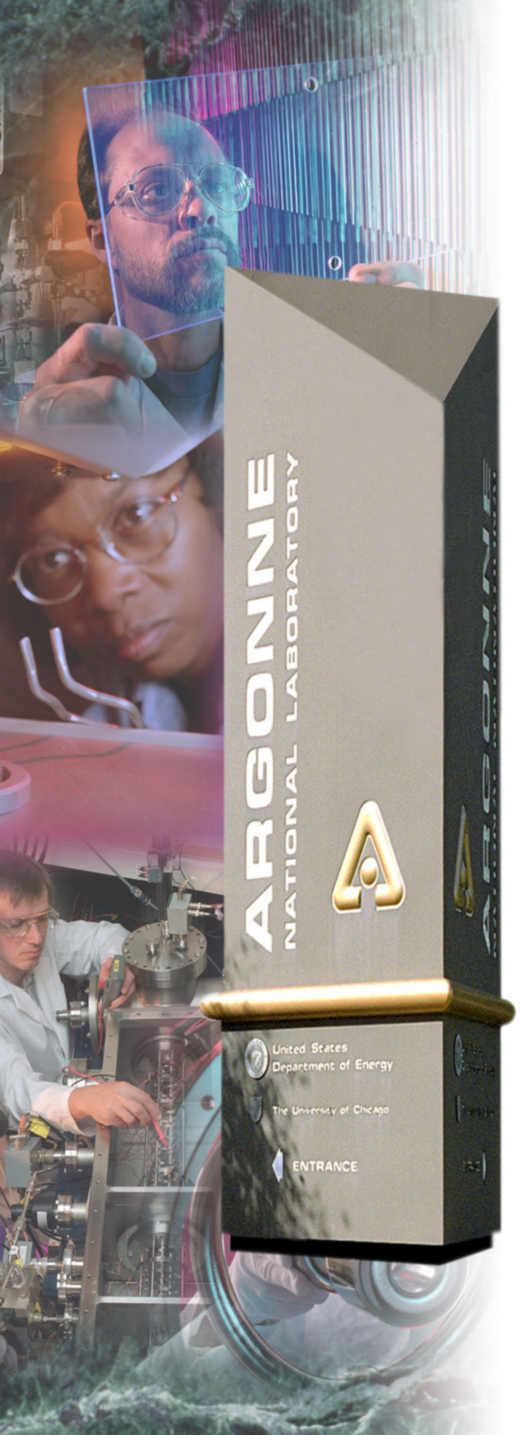

Comment on the Invariant-Envelope Solution in rf Photo-injectors

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Work in Progress

Beam-envelope Equation with acceleration

$$\boxed{\sigma'' + \frac{\gamma'}{\beta^2\gamma}\sigma' + K_r\,\sigma - \frac{\kappa_s}{\beta^3\gamma^3\,\sigma} - \frac{\epsilon_n^2}{\beta^2\gamma^2\,\sigma^3} = 0}$$

$$K_r = \left(\frac{\gamma'}{\gamma}\right)^2\Omega^2\,, \qquad \Omega^2 = \frac{1}{\sin^2\phi}\left[\frac{\eta}{8} + \left(\frac{B_z c}{E_0}\right)^2\right]\,, \qquad \kappa_s = Ig(\zeta)/2I_0$$

- Complications due to acceleration (adiabatic damping)
- Cauchy coordinate to remove the damping term

$$\boxed{y = \ln(\gamma/\gamma_0)} \qquad \sigma'' + \frac{\gamma'}{\beta^2\gamma}\sigma' = \left(\frac{\gamma'}{\gamma}\right)^2\left[\frac{d^2\sigma}{dy^2} + \left(\cancelto{0}{\frac{1}{\beta^2} - 1 + \frac{\gamma\gamma''}{\gamma'^2}}\right)\frac{d\sigma}{dy}\right]$$

- Cauchy-space envelope equation

laminar

$$\boxed{\frac{d^2\sigma}{dy^2} + \Omega^2\sigma - \frac{S}{\gamma_0}\frac{e^{-y}}{\sigma} + \cancelto{0}{\frac{\epsilon_n^2}{\gamma'^2\sigma^3}} = 0} \qquad S = \kappa_s/\gamma'^2$$

- Invariant envelope solution (Serafini & Rosenzweig, 97)

$$\boxed{\sigma_{\text{inv}} = \sqrt{\frac{S}{(\Omega^2 + 1/4)\gamma_0}}\,e^{-y/2} = \sqrt{\frac{S}{(\Omega^2+1/4)\gamma}}}\,, \qquad \boxed{\frac{\sigma'_{\text{inv}}}{\sigma_{\text{inv}}} = -\frac{\gamma'}{2\gamma}}$$

Reduced Beam-envelope Equation

- Reduced coordinate to remove the damping term (more standard approach)

$$\boxed{\hat{\sigma} = \sqrt{\beta\gamma}\,\sigma}$$

$$\sigma'' + \frac{\gamma'}{\beta^2\gamma}\sigma' = \frac{1}{\sqrt{\beta\gamma}}\left(\hat{\sigma}'' - \frac{\sqrt{\beta\gamma}''}{\sqrt{\beta\gamma}}\hat{\sigma}\right)$$

- Reduced envelope equation (e.g. Lawson)

$$\boxed{\hat{\sigma}'' + \left(K_r - \frac{\sqrt{\beta\gamma}''}{\sqrt{\beta\gamma}}\right)\hat{\sigma} - \frac{\kappa_s}{\beta^2\gamma^2\,\hat{\sigma}} - \frac{\epsilon_n^2}{\hat{\sigma}^3} = 0}$$

exact

Pseudo focusing accounts for most of the complications due to acceleration

$$-\frac{\sqrt{\beta\gamma}''}{\sqrt{\beta\gamma}} = \frac{1}{4}\left(1 + \frac{2}{\gamma^2}\right)\left(\frac{\gamma'}{\beta^2\gamma}\right)^2 - \frac{\gamma''}{2\beta^2\gamma} \simeq \frac{1}{4}\left(\frac{\gamma'}{\beta\gamma}\right)^2$$

$$\boxed{\hat{\sigma}'' + \left(\frac{\gamma'}{\beta\gamma}\right)^2\left(\Omega^2 + \frac{1}{4}\right)\hat{\sigma} - \left(\frac{\gamma'}{\beta\gamma}\right)^2\frac{S}{\hat{\sigma}} - \frac{\epsilon_n^2}{\hat{\sigma}^3} = 0}$$

- Invariant envelope solution is obvious

$$\boxed{\hat{\sigma}_{\text{inv}} = \sqrt{\frac{S}{\Omega^2 + 1/4}},}$$

$$\hat{\sigma}' = \hat{\sigma}'' = 0$$

Envelope Hamiltonian

$$\hat{\sigma}'' + \left(\frac{\gamma'}{\beta\gamma}\right)^2 \left(\Omega^2 + \frac{1}{4}\right)\hat{\sigma} - \left(\frac{\gamma'}{\beta\gamma}\right)^2 \frac{S}{\hat{\sigma}} - \frac{\epsilon_n^2}{\hat{\sigma}^3} = 0$$

Envelope Hamiltonian $\boxed{H(\hat{\sigma}, p_\sigma, s) = \frac{p_\sigma^2}{2} + V(\hat{\sigma}, s)}$

$$V = \left(\frac{\gamma'}{\beta\gamma}\right)^2 \left(\Omega^2 + \frac{1}{4}\right)\frac{\hat{\sigma}^2}{2} - \left(\frac{\gamma'}{\beta\gamma}\right)^2 S \ln\frac{\hat{\sigma}}{\hat{\sigma}_{\rm inv}} + \frac{\epsilon_n^2}{2\hat{\sigma}^2}$$

$$= \left(\frac{\epsilon_n}{\hat{\sigma}_{\rm inv}}\right)^2 \frac{1}{2\rho}\left(\frac{\hat{\sigma}^2}{\hat{\sigma}_{\rm inv}^2} - \ln\frac{\hat{\sigma}^2}{\hat{\sigma}_{\rm inv}^2} + \rho\frac{\hat{\sigma}_{\rm inv}^2}{\hat{\sigma}^2}\right)$$

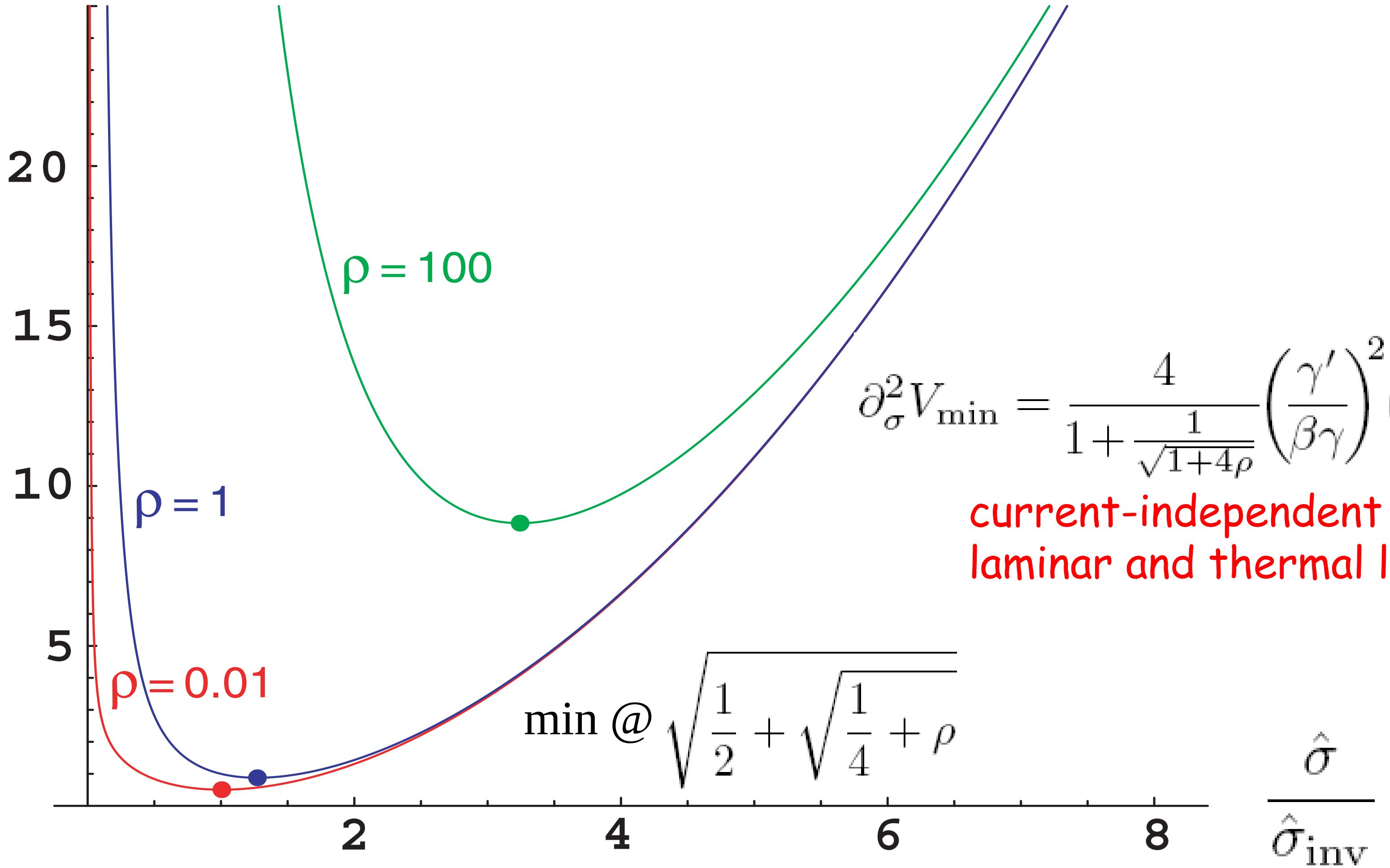
$$\rho \equiv \left(\frac{\beta\gamma\gamma'\epsilon_n}{\kappa_s}\right)^2 \left(\Omega^2 + \frac{1}{4}\right)$$

0 laminar

∞ thermal

Time-dependent Potential

$$\tilde{V} = \left(\frac{\hat{\sigma}_{\rm inv}}{\epsilon_n}\right)^2 \rho\, V = \frac{1}{2}\left(\frac{\hat{\sigma}^2}{\hat{\sigma}^2_{\rm inv}} - \ln\frac{\hat{\sigma}^2}{\hat{\sigma}^2_{\rm inv}} + \rho\,\frac{\hat{\sigma}^2_{\rm inv}}{\hat{\sigma}^2}\right)$$



From laminar to thermal regimes

Potential $\mathbf{V}$ reaches minimum at

$$\boxed{\hat{\sigma}_\rho = \hat{\sigma}_{\mathrm{inv}}\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+\rho}} = \hat{\sigma}_\infty\sqrt{\frac{\nu}{2}+\sqrt{1+\left(\frac{\nu}{2}\right)^2}}}$$

$\rho \rightarrow 0$

laminar

$\rho \rightarrow \infty$

thermal

$$\sigma_\infty = \left(\Omega^2+\frac{1}{4}\right)^{-\frac{1}{4}}\sqrt{\frac{\epsilon_n}{\gamma'}}$$

$\hat{\sigma}_{\mathrm{inv}}$

$\hat{\sigma}_\infty = \hat{\sigma}_{\mathrm{inv}}\rho^{1/4}$

exact solution

approximate solution

📖 "Laminar Flow in Non-relativistic Intense Proton Beams", Serafini & Rosenzweig (EPAC98)

📖 "New generation issues in the beam physics of RF laser-driven electron photoinjectors", Serafini & Ferrario (SPIE-LASER'99)

Invariant Envelope solution

$$\hat{\sigma}_{\mathrm{inv}} = \sqrt{\frac{S}{\Omega^2 + 1/4}}, \qquad \hat{\sigma}'_{\mathrm{inv}} = 0\,, \quad \frac{\hat{\sigma}'_{\mathrm{inv}}}{\hat{\sigma}_{\mathrm{inv}}} = 0$$

Invariant size through time   Invariant slope across slices

- A unique equilibrium solution seating on the potential minimum
- Defocusing space-charge force is canceled by focusing forces:
  1) ponderomotive force from rf
  2) magnetic force from solenoid
  3) pseudo force due to acceleration

Brillouin flow

- Straight flow lines parallel to the propagation direction
- Oscillations around it have the same frequency for all currents
- Minimize the maximum beam size, thus less nonlinear degradation
- Zero transverse momentum significantly reduces the number of terms due to nonlinear forces

Oscillations around invariant envelope

Cauchy coordinates
non-canonical

$$\begin{bmatrix} \sigma \\ \dot\sigma \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\gamma}} & 0 \\ \frac{-1}{2\sqrt{\gamma}} & \frac{\sqrt{\gamma}}{\gamma'} \end{bmatrix} \begin{bmatrix} \hat\sigma \\ \hat\sigma' \end{bmatrix} \equiv A \begin{bmatrix} \hat\sigma \\ \hat\sigma' \end{bmatrix}$$

reduced coordinates
canonical

$$\delta\ddot\sigma + \omega^2 \delta\sigma = 0$$

$$\delta\hat\sigma'' + 2(\gamma'/\beta\gamma)^2(\Omega^2 + 1/4)\delta\hat\sigma = 0$$

$$R_C = \begin{bmatrix} \cos\omega y & \frac{1}{\omega}\sin\omega y \\ -\omega\sin\omega y & \cos\omega y \end{bmatrix}$$

$$\hat R = A(\gamma)^{-1} R_C A(\gamma_0) =$$

$$\begin{bmatrix} \sqrt{\frac{\gamma}{\gamma_0}}\left(\cos u - \frac{\sin u}{2\omega}\right) & \frac{\sqrt{\gamma_0\gamma}}{\omega\gamma'}\sin u \\ -\frac{\omega\gamma'}{\sqrt{\gamma_0\gamma}}\left(1 + \frac{1}{4\omega^2}\right)\sin u & \sqrt{\frac{\gamma_0}{\gamma}}\left(\cos u + \frac{\sin u}{2\omega}\right) \end{bmatrix}$$

$$\sigma = \sigma_{\rm inv} + \frac{\delta\sigma(0)}{\cos\theta}\cos(u+\theta)$$

$$\sigma' = \sigma'_{\rm inv} - \frac{\delta\sigma(0)}{\cos\theta}\frac{\omega\gamma'}{\gamma}\sin(u+\theta)$$

$$\hat\sigma = \hat\sigma_{\rm inv} + \sqrt{\frac{\gamma}{\gamma_0}}\frac{\delta\hat\sigma(0)}{\cos\theta}\cos(u+\theta),$$

$$\hat\sigma' = -\sqrt{\frac{\gamma'^2}{\gamma_0\gamma}}\left(\omega^2 + \frac{1}{4}\right)\frac{\delta\hat\sigma(0)}{\cos\theta}\sin(u+\theta-\theta_0)$$

$$\omega = \sqrt{2\Omega^2 + 1/4} \qquad u = \omega y = \omega\ln(\gamma/\gamma_0)$$

$$\theta = \tan^{-1}\left[\frac{1}{2\omega} - \frac{\gamma_0\delta\hat\sigma'(0)}{\omega\gamma'\delta\hat\sigma(0)}\right]$$

$$\theta_0 = \tan^{-1}(1/2\omega)$$

Emittance compensation

- In laminar regime each slice is a thick-less segment in the phase space
- Bunch emittance results from spread of tilt angles of all slice segments
- Goal of emittance compensation:

  alignment of all slice segments in phase space, i.e.,

  all slices have the same angle $\frac{\hat{\sigma}'}{\hat{\sigma}} = \frac{\gamma'}{2\gamma} + \frac{\sigma'}{\sigma} = \frac{\gamma'}{2\gamma} + \frac{\omega\gamma'}{\gamma}\frac{\dot{\sigma}}{\sigma}$

- Slices matched to the invariant envelopes

$$\boxed{\frac{\hat{\sigma}'_{\mathrm{inv}}}{\hat{\sigma}_{\mathrm{inv}}} = 0}$$

- Almost the same picture as the much simpler model without acceleration, (pseudo-focusing taking care of the acceleration)

Emittance compensation

Estimate the emittance with two-slice

$$\epsilon = \frac{1}{2}\left|\hat{\sigma}_+\hat{\sigma}'_- - \hat{\sigma}_-\hat{\sigma}'_+\right|$$

yields

$$\epsilon = \hat{\sigma}_{\mathrm{rms}}\sqrt{\frac{\gamma'^2}{\gamma_0\gamma}\left(\omega^2+\frac{1}{4}\right)}\left|\frac{\delta\hat{\sigma}_{\mathrm{edge}}(0)}{\cos\theta}\sin\left(u+\theta-\theta_0\right)\right|$$
$$\propto \boxed{\frac{1}{\sqrt{\gamma}}\left|\sin\left(\omega\ln\frac{\gamma}{\gamma_0}+\theta-\theta_0\right)\right|}.$$

It clearly shows that the correlated emittance is damped by the square root of γ and periodically returns to zero --- the behavior of an emittance-compensated beam. The focusing solenoid controls the emittance oscillation through ω