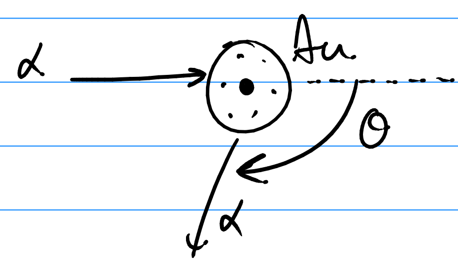
Electron scattering experiments:

* Historical overview:

- J.J. Thomson model of atomic structure (1904)
  "plum pudding" with electrons embedded in positively charged sphere
  (neutrons, protons not discovered yet)

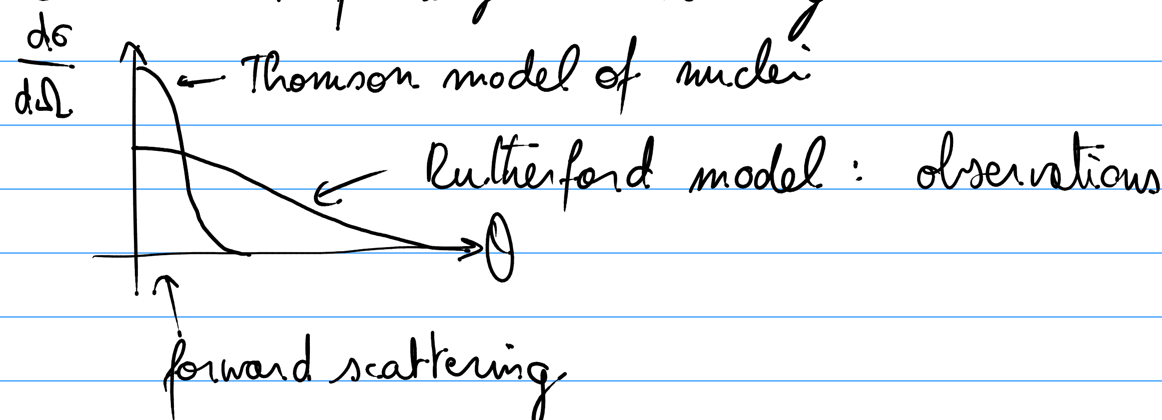
  → expect beams of α particles to be scattered by positive matter such that only small scattering angles will occur

- Rutherford experiment = elastic α, Au scattering



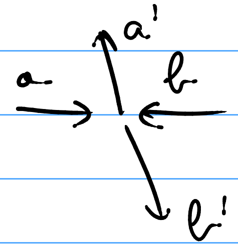
1908 - 1913: experiments by Ernest Rutherford, Hans Geiger, Ernest Marsden

Observation of large θ scattering:

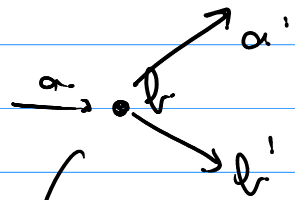


* Kinematic parameters

- elastic scattering : $a + b \to a' + b'$



in beam-beam configuration



in beam-fixed target configuration

four-momenta : $a(k)$, $a'(k')$, $k = (E, \vec{k})$
$b(p)$, $b'(p')$

conservation : $p + k = p' + k'$

masses : $\begin{cases} p^2 = p'^2 = M & \text{(target particle)} \\ k^2 = k'^2 = m & \text{(small, beam particle)} \end{cases}$

working in $\hbar = c = 1$ units

Question: Calculate E' in terms of E, $\cos\theta$ if $m \ll E$ for fixed target

$$p + k = p' + k'$$
$$\rightarrow (p+k)^2 = (p'+k')^2$$
$$\rightarrow p \cdot k = p' \cdot k'$$
$$\rightarrow p \cdot k = (p + k - k') \cdot k'$$
$$\rightarrow M \cdot E = p \cdot k' + k \cdot k' - m^2$$
$$\rightarrow M \cdot E = M \cdot E' + E \cdot E' (1 - \cos\theta) - \cancelto{0}{m^2}$$
$$\rightarrow E' = \frac{E}{1 + \frac{E}{M}(1 - \cos\theta)}$$

or $E = E' + \frac{E \cdot E'}{M}(1 - \cos\theta)$

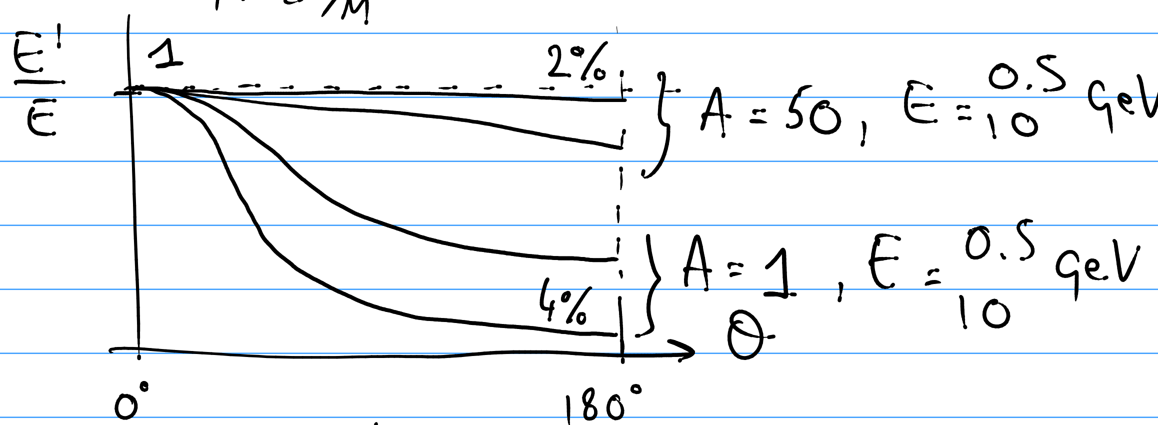
original energy — scattered energy — recoil energy : large for small M, large E

$$\nu = E - E' = \frac{E E'}{M}(1 - \cos\theta)$$

ν = energy transfer from electron to target

$E' = E$ at $\theta = 0°$

$E' = E \frac{1}{1 - 2E/M} < E$ at $\theta = 180°$



→ heavy targets, low energy beam : recoil can be ignored

$\vec{q} = \vec{k} - \vec{k}'$ = momentum transfer

$q = k - k'$ = four-momentum transfer

- inelastic scattering : $a + b \to c + d + e$

E' not function of E, θ anymore

* Rutherford scattering cross section:

Scattering of point particles by $V(r) \propto \frac{1}{r}$ potential

$$\left(\frac{d\sigma}{d\Omega}\right)^{*}_{Ruth} = \frac{\alpha^2}{4E^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$

Derived from classical mechanics when recoil can be ignored (hence the asterisk) and when charge distribution $\rho(r) = \delta(r)$

$\Rightarrow$ describes observed scattering seen by Rutherford for low energy α particles off heavy Au atoms when de Broglie wavelength is large compared to Au charge radius

We must modify the Rutherford results for electron scattering:

- classical $\rightarrow$ quantum description
- include relativistic effects since $k \gtrsim O(m)$
- include effects from non-negligible recoil
- include effects from spin of both particles
- include extended target charge distributions

Taking a descriptive step-wise approach here, then will revisit using quantum field theory.

* Quantum-mechanical calculation of Rutherford scattering cross section:

proceed via Fermi Golden Rule using initial and final state wave functions and matrix element

$$\frac{d\sigma}{d\Omega} \propto |\langle \psi_f | H_{int} | \psi_i \rangle|^2$$

Born approx

$$\Big\downarrow \; H_{int} \propto V(r), \; \psi \propto e^{ip\cdot r}$$

$$\int \rho(r)\, e^{iq\cdot r}\, d^3r \propto F(q)$$

For $\rho(r) \propto \delta(r) \longrightarrow F(q) = 1$
(point particles)

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)^*_{point} = \frac{\alpha^2}{4E^2} \frac{1}{\sin^4\frac{\theta}{2}} = \left(\frac{d\sigma}{d\Omega}\right)^*_{Ruth} \text{ as before}$$

For $\rho(r) \neq \delta(r) \rightarrow$ spatial distribution of charge of target, with Fourier transform $F(q)$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)^*_{exp} = \left(\frac{d\sigma}{d\Omega}\right)^*_{point} F(q)$$

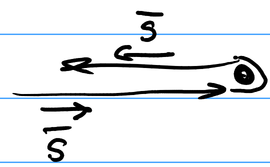
* Including effects of beam spin and relativistic motion:
  - relativistic, $\beta = v/c$
  - spin for beam particle, unpolarized target

$$\left(\frac{d\sigma}{d\Omega}\right)^*_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)^*_{Ruth} \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right)$$

For $\beta \approx 1$: $$\left(\frac{d\sigma}{d\Omega}\right)^*_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)^*_{Ruth} \cos^2\frac{\theta}{2}$$

$$= \frac{\alpha^2}{4E^2} \frac{\cos^2\theta/2}{\sin^4\theta/2}$$

helicity conservation
angular momentum conservation ensures that cross section is zero at $\theta = 180°$



is impossible for unpolarized target

* Including effects of recoil on light targets:

When the target is light or beam energy is large, we cannot ignore the recoil:
- nucleons have size of ~ 0.8 fm or need hundreds of MeV or several GeV
- mass of nucleon is $938\,\text{MeV} \simeq 1\,\text{GeV}$

$\Rightarrow$ recoil energy of order $\frac{E}{M}$ is not small anymore

Taking into account recoil gives:

$$\underbrace{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}}_{\text{with recoil}} = \underbrace{\left(\frac{d\sigma}{d\Omega}\right)^{*}_{\text{Mott}}}_{\text{no recoil}} \frac{k'}{k} = \frac{\alpha^2}{4k^2} \frac{k'}{k} \frac{1}{\sin^4\frac{\theta}{2}} \cos^2\frac{\theta}{2}$$

correction

Introduce four-momentum transfer:

$$q = k' - k \rightarrow q^2 = (k'-k)^2 = 2m^2 - 2k'\cdot k$$

$$Q^2 = -q^2 \approx 2k'\cdot k \approx 2EE'(1-\cos\theta)$$

$$\Rightarrow \quad Q^2 = 4EE'\sin^2\frac{\theta}{2}$$

* Scattering off spin-$\frac{1}{2}$ targets :

Until now we have only considered the electric charge of the nucleon, for point particles with $\rho(r) = \delta(r)$ and $F(q) = 1$ or more generally for extended charges.

The spin-$\frac{1}{2}$ nature of the nucleon corresponds to a magnetic moment in addition to the electric charge :

$$\mu = g \frac{e}{2M} \frac{\hbar}{2}$$

with $g \equiv 2$ for fundamental spin-$\frac{1}{2}$ particles

This results in cross section :

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{magnetic} \\ \text{spin-}\frac{1}{2}}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}} \sin^2\frac{\theta}{2} \qquad \left(\text{cf. } \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}} \cos^2\frac{\theta}{2} \right)$$

scattering off magnetic moment suppresses scattering at $\theta = 0°$, maximum at $\theta = 180°$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{spin-}\frac{1}{2}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left(1 + 2\tau \tan^2\frac{\theta}{2} \right)$$

with $\tau = \dfrac{Q^2}{4M^2}$ kinematic variable

* Charge and magnetic form factors of the nucleon:

However, g can have any value for composite structures such as the nucleon

$$\left.\begin{aligned} \mu_p &= +2.79\, \mu_N \\ \mu_n &= \underbrace{-1.91}_{g/2}\, \mu_N \end{aligned}\right\} \mu_N = \frac{e}{M_p}\frac{\hbar}{2}$$

And just like the electric charge distribution $\rho(r) \rightarrow F(q)$ there is a magnetic moment distribution:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rosenbluth}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left(\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2) \tan^2\frac{\theta}{2} \right)$$

with $G_E(Q^2)$ and $G_M(Q^2)$ the electric and magnetic form factors

Limit for $Q^2 \rightarrow 0$: consider the entire nucleon at an energy resolution where it is a point particle

$$G_E^p(Q^2 \rightarrow 0) = 1 \quad \text{(normalized to } e\text{)}$$
$$G_E^n(Q^2 \rightarrow 0) = 0$$

$$G_M^p(Q^2 \rightarrow 0) = +2.79 \quad \text{(normalized to } \mu_N\text{)}$$
$$G_M^n(Q^2 \rightarrow 0) = -1.91$$

But the detailed Q^2 behavior depends on the substructure of the nucleon: QCD dynamics.

Common shape to $G_E(Q^2)$ and $G_M(Q^2)$ is described by the dipole form factor $G_D(Q^2)$

$$G_E^p(Q^2) \approx \frac{G_M^p(Q^2)}{2.79} \approx \frac{G_M^n(Q^2)}{-1.91} \approx G_D(Q^2)$$

with $G_D(Q^2) = \dfrac{1}{\left(1+\dfrac{Q^2}{0.71}\right)^2}$

while $G_E^n(Q^2) \approx 0$

Dipole form factor is Fourier transform of exponential distribution, e.g. $\rho(r) = \rho(0)\, e^{-ar}$ with $a = 4.27\ fm^{-1}$

Root mean square of $\rho(r)$ is related to a:

$$\langle r^2 \rangle_D = -6 \left.\frac{dG_D(Q^2)}{dQ^2}\right|_{Q^2=0} = \frac{12}{a^2}$$

$\rightarrow \sqrt{\langle r^2 \rangle_D} = 0.81\ fm$ = charge radius of the dipole form factor

$\rightarrow$ proton radius puzzle: requires accurate measurement of $G_E^p(Q^2 \to 0)$ and determination of derivative

* Elastic scattering $e^- p \to e^- p$ $\to k, k', \theta$: pick 2

$$d\sigma = \frac{(2\pi)^4 \delta^4(\Sigma p)}{4Mk} \frac{d^3 k'}{(2\pi)^3 2E'_k} \frac{d^3 p'}{(2\pi)^3 2E'_p} \frac{e^4}{q^4} L_e^{\mu\nu} L_{p\,\mu\nu}$$

with leptonic tensor $L_e^{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\gamma^\mu (\not k + m) \gamma^\nu (\not k' + m) \right]$

and hadronic tensor $L_{p\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\gamma_\mu (\not p + M) \gamma_\nu (\not p' + M) \right]$

$$L_e^{\mu\nu} = 2\left[k^\mu k'^\nu + k'^\mu k^\nu + g^{\mu\nu} \frac{q^2}{2} \right.$$

$$L_{p\mu\nu} = 2\left[p_\mu p'_\nu + p'_\mu p_\nu + g_{\mu\nu} \frac{q^2}{2} \right]$$

$$\longrightarrow \frac{d\sigma}{dk'\, d\Omega} = \frac{\alpha^2}{q^4} \frac{k'}{k} L_e^{\mu\nu} W_{\mu\nu}$$ for specific $W_{\mu\nu}$ tensor which contains all info about proton

δ function requiring elasticity

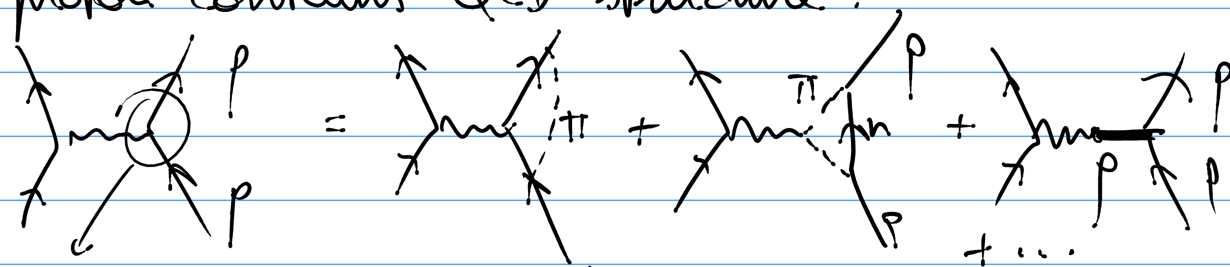
If proton is elementary in pure QED:

$$M = (ie)\bar u(k') \gamma^\mu u(k) \left(\frac{-i g_{\mu\nu}}{q^2} \right) (-ie) \bar u(p') \gamma^\nu u(p)$$

and $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{q^4} \frac{k'}{k} \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right)$$

actually $4k^2 \sin^4\frac{\theta}{2}$

But proton contains QCD structure:



$-ie\gamma^\nu \to -ie\Gamma$ with

$$\Gamma^\mu = \gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M} q_\nu F_2^p(q^2)$$
$$+ q^\mu F_3^p(q^2)$$
$$+ \gamma^\mu\gamma^5 g_1^p(q^2) + \frac{i\sigma^{\mu\nu}\gamma^5}{2M} q_\nu g_2^p(q^2) + q^\mu\gamma^5 g_3^p(q^2)$$

g_1^p, g_2^p, g_3^p require parity violation → not present in QED + QCD but allowed by weak interaction in proton

Because $q_\mu J^\mu = q_\mu \bar{u}(p') \Gamma^\mu u(p) = 0$

$$\to \begin{cases} F_3^p(q^2) = 0 \\ \text{and} \\ 2M g_1^p(q^2) + q^2 g_3^p(q^2) = 0 \end{cases}$$

⇒ QED + QCD elastic scattering:

$$\Gamma^\mu = \gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M} q_\nu F_2^p(q^2)$$

$$\Gamma^\mu = \gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M} q_\nu F_2^p(q^2)$$

with electromagnetic form factors

$$\begin{cases} G_E = F_1 - \tau F_2 \\ G_M = F_1 + F_2 \end{cases}$$ are Sachs form factors (electric / magnetic)

with $\tau = \frac{Q^2}{4M}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha}{q^4}\frac{k'}{k}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right]$$

↳ Rosenbluth formula

with $G_E^p(0) = 1$, $G_M^p(0) = 1 + \kappa$

* Experimental access to G_E, G_M:

"Rosenbluth separation":
- measure elastic scattering cross section over a range of Q^2 and θ (i.e. k and θ)
- normalize by theoretical Mott cross section
- fit the dependence on $\tan^2\frac{\theta}{2}$ for each set of Q^2 data points
  - ↳ slope of line gives $G_M(Q^2)$
  - intercept of line with $\tan^2\frac{\theta}{2}$ gives $\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau}$ and thus $G_E(Q^2)$

Proton targets: $LH_2 \rightarrow G_E^p, G_M^p$ results

Neutron targets require a neutron embedded in a nucleus $\longrightarrow$ quasi-elastic scattering

Proton charge radius:

$$G_E^p(Q^2) \approx 1 - \frac{1}{6}R_p^2\, Q^2$$

with $R_p^2 = \langle r^2\rangle = -6\left.\frac{dG_E^p(Q^2)}{dQ^2}\right|_{Q^2=0}$

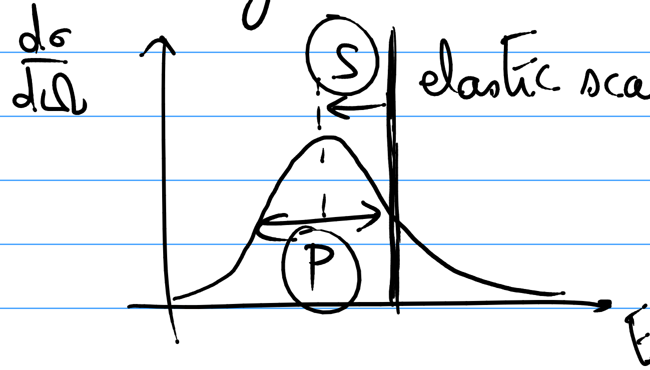
$$= \int d^3r\, |\bar{r}|^2\, |\psi(\bar{r})|^2 = \text{RMS charge radius}$$

* Quasi-elastic scattering:

Neutron targets require a neutron embedded in a nucleus $\rightarrow$ some complications arise ...

- bound nucleon, binding energy S
- moving nucleon, fermi momentum P

$\Rightarrow$ elastic scattering turns into <u>quasi-elastic</u> scattering:

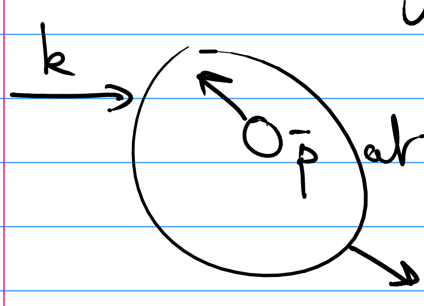


elastic scattering, δ function in E' because $E'(E,\theta)$

Inside nucleus the nucleons move isotropically with moment P

$$\nu = E - E' = \text{energy transfer}$$

$$= \frac{\bar{q}^2}{2M} + \underbrace{S}_{\substack{\text{additional}\\ \text{binding energy}}} + \underbrace{\frac{2\,\bar{p}\cdot\bar{q}}{2M}}_{\substack{\text{depends on angle}\\ \text{between internal motion}\\ \text{and momentum transfer}}}$$



$\bar{p}$ at angle α with k

$-\bar{p}$ (residual nucleus)

Binding energy increases with A, from ~10 MeV up to ~50 MeV

Width of ν given by root mean square of the variable term;

$$\nu_0 = \langle \nu \rangle = \frac{q^2}{2M} + S$$

$$\sigma_\nu = \sqrt{\langle (\nu - \nu_0)^2 \rangle} = \frac{|\bar{q}|}{M} \sqrt{\frac{1}{3} \langle \bar{p} \rangle}$$

Nuclear dynamics can be used to determine the root mean square of $\bar{p}$

$$p_F = \text{Fermi momentum} \approx 250 \text{ MeV}$$

with $p_F^2 = \frac{5}{3} \langle p^2 \rangle$

* Recap of elastic scattering:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rosenbluth}} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{k'}{k} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right]$$

with G_E, G_M electric and magnetic form factors

$$\begin{cases} G_E^p, G_M^p, G_M^n \propto G_{\text{Dipole}} = \dfrac{1}{\left(1+\dfrac{Q^2}{0.71\,\text{GeV}^2}\right)^2} \sim \dfrac{1}{Q^4} \\ G_E^n \approx 0 \end{cases}$$

$$G^2_{\text{Dipole}} \propto \frac{1}{Q^8}$$

Both G_E and G_M are related to the electric and magnetic charge distributions in the nucleon, with

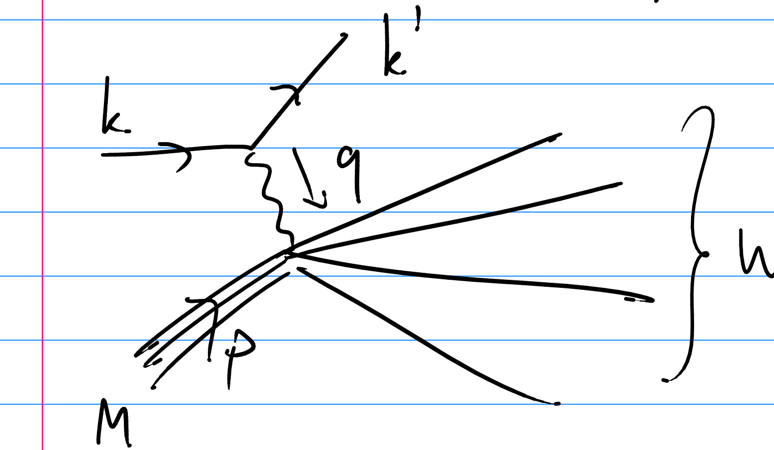
$$G_E(Q^2 \to 0) = \text{electric charge} \begin{cases} 0 & \text{for } n \\ 1 & \text{for } p \end{cases}$$

$$G_M(Q^2 \to 0) = \text{magnetic moment} \begin{cases} -1.91 & \text{for } n \\ +2.79 & \text{for } p \end{cases}$$

Also: $\gamma^\mu \to \gamma^\mu F_1(Q^2) + \dfrac{i\sigma^{\mu\nu}}{2M} q_\nu F_2(Q^2)$

with $G_E = F_1 - \tau F_2$ and $G_M = F_1 + F_2$

* Kinematic variables of inelastic scattering



$q = k - k'$

$\hookrightarrow Q^2 = -q^2$

$q(E - E', \bar{k} - \bar{k}')$

$p(M, \bar{0})$

for elastic scattering:

$W^2 = (p+q)^2 = p'^2 = M^2$

inelastic scattering:

$W^2 = (p+q)^2 = M^2 + \underbrace{2M\nu} + q^2$

$$\nu = \frac{p \cdot q}{M} = E - E'$$

for elastic scattering: $\nu = \dfrac{Q^2}{2M}$, depends completely on Q^2

but for inelastic scattering: Q^2 and ν are independent variables for reaction

Also: "inelasticity" $x = \dfrac{Q^2}{2M\nu}$ $\left(\begin{array}{l} = 1 \text{ for elastic} \\ < 1 \text{ for inelastic} \end{array} \right.$

* Inelastic scattering: $e^- p \to e^- X$

Same form for $\frac{d\sigma}{dk' d\Omega}$ can be used, with $W_{\mu\nu}$

$W_{\mu\nu}$ can only depend on p and q ($p' = p + q$)

- symmetric tensors: $p^\mu p^\nu$, $q^\mu q^\nu$, $p^\mu q^\nu + p^\nu q^\mu$, $g^{\mu\nu}$
- anti-symmetric tensors: $p^\mu q^\nu - p^\nu q^\mu$, $\varepsilon^{\mu\nu\rho\sigma} g_{\rho\sigma}$

But $L_e^{\mu\nu}$ is symmetric $\to$ $W_{\mu\nu}$ must be too

Most general form after momentum conservation

$\hookrightarrow q_\mu W^{\mu\nu} = 0$

$$W_{\mu\nu} = \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] W_1(Q^2, \nu) + \frac{1}{M^2}\left[p_\mu - \frac{p\cdot q}{q^2} q_\mu\right]\left[p_\nu - \frac{p\cdot q}{q^2} q_\nu\right] W_2(Q^2, \nu)$$

W_1, W_2 are structure functions

$$\frac{d\sigma}{dk' d\Omega} = \frac{\alpha^2}{q^4}\left[\underset{\text{(electric)}}{W_2(Q^2,\nu)\cos^2\frac{\theta}{2}} + \underset{\text{(magnetic)}}{2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}}\right]$$

$\to$ W_1, W_2 generalize F_1, F_2 for elastic scatt. but have dependence on ν as well or x

$$\begin{cases} F_1(x, Q^2) = M W_1(Q^2, \nu) \\ F_2(x, Q^2) = \nu W_2(Q^2, \nu) \end{cases}$$ notations

But observed data nearly independent of Q^2

Bjorken scaling: as $Q^2 \to \infty$

$F_1(x, Q^2) \to F_1(x)$ (constant charge)

instead of $F_1(x, Q^2) \to 0$ (negligible charge)

$\Rightarrow$ scattering independent of small scale $\sim \frac{1}{Q^2}$

$\to$ hard partons inside proton off which the electrons, virtual photons scatter

Contrast:

elastic scattering $\Longleftrightarrow$ inelastic scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rosenbluth}} \propto \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \times \frac{1}{Q^8} \qquad \left(\frac{d^2\sigma}{d\Omega dk'}\right) \propto \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \times 1$$

due to e^{-ar} spatial distribution

no dependence on spatial resolution

$\downarrow$

point particles

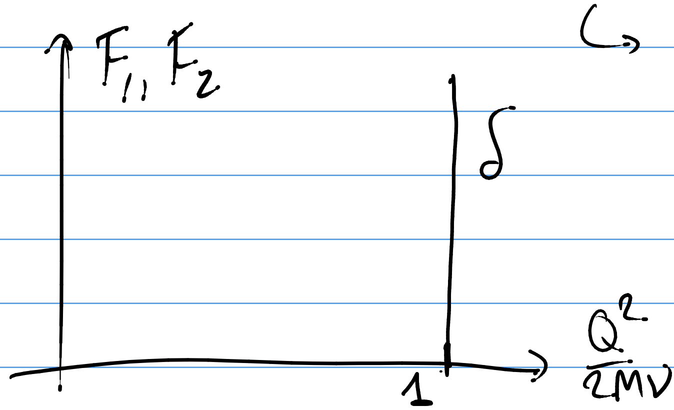
* Quark parton model (as the precursor to a full understanding of QCD)

- elastic scattering:

$$\begin{cases} W_1(Q^2,\nu) = \dfrac{Q^2}{4M^2}\,\delta\left(\nu - \dfrac{Q^2}{2M}\right) = \dfrac{Q^2}{4M^2\nu}\,\delta\left(1 - \dfrac{Q^2}{2M\nu}\right) \\ W_2(Q^2,\nu) = \delta\left(\nu - \dfrac{Q^2}{2M}\right) = \dfrac{1}{\nu}\,\delta\left(1 - \dfrac{Q^2}{2M\nu}\right) \end{cases}$$

$$F_1 = MW_1 = \frac{1}{2}\,\delta\left(1 - \frac{Q^2}{2M\nu}\right) \quad \& \quad F_2 = \delta\left(1 - \frac{Q^2}{2M\nu}\right)$$

↳ notice $F_2 = 2F_1$



With this δ function we get

$$\frac{d^2\sigma}{dk'\,d\Omega} \longrightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\,\frac{k'}{k}\left(\cos^2\frac{\theta}{2} - \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

- scattering off point-like partons:

$M \to x_i M$ (momentum fraction), charge fraction e_i

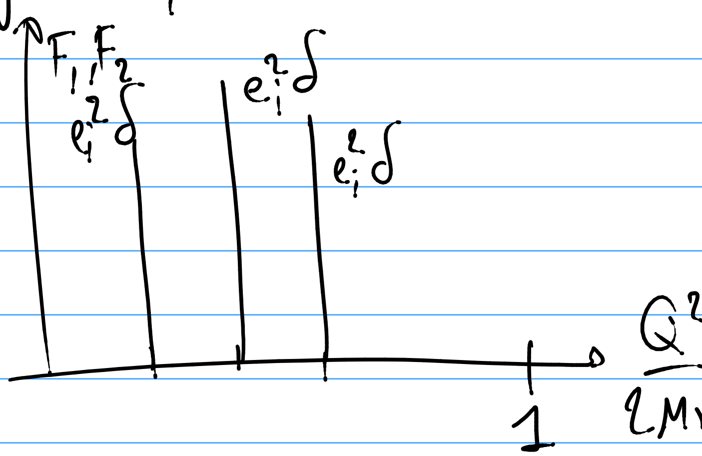
$$\begin{cases} W_1^i(Q^2,\nu) = e_i^2 \dfrac{Q^2}{4x_i M^2 \nu} \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \\ W_2^i(Q^2,\nu) = e_i^2 \dfrac{x_i}{\nu} \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \end{cases}$$

x_i here moved outside of δ

$$\to \begin{cases} F_1^i = MW_1^i = \frac{1}{2} e_i^2 \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \\ F_2 = \nu W_2^i = e_i^2 x_i \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \end{cases} \to x_i = \frac{Q^2}{2M\nu}$$

$\parallel$ x $\parallel$ inelasticity of earlier

e.g. 3 quarks



- distribution of valence quarks inside proton:

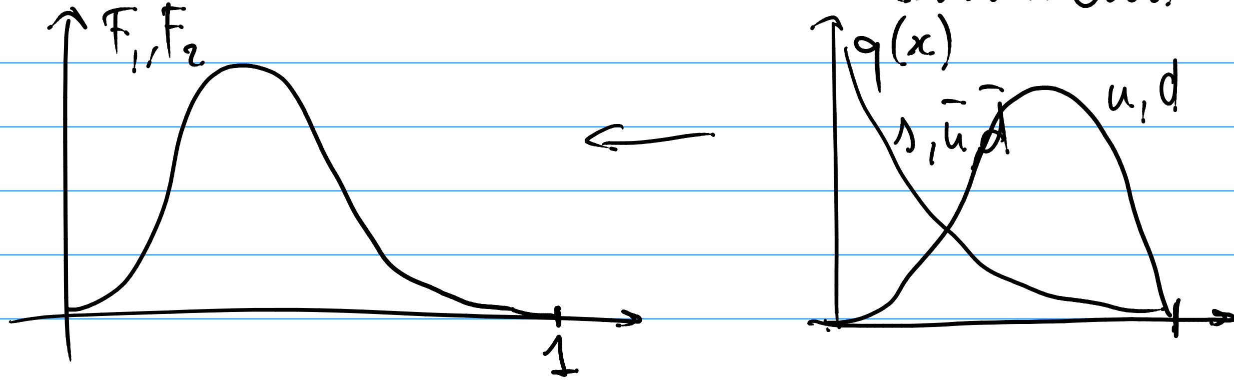
$q_i(x_i)$ probability density = parton distribution function PDF

$$F_2(x,Q^2) = \sum_i \int_0^1 dx_i\, q_i(x_i)\, e_i^2 x_i\, \delta\left(x_i - \frac{Q^2}{2M\nu}\right)$$

$$= 2xF_1(x,Q^2) = \sum_i e_i^2 x\, q_i(x) \quad , x = \frac{Q^2}{2M\nu}$$

Callan-Gross relation: $F_2(x, Q^2) = 2xF_1(x, Q^2)$

↪ indicates scattering off point-like constituents



⇒ started with x = convenient dimensionless kinematic variable

now x has interpretation as momentum fraction carried by struck quark

$q(x)$ = non-perturbative quantity that is messy to calculate

- distribution of sea quarks inside the proton

$$F_2(x, Q^2) = \sum_i e_i^2 \left(q_i(x) + \bar{q}_i(x) \right)$$

valence + sea   sea

$$\int_0^1 u_v(x)\, dx = 2, \quad \int_0^1 d_v(x)\, dx = 1 \quad \text{for proton}$$

$$\sum_i \int_0^1 x\left(q_i(x)+\bar{q}_i(x)\right)dx + \int_0^1 xG(x)\,dx = 1$$

$\to$ entire momentum fraction explained by all partons

In proton: $\underbrace{u_v(x), d_v(x)}_{\text{valence}}$, $\left.\begin{matrix} u_S(x), d_S(x) \\ \bar{u}_S(x), \bar{d}_S(x) \end{matrix}\right\}$ sea

$$u_S(x) = d_S(x) = \bar{u}_S(x) = \bar{d}_S(x)$$

$$s_S(x) = \bar{s}_S(x)$$

In neutron: $u_v^n(x) = d_v^p(x)$

$d_v^n(x) = u_v^p(x)$

$$\frac{1}{x}\left(F_2^{ep}(x) - F_2^{en}(x)\right) = \frac{1}{3}\left(u_v(x) - d_v(x)\right)$$

$\hookrightarrow$ peaks at $\frac{1}{3}$ $\left(\underbrace{u_v(x) - d_v(x)}_{=1}\right)$

Momentum balance (ignoring strange quarks)

p: $\int dx \; F_2^{ep}(x) = \left(\frac{2}{3}\right)^2 \int_0^1 dx \; x(u+\bar{u})$

$+\left(-\frac{1}{3}\right)^2 \int_0^1 dx \; x(d+\bar{d})$

$= \frac{4}{9}\varepsilon_u + \frac{1}{9}\varepsilon_d \overset{\text{(data)}}{=} 0.18$

ε_u = fraction of momentum carried by u quarks

ε_d = ... by d quarks

n: $\int dx \; F_2^{en}(x) = \frac{1}{9}\varepsilon_u + \frac{4}{9}\varepsilon_d \overset{\text{(data)}}{=} 0.12$

$\rightarrow \quad \varepsilon_u = 0.36 \qquad \varepsilon_d = 0.18$

$\rightarrow \quad \varepsilon_g = 0.46$: ~50% of nucleon momentum carried by gluons

Polarized pdfs : $\Delta q(x) = q^+(x) - q^-(x)$

$\hookrightarrow$ $q^+(x)$ = probability of finding quark with spin along proton spin

$q^-(x)$ = ... against

Spin puzzle: $\frac{1}{2} = \frac{1}{2}\underbrace{\Delta\Sigma}_{\sum_q \Delta q} + L_q + L_g$

Spin structure functions g_1, g_2, g_3

$\rightarrow g_1^p = \sum e_q^2 \left(q(x) - \bar{q}(x) \right)$

* Dirac equation and symmetry properties of bilinears

Fermions described by $(i\not{\partial} - m)\psi = 0$

ψ = four-component spinor fields

$\bar{\psi} = \psi^{+}\gamma^{0}$

γ^{μ} = set of 4x4 matrices operating on ψ components

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}$$

and $\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$

→ electromagnetic current (density)

$$j^{\mu} = -e\bar{\psi}\gamma^{\mu}\psi \quad \text{with } \partial_{\mu}j^{\mu} = 0$$
(conservation)

Bilinears: $\bar{\psi}$. 4x4 matrix . ψ → j^{μ} ~ vector quantity

Symmetries: P = parity operation, $\vec{r} \to -\vec{r}$

C = charge conjugation, $e^{-} \to e^{+}$

T = time reversal, $t \to -t$

→ focus on parity for our purposes:
helicity $= \frac{\vec{s}\cdot\vec{p}}{|\vec{p}|}$ changes sign under P operation

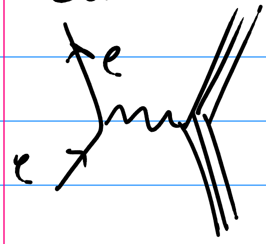
Bilinear transformations:

	P	C	T
scalar, $\bar{\psi}\psi$	$+1$	$+1$	$+1$
pseudo scalar, $\bar{\psi}\gamma^5\psi$	$-\gamma^5$	$+\gamma^5$	$+\gamma^5$
vector, $\bar{\psi}\gamma^\mu\psi$	γ_μ	$-\gamma^\mu$	γ_μ
axial vector, $\bar{\psi}\gamma^\mu\gamma^5\psi$	$-\gamma_\mu\gamma^5$	$-\gamma^\mu\gamma^5$	$\gamma_\mu\gamma^5$
tensor, $\bar{\psi}\sigma^{\mu\nu}\psi$	$\sigma_{\mu\nu}$	$-\sigma^{\mu\nu}$	$-\sigma_{\mu\nu}$

$\Rightarrow$ electromagnetic current transforms as vector under P, C, T

Parity on vector $\times$ vector $\rightarrow +1$
axial $\times$ axial $\rightarrow +1$
axial $\times$ vector $\rightarrow -1$ $\longrightarrow$ observable changes sign when parity is applied

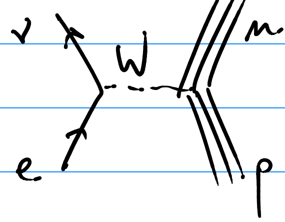
Interactions of electrons with nucleon written using currents:



$$\left(-e\bar{\psi}_e\gamma^\mu\psi_e\right)\frac{g_{\mu\nu}}{Q^2}\left(e\bar{\psi}_p\gamma^\nu\psi_p\right) = J^\mu_{\gamma,e}\,\frac{g_{\mu\nu}}{Q^2}\,J^\nu_{\gamma,p}$$

$$V \quad \times \quad V$$

Weak interactions : charged currents, β-decay



$$\left(\bar{\psi}_\nu \gamma^\mu (1-\gamma^5) \psi_e\right) G_F \left(\bar{\psi}_n \gamma_\mu (1-\gamma^5) \psi_p\right)$$

$$(V-A) \qquad \times \qquad (V-A)$$

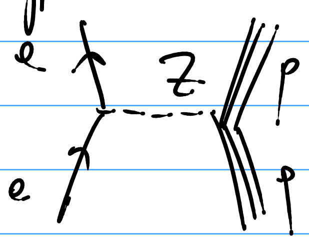
$$= J_W^\mu \; G_F \; J_{W\mu}$$

$\uparrow$ is actually $\frac{1}{Q^2+M_W^2}$ but small Q^2

$(V-A)\times(V-A)$ will have parity -1 terms

$\longrightarrow$ parity violation

Hypothesis Zel'Dovich (1958) : neutral currents



$$\left(\bar{\psi}_e \gamma^\mu (g_V^e - \gamma^5 g_A^e) \psi_e\right) \times G_F \times$$
$$\left(\bar{\psi}_p \gamma_\mu (g_V^p - \gamma^5 g_A^p) \psi_p\right)$$

↳ first observed at CERN in 1973 in $\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$

$\underbrace{g_V^e, g_A^e, g_V^p, g_A^p}$ = vector and axial weak couplings of e and p

not equal to 1 as in charged current due to electroweak symmetry breaking, Higgs mechanism

$$\longrightarrow Z^\mu = \cos\theta_W \, W_3^\mu - \sin\theta_W \, B$$

Weak mixing angle "runs" with energy.

$\hookrightarrow$ running plot and existing experimental results

$$J^\mu_Z = \sum_f \bar{\psi} \gamma^\mu \left(g_V^f - g_A^f \gamma^5 \right) \psi$$

$$\begin{cases} g_V^f = t_{3f} - 2 q_f \sin^2\theta_W & \text{(mixture of weak and EM)} \\ g_A^f = t_{3f} & \text{(pure weak isospin)} \end{cases}$$

	electrical	weak charges g_V	g_A
u	$+\frac{2}{3}$	$1 - \frac{8}{3}\sin^2\theta_W$	-1
d	$-\frac{1}{3}$	$-1 + \frac{4}{3}\sin^2\theta_W$	$+1$
s	$-\frac{1}{3}$	$-1 + \frac{4}{3}\sin^2\theta_W$	$+1$
p	$+1$	$1 - 4\sin^2\theta_W$	-1
n	0	-1	$+1$
e	-1	$-1 + 4\sin^2\theta_W$	$+1$

* Elastic scattering: $G_E(Q^2)$ and $G_M(Q^2)$ as Fourier transforms of electric/magnetic distributions

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{k'}{k} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right]$$

↳ Rosenbluth separation to get G_E, G_M

(Modern approach: recoil polarization, $\frac{G_E}{G_M}$)

Introduce $\varepsilon = \left(1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right)^{-1}$, $\tau = \frac{Q^2}{4M^2}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \cdots \left[\varepsilon G_E^2 + \tau G_M^2\right]$$

Can we say more about what makes up G_E, G_M?

flavor decomposition

$$G^{\gamma,p}_{E,M} = \frac{2}{3} G^{u,p}_{E,M} - \frac{1}{3} G^{d,p}_{E,M} \underbrace{- \frac{1}{3} G^{s,p}_{E,M}}_{\text{strange quark form-factors in the proton}}$$

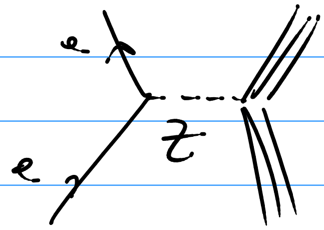
$$G^{\gamma,n}_{E,M} = \frac{2}{3} G^{u,n}_{E,M} - \frac{1}{3} G^{d,n}_{E,M} \underbrace{- \frac{1}{3} G^{s,n}_{E,M}}_{\text{in the neutron}}$$

Isospin symmetry : $u \leftrightarrow d$, $p \leftrightarrow n$

$$G^{u,n}_{E,M} = G^{d,p}_{E,M}, \quad G^{d,n}_{E,M} = G^{u,p}_{E,M}, \quad G^{s,p}_{E,M} = G^{s,n}_{E,M}$$

↳ 3 unknowns, but only 2 measurements

→ add third measurement using weak interaction



$\frac{d\sigma}{d\Omega}$ described by $G^{Z,p}_{E,M} \times G^{\gamma,p}_{E,M}$

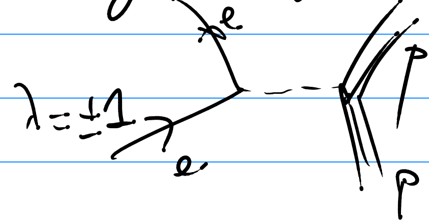
$$G^{Z,p}_{E,M} = \left(1 - \frac{8}{3}\sin^2\theta_W\right) G^{u,p}_{E,M} + \left(-1 + \frac{4}{3}\sin^2\frac{\theta}{2}\right) G^{d,p}_{E,M} + \left(-1 + \frac{4}{3}\sin^2\frac{\theta}{2}\right) G^{s,p}_{E,M}$$

different charges, but same quark form factors

↳ 3 measurements, 3 unknowns

Experimental access through asymmetry in elastic scattering

$$A_{PV} = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-}$$



→ change helicity of incoming electrons

$$\Rightarrow A_{PV} = -G_F \frac{Q^2}{4\pi\alpha\sqrt{2}} \left[A_E + A_M + A_A\right] \frac{1}{2\sigma_{unpolarized}}$$

$$\begin{cases} A_E = \varepsilon\, G_E^{\gamma,p}\, G_E^{Z,p} & \text{electric} \\ A_M = \tau\, G_M^{\gamma,p}\, G_M^{Z,p} & \text{magnetic} \\ A_A = -\underbrace{(1-4\sin^2\frac{\theta}{2})}_{\text{suppressed}}\varepsilon'\, G_A^e\, G_M^{\gamma,p} & \text{axial} \end{cases}$$

form factor

Again we have 3 unknowns $\to$ need 3 measurements

forward $\vec{e}+p$
backward $\vec{e}+p$
backward $\vec{e}+d$

$G_M^s(Q^2\to 0) = \mu_s \approx 0$ from experiments (HAPPEX, G0, PVA4)

$$\left.\frac{dG_E^s}{dQ^2}\right|_{Q^2=0} = r_s^2$$

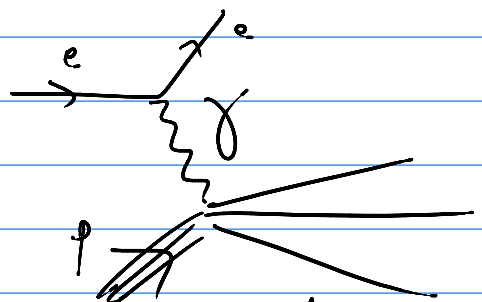
* Deep-inelastic scattering :

$$\frac{d^2\sigma}{dk'\,d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \, \frac{k'}{k} \, L_\gamma^{\mu\nu} \, W_{\mu\nu}$$

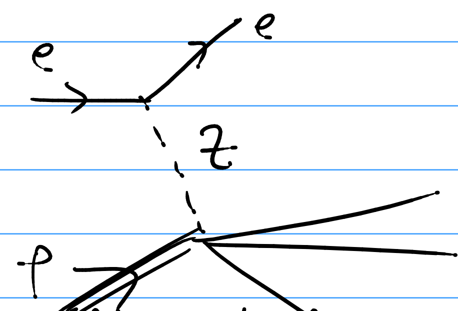
leptonic   hadronic tensors

$L_\gamma^{\mu\nu}$: leptonic tensor for photon exchange

$\sigma \propto \Big|$



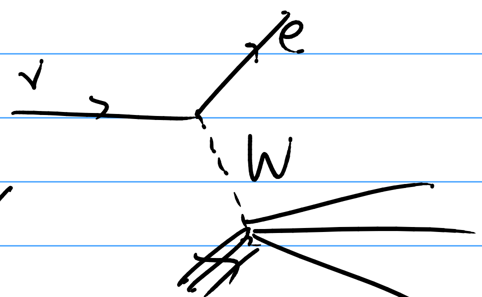
$+$



$\Big|^2$

electromagnetic   weak neutral current, NC

propagators $\frac{1}{Q^2} \longleftrightarrow \frac{1}{Q^2+M_Z^2}$



$\frac{1}{Q^2+M_W^2}$ charged neutral current, CC

In deep-inelastic (electron) scattering:

$$\sigma \propto |M_\gamma|^2 + 2\,\mathrm{Re}\,(M_\gamma^* M_Z) + |M_Z|^2$$

$$|M_\gamma|^2 \rightarrow L_\gamma^{\mu\nu} \qquad \lambda = \text{helicity}$$

$$2\,\mathrm{Re}\,(M_\gamma^* M_Z) \longrightarrow L_{\gamma Z}^{\mu\nu} = L_\gamma^{\mu\nu}\,(g_V^e - \lambda g_A^e)\ \eta_{\gamma Z}$$

$$|M_Z|^2 \rightarrow L_Z^{\mu\nu} = L_\gamma^{\mu\nu}\,(g_V^e - \lambda g_A^e)^2\ \eta_{\gamma Z}^2$$

with $\eta_{\gamma Z} = \dfrac{\text{propagator for } Z}{\text{propagator for } \gamma}$

or $\eta_{\gamma Z} = \dfrac{G_F M_Z^2}{2\sqrt{2}\,\pi\alpha}\,\dfrac{1}{Q^2 + M_Z^2} \Big/ \dfrac{1}{Q^2}$

For γ exchange: structure functions $F_1^\gamma, F_2^\gamma, F_3^\gamma$
polarized structure functions $g_1^\gamma, \dots, g_5^\gamma$
expressed in terms of $q(x)$, $\Delta q(x)$

e.g. $F_2^\gamma = \sum_i e_i^2 \,(q(x) + \bar{q}(x))$

$$g_1^\gamma = \sum_i e_i^2 \,(\Delta q(x) + \Delta\bar{q}(x))$$

γZ interference: $e_i^2 \rightarrow 2\, e_i\, g_V^q$

Z exchange: $e_i^2 \rightarrow {g_V^q}^2 + {g_A^q}^2$

→ access to $g_1^{\gamma Z}$, $F_2^{\gamma Z}$ gives us access to different linear combination of $q(x)$, $\Delta q(x)$