# Nucleon and Nuclear Structure Functions

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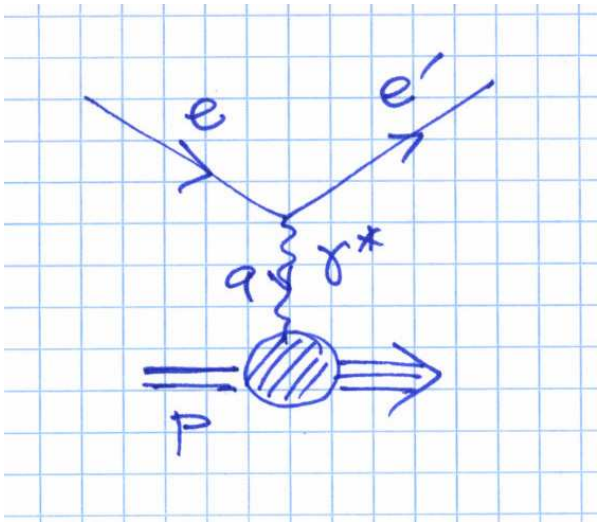
# Outline

- Kinematics, cross sections and structure functions of deep inelastic scattering lepton scattering.
- Unpolarized scattering and spin effects.
- Elastic scattering and resonance production.
- Space-time scales of DIS.
- Data on structure functions.
- Parton model (and corrections to that) and PDF phenomenology.

# Inelastic lepton scattering

## Kinematics & cross sections

Inclusive inelastic lepton scattering $e^-(k) + A(p) \to e^-(k') +$ anything. Measured are the electron (muon) scattering angle $\theta$ and the lepton energy $E'$ (energy transfer $\omega = E - E'$) and the differential cross section $\mathrm{d}\sigma/\mathrm{d}\omega\mathrm{d}\Omega$. The leading order contribution to the scattering amplitude is described by one-photon-exchange approximation.




$$\mathcal{M}_n = \frac{e^2}{q^2}\,\bar{u}(k')\gamma_\mu u(k)\,\langle n|\,J_\mu(0)\,|p,S\rangle$$

$$q = k - k', \quad Q^2 = -q^2 = 4EE'\sin^2(\theta/2)$$

$$\mathrm{d}\sigma = \frac{1}{4M|\boldsymbol{k}_{\mathrm{Lab}}|}\sum_n |\mathcal{M}_n|^2(2\pi)^4\delta(p+q-p_n)\frac{\mathrm{d}^3k'}{(2\pi)^3 2E'} =$$

$$= \frac{|\boldsymbol{k}'|}{ME}\,\frac{\alpha^2}{Q^4}\,L^{\mu\nu}H_{\mu\nu}\,\mathrm{d}\omega\mathrm{d}\Omega$$

# Leptonic tensor

Leptons are point-like particles and $L_{\mu\nu}$ is easy to calculate.

$$L_{\mu\nu} = \sum_{s'} (\bar{u}(k',s')\gamma_\mu u(k,s))^* \bar{u}(k',s')\gamma_\nu u(k,s)$$
$$= 2\left(k_\mu k'_\nu + k_\nu k'_\mu\right) + q^2 g_{\mu\nu} + 2i\, m_l \varepsilon_{\mu\nu\alpha\beta} q^\alpha s_l^\beta$$

If the incoming lepton beam is not polarized then $L_{\mu\nu}$ has only symmetric $\mu \leftrightarrow \nu$ tensor part. In case of polarized lepton beam $L_{\mu\nu}$ has antisymmetric part as well which depends on lepton polarization axial 4-vector $m_l s_l = (\boldsymbol{k}\cdot\boldsymbol{\zeta}, m_l\boldsymbol{\zeta} + \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\zeta})/(E+m_l))$ ($\boldsymbol{\zeta}$ is a unit vector along the direction of polarization in lepton rest frame). General properties of polarization vector: $s_l^2 = -1$, $s\cdot k = 0$. For a lepton beam polarized along the beam momentum in the laboratory frame $m_l s_l = (|\boldsymbol{k}|, E\boldsymbol{k}/|\boldsymbol{k}|)$ (neglecting lepton mass).

# Hadronic tensor

Calculation of hadronic tensor $H_{\mu\nu}$ is not as simple as $L_{\mu\nu}$.

$$H_{\mu\nu} = \frac{1}{4\pi} \sum_n \langle p, S| J_\mu(0) |n\rangle \langle n| J_\nu(0) |p, S\rangle (2\pi)^4 \delta(p + q - p_n)$$
$$= \frac{1}{4\pi} \int \mathrm{d}^4\xi \exp(iq \cdot \xi) \langle p, S| [J_\mu(\xi), J_\nu(0)] |p, S\rangle$$

Tensor $H_{\mu\nu}$ can be parameterized in terms of Lorentz-invariant and dimensionless functions (structure functions), which characterize the structure of the target and measure the reaction probability. General requirements on hadronic tensor:

Electromagnetic current conservation $\Rightarrow q_\mu H_{\mu\nu} = H_{\mu\nu} q_\nu = 0$
Parity conservation $\Rightarrow H_{\mu\nu}(p, q) = H_{\mu\nu}(\tilde{p}, \tilde{q})$,
$\tilde{p} = (p_0, -\boldsymbol{p}),\ \tilde{q} = (q_0, -\boldsymbol{q})$
Time reversal symmetry $\Rightarrow H_{\mu\nu}(p, q) = H^*_{\mu\nu}(\tilde{p}, \tilde{q})$

For a spin-1/2 target there are only 2 symmetric and 2 antisymmetric tensors satisfying these requirements.

$$H_{\mu\nu} = H_{\{\mu\nu\}} + H_{[\mu\nu]},$$

$$H_{\{\mu\nu\}}(p,q) = F_1 \left( \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) + \frac{F_2}{p\cdot q} \left( p_\mu - q_\mu \frac{p\cdot q}{q^2} \right) \left( p_\nu - q_\nu \frac{p\cdot q}{q^2} \right),$$

$$H_{[\mu\nu]}(p,q) = \frac{M}{p\cdot q} \left[ \varepsilon_{\mu\nu\alpha\beta} q^\alpha S^\beta (g_1 + g_2) - \varepsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta \frac{S\cdot q}{p\cdot q} g_2 \right]$$

$$S = \left( \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{M},\ \boldsymbol{\zeta} + \frac{\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\zeta})}{M(E(\boldsymbol{p}) + M)} \right)$$

If target is unpolarized $H_{\mu\nu}$ is symmetric tensor. For polarized target $H_{\mu\nu}$ involves antisymmetric tensor which depends on the target polarization 4-vector $S^\beta$. Information about hadron internal structure is encoded in Lorentz invariant and dimensionless structure functions, which can be extracted from cross section data. Structure functions are usually considered as functions of $Q^2$ and the Bjorken dimensionless variable $x_{\mathsf{Bj}} = Q^2/2p\cdot q = Q^2/2M\omega$.

# Kinematical boundaries

$W^2 = (p+q)^2 = M^2 + 2p\cdot q + q^2 \geqslant M^2$ $\Longrightarrow$ $x_{\mathsf{Bj}} \leqslant 1$ ($x_{\mathsf{Bj}} = 1$ corresponds to elastic scattering).

The invariant mass is bound by the lepton beam energy:

$W^2 < (p+k)^2 = M^2 + 2ME$ (about $25\ \mathrm{GeV}^2$ for Jlab12 GeV and about $10^3\ \mathrm{GeV}^2$ for fixed target experiments at FNAL).

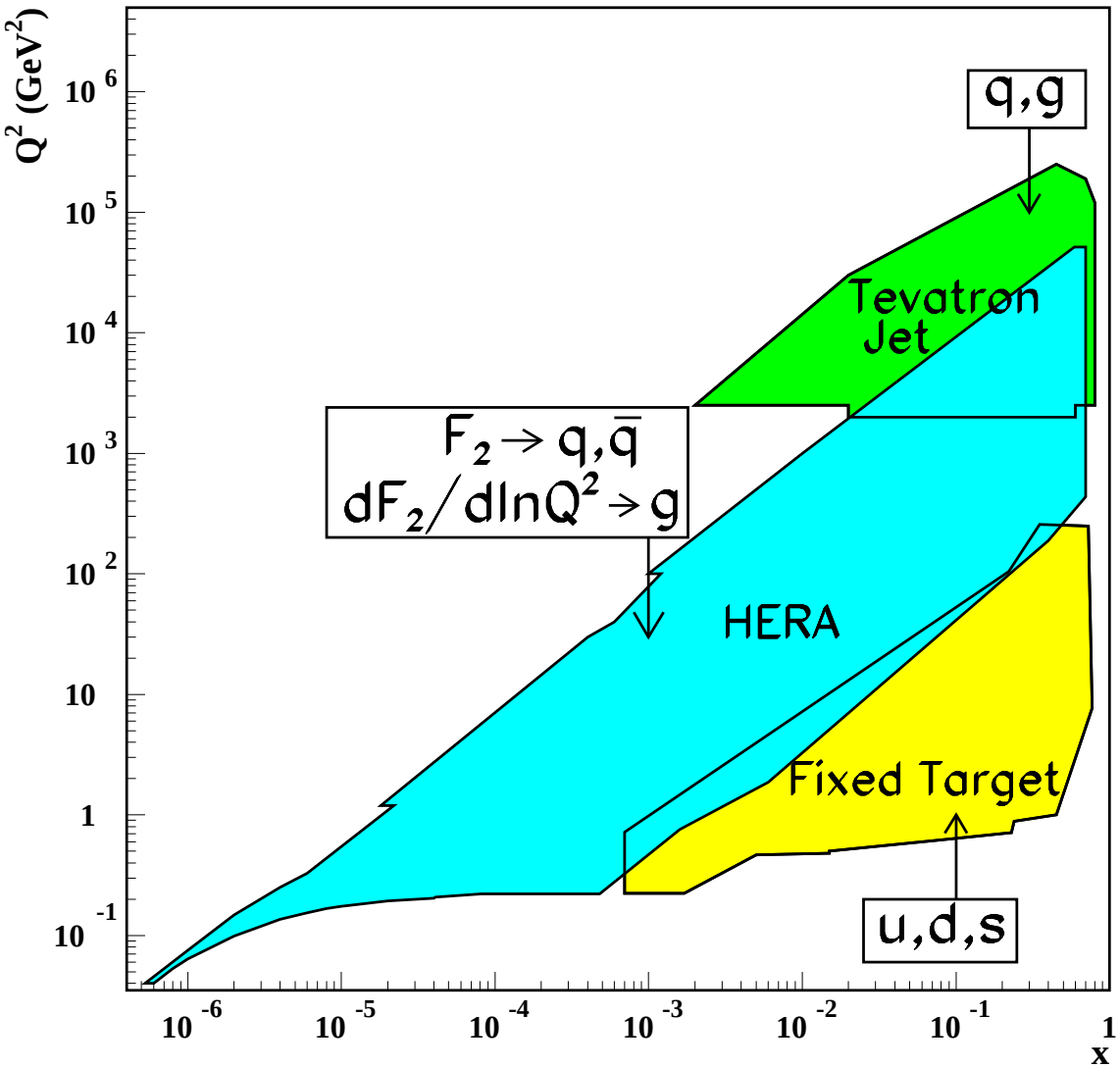
Inelasticity $y = p\cdot q/p\cdot k = (E-E')/E$.

Kinematical limits: $0 \leqslant y < 1$.

Note $x$, $y$, and $Q^2$ are correlated:

$$xy = \frac{Q^2}{2p\cdot q}\,\frac{p\cdot q}{p\cdot k} = \frac{Q^2}{2ME}$$

$$x_{\mathsf{min}}(Q^2) = \frac{Q^2}{2ME}$$



# Cross sections in terms of structure functions

The polarization averaged differential cross section is determined by the structure functions $F_{1,2}$. In terms of variables $x$ and $Q^2$ the cross section is

$$\frac{\mathrm{d}^2\sigma^{\text{unpol}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{Mxy}{2E}\right)\frac{F_2}{x} + y^2\left(1 - \frac{2m_l^2}{Q^2}\right)F_1\right],$$

Here $y = p \cdot q / p \cdot k$, the inelasticity ($y$ is not independent variable and related to $x$ and $Q^2$ via the equation $xy = Q^2/2p \cdot k$).

If the beam and the target are both polarized, then the cross section must be supplimented by the spin dependent part, which is given in terms of the spin structure functions $g_{1,2}$:

$$\frac{\mathrm{d}^2\sigma^{\text{spin}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\frac{2m_l y}{E}\left[-s_l \cdot S\, g_1 + \frac{s_l \cdot p\, S \cdot q}{p \cdot q}\, g_2\right].$$

# Helicity cross sections

Structure functions $F_{1,2}$ can be related to absorption cross sections of virtual transversely and longitudinaly polarized photon.

| Polarization state | Structure function |
|---|---|
| $e_T = (0, 1, i, 0)/\sqrt{2}$, | $F_T = e_T^{\mu *} H_{\mu\nu} e_T^{\nu} = F_1$ |
| $e_L = (q_z, \mathbf{0}_\perp, q_0)/Q$, | $F_L = e_L^{\mu *} H_{\mu\nu} e_L^{\nu} = -F_1 + \gamma^2 F_2/(2x)$ |

The $z$ axis is along the momentum transfer $\boldsymbol{q}$, $Q = \sqrt{Q^2}$, $\gamma^2 = \boldsymbol{q}^2/q_0^2 = 1 + (2Mx/Q)^2$.
Properties of polarization vectors: $e_T \cdot q = e_L \cdot q = e_T \cdot e_L = 0$, $e_T^* \cdot e_T = -1$, $e_L^2 = 1$.

The properties of $F_L$ and $F_T$ are closely related to the spin of partons (elementary constituents of the target). In the limit of large $Q^2$:

- $F_T = 0$ for the scattering from a point-like scalar particle with spin $0$.
- $F_L = 0$ for point-like fermion with spin $1/2$.

The ratio $R = F_L/F_T$ is a sensetive probe of the spin of target's constituents.

# Elastic scattering

Elastic scattering from a nucleon = the final hadronic state is the nucleon (proton or neutron). The hadronic tensor reads

$$
\begin{aligned}
W^{el}_{\mu\nu} &= \frac{1}{4\pi}\sum_{s'}\int\frac{\mathrm{d}^3p'}{(2\pi)^3\,2E(\boldsymbol{p}')}(2\pi)^4\delta(p+q-p')\,\langle p,s|\,J_\mu(0)\,|p',s'\rangle\,\langle p,'s'|\,J_\nu(0)\,|p,s\rangle \\
&= \frac{1}{2}\sum_{s'}\langle p,s|\,J_\mu(0)\,|p',s'\rangle\,\langle p,'s'|\,J_\nu(0)\,|p,s\rangle\,\delta(W^2-M^2),
\end{aligned}
$$

$W^2=(p+q)^2$ is invariant reaction mass.
Nucleon matrix element of electromagnetic current are parametrized in terms of two form factors

$$
\langle p,'s'|\,J_\mu(0)\,|p,s\rangle = \bar{u}(p',s')\left[F_E(Q^2)\gamma_\mu + i\frac{F_M(Q^2)}{2M}\sigma_{\mu\nu}q^\nu\right]u(p,s)
$$

$F_E(Q^2)$ and $F_M(Q^2)$ are electric and magnetic Pauli form factors. The Sachs form factors

are often used which are related to Pauli form factors as

$$\begin{aligned} G_E(Q^2) &= F_E(Q^2) - \frac{Q^2}{(2M)^2} F_M(Q^2), \\ G_M(Q^2) &= F_E(Q^2) + F_M(Q^2). \end{aligned}$$

At zero momentum transfer the form factors are normalized as $G_E^p(0) = 1,\ G_E^n(0) = 0$, and $G_M^{p,n}(0) = \mu^{p,n}$, where $\mu^{p,n}$ is magnetic moment of the proton (neutron). Empirical parameterization by a dipole formula $G_E^p(Q^2) = (1 + Q^2/m_V^2)^{-2}$ with $m_V^2 = 0.71\,\mathrm{GeV}^2$ good approximation up to $Q^2 = 30\,\mathrm{GeV}^2$ with a few % uncertainty. Experimentally (with the error less than 20%) there observed universality of the proton and neutron f.f.:

$$G_E = G_M^p/\mu^p = G_M^n/\mu^n = (1 + Q^2/m_V^2)^{-2}, \quad \text{and } G_E^n \approx 0.$$

Elastic structure functions. Note $W^2 = (p+q)^2 = M^2 + 2p \cdot q + q^2 = M^2 + Q^2(1/x - 1)$ and also $\delta(W^2 - M^2) = (2p \cdot q)^{-1}\delta(x-1)$. Use notation $\tau = Q^2/(2M)^2$ then

$$F_1^{el} = \frac{G_M^2}{2}\delta(x-1), \qquad F_2^{el} = \frac{G_E^2 + \tau G_M^2}{1+\tau}\delta(x-1)$$

# Resonances

What should we expect from resonace contributions?
Pion production theshold $W^2 = (M + m_\pi)^2 \quad \Rightarrow \quad 1/x \approx 1 + 2m_\pi M/Q^2$.
Production of a resonace with the mass $M_R$ corresponds $W^2 = M_R^2$ and $1/x_R = 1 + (M_R^2 - M^2)/Q^2$.
Resonance structure function $F_2^{(R)} \sim G_M^2(Q^2)\delta(x - x_R)$

# Bloom–Gilman duality

Corresponding principle (Bloom–Gilman duality) says that there are two ways of equivalent description:
(i) Averaged $F_2$ extrapolated from scaling region and
(ii) sum of (f.f.)$^2$ of hadronic resonances

$$\int_{x_*}^{1} \mathrm{d}x F_2^{\text{scaling}}(x, Q^2) = \sum_R c_R {G_M^{(R)}}^2(Q^2)$$

$$x_*^{-1} = 1 + \frac{M_*^2 - M^2}{Q^2}, \quad M^2 < M_R^2 < M_*^2$$

Relation between $x \to 1$ asymptotics of structure function $F_2(x) \sim C(1-x)^p$ and $Q^2 \to \infty$ asymptotics of f.f. $G_M \sim (Q^2)^{-n}$

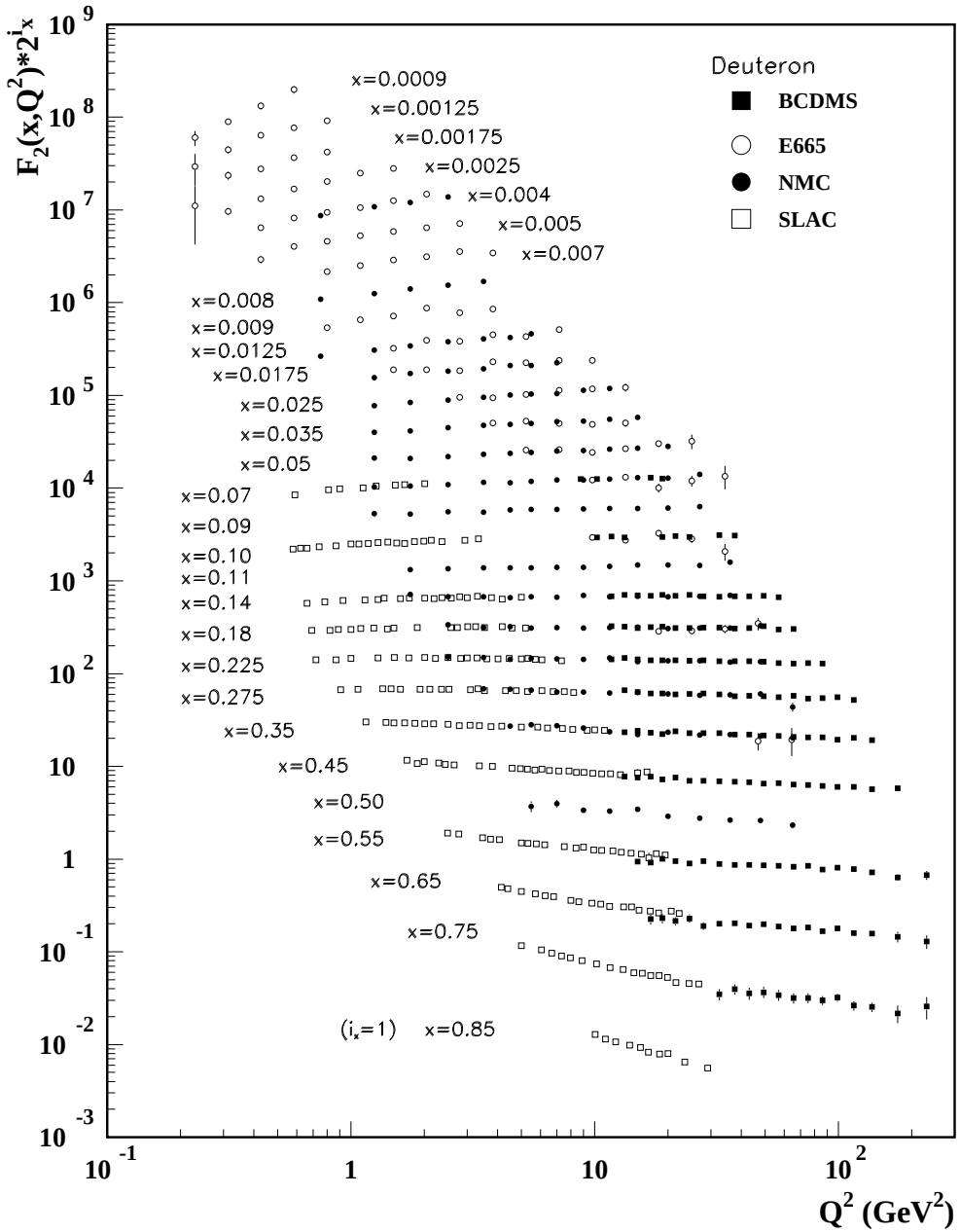
$$p + 1 = 2n$$

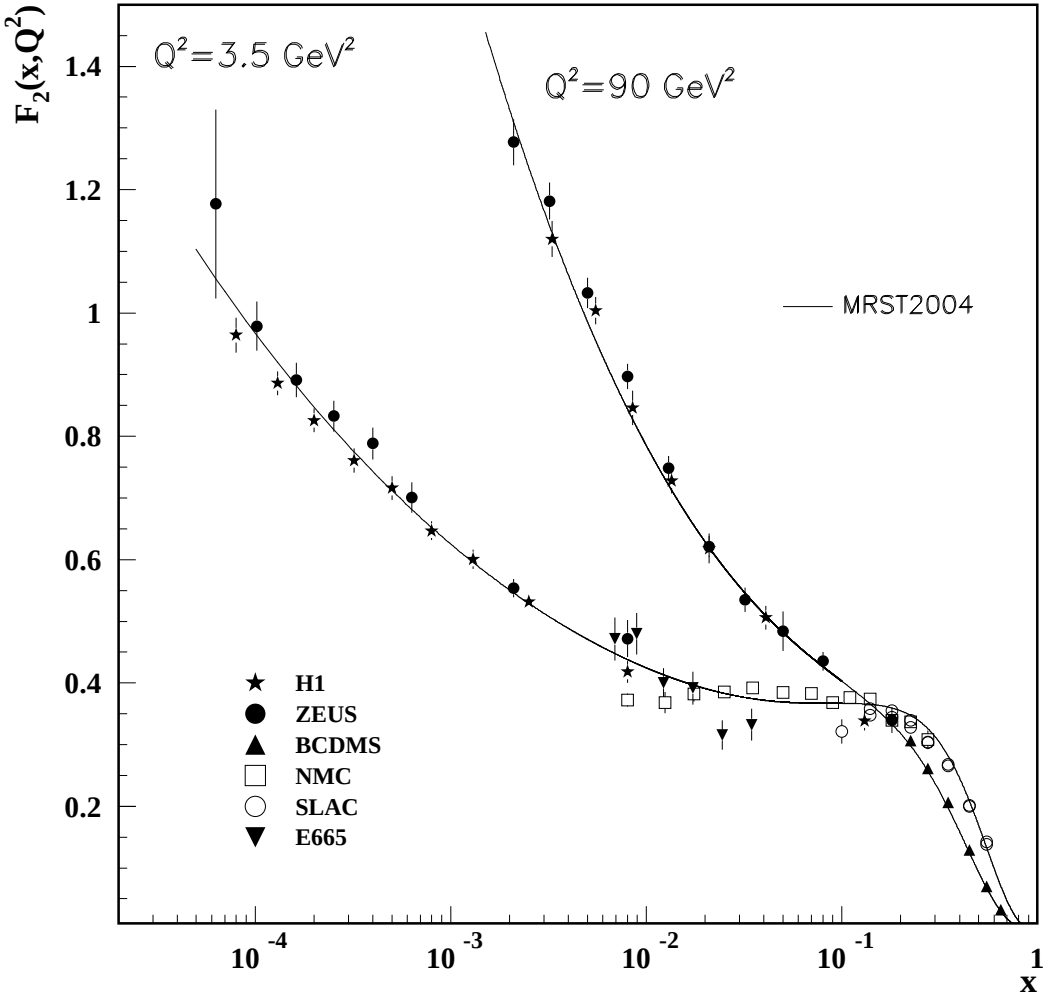
For $n = 2$ (dipole f.f.) we have $p = 3$ in agreement with the parton model result (so called, quark counting rules).

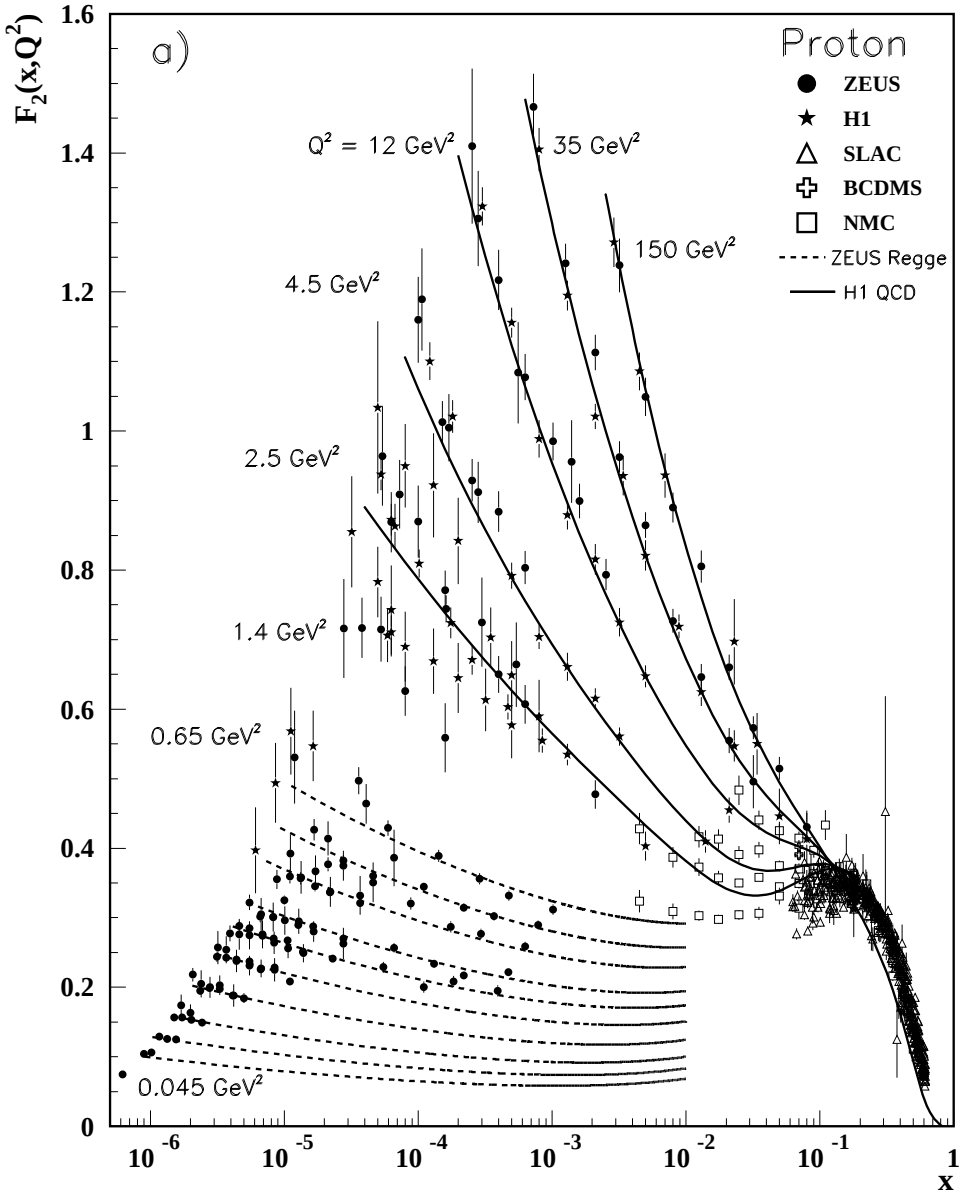
# DIS Data

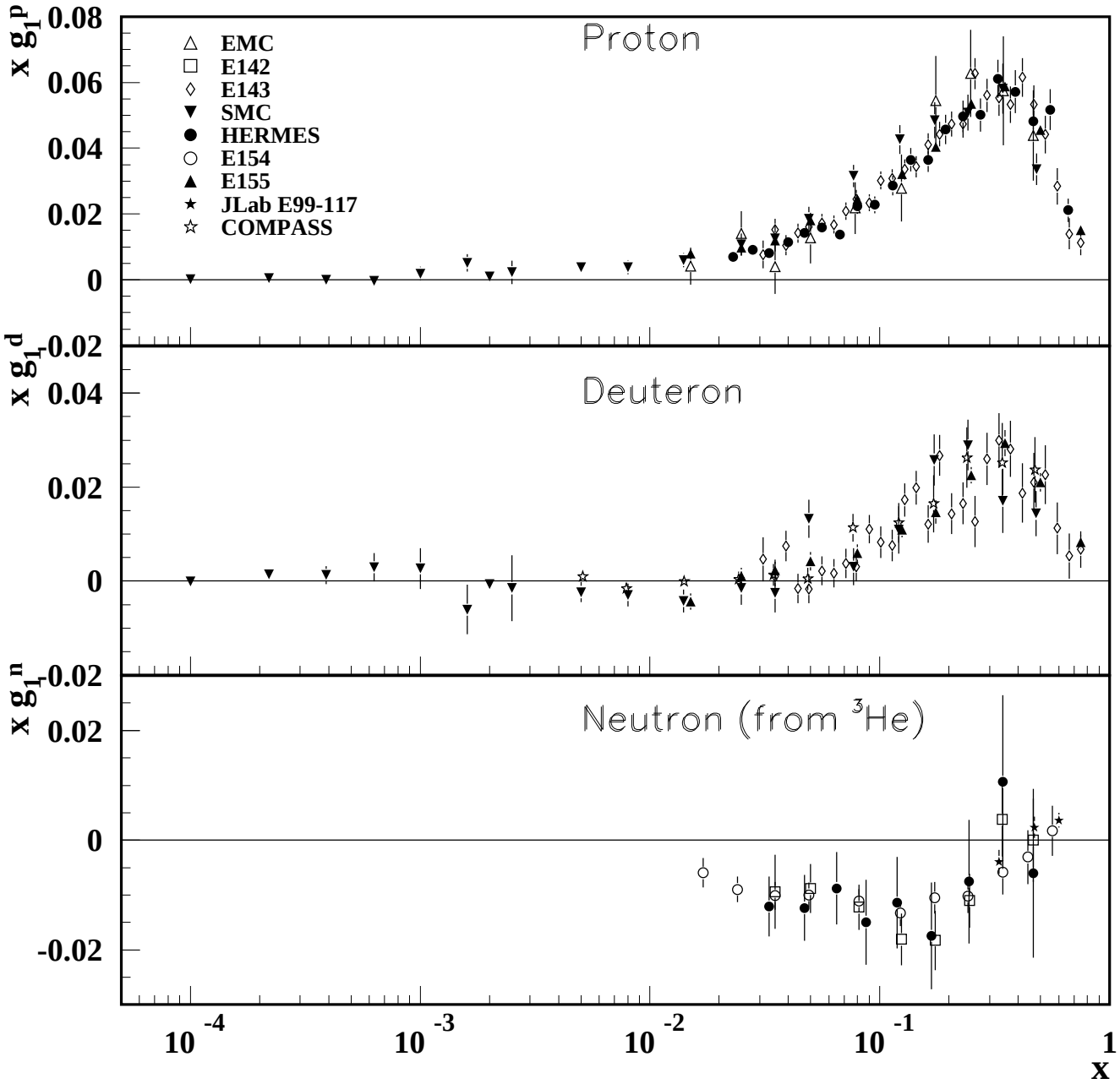
- To a very good approximation SFs are independent of $Q^2$ and depend only on $x$ (scaling phenomenon). The scaling phenomenon suggests that proton must contain point-like particles (because any object with a finite size has a form factor which introduces some $Q^2$ dependence).

- An accurate measurement of $Q^2$ variations provides an important test on the predictions of Quantum Chrodynamics (QCD, theory of strong interaction). On the other hand, QCD phenomenology of DIS helps to determine the strength of the running coupling constant of strong interaction $\alpha_s(Q^2)$.




$F_2(x,Q^2)$
$Q^2$=3.5 GeV$^2$
$Q^2$=90 GeV$^2$
MRST2004
H1
ZEUS
BCDMS
NMC
SLAC
E665
x


$F_2(x,Q^2)$
a)
Proton
ZEUS
H1
SLAC
BCDMS
NMC
ZEUS Regge
H1 QCD
$Q^2$ = 12 GeV$^2$
35 GeV$^2$
150 GeV$^2$
4.5 GeV$^2$
2.5 GeV$^2$
1.4 GeV$^2$
0.65 GeV$^2$
0.045 GeV$^2$
x


Proton
Deuteron
Neutron (from $^3$He)
EMC
E142
E143
SMC
HERMES
E154
E155
JLab E99-117
COMPASS
$x\,g_1^p$
$x\,g_1^d$
$x\,g_1^n$
0.08
0.06
0.04
0.02
0
-0.02
$10^{-4}$
$10^{-3}$
$10^{-2}$
$10^{-1}$
1
x

# Space-time scales in DIS

$$H_{\mu\nu} = \int \mathrm{d}x \exp(iq \cdot x) \langle p| [J_\mu(x), J_\nu(0)] |p\rangle$$

This commutator vanishes for $x^2 < 0$ (causality principle).

Charecteristic region which dominates $H_{\mu\nu}$ at high $Q^2$ is the region about light cone $x^2 \sim Q^{-2}$

Choose $z$ axis along momentum transfer, $q = (q_0, \mathbf{0}_\perp, |\boldsymbol{q}|)$. High $Q^2$ means $Q^2 = \boldsymbol{q}^2 - q_0^2 \gg M^2$. For the DIS kinematics $q_0 \geq Q^2/2M \gg Q$ and also $|\boldsymbol{q}| \sim q_0 \gg Q$.

$$q \cdot x = q_0 t - |\boldsymbol{q}| z = q_0 t - \sqrt{q_0^2 + Q^2} z \simeq q_0(t - z) - \frac{Q^2}{2q_0} z$$

Characteristic regions in the integral: $t - z \sim 1/q_0$ ($t$ and $z$ are correlated) and $z \sim 2q_0/Q^2 = 1/(Mx_{\mathrm{Bj}})$.

Note that characteristic $z$ is not small at all! DIS can proceed at the distances of about 1 Fm ($x_{\mathrm{Bj}} \sim 0.2$) and even at 5 Fm ($x_{\mathrm{Bj}} \sim 0.05$).

What about interval?

$$x^2 = (t-z)(t+z) - \boldsymbol{r}_\perp^2$$

$$(t-z)(t+z) \sim \frac{1}{q_0}\frac{q_0}{Q^2} \sim Q^{-2}$$

$$x^2 \geq 0 \quad \Rightarrow \quad \boldsymbol{r}_\perp^2 \lesssim Q^{-2}$$

Therefore $0 \leq x^2 \lesssim Q^{-2}$.

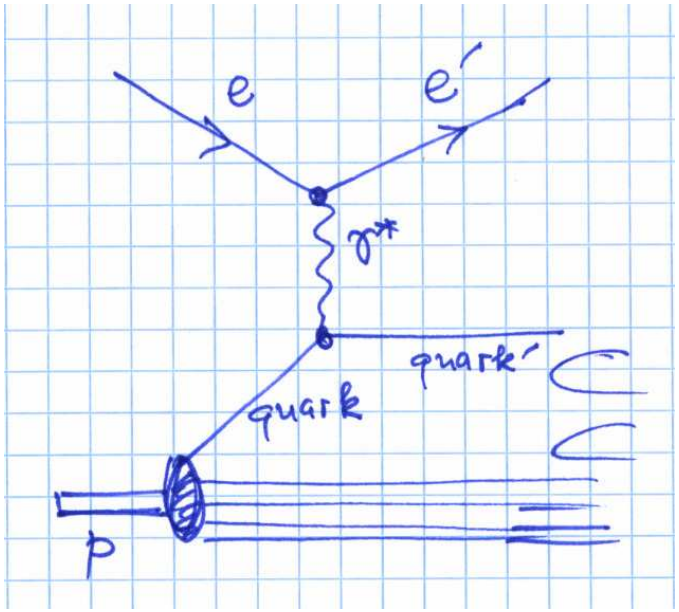
Interval is Lorentz-invariant characteristics and independent of reference frame. However, discussion of the time and space distances does depend on reference frame.

Momentum transfer is space-like 4-vector ($q^2 < 0$). We can choose a reference frame in which $q$ only has space components. Breit frame: $q = (0, \mathbf{0}_\perp, Q)$ (often called as infinite momentum frame, IMF). In the Breit frame nucleon moves antiparallel to the axis $z$ with momentum $p^{\text{Breit}} = Q/(2x_{\text{Bj}})$.
$t \sim z \sim 1/Q$, $\boldsymbol{r}_\perp^2 \sim Q^{-2}$.

The smallness of the time-space scales in IMF is a physics background of Parton model.

# SFs are quark distributions

## Parton model




Parton model: DIS $\approx$ incoherent scattering from elementary constituents of hadrons. In QCD the elementary (point-like) charged fields are (anti)quarks.

$$H_{\mu\nu}(P,q) = \sum_{q=u,d,\ldots} \int \mathrm{d}^3k\, n_q(k,P)\; h^q_{\mu\nu}(k,q),$$

$$F_2 = x \sum_{q=u,d,\ldots} e_q^2 \left[q(x) + \bar{q}(x)\right],$$

$e_u = 2/3,\ e_d = -1/3, \ldots$ are the quark charges in units of the proton charge. The distributions $q(x)$ and $\bar{q}(x)$ have a simple interpretation: $q(x)\mathrm{d}x$ is the number of quarks (or antiquarks) that carry momentum fraction between $x$ and $x + \mathrm{d}x$ of the target's momentum $P$ in a frame in which this momentum is large.

$$q(x) = \int \mathrm{d}^3k\, n_q(k,P)\delta(x - k_z/P_z)$$

# Comments on PDFs

- Quark distributions depend on the quark flavour, i.e. different for $u$, $d$, ... quarks. The distributions of the quarks and antiquarks are different.
  $q(x) - \bar{q}(x) = q_{\rm val}$ $C$-odd nonsinglet (valence) quark distribution
  $q(x) + \bar{q}(x) = q^S$ $C$-even singlet quark distribution

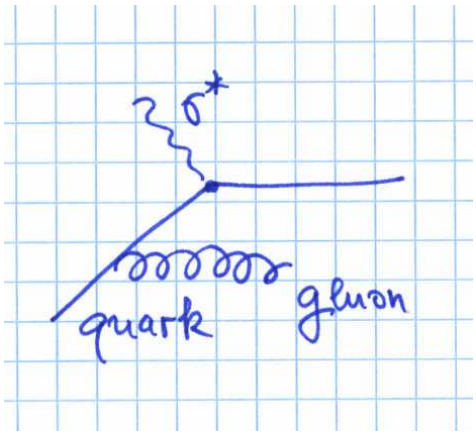
- The parton densities increase infinitely as $x \to 0$ (at very high energies) and the total number of (anti)quarks diverge. However, the difference of the quark number and the antiquark number is finite and gives the valence quark number ($\frac{1}{3}$ of the baryon number) $\int_0^1 {\rm d}x\,(q - \bar{q})(x) = 3$.

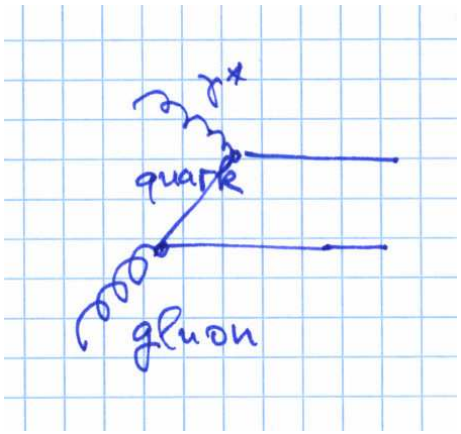
- $\langle x \rangle_q = \int_0^1 {\rm d}x\, x(q + \bar{q})(x)$ the total momentum carried by quarks & antiquarks. Experimentally $\langle x \rangle \approx 0.5$ indicating the presence of neutral fields (gluons) responsible for binding which balance the missing momentum $\langle x \rangle_q + \langle x \rangle_g = 1$.

# SFs in QCD beyond the parton model

## $Q^2$ evolution

Quark-gluon interaction leads to the mixing of quark and gluon distributions and $Q^2$ dependence of PDFs (violation of scaling).




$$\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_S(Q^2)}{2\pi}\left[\int_x \frac{\mathrm{d}y}{y} P_{qq}(\frac{x}{y},\alpha_S) q(y,Q^2) + \int_x \frac{\mathrm{d}y}{y} P_{qg}(\frac{x}{y},\alpha_S) g(y,Q^2)\right]$$

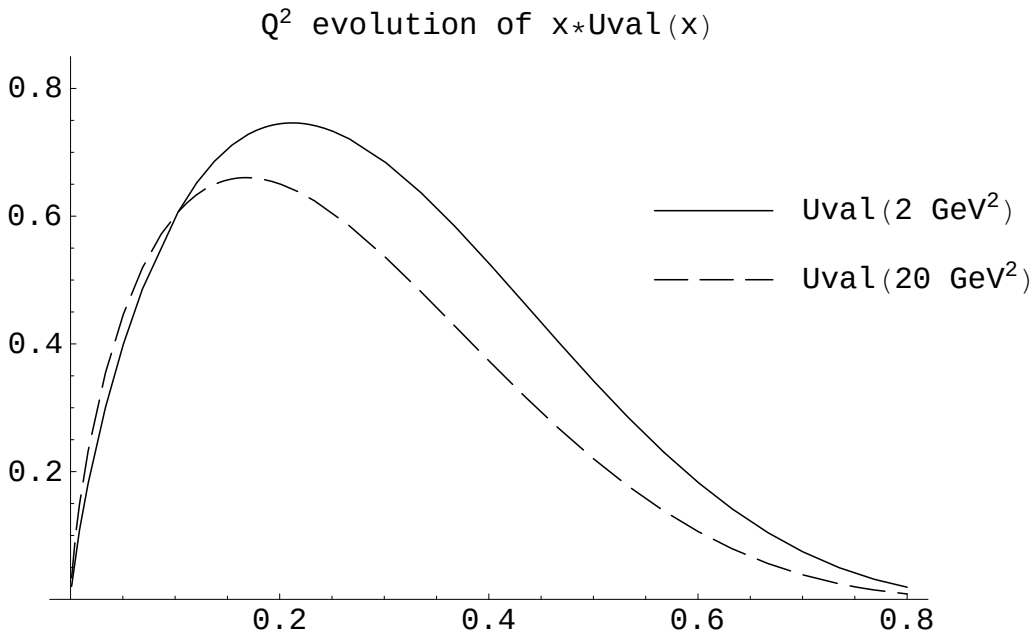
Splitting functions $P_{qq}$ and $P_{qg}$ can be calculated in perturbation theory in the strong coupling constant $\alpha_s$. PDFs evolve with $Q^2$ due to quark-gluon interaction.

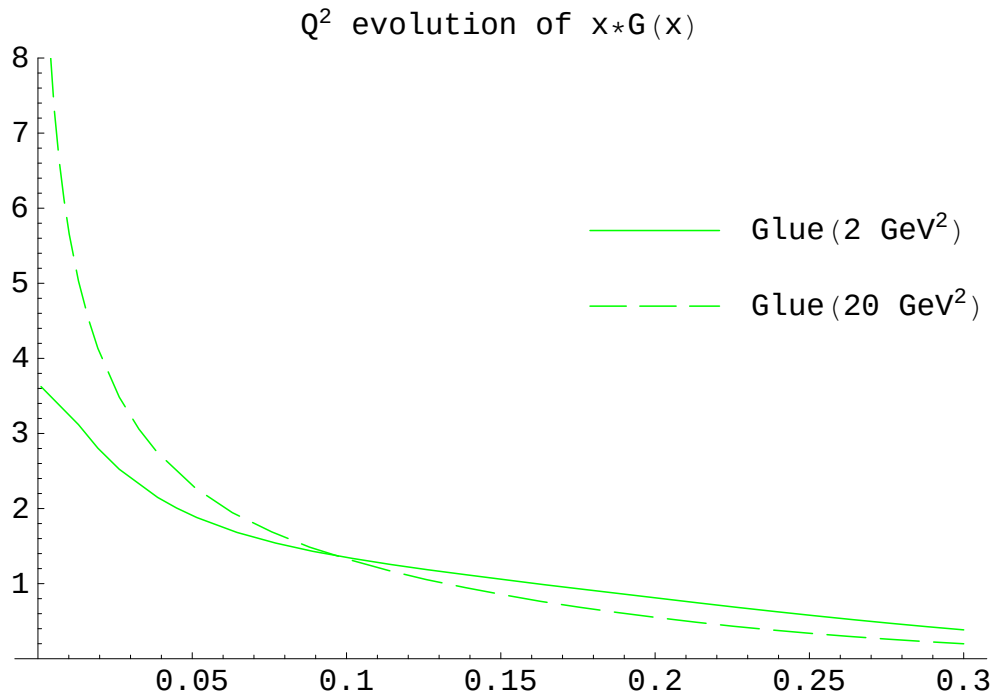
Quark-gluon interaction also lead to modification of the parton model relations between the SFs and PDFs. This effect is driven by coefficient functions.

$$F_2(x,Q^2) = \int_x \mathrm{d}y\, C_2(x/y,\alpha_S) F_2^{(\mathrm{PM})}(y,Q^2)$$

# Illustration of $Q^2$ evolution of Pdfs

The valence $u$ quark and gluon distributions calculated for $Q^2 = 2$ and $20\,\mathrm{GeV}^2$ using Alekhin's PDFs.



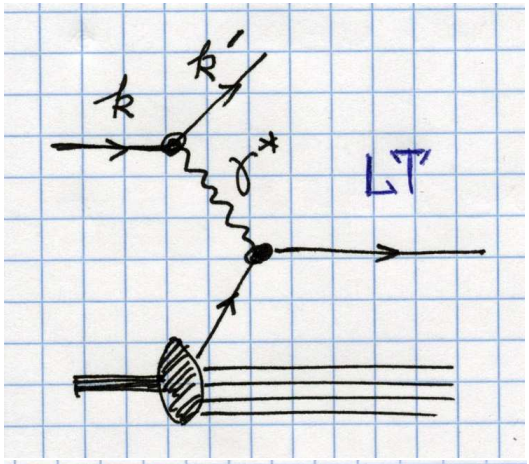



# SFs in QCD beyond the parton model

## Power corrections

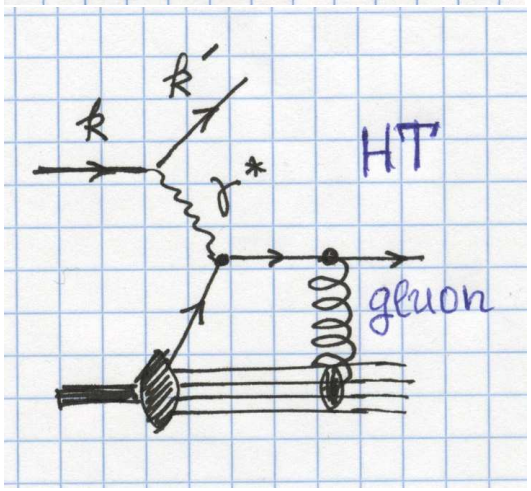
In QCD at high $Q^2$ the SFs can be expanded in inverse powers of $Q^2$ (twist expansion)

$$F_i(x,Q^2) = F_i^{\rm LT}(x,Q^2) + \frac{H_i^{(4)}(x,Q^2)}{Q^2} + \frac{H_i^{(6)}(x,Q^2)}{Q^2} + \dots$$




The leading twist term (LT) scales and is given in terms of PDFs.
Two kinds of power corrections:
(i) Corrections due to finite mass of the target
(ii) Dynamical corrections due to quark-gluon interaction

Target mass can be taken into account (actually absorbed in the LT) by introducing a modified scaling (Nachtmann) variable
$\xi = 2x/(1+(1+M^2x^2/Q^2)^{1/2})$ (Nachtmann, Georgi & Politzer).




A little is known about dynamical HT effects. However, recently there is some progress in phenomenological determination of HT (Alekhin).

# PDF phenomenology

PDFs are extracted from data by performing global QCD analysis of DIS data and DY reactions. Different PDF sets are available (Alekhin, CTEQ, GRV, MRST,...). The results are not fully identical (especially in the region which is not constrained by data). This is because the analyses use different approximations to solve evolution equations and somewhat different subsets of data.