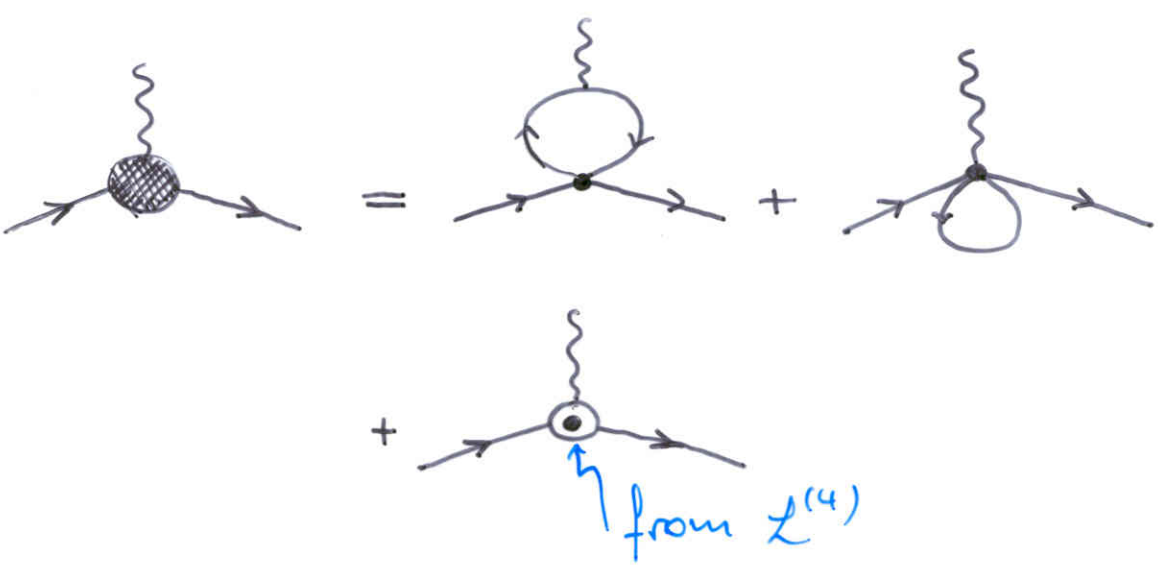
$t \equiv q^2$

Weak decays K_{ℓ_3}



Crucial for precision extraction of $|V_{us}|$ from K_{ℓ_3}

χPT:



At low t :

$$F(t) = F(0)\left(1 + \frac{1}{6}\langle r^2\rangle t + \cdots\right)$$

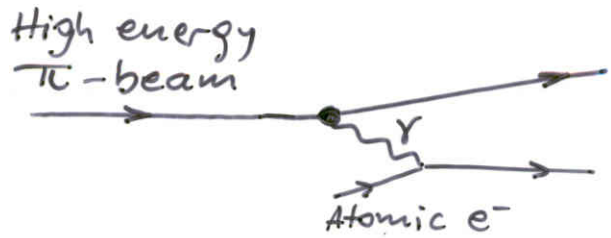
EM Form Factors

$$\langle \pi^+, p' | V_\mu^{EM} | \pi^+, p\rangle = f_+^{\pi^+}(t)\,(p+p')_\mu$$

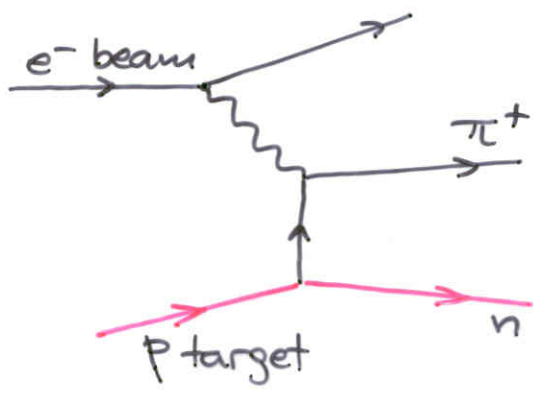
$$f_+(0) = 1$$

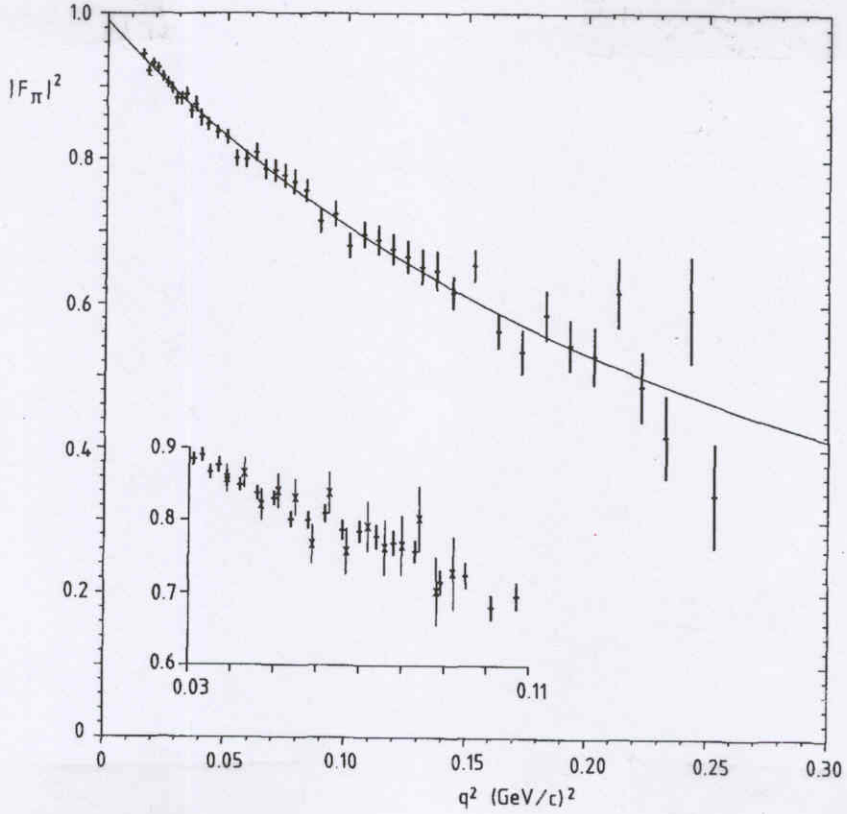
- Experimental determination of FF.

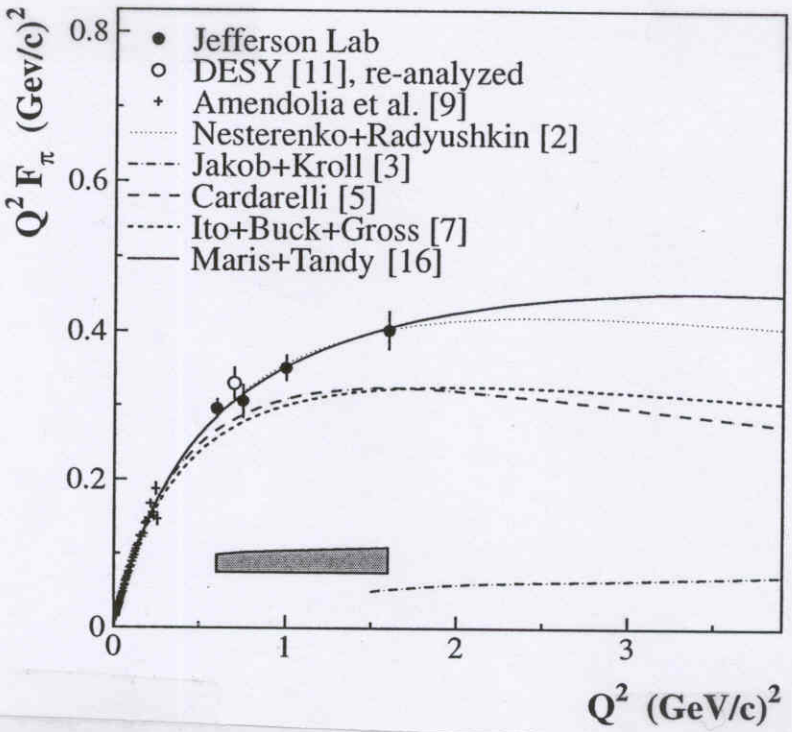
Low t :



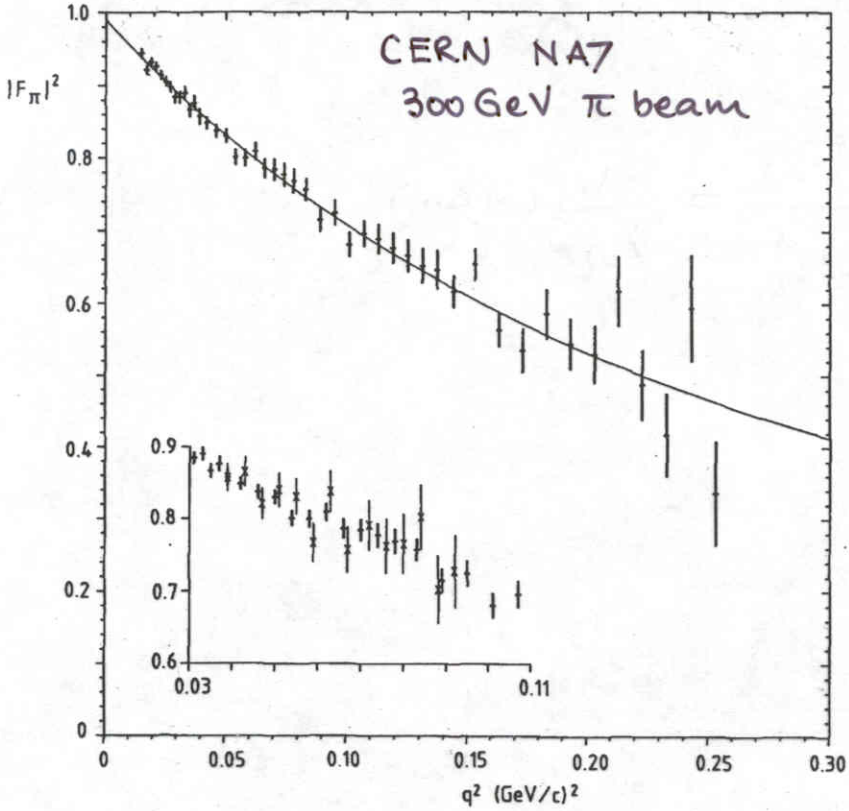
Higher t : e.g. JLab Exp. E93-021



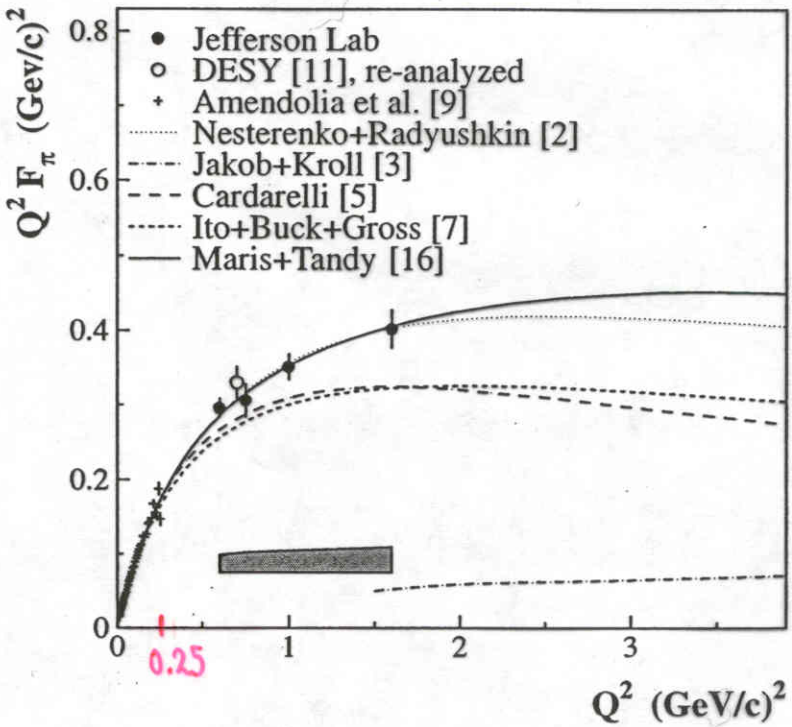

$|F_\pi|^2$
1.0
0.8
0.6
0.4
0.2
0
0.9
0.8
0.7
0.6
0.03
0.11
0
0.05
0.10
0.15
0.20
0.25
0.30
q^2 (GeV/c)^2


$Q^2 F_\pi$ (Gev/c)^2
Jefferson Lab
DESY [11], re-analyzed
Amendolia et al. [9]
Nesterenko+Radyushkin [2]
Jakob+Kroll [3]
Cardarelli [5]
Ito+Buck+Gross [7]
Maris+Tandy [16]
0.8
0.6
0.4
0.2
0
0
1
2
3
Q^2 (GeV/c)^2

Amendolia et. al (1986)



Volmer et. al (2001)



Amendolia et. al : $\langle r_\pi^2 \rangle_{EM}^{Exp} = 0.431 \pm 0.010 \text{ fm}^2$

- $$\langle r_{\pi^\pm}^2 \rangle_{EM} = 12 \frac{L_9^r(\mu)}{F_0^2} - \frac{1}{32\pi^2 F_0^2}\left(3 + 2\log\frac{M_\pi^2}{\mu^2} + \log\frac{M_K^2}{\mu^2}\right)$$

diverges logarithmically as $M_\pi^2 \to 0$.

for $\mu \sim M_\eta$ second term gives $+0.038 \text{ fm}^2$

$$L_9^r(M_\eta) = (7.2 \pm 0.2) \times 10^{-3}$$

- $$\langle r_{K^0}^2 \rangle_{EM} = -\frac{1}{32\pi^2 F_0^2} \log\frac{M_K^2}{M_\pi^2} \quad !$$

$$\cong -0.04 \text{ fm}^2$$

- L_9^r allows to pin down $\langle r_{K^\pm}^2 \rangle_{EM} = 0.394 \pm 0.010 \text{ fm}^2$

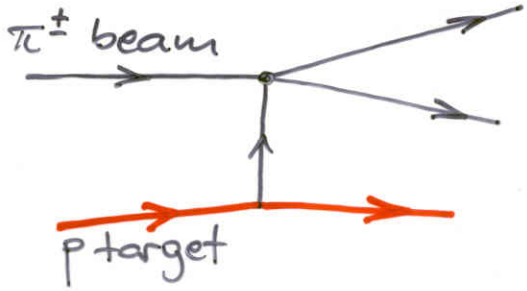
Experimentally (Amendolia et al.)

$$\langle r_{K^\pm}^2 \rangle_{EM}^{Exp} = 0.34 \pm 0.05 \text{ fm}^2$$

$\pi\pi$ Scattering @ $\mathcal{O}(p^4)$

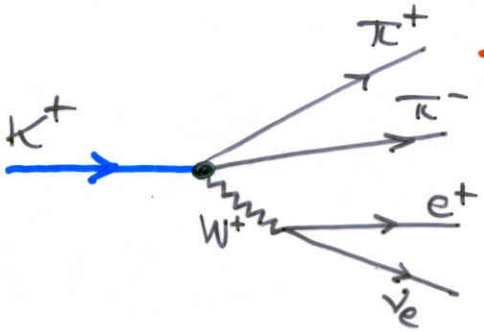
Types of experiments

Peripheral π-production



All partial waves

K_{e4} decay



$\pi\pi$ final state interaction

BNL E865 (2003) Sensitive to combination

$$\delta_0^0 - \delta_1^1$$

Scattering lengths from data

Global analysis consistent with unitarity and crossing using dispersion relations (Roy equations) [Colangelo et al]

	Imposing Constraints of χPT to $\mathcal{O}(p^6)$!	No χPT constraints	LO χPT
a_0^0	0.220 ± 0.005	0.24 ± 0.06	0.16
$M_\pi^2 b_0^0$	0.276 ± 0.006	0.26 ± 0.02	0.17
a_0^2	-0.0444 ± 0.0010	-0.036 ± 0.013	-0.044
$M_\pi^2 b_0^2$	-0.0803 ± 0.0012	-0.082 ± 0.008	-0.085
$M_\pi^2 a_1^1$	0.0379 ± 0.0005	0.038 ± 0.002	0.029
$M_\pi^4 b_1^1$	$(0.567 \pm 0.013) \times 10^{-2}$	$(0.54 \pm 0.04) \times 10^{-2}$	0
$M_\pi^4 a_2^0$	$(0.175 \pm 0.003) \times 10^{-2}$	$(0.17 \pm 0.01) \times 10^{-2}$	0
$M_\pi^6 b_2^0$	$(-0.355 \pm 0.014) \times 10^{-3}$	$(-0.35 \pm 0.06) \times 10^{-3}$	⋮
⋮	⋮	⋮	

precision: few %! precision: 5 to 25%

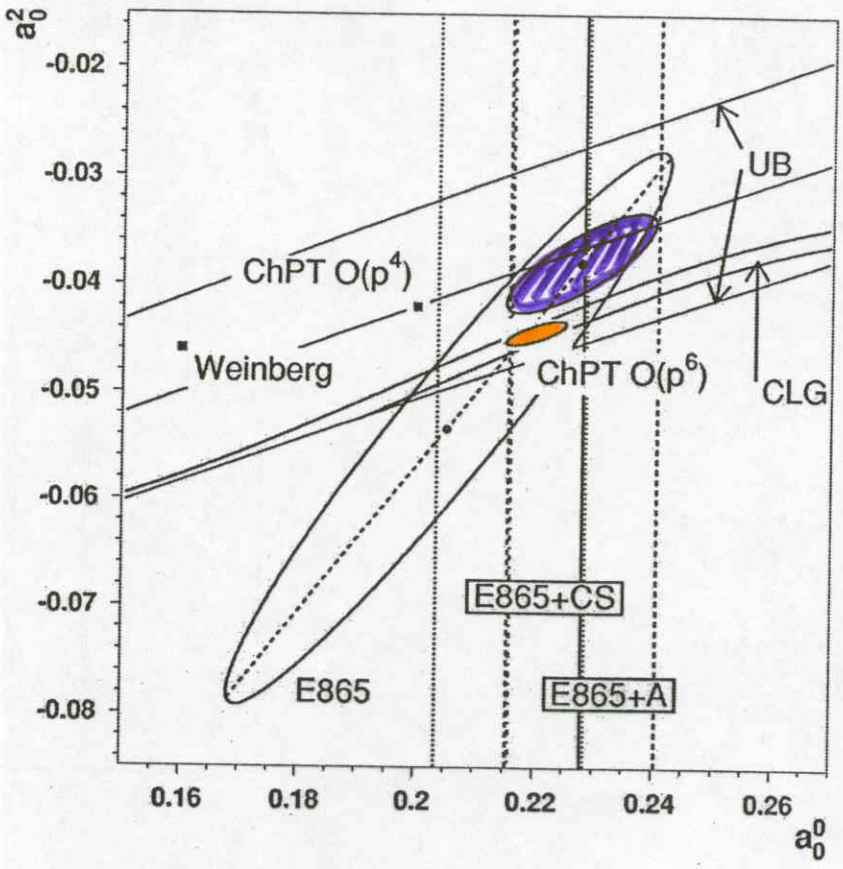
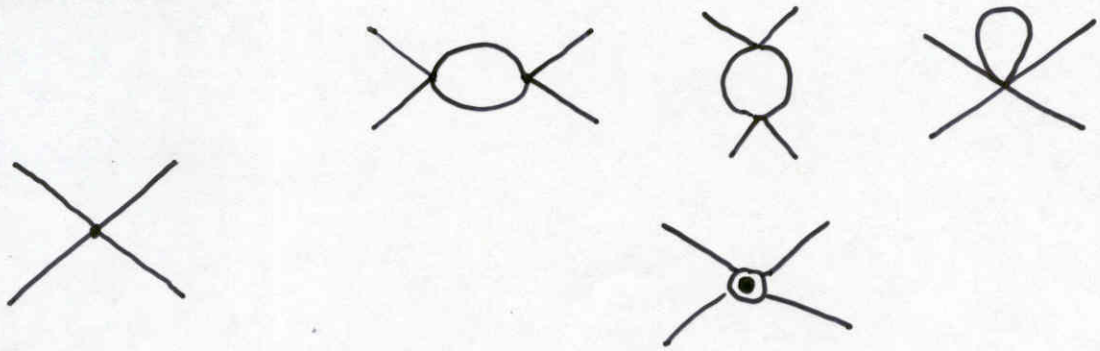


FIG. 10. (Color online) Results for the $\pi\pi$ scattering lengths a_0^0 and a_0^2 obtained from fits to the K_{e4} data directly or from fits to the phase shifts obtained in this experiment. Large ellipse labeled E865: fit to our K_{e4}-data leaving both a_0^2 and a_0^0 as free parameters using Eq. (13) with the parameters of Ref. [24] (1σ contour, see text for remark concerning the region outside the universal band). Medium size ellipse without label: fit of Ref. [55] (1σ contour) to our phase shifts. Theoretical predictions: [18] (Weinberg, square), [19] [ChPT O(p^4), square], and [21] [ChPT O(p^6), small ellipse]. Solid curves labeled UB: universal band of allowed values based on Eq. (14). Solid curves labeled CLG: narrow band of allowed values based on Eq. (15). Solid vertical line labeled E865 (+A $\equiv$ analyticity constraints): fit to K_{e4} data using Eq. (14) with 1 σ error limits given by dashed vertical lines. Dashed-dotted line labeled E865 (+CS $\equiv$ analyticity and chiral symmetry constraints) fit to K_{e4} data using Eq. (15) with 1 σ error limits given by dotted vertical lines.

χPT to $\mathcal{O}(p^4)$

S, P, D wave scattering



$\mathcal{O}(p^2)$ $\mathcal{O}(p^4)$

Dependence L_1^r, L_2^r, L_3, L_4^r, L_5^r and L_8^r in three combinations.

If use of $1/N_c$ expansion is made:

$$L_i^r = \begin{cases} \mathcal{O}(N_c) \text{ if } i = 1,2,3,5,8,9,10 \\ \mathcal{O}(1) \text{ if } i = (2\times 1 - 2), 4, 6 \\ \mathcal{O}(N_c^2) \text{ if } i = 7 \end{cases}$$

Eliminating $\mathcal{O}(1)$ combinations, and knowing L_i^r for $i = 5, 8$ one can pin down L_1^r, L_3.

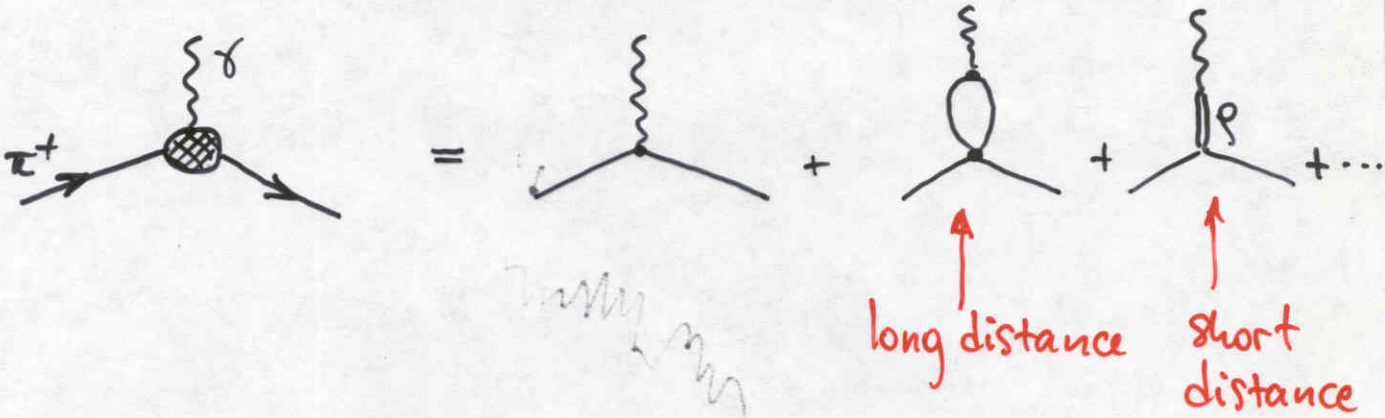
$$L_1^r(M_\eta) \sim \frac{1}{2} L_2^r(M_\eta) = (4 \pm 2) \times 10^{-4}$$

$$L_3 = (-2.5 \pm 1.3) \times 10^{-3}$$

What is the Physics behind the LECs?

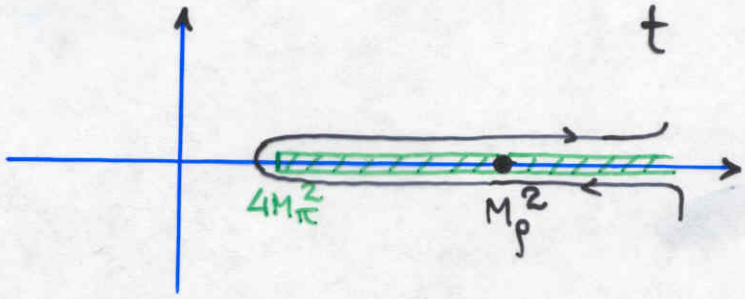
- LECs contain the short distance physics that the dynamics of the effective d.o.f. (GB's) cannot capture.
- They are important, even dominant in some cases, e.g. L_5, L_9.

Disecting L_9



Dispersion Relation : Cauchy's Th. for f_+

$$f_+(t) = f_+(0) + \frac{t}{\pi} \int \frac{\mathrm{Im}\, f_+(t')}{t'(t'-t)}\, dt'$$



$$\mathrm{Im}\, f_+(t)\Big|_{\rho\text{-pole}} = -\,\delta(t-M_\rho^2)\, F_\rho\, G_{\rho\pi}$$

$$f_+(t) \cong f_+(0) + \delta f^{\text{cont}}(t) + \delta f^{\rho\text{-pole}}(t)$$

$$\delta f^{\rho\text{-pole}}(t) = -\,F_\rho\, G_{\rho\pi}\, \frac{t}{M_\rho^2(M_\rho^2-t)}$$

Contribution to charge radius:

XPT: $$\langle r^2\rangle = \frac{12}{F_0^2} L_9^r(\mu) - \overbrace{\frac{1}{32\pi^2 F_0^2}\left(3 + 2\log\frac{M_\pi^2}{\mu^2} + \log\frac{M_K^2}{\mu^2}\right.}^{*}$$

Dispersion Rel.: $$\langle r^2\rangle = \frac{6}{\pi}\int \frac{\mathrm{Im}\, f_+^{\text{cont}}(t')}{t'^2}\, dt' - 6\,\frac{F_\rho\, G_{\rho\pi}}{M_\rho^4}$$

$$G_V \equiv \frac{F_0^2}{M_V^2}\, G_{\rho\pi}$$

Take $\mu \sim M_\eta$, and assume continuum contribution $\sim$ given by XPT long distance *.

Then: $$L_9^r(\mu \sim M_\eta) = \frac{1}{2}\, F_\rho\, G_V / M_\rho^2$$

from $\rho^0 \to e^+e^-$,

$$|F_\rho| \sim 154 \text{ MeV}$$

from $\rho \to \pi\pi$

$$|G_V| \sim 69 \text{ MeV}$$

$$\Downarrow$$

$$L_9^{VMD} \simeq 8.9 \times 10^{-3}$$

Compare with value extracted with χPT

$$L_9^r(M_\eta) = 7.2 \times 10^{-3}$$