lattice QCD and the hadron spectrum

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the light meson spectrum

relatively simple models of hadrons:

bound states of **constituent** quarks and antiquarks

"the quark model"

$$M \sim q\bar{q} \qquad B \sim qqq$$

empirical meson flavour systematics

I=1, S=0 : π,ρ,b₁,a_J …

$$u\bar{d},\ \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}),\ d\bar{u}$$

I=½,S=±1 : K,K* …

$$u\bar{s},\ d\bar{s},\ s\bar{u},\ s\bar{d}$$

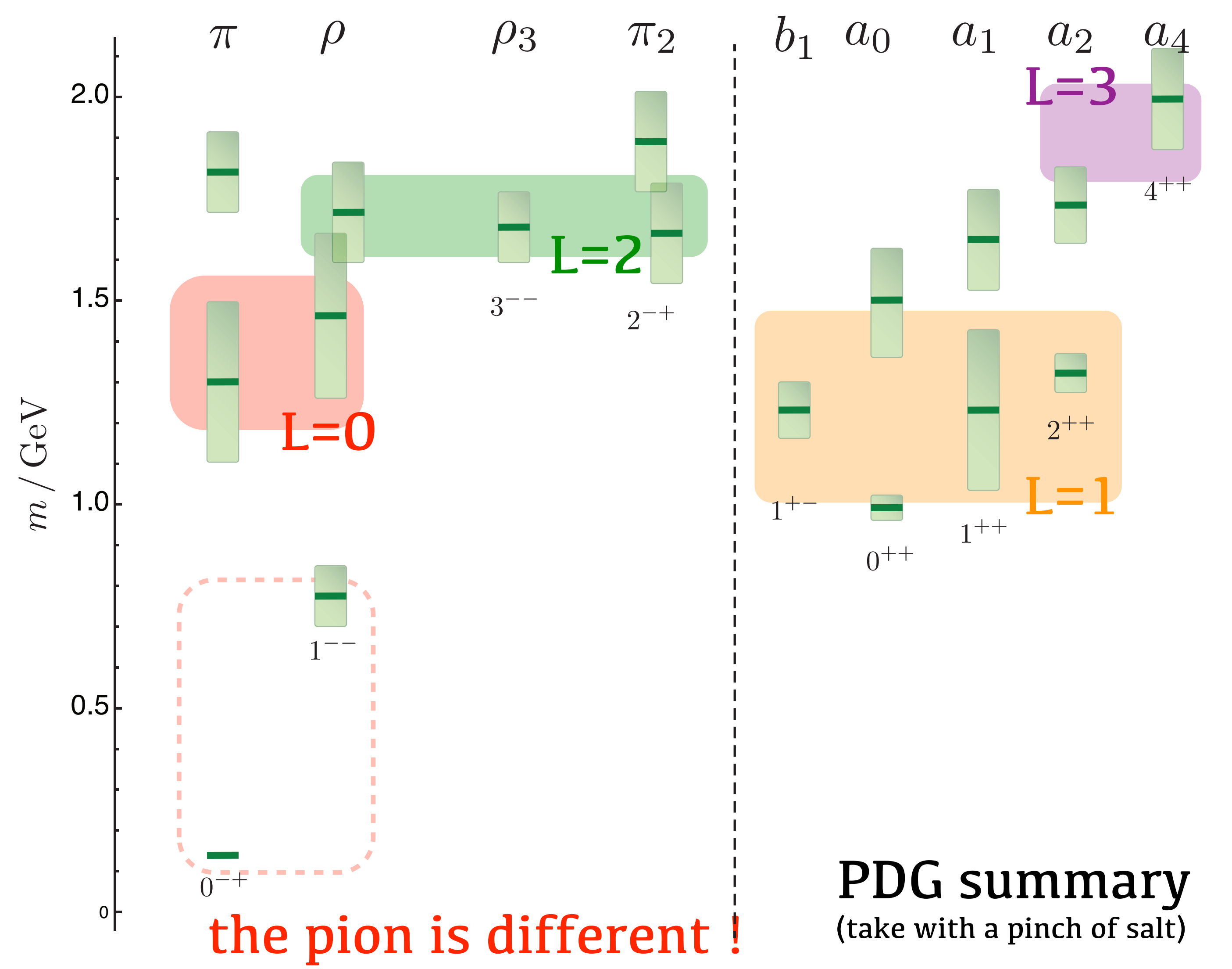
I=0, S=0 : η,φ,ω,f_J …

$$A\left[\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\right] + B\left[s\bar{s}\right]$$

I≯1, |S|≯1

$$\cancel{ud\bar{s}\bar{s}},\ \ldots$$

empirical J^{PC} distribution



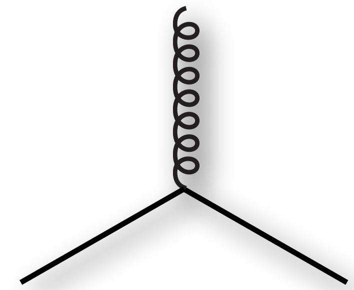
light quarks with mass ~ 400 MeV ?

Quantum Chromodynamics

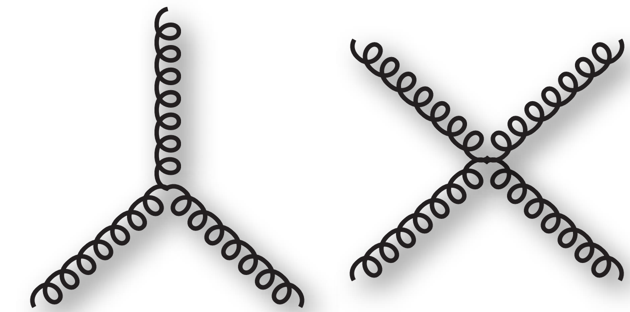
$$\mathcal{L}_{\mathrm{QCD}} = \sum_{q=uds} \bar{\psi}_q \left(i\gamma^\mu \partial_\mu - m_q\right) \psi_q + g\, \bar{\psi}_q \gamma^\mu t^a \psi_q\, A^a_\mu - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

massless gluons
light quarks

$m_{u,d} \sim O(1)$ MeV

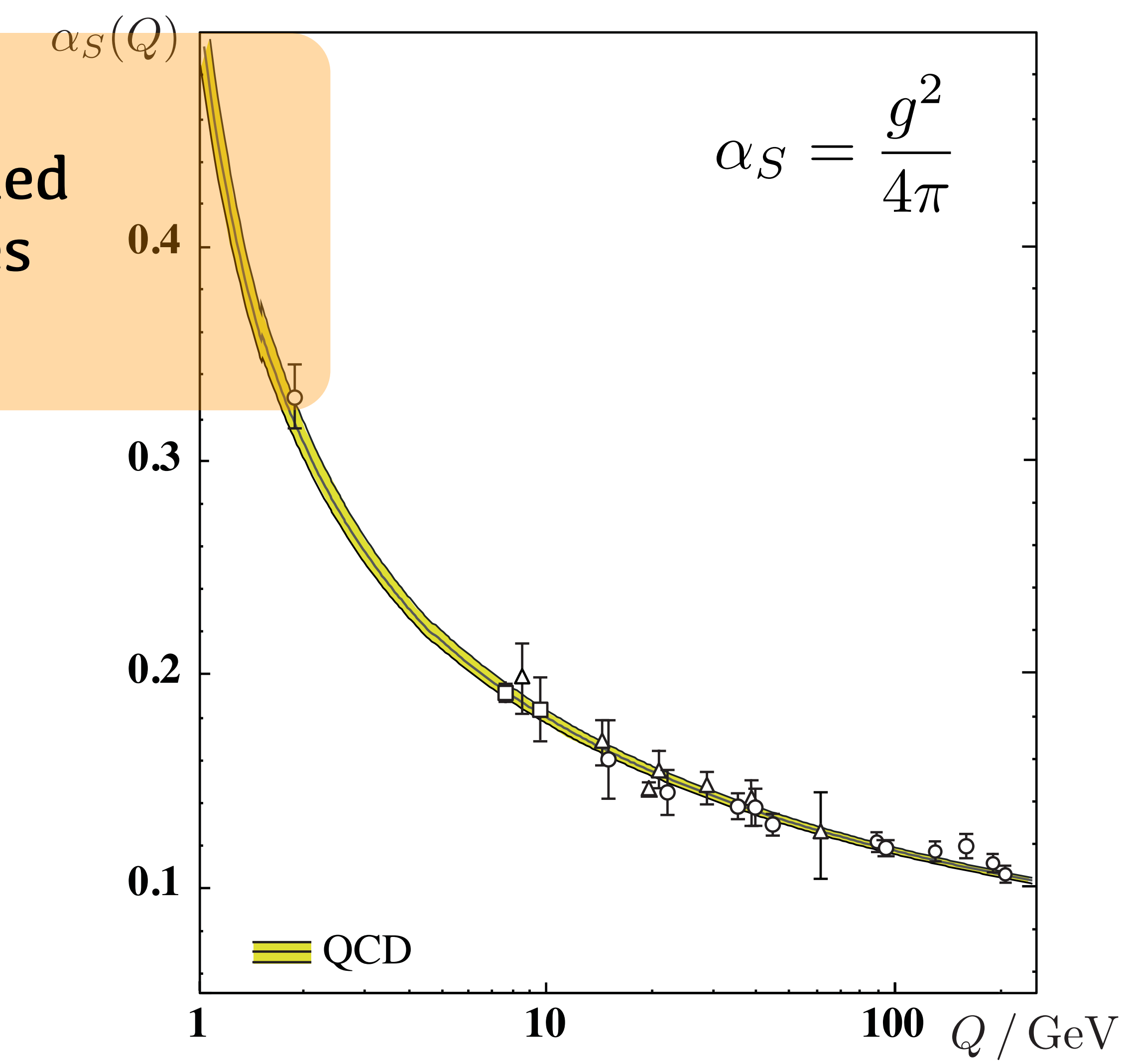


quark-gluon coupling



gluonic self-couplings

strongly coupled at low energies



➟ so why the (heavy) ‘constituent quark’ pattern observed in experiment ?

➟ what is the role of strongly coupled glue in the spectrum ?

glueballs, hybrid mesons ?

the light meson spectrum

an example of states beyond minimal quark model configurations **hybrid mesons**

states in which a gluonic excitation is present

smoking gun signature - J^{PC} outside the set accessible to $q\bar{q}$

$$J^{PC}_{q\bar{q}} \neq 0^{--}, 0^{+-}, 1^{-+}, 2^{+-} \ldots$$

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Model of mesons with constituent gluons*

California Institu

Massachusetts Institut

A model of mesons composed of a constituents is provided by a confining li the model are described, and their relativ states.

NEW MESON CONFIGURATION IN THE BAG MODEL
(I). First order energy spectrum of $q\bar{q}g$ states

F. DE VIRON and J. WEYERS

Département de Physique Théorique, Université Catholique de Louvain, B–1348 Louvain-la-Neuve,

(Fina

Perturbative QCD in the The meson spectrum is comp $J^{PC} = 1^{-+}, 0^{+-}, 0^{--}$ are found

A LIGHT EXOTIC $q\bar{q}g$ HERMAPHRODITE MESON?

Ted BARNES and F.E. CLOSE
Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, UK

Received 15 April 1982

We suggest that $q\bar{q}g$ mesons may exist as low as 1 GeV in mass. The exotic $J^{PC} = 1^{-+}$ multiplet will have distinctive decay modes and perhaps be relatively stable. The bag model spectrum of the lowest lying $q\bar{q}g$ multiplet including hyperfine splittings is computed analogously to Jaffe's $q\bar{q}q\bar{q}$ bag model multiplets. Relevance to light meson phenomenology is discussed.

several models, several different spectrum predictions ...

calculating in QCD

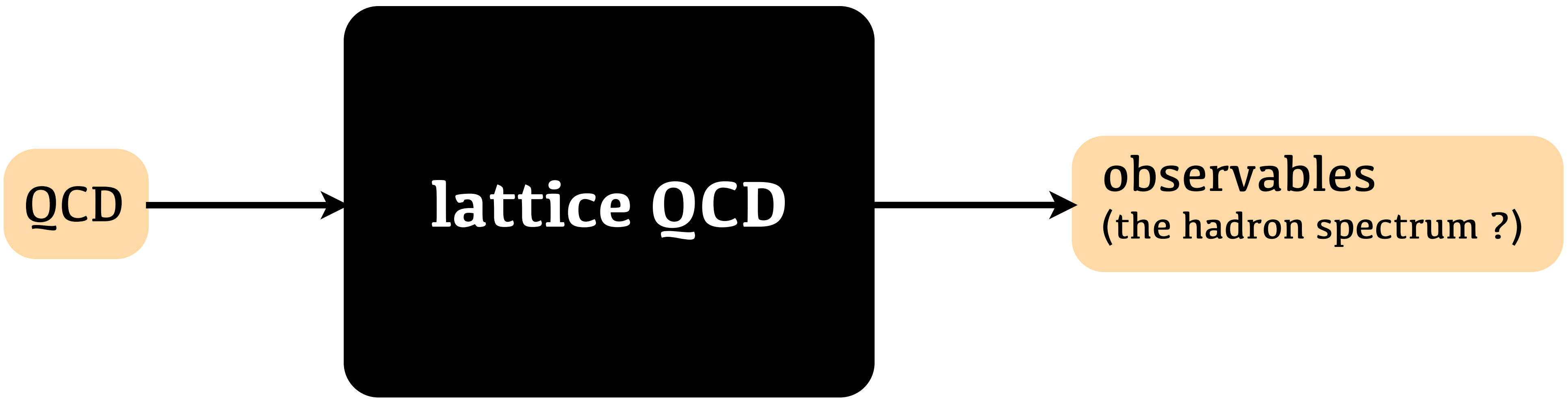
QCD $\xrightarrow{\text{some theoretical technique to perform calculations}}$ observables (the hadron spectrum ?)

computational approach ('simulation')

QCD

observables
(the hadron spectrum ?)

a black box ?



in these lectures,
hope to give you a look inside the box

these lectures

how's it done ?

- ➟ field theory on a lattice
- ➟ numerical approach
- ➟ QCD, quarks and gluons
- ➟ lattice QCD calculation workflow

if hadrons were stable . . .

- ➟ extracting an excited state spectrum
- ➟ quark-gluon bound-state interpretation
- ➟ a QCD phenomenology of hybrid mesons

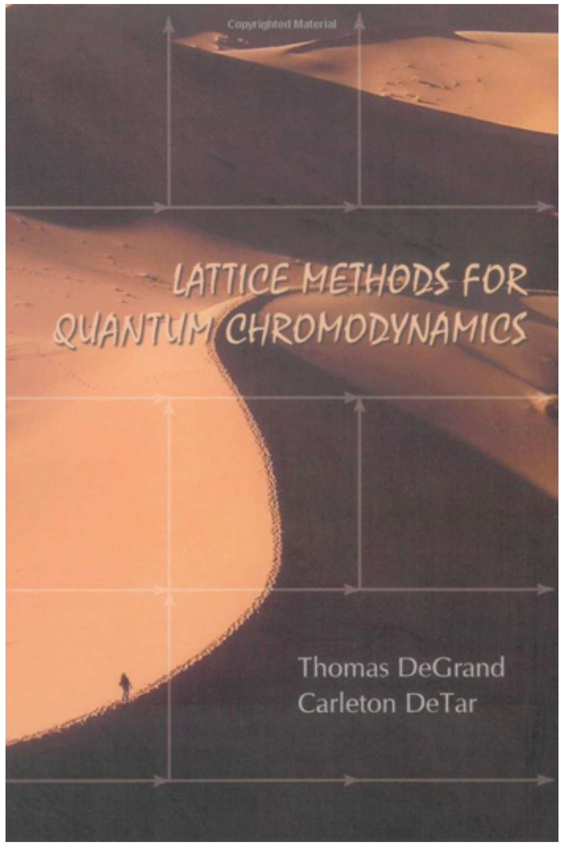
hadron scattering

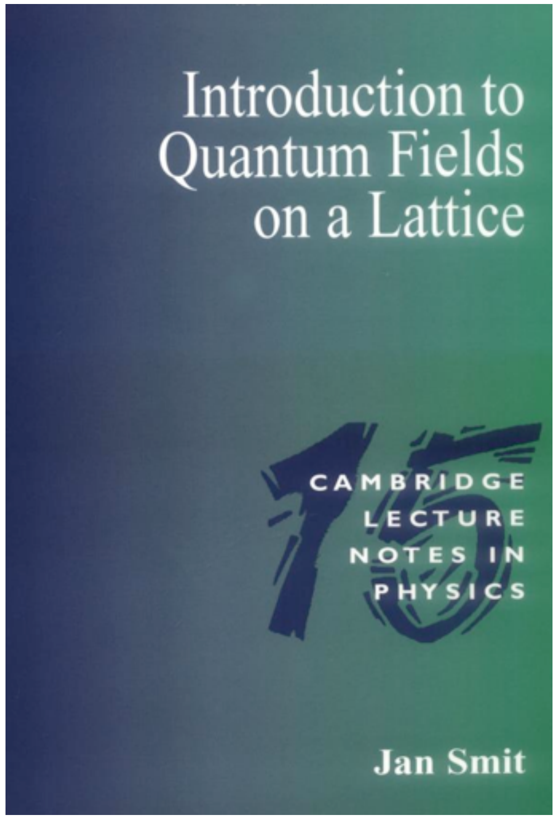
- ➟ continuum multi-hadron spectrum
- ➟ QCD in a finite-volume
- ➟ $\pi\pi$ scattering
- ➟ resonance calculations

some pedagogic books

deGrand and deTar

Lattice Methods for Quantum Chromodynamics





Smit

Introduction to Quantum Fields
on a Lattice

the technique ... lattice field theory

need a technique to perform non-perturbative calculations in field theories

lattice regularization leads to a numerical approach

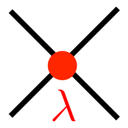
first some reminders of the contents of a field theory

Lagrangian density for a certain interacting scalar field theory

$$\mathcal{L} = \underbrace{\tfrac{1}{2}\,\partial_t\hat{\varphi}(x)\,\partial_t\hat{\varphi}(x) - \tfrac{1}{2}\vec{\nabla}\hat{\varphi}(x)\cdot\vec{\nabla}\hat{\varphi}(x)}_{\text{kinetic term}} - \underbrace{\tfrac{1}{2}\mu^2\hat{\varphi}(x)^2}_{\text{mass term}} - \underbrace{\tfrac{1}{4}\lambda\hat{\varphi}(x)^4}_{\text{interaction}}$$

can derive Feynman rules

free propagators $\dfrac{1}{p^2-\mu^2}$

interaction vertex 

the 'action'

$$S = \int d^4x\,\mathcal{L}$$

c.f. classical physics
- principal of least action determines motion

the technique ... lattice field theory

correlation functions - determine correlation between fields at different space-time points

e.g. $\langle 0|\hat{\varphi}(\vec{y},t')\hat{\varphi}(\vec{x},t)|0\rangle$

e.g. information about the energy spectrum is embedded in the 'two-point' correlation function

$$\begin{aligned}\langle 0|\hat{\varphi}(t)\hat{\varphi}(0)|0\rangle &= \langle 0|e^{i\hat{H}t}\hat{\varphi}(0)e^{-i\hat{H}t}\hat{\varphi}(0)|0\rangle \\ &= \langle 0|\hat{\varphi}(0)e^{-i\hat{H}t}\sum_{\mathfrak{n}}|\mathfrak{n}\rangle\langle\mathfrak{n}|\;\hat{\varphi}(0)|0\rangle \\ &= \sum_{\mathfrak{n}} e^{-iE_{\mathfrak{n}}t}\langle 0|\hat{\varphi}(0)|\mathfrak{n}\rangle\langle\mathfrak{n}|\hat{\varphi}(0)|0\rangle\end{aligned}$$

eigenstates of the Hamiltonian

$$\hat{H}\,|\mathfrak{n}\rangle = E_{\mathfrak{n}}|\mathfrak{n}\rangle$$

complete set

$$1 = \sum_{\mathfrak{n}}|\mathfrak{n}\rangle\langle\mathfrak{n}|$$

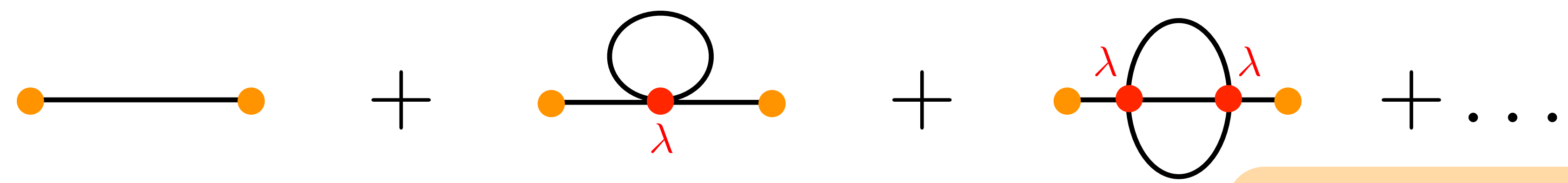
might include an **integral** over continuum of states

the technique ... lattice field theory

we need a technique to calculate the correlation functions $\langle 0|\hat{\varphi}(t)\hat{\varphi}(0)|0\rangle$

$$\mathcal{L} = \tfrac{1}{2}\,\partial_t\hat{\varphi}(x)\,\partial_t\hat{\varphi}(x) - \tfrac{1}{2}\vec{\nabla}\hat{\varphi}(x)\cdot\vec{\nabla}\hat{\varphi}(x) - \tfrac{1}{2}\mu^2\hat{\varphi}(x)^2 - \tfrac{1}{4}\lambda\hat{\varphi}(x)^4$$

➟ if λ is small - power series expansion - `perturbation theory’



integrals over products of free-particle propagators $\dfrac{1}{p^2-\mu^2}$

➟ if coupling strength is not weak, need to find another method ...

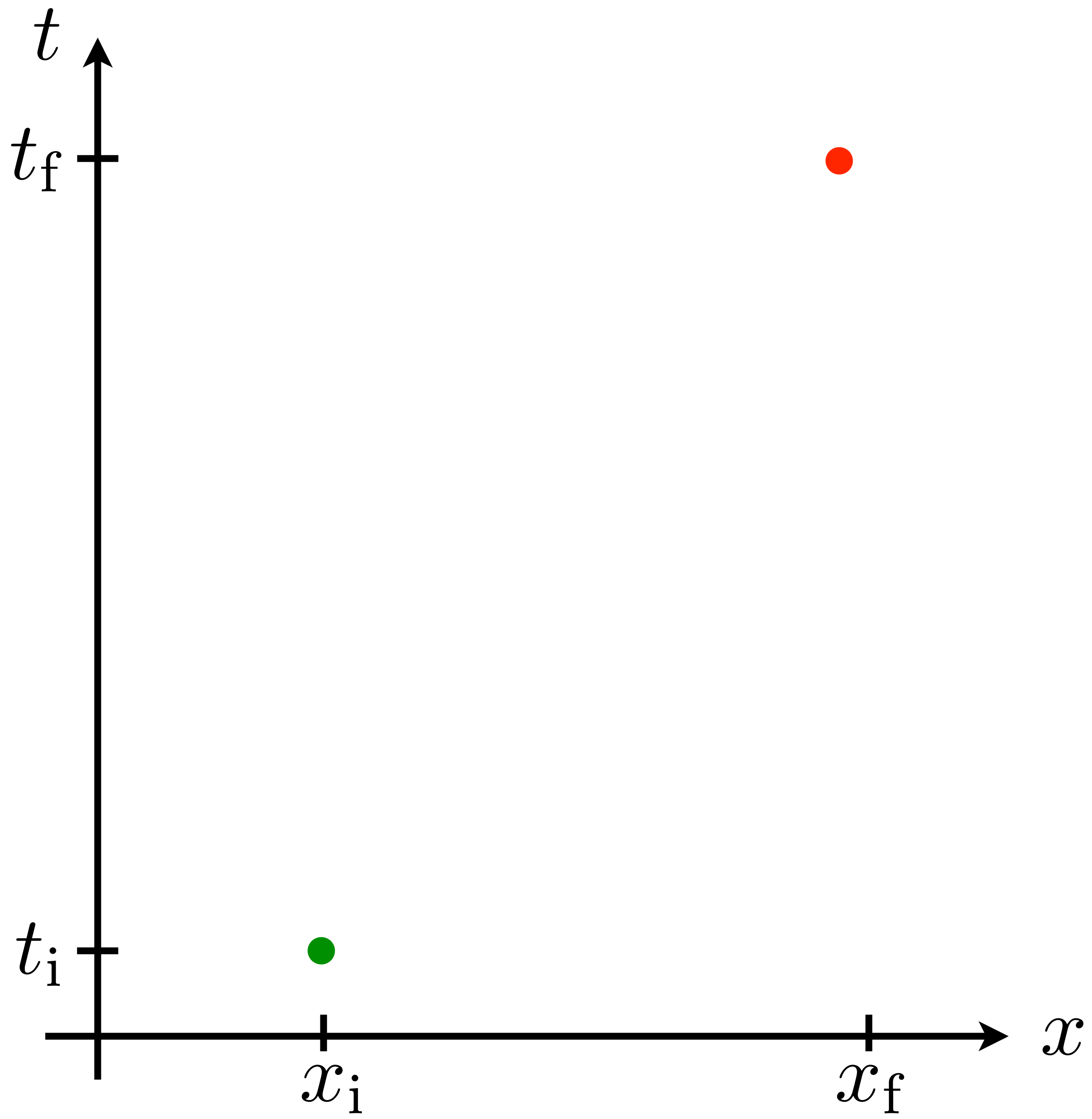
... start with the `path integral’ representation of the theory

path integrals in quantum mechanics

e.g. a free particle moving between a fixed initial position (x_i,t_i) and a fixed final position (x_f,t_f)

$$\langle x_{\rm f}| \, e^{-i\hat{H}(t_{\rm f}-t_{\rm i})/\hbar} \, |x_{\rm i}\rangle$$

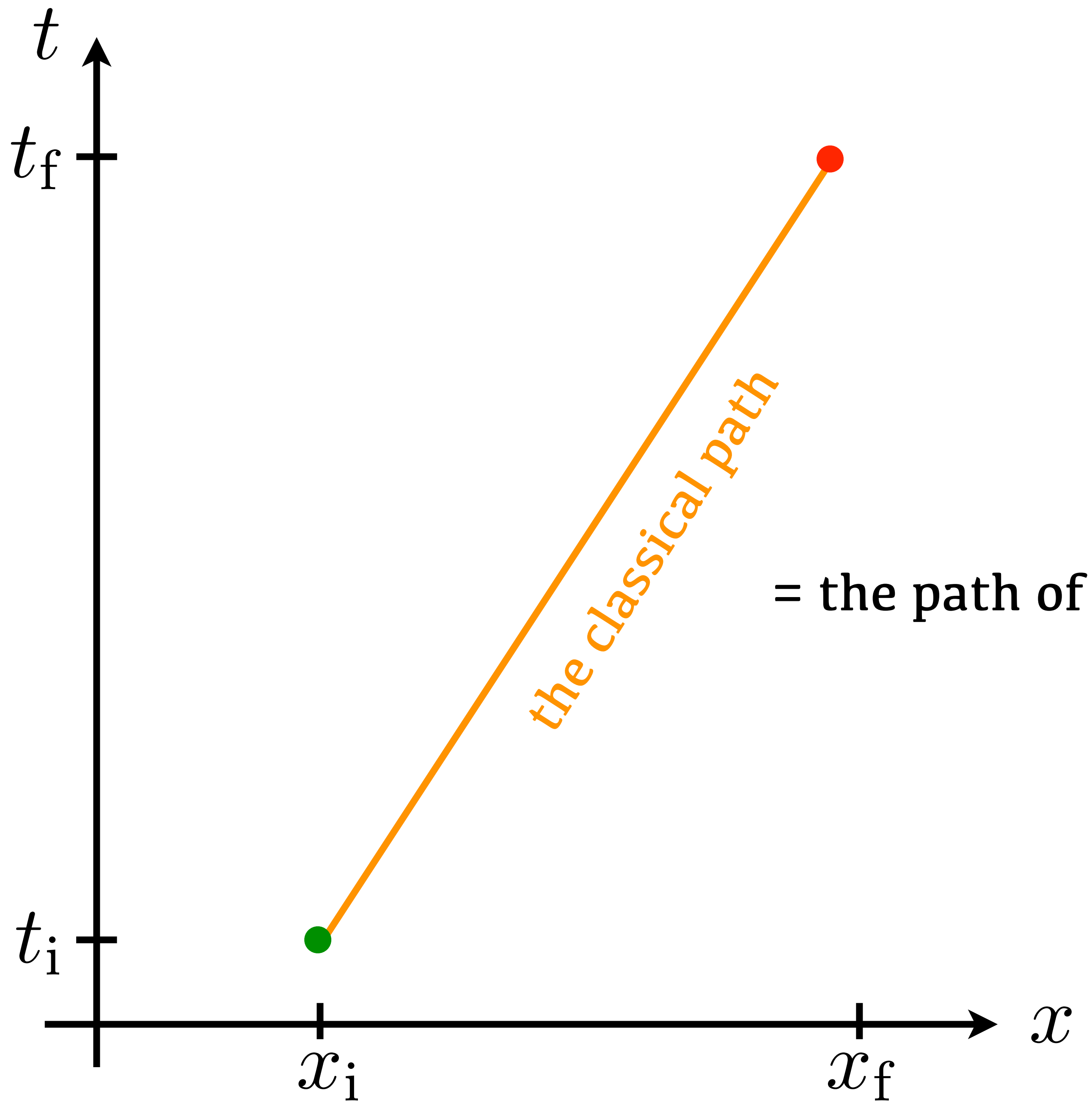
QM amplitude for propagation



path integrals in quantum mechanics

$$\langle x_\mathrm{f}|\, e^{-i\hat{H}(t_\mathrm{f}-t_\mathrm{i})/\hbar}\,|x_\mathrm{i}\rangle = \int \mathcal{D}x\; e^{-iS(x)}$$

in QM need to sum over **all** paths,
not just the classically allowed one



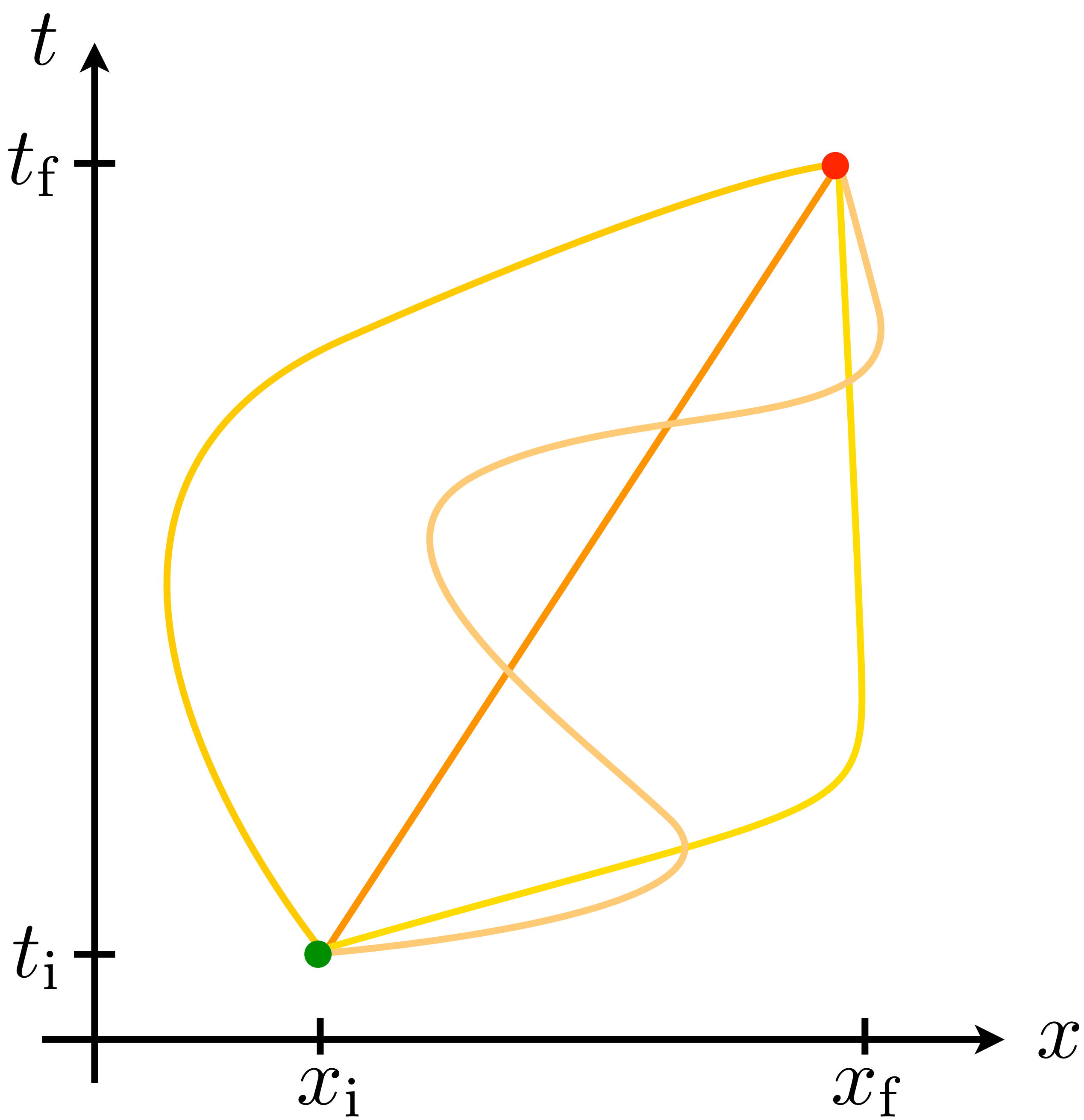
= the path of minimum action

$$S = \int dt\; \tfrac{1}{2} m \left(\tfrac{dx}{dt}\right)^2$$

path integrals in quantum mechanics

$$\langle x_{\mathrm{f}} | \, e^{-i\hat{H}(t_{\mathrm{f}}-t_{\mathrm{i}})/\hbar} \, | x_{\mathrm{i}} \rangle = \int \mathcal{D}x \; e^{-iS(x)}$$

in QM need to sum over **all** paths, not just the classically allowed one



each path has an action associated with it $S\big(x(t)\big)$

$$S = \int dt \; \tfrac{1}{2} m \left(\tfrac{dx}{dt}\right)^2$$

$\int \mathcal{D}x$ implements the continuous 'sum' over possible paths

usual rules of QM follow from this formalism

path integrals in quantum field theory

the path-integral for our scalar field theory $$Z = \int \mathcal{D}\varphi(x)\; e^{-iS[\varphi(x)]}$$

action $S[\varphi(x)] = \int d^4x\; \mathcal{L}[\varphi(x)]$

correlation functions have a path-integral representation

e.g. $$\langle 0|\hat{\varphi}(x'')\hat{\varphi}(x')|0\rangle = \int \mathcal{D}\varphi(x)\; \varphi(x'')\varphi(x')\; e^{-iS[\varphi(x)]}$$

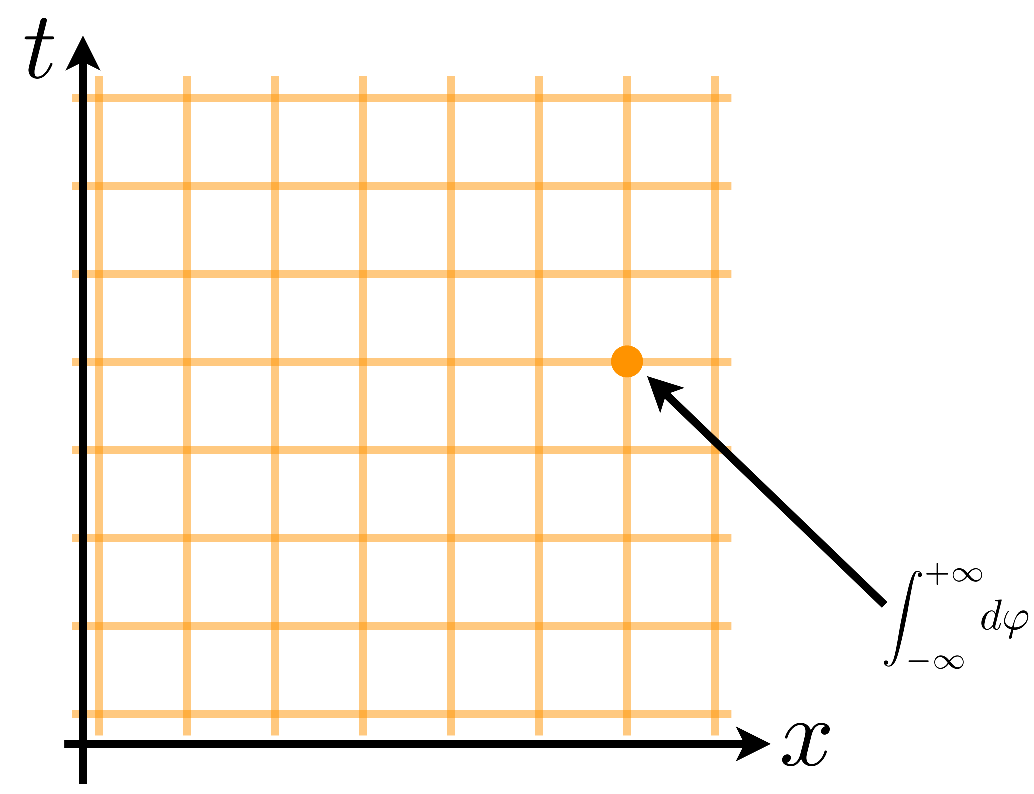
now the ‘sum’ is over all possible field configurations

a concrete representation of $\mathcal{D}\varphi(x)$

is given if we represent space-time by a grid of points

then $$\mathcal{D}\varphi(x) = \prod_x \int d\varphi_x$$

‘do an integral over field value at each site’



path integrals in quantum field theory

a very convenient mathematical operation - analytic continuation to imaginary time

$$t \to -i\tilde{t}$$

e.g. the exponent in the path-integral :

$$-iS = -i\int d^3x\, dt\, \mathcal{L} \to -\int d^3x\, d\tilde{t}\, \tilde{\mathcal{L}} = -\tilde{S}$$

$$\tilde{Z} = \int \mathcal{D}\varphi(x)\; e^{-\tilde{S}[\varphi(x)]}$$

geometric interpretation: $g_{\mu\nu} = \begin{pmatrix} +1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} \to \delta_{\mu\nu} = \begin{pmatrix} +1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & +1 \end{pmatrix}$

Minkowski SO(3+1) → Euclidean SO(4)

the 'Euclidean' path-integral

path integrals in quantum field theory

a very convenient mathematical operation - analytic continuation to imaginary time

$$t \to -i\tilde{t}$$

this is not as strange as it seems,
some operation of this type is required

➠ $Z = \int \mathcal{D}\varphi(x)\, e^{-iS[\varphi(x)]}$ doesn't converge, c.f. $\int_{-\infty}^{\infty} dx\, e^{i\alpha x^2}$

e.g. $t \to t - i\epsilon$ regulates the integral

➠ $\dfrac{1}{p^2 - \mu^2}$ needs regularisation ➞ $\dfrac{1}{p^2 - \mu^2 + i\epsilon}$

ensures 'outgoing' boundary conditions

we'll proceed with the Euclidean theory ...

(and watch out for quantities that may be affected by the analytic continuation)

euclidean path integrals in quantum field theory

$$\tilde{Z} = \int \mathcal{D}\varphi(x)\, e^{-\tilde{S}[\varphi(x)]}$$

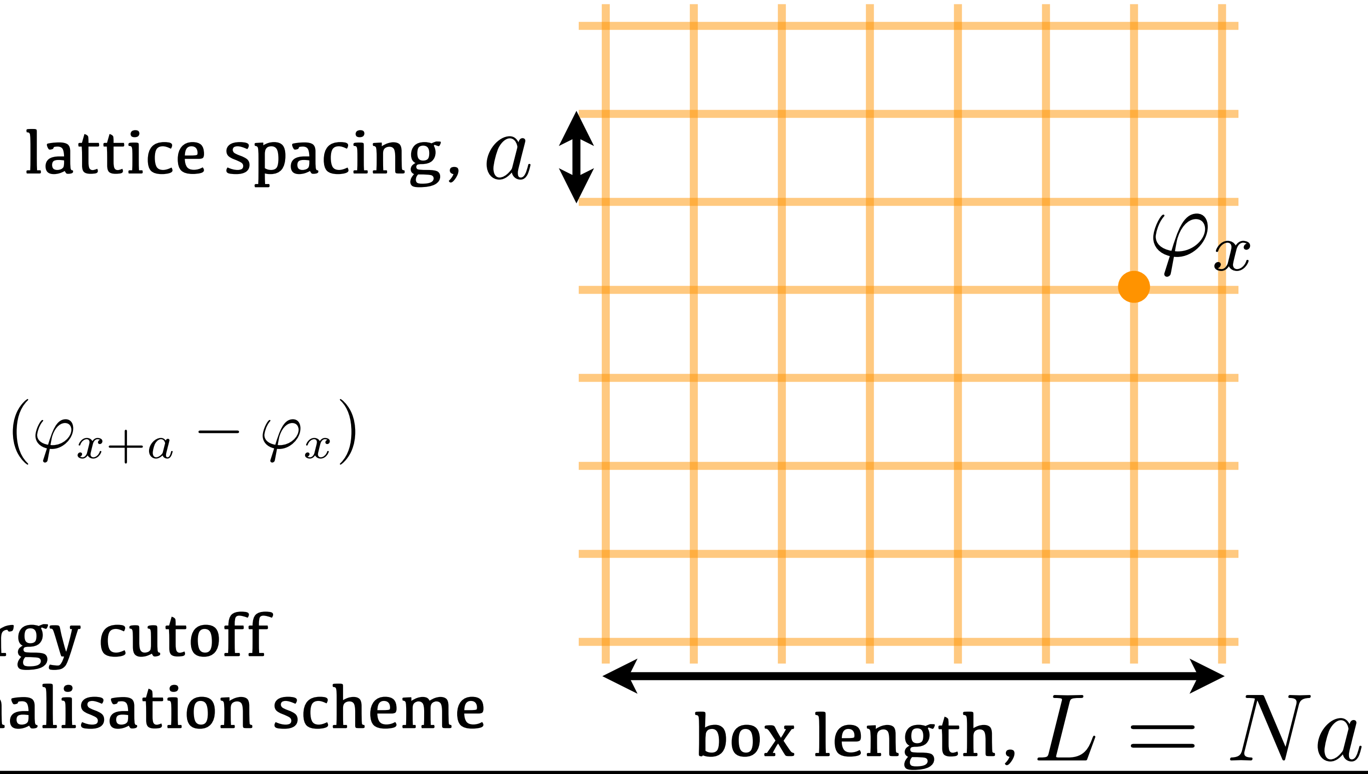
treat this like a probability ?

importance sampled Monte Carlo ?

but $\varphi(x)$ is a continuous function over an infinite space !

one possible approach:
discretise the field on a space-time grid of finite extent

lattice field theory



e.g. $\partial\varphi \to \frac{1}{a}\left(\varphi_{x+a} - \varphi_x\right)$

minimum length scale acts as a high-energy cutoff
- thus we automatically have a renormalisation scheme

lattice field theory

$$\tilde{Z} = \int \mathcal{D}\varphi(x)\; e^{-\tilde{S}[\varphi(x)]}$$

importance sampled Monte Carlo

generate field configurations $\left\{\varphi_x\right\}$ (a value of φ at each lattice site x)

according to the probability distribution $e^{-\tilde{S}[\varphi_x]}$

obtain an ensemble of N configurations $\left\{\varphi_x\right\}^{(\mathsf{i}=1\ldots\mathsf{N})}$

then an observable function of the field $\langle O[\hat{\varphi}]\rangle = \int \mathcal{D}\varphi\; O[\varphi]\, e^{-\tilde{S}[\varphi]}$

is approximated by the average over configurations

$$\langle O\rangle \approx \overline{O} \equiv \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} O[\varphi^{(\mathsf{i})}]$$

lattice field theory

then an observable function of the field $\langle O[\hat{\varphi}]\rangle = \int \mathcal{D}\varphi \, O[\varphi] \, e^{-\tilde{S}[\varphi]}$

is approximated by the average over configurations

$$\langle O \rangle \approx \overline{O} \equiv \tfrac{1}{\mathsf{N}} \textstyle\sum_{\mathsf{i}=1}^{\mathsf{N}} O[\varphi^{(\mathsf{i})}]$$

and an estimate of the precision of the approximation
comes from the variance of the mean

$$\mathrm{var}(O) \equiv \tfrac{1}{\mathsf{N}(\mathsf{N}-1)} \textstyle\sum_{\mathsf{i}} \left(O[\varphi^{(\mathsf{i})}] - \overline{O} \right)^2$$

$$\epsilon(O) = \sqrt{\mathrm{var}(O)} \sim \tfrac{1}{\sqrt{\mathsf{N}}}$$

$$\Longrightarrow \boxed{\langle O \rangle = \overline{O} \pm \epsilon(O)}$$

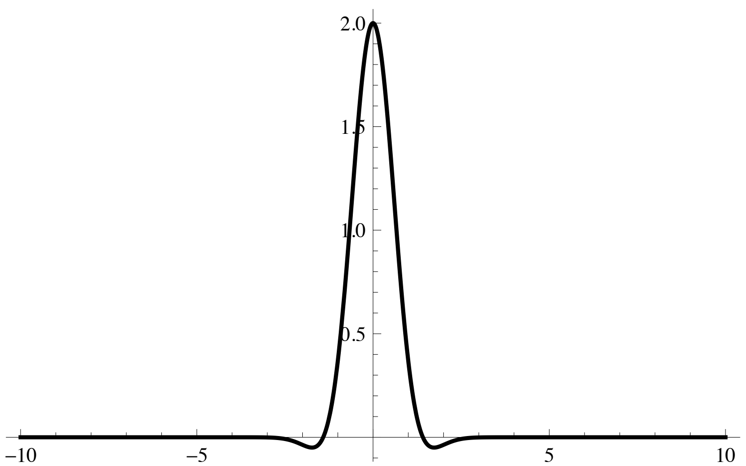
'statistical error'

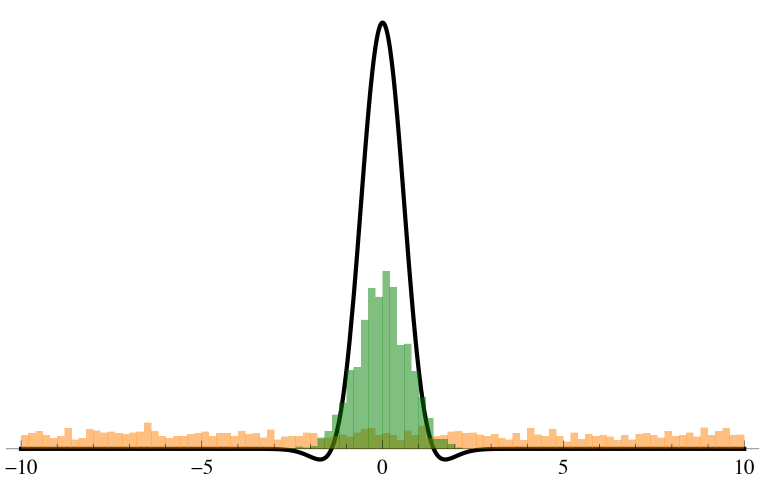
a 1-D integral example

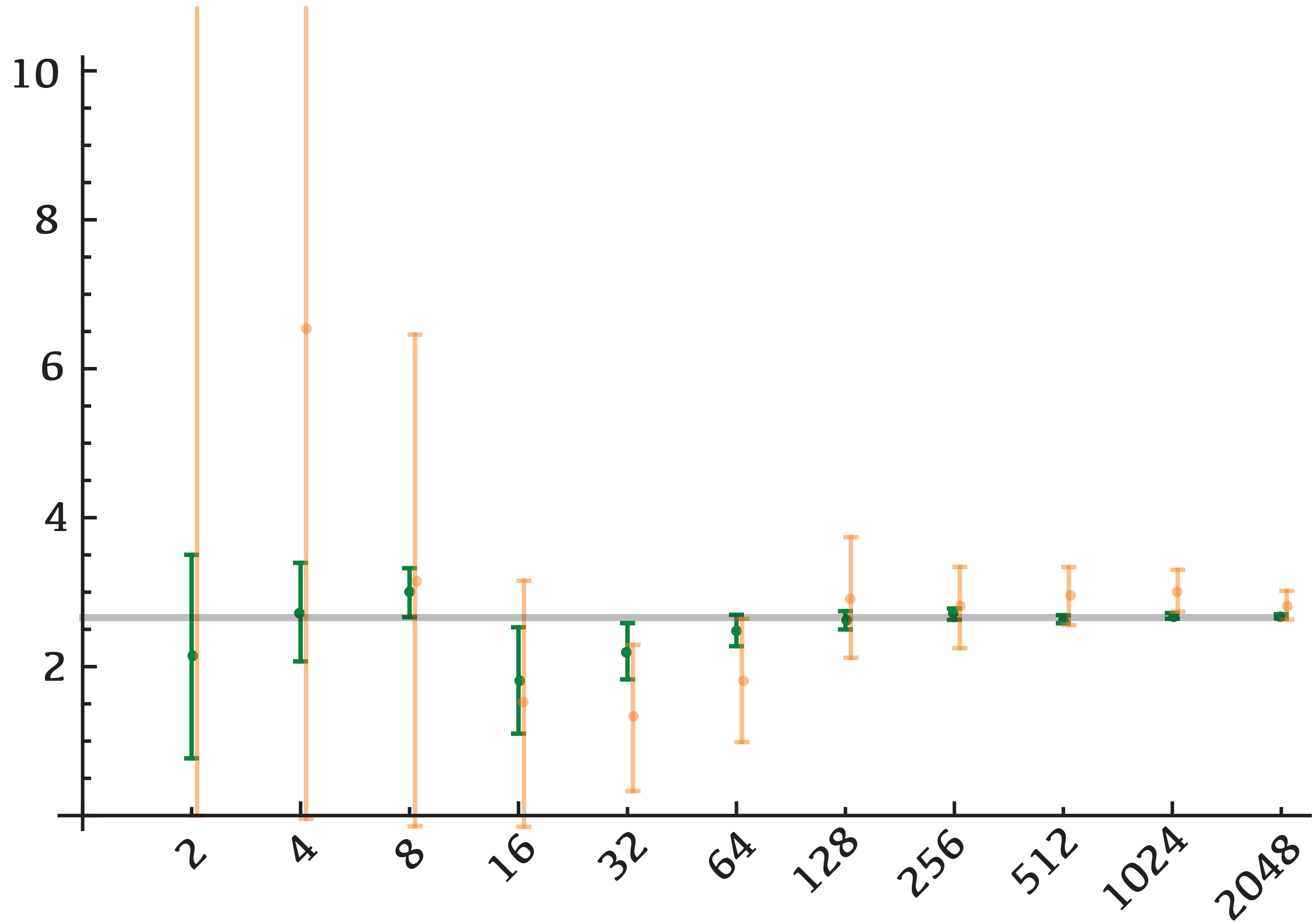
$$\int_{-10}^{+10} dx \ \ (2 - x^2) e^{-x^2}$$

$$\approx \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} (2 - x_{\mathsf{i}}^2)$$ x_{i} generated with $P(x) \propto e^{-x^2}$

$$\approx \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} (2 - x_{\mathsf{i}}^2) e^{-x_{\mathsf{i}}^2}$$ x_{i} generated with $P(x) \propto 1$







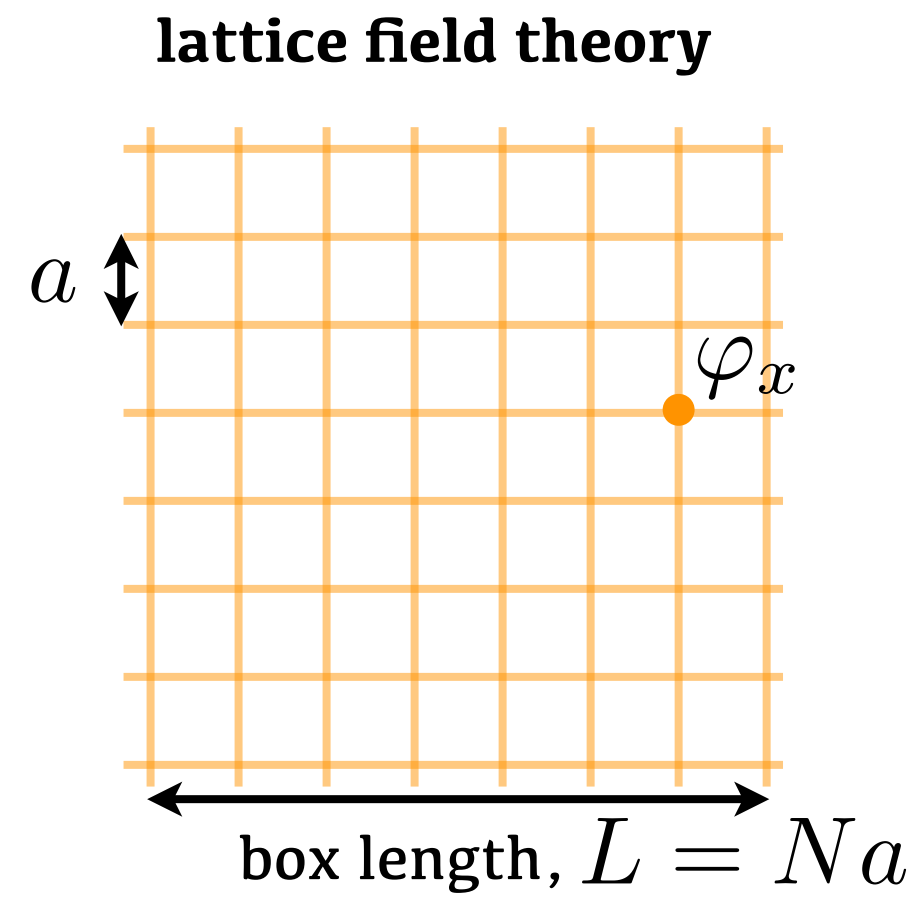
lattice field theory

observables will also depend upon the lattice spacing and the volume (in general)

ideally re-do the calculation for a number of lattice spacings and extrapolate $a \to 0$

c.f. take the energy cut-off to infinity

volume dependence will turn out to be quite interesting …



to the problem in hand . . . QCD

Quantum ChromoDynamics

QCD field content

quark fields $\psi^{i}_{f\,\alpha}(x)$

four component spinor operator (relativistic spin-1/2 field)
carries a color label (i=1,2,3)
independent fields for each quark flavour (up, down, strange ...)

gluon field $A^{ij}_{\mu}(x) = \sum_{a=1}^{8} A^{a}_{\mu}(x)\, t^{a}_{ij}$

vector field of 3×3 matrices
8 independent real numbers (adjoint representation)

t^{a}_{ij} generators of SU(3) color

QCD Lagrangian (Euclidean)

$$\tilde{\mathcal{L}} = -\tfrac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \sum_{f} \bar{\psi}_{f}(x) \left(\gamma_{\mu} D_{\mu} + m_{f}\right) \psi_{f}(x)$$

gauge covariant derivative $D_{\mu} = \partial_{\mu} - ig A_{\mu}(x)$

required if the theory is to be **locally** gauge invariant

$\bar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m\right)\psi$ free massive quark field

$-ig\, A^{a}_{\mu}\left(\bar{\psi}\gamma_{\mu} t^{a} \psi\right)$ gluon field couples to quark color vector current

Quantum ChromoDynamics

QCD field content

quark fields $\psi^{i}_{f\,\alpha}(x)$

gluon field $A^{ij}_{\mu}(x) = \sum_{a=1}^{8} A^{a}_{\mu}(x)\, t^{a}_{ij}$

QCD Lagrangian (Euclidean)

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gauge covariant derivative $D_{\mu} = \partial_{\mu} - igA_{\mu}(x)$

field-strength tensor
$$F_{\mu\nu} = \frac{i}{g}\big[D_{\mu}, D_{\nu}\big] = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig\big[A_{\mu}, A_{\nu}\big]$$

gluonic kinetic term &
gluon-gluon interactions

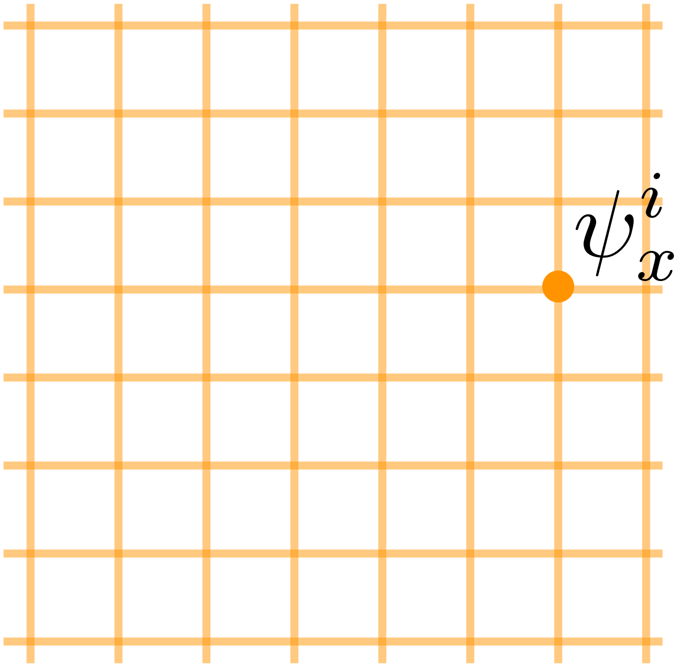
QCD on a lattice

QCD Lagrangian (Euclidean)

$$\tilde{\mathcal{L}} = -\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \sum_f \bar{\psi}_f(x) \left(\gamma_\mu D_\mu + m_f\right) \psi_f(x)$$

$$D_\mu = \partial_\mu - igA_\mu(x)$$

fermion fields take values on the sites of the lattice ψ_x^i



what shall we do with the gluon field?

QCD on a lattice

QCD Lagrangian (Euclidean)

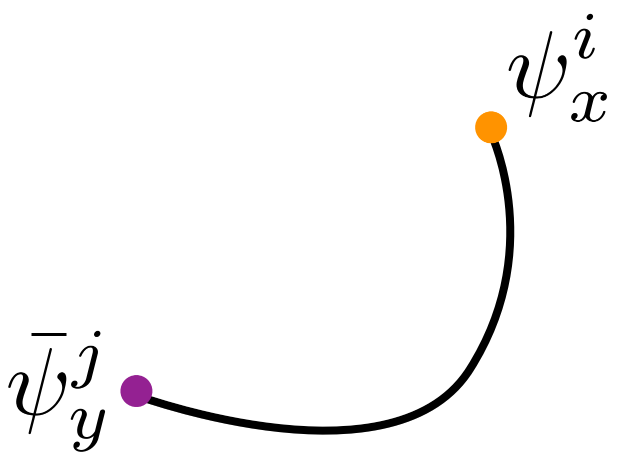
$$\tilde{\mathcal{L}} = -\tfrac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \sum_f \bar{\psi}_f(x) \left(\gamma_\mu D_\mu + m_f\right) \psi_f(x)$$

$D_\mu = \partial_\mu - igA_\mu(x)$

fermion fields take values on the sites of the lattice ψ^i_x

what shall we do with the gluon field?

in the continuum theory - consider a fermion anti-fermion pair separated by some distance



the combination $\bar{\psi}^j_y \, \delta_{ji} \, \psi^i_x$ is not gauge-invariant

(can make **different** local gauge transformations at x and y)

a gauge-invariant combination is $\bar{\psi}^j_y \boxed{\left[e^{ig \int_x^y dz_\mu \cdot A_\mu(z)} \right]_{ji}} \psi^i_x$

"Wilson line" transports the color

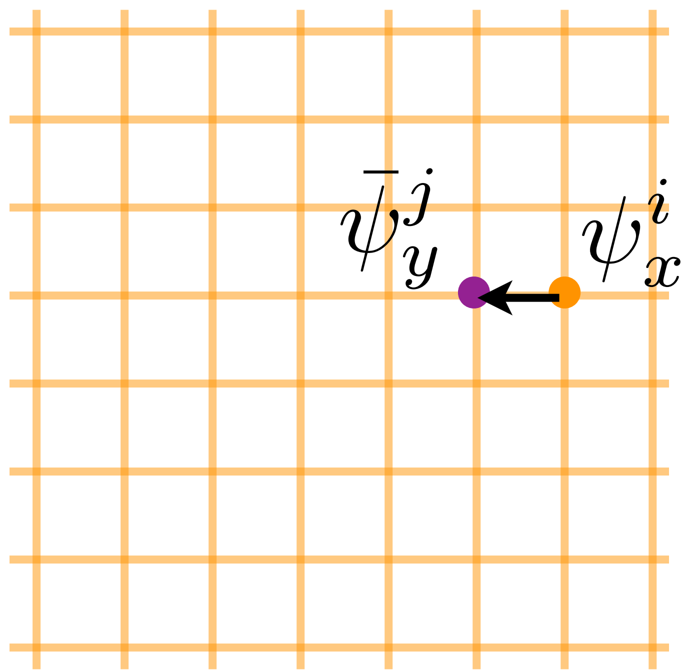
QCD on a lattice

QCD Lagrangian (Euclidean)

$$\tilde{\mathcal{L}} = -\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \sum_f \bar{\psi}_f(x) \left(\gamma_\mu D_\mu + m_f\right) \psi_f(x)$$

$$D_\mu = \partial_\mu - igA_\mu(x)$$

fermion fields take values on the sites of the lattice ψ_x^i



a gauge-invariant fermion bilinear is

$$\bar{\psi}_y^j \left[e^{ig\int_x^y dz_\mu \cdot A_\mu(z)} \right]_{ji} \psi_x^i$$

the shortest possible path on the lattice is between neighbouring sites

$$\bar{\psi}_{x+a\hat{\mu}}^j \left[e^{igaA_{x\mu}} \right]_{ji} \psi_x^i$$

SU(3) matrix

$$\bar{\psi}_{x+a\hat{\mu}} U_{x\mu} \psi_x$$

an SU(3) matrix for the ‘link’

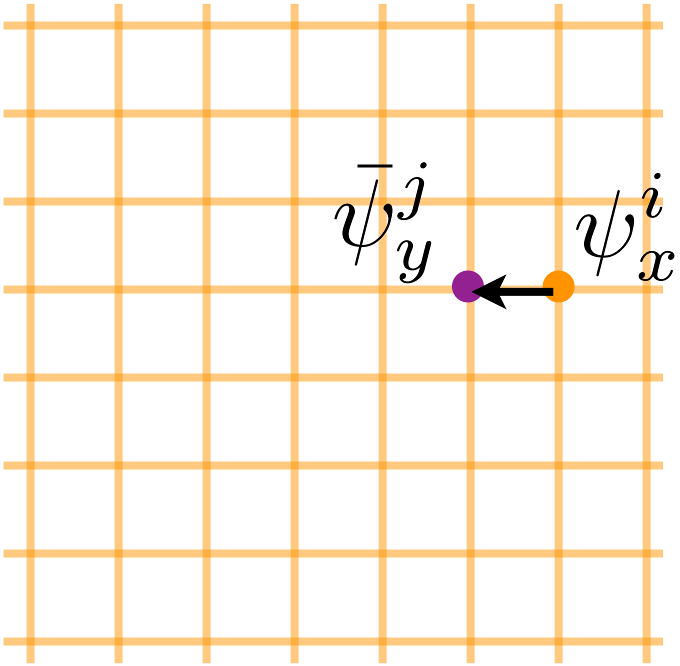
QCD on a lattice

QCD Lagrangian (Euclidean)

$$\tilde{\mathcal{L}} = -\tfrac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x) + \sum_f \bar{\psi}_f(x)\left(\gamma_\mu D_\mu + m_f\right)\psi_f(x)$$

fermion fields take values on the sites of the lattice ψ_x^i

gluon fields represented by SU(3) matrices on the links of the lattice $[U_{x\mu}]^{ij}$



$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_g\big(U_{x\mu}\big) + \tilde{\mathcal{L}}_q\big(\bar{\psi}_x, \psi_x, U_{x\mu}\big)$$

in principle can choose any discretisation
as long as it becomes correct as $a \to 0$

$$\tfrac{1}{2a}\left(\bar{\psi}_x\,\gamma_\mu U^\dagger_{x,\mu}\,\psi_{x+\mu a} - \bar{\psi}_x\,\gamma_\mu U_{x-\mu a,\mu}\,\psi_{x-\mu a}\right)$$
$$\xrightarrow[a\to 0]{} \bar{\psi}(x)\gamma_\mu D_\mu \psi(x)$$

QCD on a lattice

how you choose to discretise the fermions gives rise to much of the jargon you'll hear

- ➟ clover (or improved Wilson) quarks
- ➟ staggered quarks
- ➟ overlap, domain-wall … quarks

each have their own advantages and disadvantages
- won't go into them here

generic form of the quark action : $S_q = \sum_{xy} \bar{\psi}_y \, Q[U]_{yx} \, \psi_x$

$$S_q = \sum \bar{\psi}_{\alpha i y} \, Q[U]_{\alpha i y; \beta j x} \, \psi_{\beta j x}$$

e.g. "naive" fermions :

$$Q[U]_{yx} = \gamma_\mu U^\dagger_{y\mu} \delta_{y,x-\mu a} - \gamma_\mu U_{x\mu} \delta_{y,x+\mu a} + (ma)\delta_{y,x}$$

a sparse matrix

fermions in path-integrals

(have been a bit sloppy here by not putting hats on field **operators**)

for scalar fields, each (commuting) field operator $\hat{\varphi}$ $[\hat{\varphi}(x), \hat{\varphi}(y)] = 0 \quad x \neq y$

was replaced in the path-integral by an ordinary commuting number φ

$$\text{e.g. } \langle 0|\hat{\varphi}(x'')\hat{\varphi}(x')|0\rangle = \int \mathcal{D}\varphi(x)\; \varphi(x'')\varphi(x')\; e^{-iS[\varphi(x)]}$$

for fermion fields this doesn't work - fermion fields **anticommute** $\{\hat{\psi}(x), \hat{\psi}(y)\} = 0$
$x \neq y$

they are replaced in the path-integral by **anticommuting** numbers

(Grassmann numbers)

ordinary numbers: $AB = BA$

Grassmann numbers: $\theta_A \theta_B = -\theta_B \theta_A$

QCD on a lattice

generic form of the quark action : $S_q = \sum_{xy} \bar{\psi}_x \, Q[U]_{xy} \, \psi_y$

we want a numerical approach - Grassmann numbers not ideal

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \; e^{-S_g[U]} e^{-S_q[\psi, \bar{\psi}, U]}$$

the simple bilinear form of the quark action is such that we can perform the fermion integral **exactly**

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\bar{\psi} Q \psi\right) = \det Q$$

QCD on a lattice

which leaves $Z = \int \mathcal{D}U \, \boxed{\det Q[U] \, e^{-S_g[U]}}$

treat this like a probability ?

importance sampled Monte Carlo ?

developing algorithms to efficiently generate gauge fields is an industry within lattice field theory

computing the determinant is slow - in ‘the olden days’ it was often ignored - the ‘quenched approximation’ - sometimes very bad !

computation of the determinant gets slower for smaller quark masses - many calculations use artificially heavy quark masses

stage 1: generate an ensemble of gauge-field configurations

$$\left\{ U_{x\mu} \right\}^{i=1\ldots N}$$

a bit more Grassmann integration

correlation functions will typically contain fermion fields at various positions (and spins, and colours)

we can still do the integration over fermion fields exactly :

e.g. $$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ \psi_y \bar{\psi}_x\ \exp\left(-\bar{\psi} Q \psi\right) = Q^{-1}_{yx} \det Q$$

the matrix inverse
of the 'Dirac operator'

in fact, with many fermion fields we can recover a familiar result :

e.g. $$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ \psi_y \bar{\psi}_x \psi_z \bar{\psi}_w\ \exp\left(-\bar{\psi} Q \psi\right)$$

$$= \left(Q^{-1}_{yx} Q^{-1}_{zw} - Q^{-1}_{yw} Q^{-1}_{zx}\right)\ \det Q$$

c.f. Wick's theorem thinking of

Q^{-1} as the 'propagator'

a bit more Grassmann integration

e.g. $\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ \psi_y \bar{\psi}_x \psi_z \bar{\psi}_w\ \exp\left(-\bar{\psi} Q \psi\right)$

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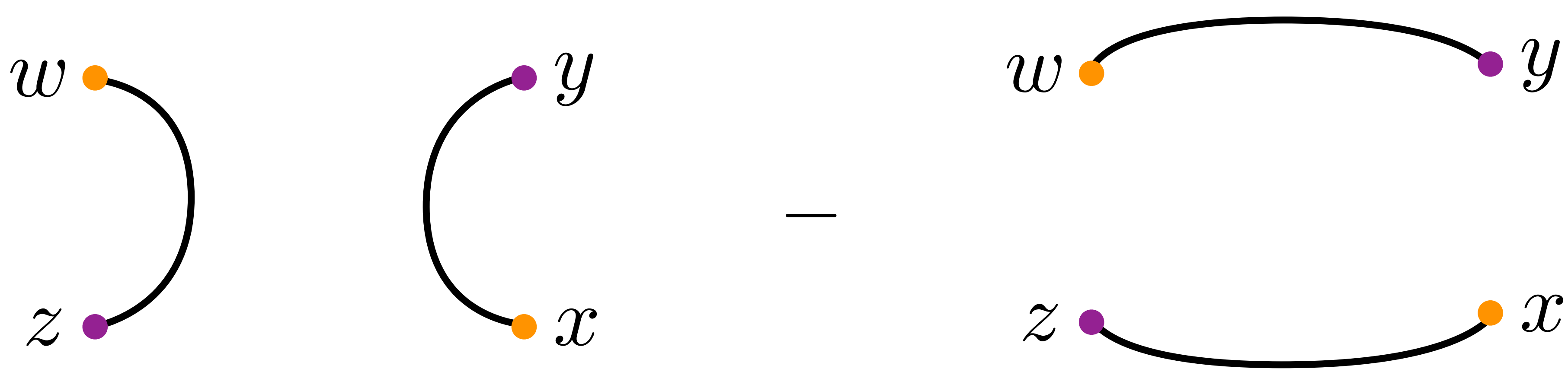


a simple lattice QCD correlation function

suppose we want to compute the mass of the **pion** in QCD

a suitable correlator might be

$$\langle 0 \Big| \sum_{\vec{x}} \bar{\psi}^{\bar{u}}_{\vec{x},t'} \gamma_5 \psi^{d}_{\vec{x},t'} \cdot \sum_{\vec{y}} \bar{\psi}^{\bar{d}}_{\vec{y},t} \gamma_5 \psi^{u}_{\vec{y},t} \Big| 0 \rangle$$

projects into zero momentum

has pseudoscalar quantum numbers

we can estimate this correlation function using a pre-computed ensemble of gauge fields

$$\{U_{x\mu}\}^{\mathsf{i}=1\ldots\mathsf{N}}$$

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \; \bar{\psi}_{\vec{x},t'} \gamma_5 \psi_{\vec{x},t'} \cdot \bar{\psi}_{\vec{y},t} \gamma_5 \psi_{\vec{y},t} \, e^{-S_q[\psi,\bar{\psi},U]} e^{-S_g[U]}$$

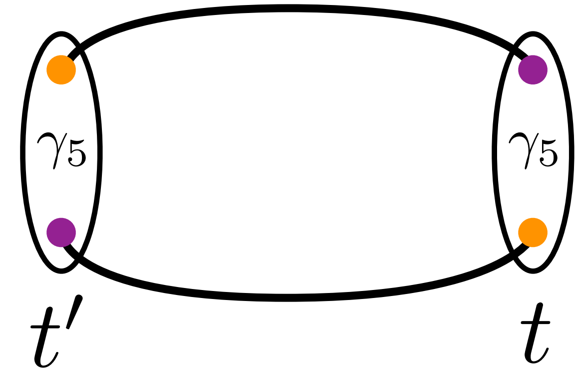
perform the fermion path integral

$$\int \mathcal{D}U \; \mathrm{tr}\left[\gamma_5 Q[U]^{-1}_{\vec{x}t',\vec{y}t} \gamma_5 Q[U]^{-1}_{\vec{y}t,\vec{x}t'} \right] \; \det Q[U] \, e^{-S_g[U]}$$

$$\approx \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} \mathrm{tr}\left[\gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{x}t',\vec{y}t} \gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{y}t,\vec{x}t'} \right]$$

a simple lattice QCD correlation function

$$\approx \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} \mathrm{tr}\Big[\gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{x}t',\vec{y}t}\gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{y}t,\vec{x}t'}\Big]$$



$$\sum \gamma_5^{\alpha\beta} \left(Q[U^{(\mathsf{i})}]^{-1}\right)_{\beta i \vec{x}t';\gamma j \vec{y}t} \gamma_5^{\gamma\delta} \left(Q[U^{(\mathsf{i})}]^{-1}\right)_{\delta j \vec{y}t;\alpha i \vec{x}t'}$$

spins (αβγδ)
colours (ij)
space (**x**,**y**)

a simple lattice QCD correlation function

$$\approx \frac{1}{\mathsf{N}} \sum_{\mathsf{i}=1}^{\mathsf{N}} \mathrm{tr}\left[\gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{x}t',\vec{y}t} \gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{y}t,\vec{x}t'}\right]$$

numerical task:

for each gauge-field configuration in the ensemble $\left\{U_{x\mu}\right\}^{\mathsf{i}=1\ldots\mathsf{N}}$

compute the inverse of the Dirac matrix $\left(Q[U^{(\mathsf{i})}]^{-1}\right)_{\alpha i\vec{x}t';\beta j\vec{y}t}$

multiply some matrices and trace $\mathrm{tr}\left[\gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{x}t',\vec{y}t} \gamma_5 Q[U^{(\mathsf{i})}]^{-1}_{\vec{y}t,\vec{x}t'}\right]$

impractical as written:

typical lattice size : $24^3 \times 128 = 1.8 \times 10^6$ sites
& 4 Dirac spins & 3 colours
⇒ matrix of size $(2\times10^7)\times(2\times10^7)$

⇒ storage space alone = 6.4 PetaBytes !

6 months of LHC data !

'all-all' propagators are impractical

point-all propagators

a minor tweak in the correlator to make it practical

$$\cancel{\langle 0 | \sum_{\vec{x}} \bar{\psi}_{\vec{x},t'} \gamma_5 \psi_{\vec{x},t'} \cdot \sum_{\vec{y}} \bar{\psi}_{\vec{y},t} \gamma_5 \psi_{\vec{y},t} | 0 \rangle}$$

$$\langle 0 | \sum_{\vec{x}} \bar{\psi}_{\vec{x},t'} \gamma_5 \psi_{\vec{x},t'} \cdot \bar{\psi}_{\vec{0},0} \gamma_5 \psi_{\vec{0},0} | 0 \rangle$$

only the 'sink' operator is explicitly projected into definite (zero) momentum - the 'source' operator is a linear superposition of all momenta, but momentum conservation projects out the zero piece

$$\mathrm{tr}\Big[\gamma_5 Q[U^{(\mathrm{i})}]^{-1}_{\vec{x}t',\vec{0}0} \gamma_5 Q[U^{(\mathrm{i})}]^{-1}_{\vec{0}0,\vec{x}t'} \Big]$$

compute the inverse of the Dirac matrix $\big(Q[U^{(\mathrm{i})}]^{-1}\big)_{\alpha i \vec{x} t'; \beta j \vec{0} 0}$

'point-all' propagators can be computed

matrix of size $(2\times 10^7)\times(12)$

all sites, colours, spins

all colours, spins on **one site**

propagators

compute the inverse of the Dirac matrix $Q[U^{(\mathsf{i})}]^{-1}_{\vec{x}t',\vec{0}0}$

'point-all' propagators can be computed

matrix of size $(2\times10^7)\times(12)$

all sites, colours, spins

all colours, spins on **one site**

stage 2: compute appropriate quark propagators
on each configuration

$$Q[U^{(\mathsf{i})}]^{-1}$$

(there are actually smarter ways to do this than using 'point-all' propagators ... perhaps later ...)

a correlator

$$C(t) = \big\langle 0 \big| \sum_{\vec{x}} \bar{\psi}_{\vec{x},t}\gamma_5\psi_{\vec{x},t} \;\cdot\; \bar{\psi}_{\vec{0},0}\gamma_5\psi_{\vec{0},0} \big| 0 \big\rangle$$

so we actually obtain an ensemble $C^{(\mathrm{i})}(t)$

one entry for each gauge-field configuration $\{U_{x\mu}\}^{\mathrm{i}=1\ldots\mathrm{N}}$

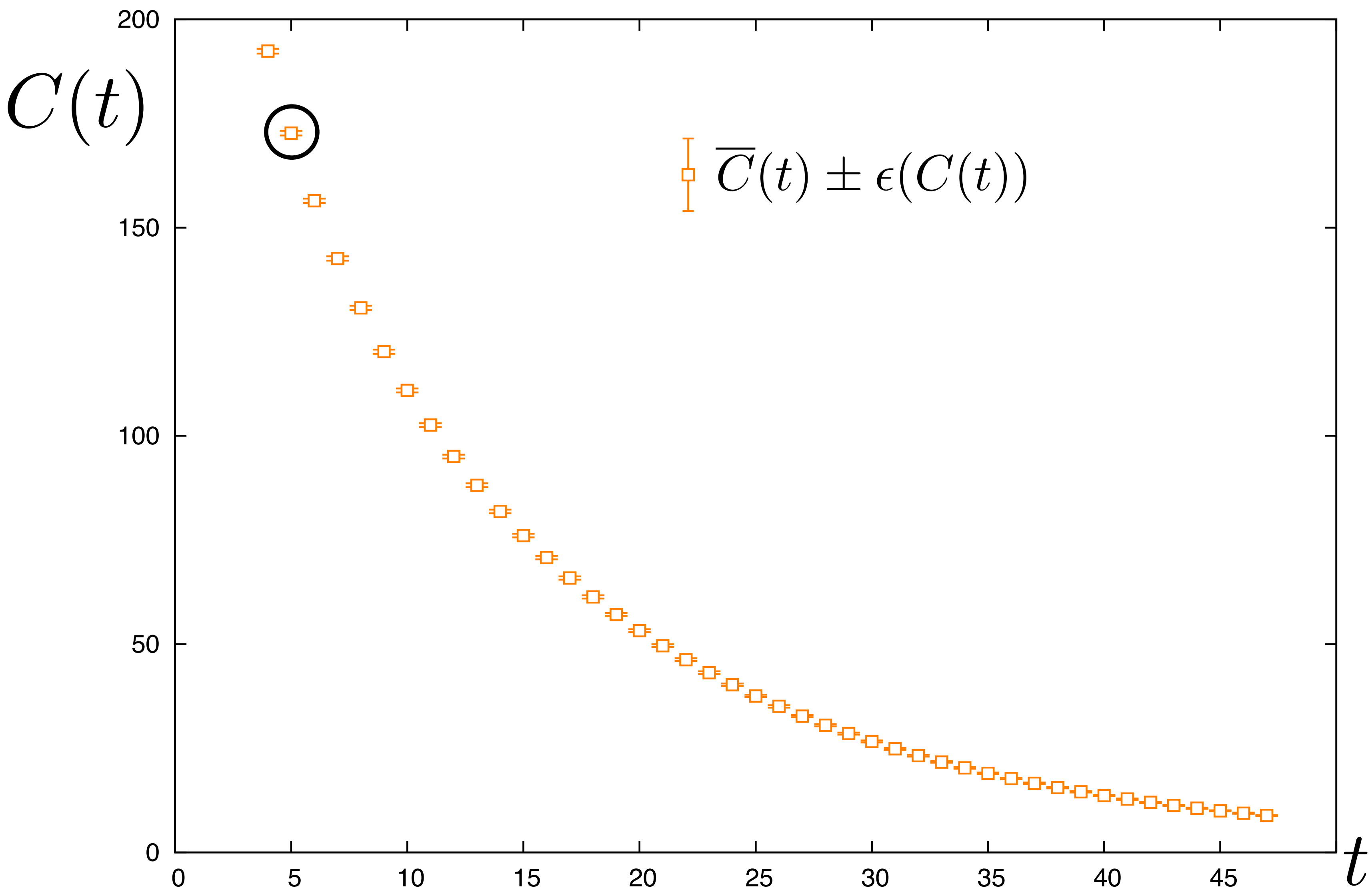
stage 3: 'contract' propagators into correlation functions evaluated on each configuration $C^{(\mathrm{i})}(t)$

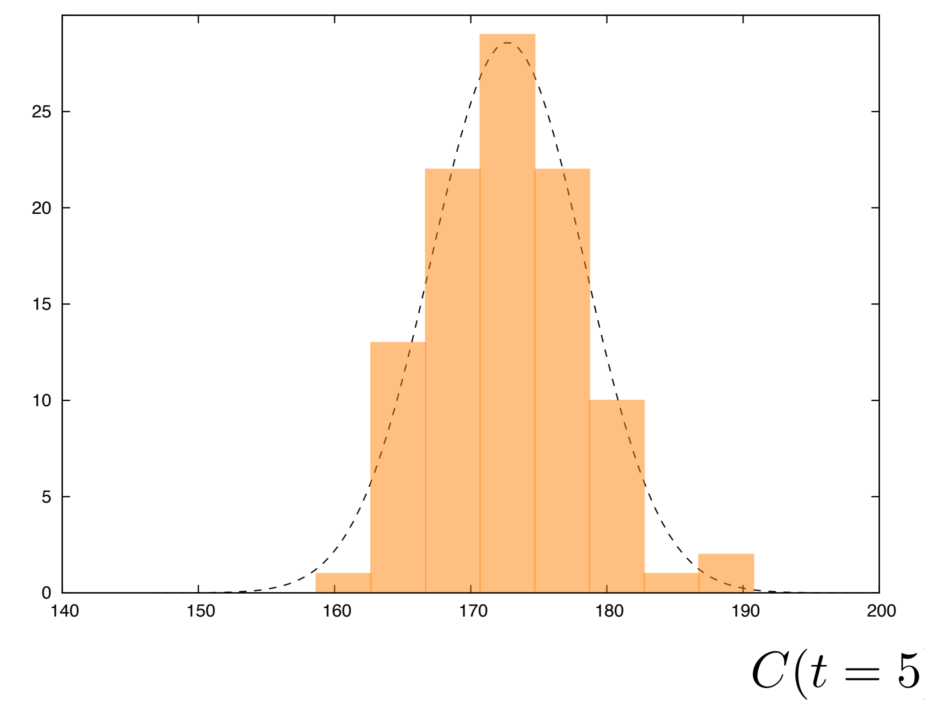
a correlator

$$C(t) = \langle 0 | \sum_{\vec{x}} \bar{\psi}_{\vec{x},t} \gamma_5 \psi_{\vec{x},t} \cdot \bar{\psi}_{\vec{0},0} \gamma_5 \psi_{\vec{0},0} | 0 \rangle$$

so we actually obtain an ensemble $C^{(\mathrm{i})}(t)$

one entry for each gauge-field configuration $\{U_{x\mu}\}^{\mathrm{i}=1\ldots\mathrm{N}}$





… how do we relate this information to the mass of the pion ?

two-point correlators & the spectrum

$$C(t) = \big\langle 0 \big| \bar{\psi}\gamma_5\psi(t) \cdot \bar{\psi}\gamma_5\psi(0) \big| 0 \big\rangle$$

insert a complete set of eigenstates of QCD $\quad H_{\mathrm{QCD}}\big|\mathfrak{n}\big\rangle = E_{\mathfrak{n}}\big|\mathfrak{n}\big\rangle$

$$1 = \sum_{\mathfrak{n}} \big|\mathfrak{n}\big\rangle\big\langle\mathfrak{n}\big|$$

$$C(t) = \sum_{\mathfrak{n}} \big\langle 0 \big| \bar{\psi}\gamma_5\psi(t) \big|\mathfrak{n}\big\rangle\big\langle\mathfrak{n}\big| \bar{\psi}\gamma_5\psi(0) \big| 0 \big\rangle$$

Euclidean time evolution

$$= \sum_{\mathfrak{n}} \big\langle 0 \big| e^{Ht} \bar{\psi}\gamma_5\psi(0) e^{-Ht} \big|\mathfrak{n}\big\rangle\big\langle\mathfrak{n}\big| \bar{\psi}\gamma_5\psi(0) \big| 0 \big\rangle$$

$$= \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}}t} \big\langle 0 \big| \bar{\psi}\gamma_5\psi(0) \big|\mathfrak{n}\big\rangle\big\langle\mathfrak{n}\big| \bar{\psi}\gamma_5\psi(0) \big| 0 \big\rangle$$

describes how effectively this operator 'interpolates' state $\big|\mathfrak{n}\big\rangle$ from the vacuum

$$= \sum_{\mathfrak{n}} A_{\mathfrak{n}}\, e^{-E_{\mathfrak{n}}t}$$

a weighted sum of exponentials

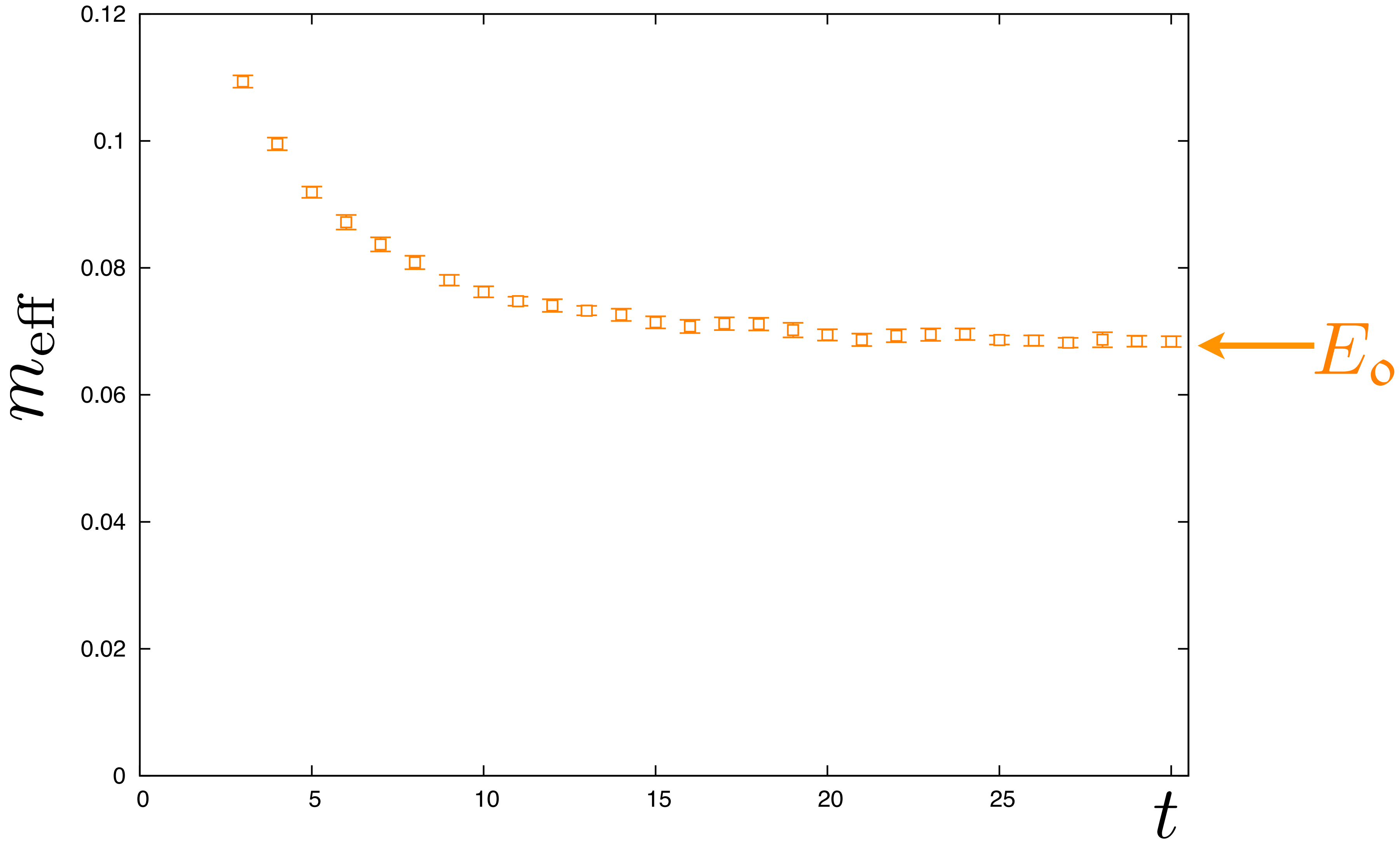
two-point correlators & the spectrum

$$C(t) = \sum_{\mathfrak{n}} A_{\mathfrak{n}}\, e^{-E_{\mathfrak{n}} t}$$

at large times $C(t \to \infty) \to A_{\mathfrak{o}}\, e^{-E_{\mathfrak{o}} t}$

only the ground state survives

can be seen in an 'effective mass plot' $m_{\text{eff}} = \frac{1}{\delta t} \log\left[\frac{C(t)}{C(t+\delta t)}\right] \sim -\frac{d}{dt}\log C(t)$



two-point correlators & the spectrum

stage 4: analyze the correlation functions in terms of
their particle content

$$C(t) = \sum_{\mathfrak{n}} A_{\mathfrak{n}}\, e^{-E_{\mathfrak{n}} t}$$

the lattice scale

in what units are we measuring the mass ?

lattice calculations usually cast everything in units of the lattice spacing

e.g. $e^{-mt} = e^{-(ma)(t/a)}$

dimensionless mass   integer timeslices

the lattice spacing is determined by computing some physical quantity & comparing to experiment

(general requirement for parameters in renormalised field theories)