Transverse Momentum of partons

Mariaelena Boglione

INFN Torino

UNIVERSITÀ DEGLI STUDI DI TORINO
ALMA UNIVERSITAS TAURINENSIS

Lecture 5

We will be talking about ...

- Deep Inelastic scattering
- Parton model
- Transverse momentum of partons

Lecture 1

- 3D kinematics (i.e. kinematics with partonic transverse momentum)
- Integrated distribution and fragmentation functions (transversity and all that ...)

Lecture 2

- **T**ransverse **M**omentum **D**ependent distribution and fragmentation functions
- Unpolarized SIDIS cross section
- Phenomenology of TMD's: extracting the Sivers distribution function from SIDIS experimental data.

Lecture 3

- Simultaneous extraction of transversity and Collins functions from SIDIS and e^+e^- scattering experimental data.

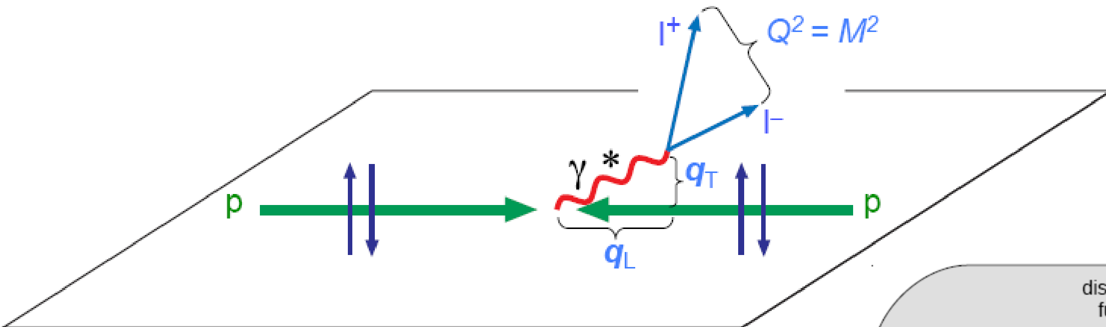
Lecture 4

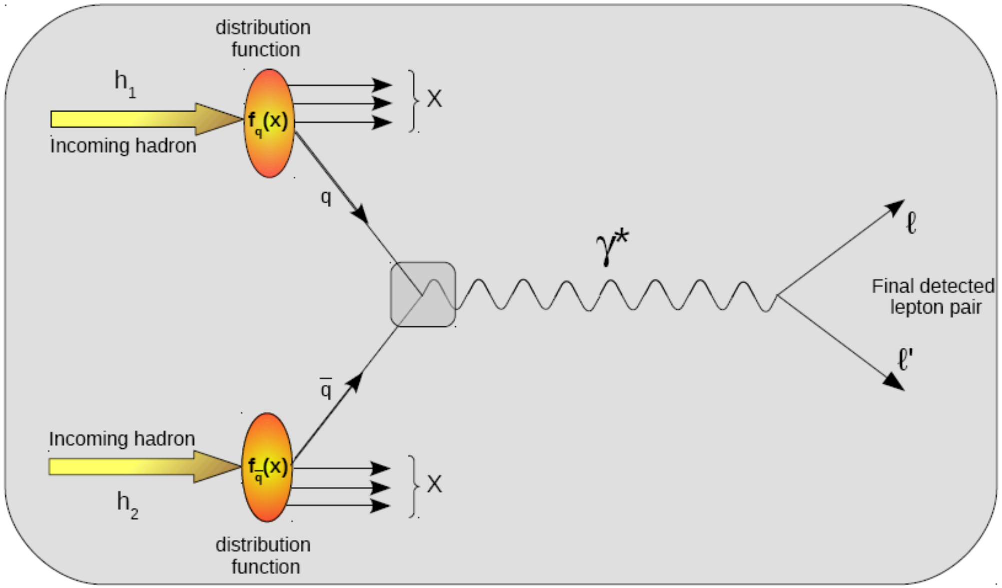
- TMDs in Drell-Yan processes
- DGLAP evolution equations and TMD evolution scheme

Lecture 5

TMD distribution functions
In Drell-Yan processes

TMD's in Drell-Yan processes





Factorization holds
Two scales: M2 and q T ,
with q T << M2

Very low q T are generated by parton intrinsic transverse momenta

The cross section is given by the convolution of two distribution functions (no fragmentation functions)

$$\sigma_{Drell-Yan} = f_q(x, k_\perp) \otimes f_{\bar{q}}(x, k_\perp) \otimes \hat{\sigma}^{q\bar{q} \to \ell\ell'}$$

Universality:
SIDIS versus Drell-Yan TMD's

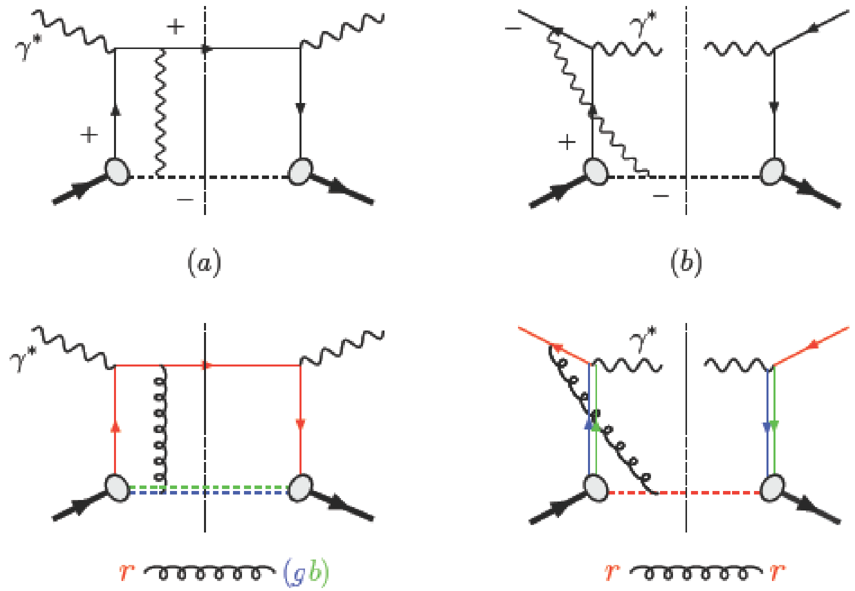
Crucial role of gauge-links in TMDs

Brodsky, Hwang, Schmidt;
Collins; Belitsky, Ji, Yuan;
Boer, Mulders, Pijlman

process-dependence of Sivers functions

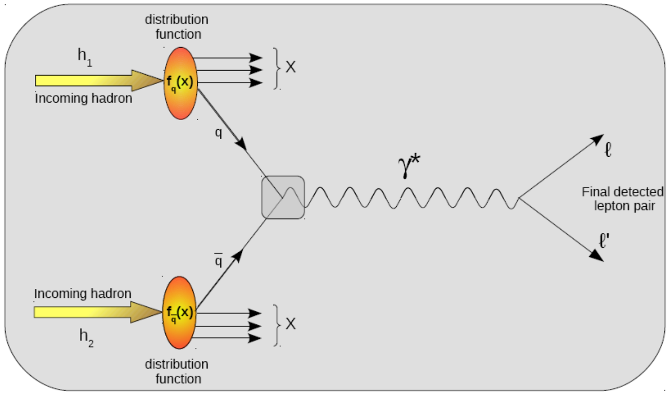
DIS:
"attractive"

D-Y:
"repulsive"



$$[f_{1T}^{q\perp}]_{\text{SIDIS}} = -[f_{1T}^{q\perp}]_{\text{DY}}$$

TMD's in Drell-Yan processes

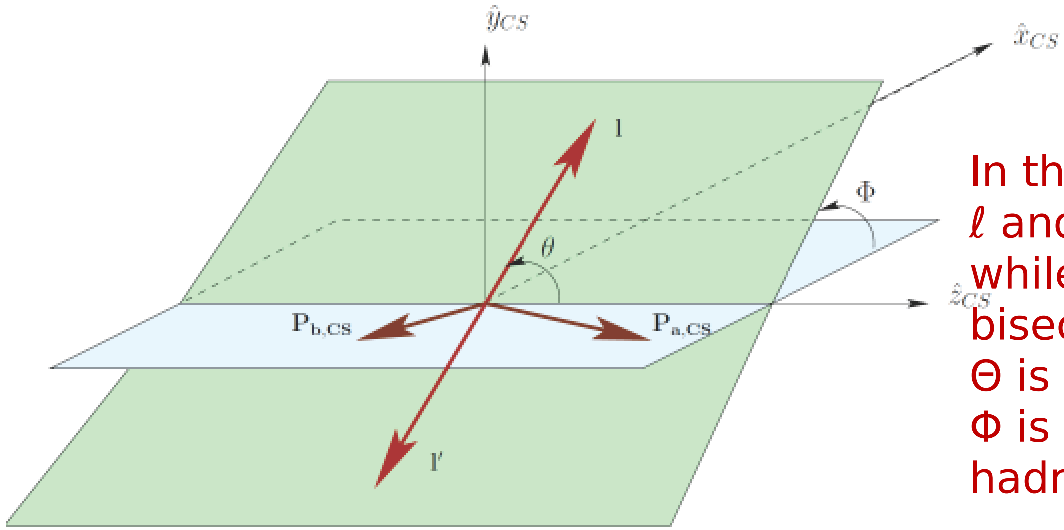


$$\sigma_{Drell-Yan} = f_q(x, k_\perp) \otimes f_{\bar{q}}(x, k_\perp) \otimes \hat{\sigma}^{q\bar{q} \to \ell\ell'}$$

proton-proton (nucleon-nucleon) Drell-Yan	the total cross section, σ, is suppressed by the antiquark parton distribution function (antiquarks from proton sea).	RHIC FERMILAB
proton-antiproton (nucleon-nucleon) Drell-Yan	the total cross section, σ, is NOT suppressed by parton distribution functions (antiquarks are valence partons in the antiproton)	PANDA PAX J-PARC
proton-pion (nucleon-pion) Drell-Yan	the total cross section, σ, is NOT suppressed by parton distribution functions (antiquarks are valence partons in pions).	COMPASS

Unpolarized Drell-Yan processes

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{4\pi}\frac{1}{\lambda+3}\left(1+\lambda\cos^2\theta+\mu\sin 2\theta\cos\phi+\frac{\nu}{2}\sin^2\theta\cos 2\phi\right)$$



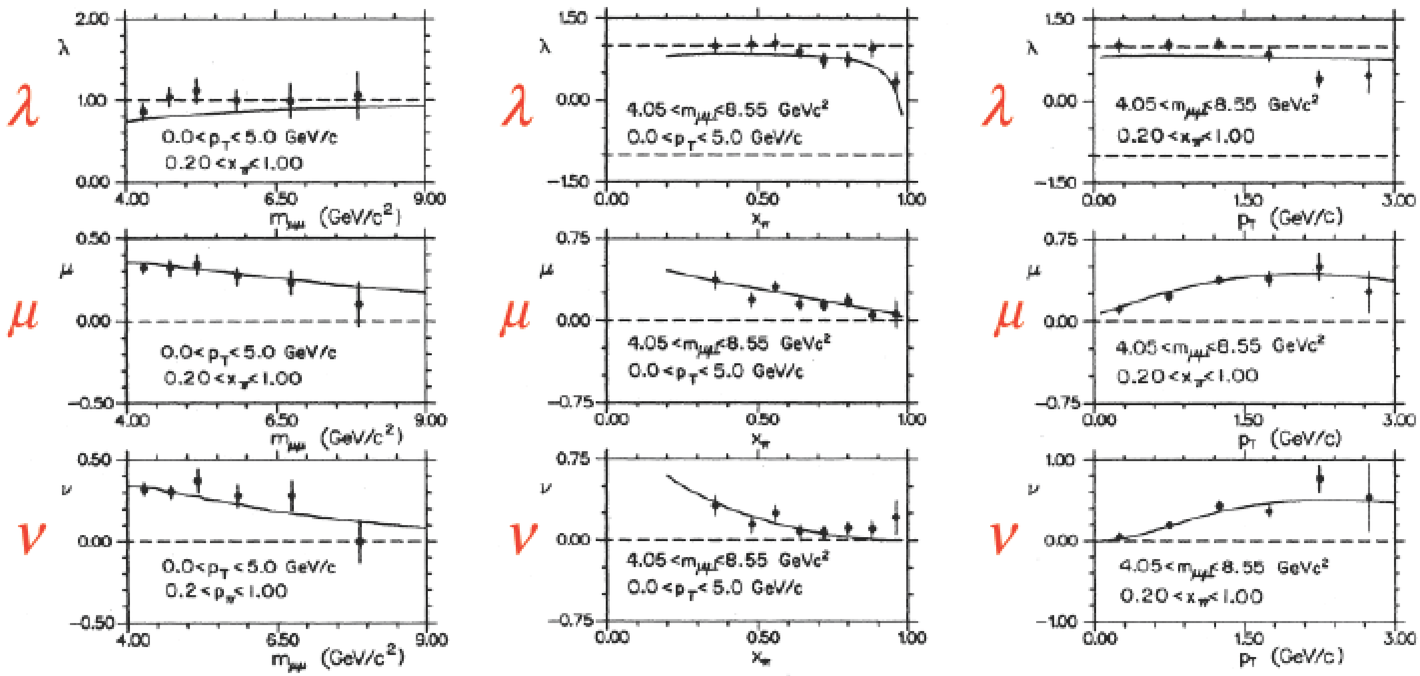
In the Collins-Soper frame
ℓ and ℓ’ are back to back,
while the ZCS axis is the
bisector of **P**a and –**P**b
Θ is the polar angle of ℓ ,
Φ is the angle between the
hadron and the lepton plane.

The unpolarized cross section is already very interesting:
the naïve parton model predicts λ=1, μ=0, ν=0, but …

Unpolarized Drell-Yan processes

Decay angular distributions in pion-induced Drell-Yan

E615 Data 252 GeV $\pi^- + W$ Phys. Rev. D 39 (1989) 92



$m_{\mu\mu}(\mathrm{GeV/c^2})$ x_π $p_T(\mathrm{GeV/c})$

$$\lambda \neq 1 \quad \mu, \nu \neq 0 \quad 1 - \lambda - 2\nu \neq 0$$

TMD's in unpolarized Drell-Yan processes

$$\sigma_{Drell-Yan} = f_q(x,k_\perp) \otimes f_{\bar{q}}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \left\{ \begin{array}{l} \sigma = f_1^q(x) \otimes f_1^{\bar{q}}(x) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \\ \\ \sigma = h_1^{\perp q}(x,k_\perp) \otimes h_1^{\perp\bar{q}}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \end{array} \right.$$

In **unpolarized** Drell-Yan processes one can study

- the **unpolarized** parton distribution function
- the **Boer-Mulders** distribution function

RECENT WORK :

Z. Lu, I. Schmidt, Phys. Rev. D81, 043023 (2010)

V. Barone, S. Melis, A. Prokudin, arXiv:1009.3423

Boer-Mulders distribution function from unpolarized Drell-Yan processes

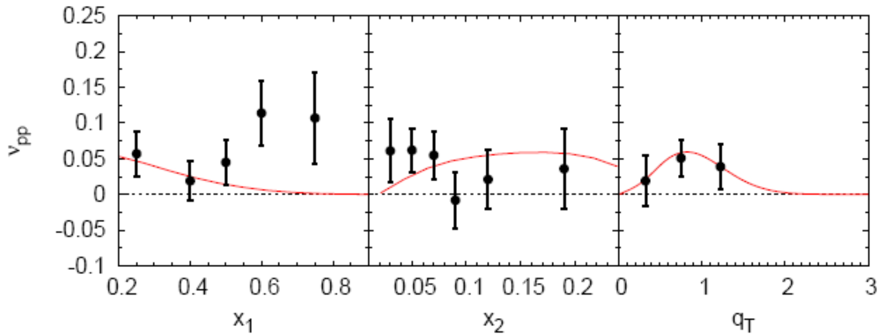
V. Barone, S. Melis, A. Prokudin, arXiv:1009.3423

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{4\pi(\lambda+3)}\left[1+\lambda\cos^2\theta+\mu\sin 2\theta\cos\phi+(\nu/2)\,\sin^2\theta\cos 2\phi\right]$$

$$\boxed{\nu \propto \frac{h_1^{\perp a}\otimes h_1^{\perp b}}{f_1^a\otimes f_1^b}}$$

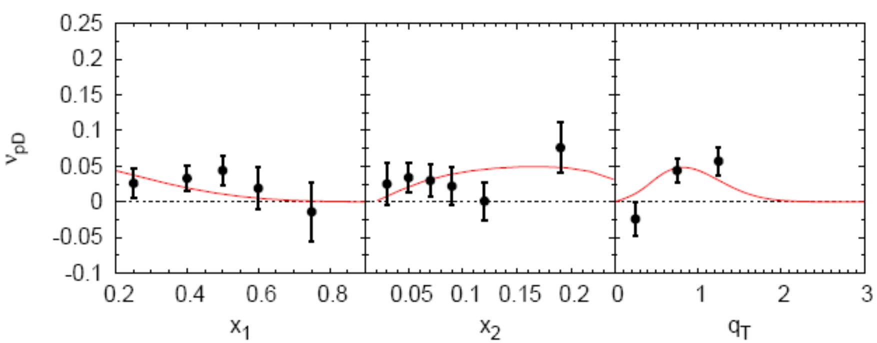
Unpolarized Drell Yan processes probe the antiquark Boer-Mulders function, which is not accessible in SIDIS.

Boer-Mulders distribution function from unpolarized Drell-Yan processes

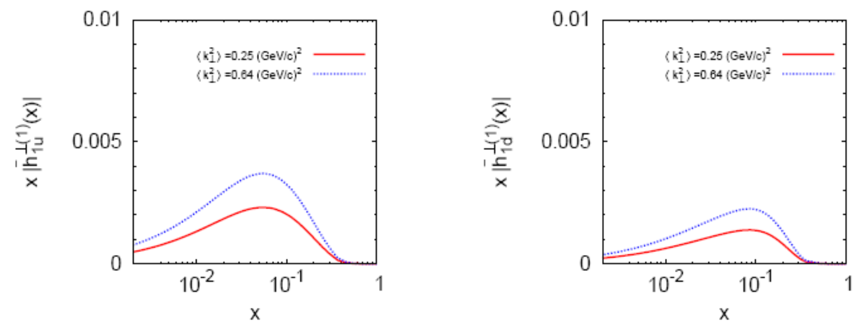


FERMILAB E866/NuSea data on pp and pD

Drell-Yan cannot distinguish between fits



BOER-MULDERS FUNCTIONS



V. Barone, S. Melis, A. Prokudin, arXiv:1009.3423

TMD's in single polarized Drell-Yan processes

$$\sigma_{Drell-Yan} = f_q(x) \otimes f_{\bar{q}} \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \quad \begin{cases} \sigma = f_{1T}^{\perp q}(x,k_\perp) \otimes f_1^{\bar{q}}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \\ \sigma = h_1^q(x,k_\perp) \otimes h_1^{\perp\bar{q}}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \end{cases}$$

In single polarized Drell-Yan processes one can study:

❖ **Sivers** parton distribution function

RECENT WORK:
M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, A. Prokudin,
Phys. Rev. D79 (2009) 054010

❖ **Boer-Mulders** distribution function

<u>*Predictions could be obtained by using transversity*</u>
<u>*as extracted from SIDIS and e+e-. I should work on that !*</u>

❖ **Transversity** and **Boer-Mulders** distribution functions

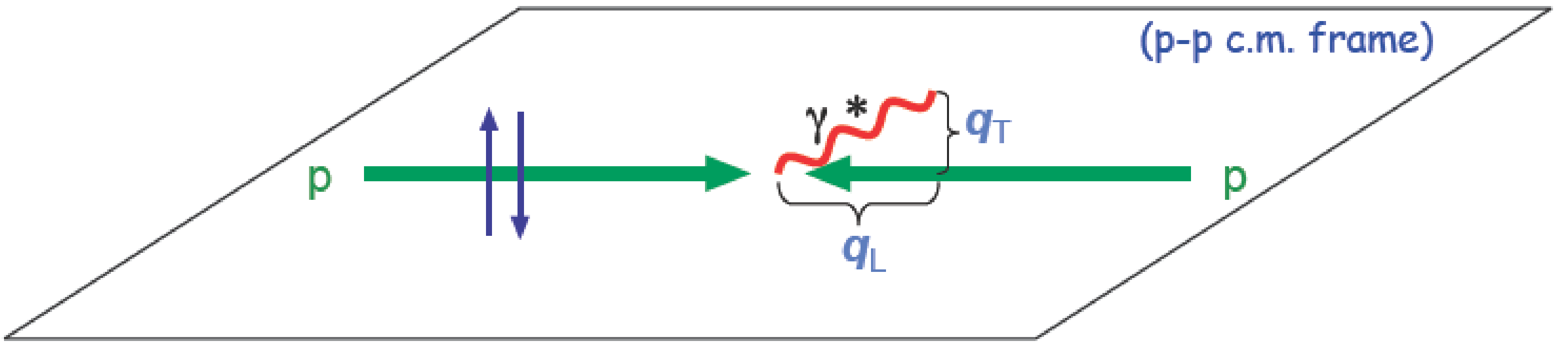
PRESENT and FUTURE EXPERIMENTS:
*RHIC, **COMPASS**, J-PARC, **PANDA**, **PAX***

Sivers distribution function in single polarized Drell-Yan processes

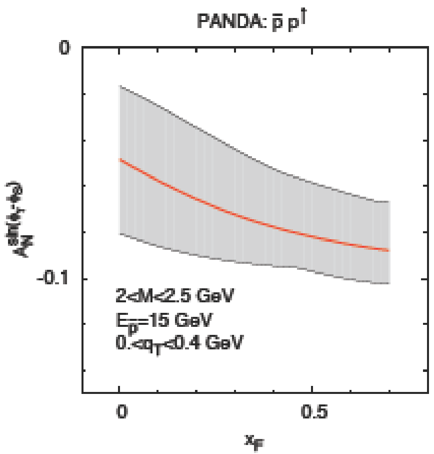
$$\mathrm{d}\sigma^{\uparrow} - \mathrm{d}\sigma^{\downarrow} \propto \sum_q \Delta^N f_{q/p^{\uparrow}}(x_1, \boldsymbol{k}_{\perp}) \otimes f_{\bar{q}/p}(x_2) \otimes \mathrm{d}\hat{\sigma}$$

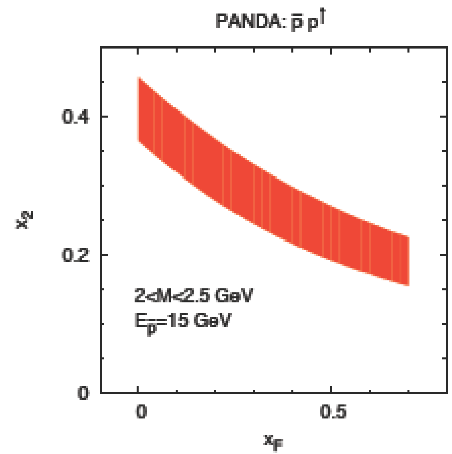
$q = u, \bar{u}, d, \bar{d}, s, \bar{s}$

$$A_N^{\sin(\phi_S - \phi_\gamma)} \equiv \frac{2\int_0^{2\pi} \mathrm{d}\phi_\gamma \, [\mathrm{d}\sigma^{\uparrow} - \mathrm{d}\sigma^{\downarrow}] \, \sin(\phi_S - \phi_\gamma)}{\int_0^{2\pi} \mathrm{d}\phi_\gamma \, [\mathrm{d}\sigma^{\uparrow} + \mathrm{d}\sigma^{\downarrow}]}$$



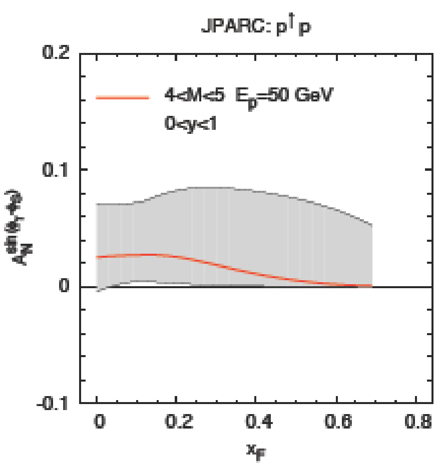
Sivers distribution function in single polarized Drell-Yan processes

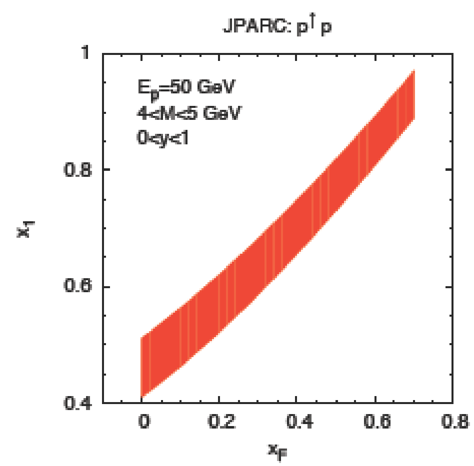




Fixed target mode
√s = 5.47 GeV
xF=x1-x2
X2 region explored is the same as in SIDIS

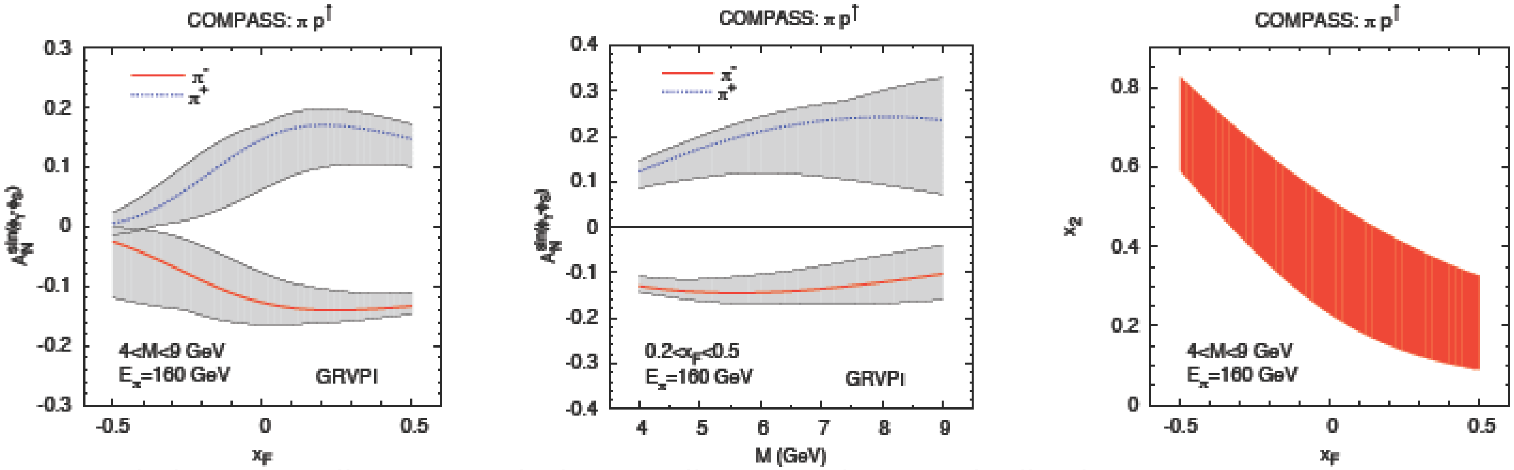
M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, A. Prokudin, Phys. Rev. D79 (2009) 054010





√s = 9.78 GeV
xF=x1-x2
X1 region explored complementary to that explored in SIDIS

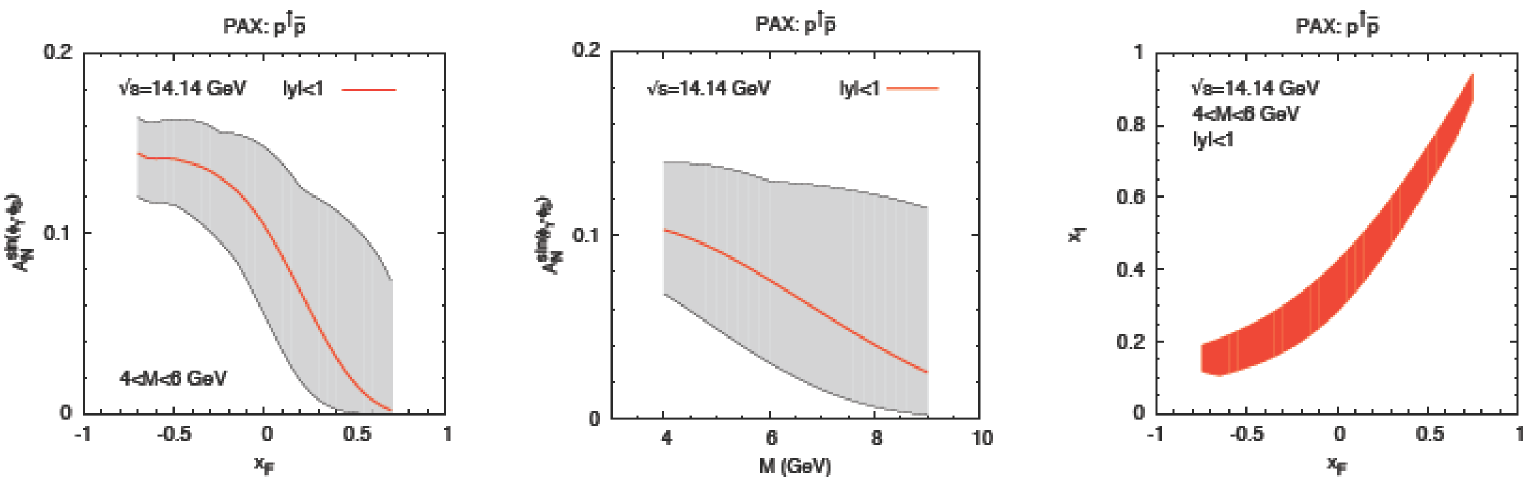
Sivers distribution function in single polarized Drell-Yan processes



M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, A. Prokudin, Phys. Rev. D79 (2009) 054010

COMPASS

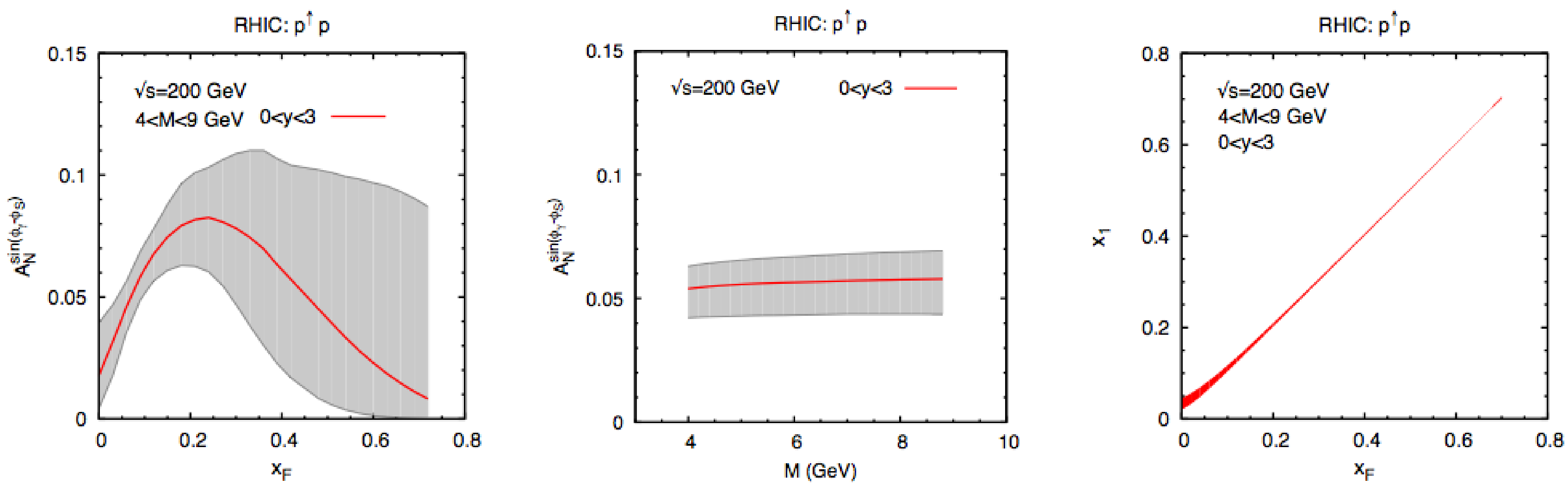
Fixed target mode
$\sqrt{s}$ = 17.4 GeV
xF=x1-x2
Large-x2 region explored at negative xF values



Polarized Antiproton eXperiments

Collider mode
$\sqrt{s}$ = 14.4 GeV
xF=x1-x2
Large-x2 region explored at positive xF values

Sivers distribution function in single polarized Drell-Yan processes



M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, A. Prokudin, Phys. Rev. D79 (2009) 054010

BRAHMS STAR PHENIX

$\sqrt{s}$ = 200 GeV
xF=x1-x2
X1 region explored extends to larger values than that explored in SIDIS

TMD's in doubly polarized Drell-Yan processes

$$\sigma_{Drell-Yan} = f_q(x) \otimes f_{\bar{q}} \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'}$$

$$\begin{cases} \sigma = h_1^q(x,k_\perp) \otimes h_1^{\bar{q}}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \\ \sigma = f_{1T}^{\perp}(x,k_\perp) \otimes f_{1T}^{\perp}(x,k_\perp) \otimes \hat{\sigma}^{q\bar{q}\to\ell\ell'} \end{cases}$$

In doubly polarized Drell-Yan processes one can study

- **Transversity** parton distribution function

At present, RHIC is the only experiment which can measure doubly polarized Drell-Yan, ***but*** *ATT is suppressed by antiquark PDF's !*
FUTURE EXPERIMENTS: COMPASS, J-PARC, PANDA, PAX

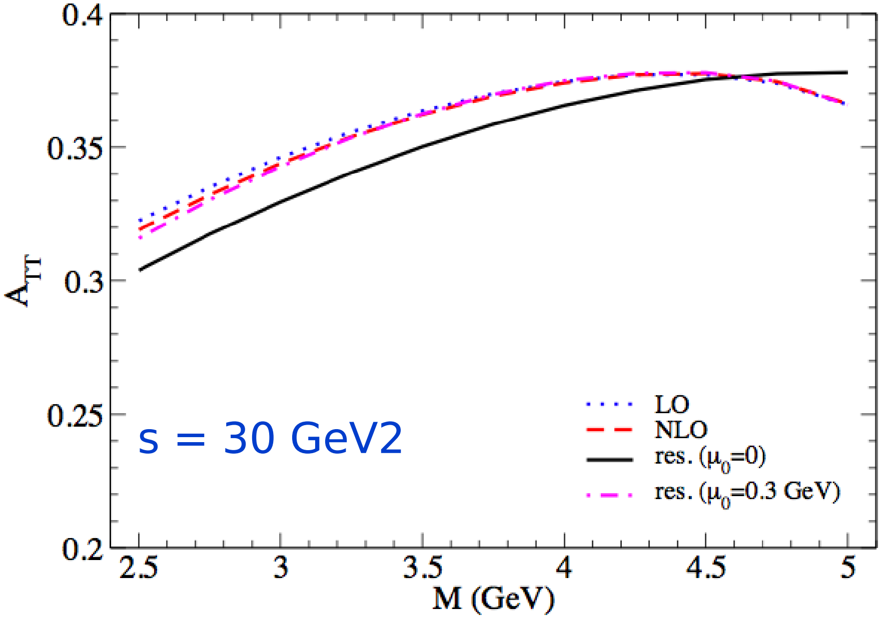
- **Sivers** distribution function

RECENT WORK:
M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, A. Prokudin, Phys. Rev. D79 (2009) 054010

Golden channel to study transversity

The dream experiment: Drell-Yan with polarized antiprotons

H. Shimizu, G. Sterman, W. Vogelsang, H. Yokoya



$$A_{TT} \equiv \frac{d\sigma^{\uparrow\uparrow} - d\sigma^{\uparrow\downarrow}}{d\sigma^{\uparrow\uparrow} + d\sigma^{\uparrow\downarrow}}$$

$$\hat{a}_{TT} = \frac{\sin^2\theta}{1+\cos^2\theta}\,\cos(2\varphi)$$

$$A_{TT} \equiv \frac{d\sigma^{\uparrow\uparrow} - d\sigma^{\uparrow\downarrow}}{d\sigma^{\uparrow\uparrow} + d\sigma^{\uparrow\downarrow}} \simeq \hat{a}_{TT}\,\frac{\sum_q e_q^2\, h_{1q}(x_1)\, h_{1q}(x_2)}{\sum_q e_q^2\, q(x_1)\, q(x_2)}$$

Conclusions

- **3-D exploration of the nucleon is starting as we speak:**
  - **Collect as much high-quality data as possible**
  - **Reconstruct the nucleon 3-D structure by "global" analyses**

- **Drell-Yan processes are very clean probes and offer the chance to pin down the transversity distribution function**

- **Ideal machines:**
  - **X-range including valence-region**
  - **Q^2 and M^2 high enough to control higher-twist corrections**
  - **P_T and q_T ranges large enough to see transition from TMD's to collinear factorization.**

Transverse Momentum of partons

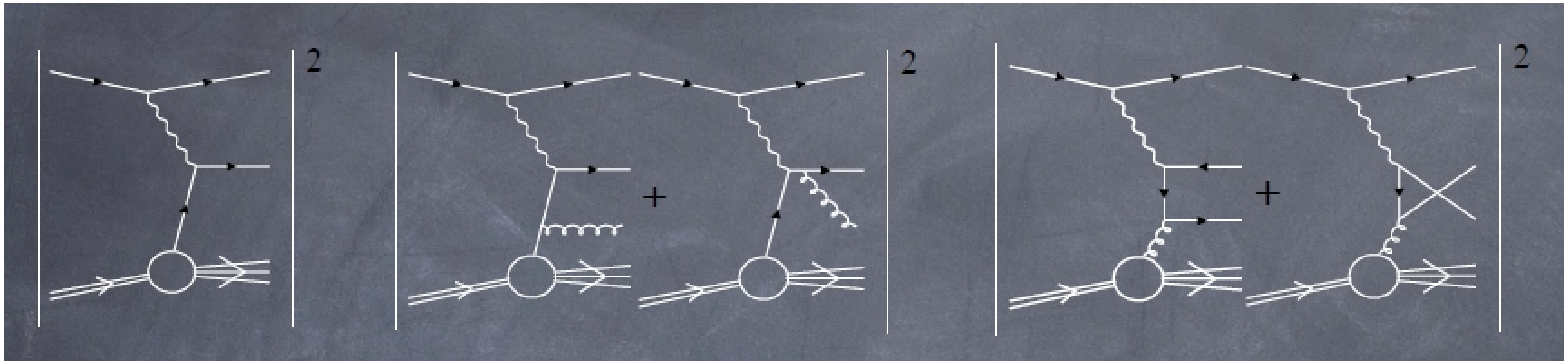
Mariaelena Boglione

INFN Torino

Lecture 6

Parton evolution equations

From Michelangelo Mangano
“Introduction to QCD”



Parton model summary

Assuming the parton picture outlined above, we can describe the cross-section for the interaction of the virtual photon with the proton as follows:

$$\boxed{\sigma_0 = \int_0^1 dx \sum_i e_i^2 \, f_i(x) \, \hat{\sigma}_0(\gamma^* q_i \to q_i', x)} \tag{133}$$

where the 0 subscript anticipates that this description represents a leading order approximation. In the above equation, $f_i(x)$ represents the density of quarks of flavour i carrying a fraction x of the proton momentum. The hatted cross-section represents the interaction between the photon and a free (massless) quark:

$$\begin{aligned} \hat{\sigma}_0(\gamma^* q_i \to q_i') &= \frac{1}{flux} \overline{\sum} \, |M_0(\gamma^* q \to q')|^2 \frac{d^3 p'}{(2\pi)^3 2p_0'} \; (2\pi)^4 \delta^4(p' - q - p) \\ &= \frac{1}{flux} \overline{\sum} |M_0|^2 2\pi \delta(p'^2) \end{aligned} \tag{134}$$

Using $p' = xP + q$, where P is the proton momentum, we get

$$(p')^2 = 2xP \cdot q + q^2 \equiv 2xP \cdot q - Q^2 \tag{135}$$

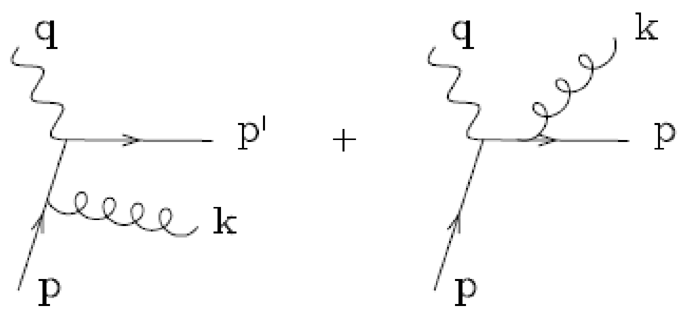
$$\hat{\sigma}_0(\gamma^* q \to q') = \frac{2\pi}{flux} \overline{\sum} |M_0|^2 \, \frac{1}{2P \cdot q} \delta(x - x_{bj}) \tag{136}$$

where $x_{bj} = \frac{Q^2}{2P \cdot q}$ is the so-called Bjorken-x variable. Finally:

$$\boxed{\sigma_0 = \frac{2\pi}{flux} \frac{\overline{\sum} |M_0|^2}{Q^2} \sum_i x_{bj} \, f_i(x_{bj}) \, e_i^2 \; \equiv \; \frac{2\pi}{flux} \frac{\overline{\sum} |M_0|^2}{Q^2} F_2(x_{bj})} \tag{137}$$

Parton evolution

Let us now study the QCD corrections to the LO parton-model description of DIS. This study will exhibit many important aspects of QCD (structure of collinear singularities, renormalization-group invariance) and will take us to an important element of the DIS phenomenology, namely scaling violations. We start from real-emission corrections to the Born level process:



(138)

The first diagram is proportional to $1/(p-k)^2 = 1/2(pk)$, which diverges when k is emitted parallel to p:

$$p \cdot k = p^0 k^0 \, (1 - \cos\theta) \overset{\cos\theta \to 1}{\longrightarrow} 0 \tag{139}$$

The second diagram is also divergent, if k is emitted parallel to p'. This second divergence turns out to be harmless, since we are summing over all possible final states. Whether the final-state quark keeps all of its energy, or whether it decides to share it with a gluon emitted collinearly, an inclusive final-state measurement will not care. The collinear divergence can then be cancelled by a similar divergence appearing in the final-state quark self-energy corrections.

Parton evolution

The first divergence is more serious, since from the point of view of the incoming photon (which only sees the quark, not the gluon) it *does* make a difference whether the momentum is all carried by the quark or is shared between the quark and the gluon. This means that no cancellation between collinear singularities in the real emission and virtual emission is possible. So let us go ahead, calculate explicitly the contribution of these diagrams, and learn how to deal with their singularities.

First of all note that while the second diagram is not singular in the region $k \cdot p \to 0$, its interference with the first one is. It is possible, however, to select a gauge for which the interference of the two diagrams is finite in this limit. You can show that the right choice is

$$\boxed{\sum \epsilon_\mu \epsilon^*_\nu(k) = -g_{\mu\nu} + \frac{k^\mu p'^\nu + k^\nu p'^\mu}{k \cdot p'} .} \qquad (140)$$

Notice that in this gauge not only $k \cdot \epsilon(k) = 0$, but also $p' \cdot \epsilon(k) = 0$. The key to getting to the end of a QCD calculation in a finite amount of time is choosing a proper gauge (which we just did) and the proper parametrization of the momenta involved. In our case, since we are interested in isolating the region where k becomes parallel with p, it is useful to set

$$k_\mu = (1-z)p_\mu + \beta p'_\mu + (k_\perp)_\mu , \qquad (141)$$

with $k_\perp \cdot p = k_\perp \cdot p' = 0$. β is obtained by imposing

$$k^2 = 0 = 2\beta(1-z)p \cdot p' + k_\perp^2 \qquad (142)$$

Defining $k_\perp^2 = -k_t^2$, we then get

$$\beta = \frac{k_t^2}{2(pp')\,(1-z)} \tag{143}$$

$$k_\mu = (1-x)p_\mu + \frac{k_t^2}{2(1-x)p\cdot p'}\,p'_\mu + (k_\perp)_\mu \tag{144}$$

$(k_\perp)_\mu$ is therefore the gluon momentum vector transverse to the incoming quark, in a frame where γ^* and q are aligned. k_t is the value of this transverse momentum. We also get

$$k\cdot p \;=\; \beta\, p\cdot p' = \frac{k_t^2}{2(1-z)} \quad \text{and} \quad k\cdot p' \;=\; (1-z)p\cdot p' \tag{145}$$

As a result $(p-k)^2 = -k_t^2/(1-z)$. The amplitude for the only diagram carrying the initial-state singularity is:

$$\boxed{M_g \;=\; ig\lambda^a_{ij}\,\bar{u}(p')\Gamma\,\frac{\hat{p}-\hat{k}}{(p-k)^2}\,\hat{\epsilon}(k)u(p)} \tag{146}$$

(where we introduced the notation $\hat{a} \equiv \not{a} \equiv a_\mu\gamma^\mu$). We indicated by Γ the interaction vertex with the external current q. It is important to keep Γ arbitrary, because we would like to get results which do not depend on the details of the interaction with the external probe. It is important that the singular part of the QCD correction, and therefore its renormalization, be process independent. Only in this way we can hope to achieve a true universality of the parton densities! So we will keep Γ generic, and make sure that our algebra does not depend on its form, at least in the $p\cdot k \to 0$ limit. Squaring the most singular part of the amplitude, and summing over colours and spins, we get:

$$\boxed{\sum_{\substack{\text{g polariz.}\\ \text{and colours}}} |M_g|^2 \;=\; g^2\, \overbrace{\sum_a \text{tr}\;(\lambda^a\lambda^a)}^{N\times C_F} \times \frac{1}{t^2} \times \sum_\epsilon \text{Tr}\;[\hat{p}'\,\Gamma(\hat{p}-\hat{k})\,\hat{\epsilon}\;\hat{p}\;\hat{\epsilon}^*\;(\hat{p}-\hat{k})\;\Gamma^+]} \tag{147}$$

with $t = (p-k)^2 = -k_t^2/(1-z)$.

Let us look first at

$$\sum_{\epsilon} \hat{\epsilon}\,\hat{p}\,\hat{\epsilon}^* \;=\; \sum \epsilon_\mu\,\epsilon^*_\nu\;\gamma^\mu \hat{p}\gamma^\nu \;=\; -\gamma^\mu \hat{p}\gamma^\mu + \frac{1}{k\cdot p'}\,(\hat{p}'\hat{p}\hat{k} + \hat{k}\hat{p}\hat{p}') \;=\; \frac{2}{1-z}\,(\hat{k} + \beta\hat{p}') \qquad (148)$$

(we used: $\hat{a}\hat{b}\hat{c} + \hat{c}\hat{b}\hat{a} = 2(a\cdot b)\,\hat{c} - 2(a\cdot c)\,\hat{b} + 2(b\cdot c)\,\hat{a}$ and some of the kinematical relations from the previous page). Then take

$$(\hat{p}-\hat{k})\,(\hat{k}+\beta\hat{p}')\,(\hat{p}-\hat{k}) = (\hat{p}-\hat{k})\,\hat{k}\,(\hat{p}-\hat{k}) + \beta(\hat{p}-\hat{k})\,\hat{p}'\,(\hat{p}-\hat{k}) \qquad (149)$$

In the second term, proportional to β, we can approximate $\hat{k} = (1-z)\hat{p}$. This is because the other pieces $(\beta\hat{p}' + \hat{k}_\perp)$ multiplied by β would cancel entirely the $\frac{1}{t^2}$ singularity, and would only contribute a non-singular term, which we are currently neglecting. So Eq. (149) becomes

$$\hat{p}\hat{k}\hat{p} + \beta z^2 \hat{p}\hat{p}'\hat{p} = 2(p\cdot k)\hat{p} + \beta z^2 2(p\cdot p')\hat{p} = 2(p\cdot k)\,(1+z^2)\hat{p} \qquad (150)$$

and

$$\boxed{\sum |M_g|^2 = 2g^2\,C_F\,\frac{(1-z)}{k_t^2}\left(\frac{1+z^2}{1-z}\right)\;N\,\mathrm{Tr}[\hat{p}'\Gamma\hat{p}\Gamma^+]} \qquad (151)$$

The last factor with the trace corresponds to the Born amplitude squared. So the one-gluon emission process factorizes in the collinear limit into the Born process times a factor which is independent of the beam's nature! If we add the gluon phase-space:

$$[dk] \equiv \frac{d^3k}{(2\pi)^3 2k^0} = \frac{dk_\parallel}{k^0}\,\frac{d\phi}{2\pi}\,\frac{1}{8\pi^2}\,\frac{dk_\perp^2}{2} = \frac{dz}{(1-z)}\,\frac{1}{16\pi^2}\,dk_\perp^2 \qquad (152)$$

Parton evolution

we get:

$$\boxed{\overline{\sum}|M_g|^2\,[dk] = \frac{dk_\perp^2}{k_\perp^2}\,dz\,\left(\frac{\alpha_s}{2\pi}\right)\,P_{qq}(z)\,\overline{\sum}|M_0|^2} \tag{153}$$

where

$$\boxed{P_{qq}(z) = C_F\,\frac{1+z^2}{1-z}} \tag{154}$$

is the so-called Altarelli-Parisi splitting function for the $q \to q$ transition (z is the momentum fraction of the original quark taken away by the quark after gluon emission). We are now ready to calculate the corrections to the parton-model cross-section:

$$\sigma_g \;=\; \int dx\, f(x)\,\frac{1}{flux}\int dz\,\frac{dk_\perp^2}{k_\perp^2}\,\left(\frac{\alpha_s}{2\pi}\right)\,P_{qq}(z)\,\overline{\sum}|M_0|^2\,2\pi\delta(p'^2) \tag{155}$$

Using $(p')^2 = (p-k+q)^2 \sim (zp+q)^2 = (xzP+q)^2$ and

$$\delta(p'^2) \;=\; \frac{1}{2P\cdot q}\,\frac{1}{z}\,\delta(x - \frac{x_{bj}}{z}) \;=\; \frac{x_{bj}}{z}\,\delta(x-\frac{x_{bj}}{z}) \tag{156}$$

we finally obtain:

$$\boxed{\sigma_g = \frac{2\pi}{flux}\left(\frac{\overline{\sum}|M_0|^2}{Q^2}\right)\sum_i e_i^2\,x_{bj}\,\frac{\alpha_s}{2\pi}\int\frac{dk_\perp^2}{k_\perp^2}\int\frac{dz}{z}\,P_{qq}(z)\,f_i\left(\frac{x_{bj}}{z}\right)} \tag{157}$$

To be compared with

$$\sigma_0 = \frac{2\pi}{flux}\,\frac{\overline{\sum}|M_0|^2}{Q^2}\sum_i x_{bj}\,f_i(x_{bj})\;e_i^2$$

Parton evolution

We then find that the inclusion of the $\mathcal{O}(\alpha_s)$ correction is equivalent to a contribution to the parton density:

$$\boxed{f_i(x) \to f_i(x) + \frac{\alpha_s}{2\pi} \int \frac{dk_\perp^2}{k_\perp^2} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x}{z}\right)} \tag{158}$$

Notice the presence of the integral $\int dk_\perp^2/k_\perp^2$. The upper limit of integration is proportional to Q^2. The lower limit is 0. Had we included a quark mass, the propagator would have behaved like $1/(k_\perp^2 + m^2)$. But the quark is bound inside the hadron, so we do not quite know what m should be. Let us then assume that we cutoff the integral at a $k_\perp$ value equal to some scale μ_0, and see what happens. The effective parton density becomes:

$$f(x, Q^2) = f(x) + \log\left(\frac{Q^2}{\mu_0^2}\right) \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f\left(\frac{x}{z}\right) \tag{159}$$

The dependence on the scale μ_0, which is a non-perturbative scale, can be removed by defining $f(x, Q^2)$ in terms of the parton density f *measured* at a large, perturbative scale μ^2:

$$f(x, \mu^2) = f(x) + \log\left(\frac{\mu^2}{\mu_0^2}\right) \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f\left(\frac{x}{z}\right) \tag{160}$$

We can then perform a subtraction, and write:

$$\boxed{f(x, Q^2) = f(x, \mu^2) + \log\left(\frac{Q^2}{\mu^2}\right) \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f\left(\frac{x}{z}\right)} \tag{161}$$

Parton evolution

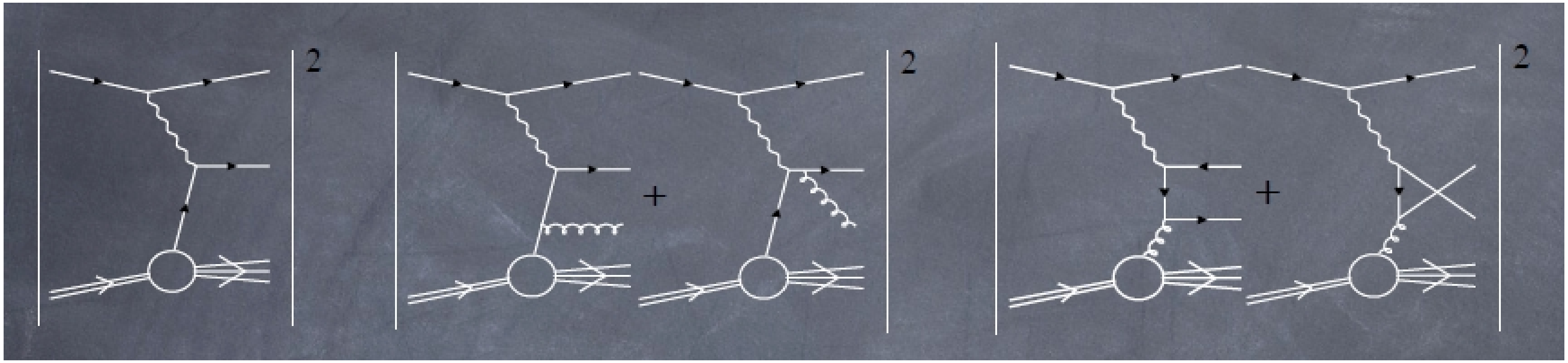
The scale μ plays here a similar role to the renormalization scale introduced in the second lecture. Its choice is arbitrary, and $f(x, Q^2)$ should not depend on it. Requiring this independence, we get the following "renormalization-group invariance" condition:

$$\frac{df(x,Q^2)}{d\ln\mu^2} = \mu^2\,\frac{df(x,\mu^2)}{d\mu^2} - \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\,P_{qq}(z)\,f\left(\frac{x}{z}\right) \equiv 0 \qquad (162)$$

and then

$$\boxed{\mu^2\frac{df(x,\mu^2)}{d\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\,P_{qq}(z)\,f\left(\frac{x}{z},\mu^2\right)} \qquad (163)$$

This equation is usually called the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equation. As in the case of the resummation of leading logarithms in $R_{e^+e^-}$ induced by the RG invariance constraints, the DGLAP equation – which is the result of RG-invariance – resums a full tower of leading logarithms of Q^2.



Parton evolution

Proof: Let us define $t = \log \frac{Q^2}{\mu^2}$. We can then expand $f(x,t)$ in powers of t:

$$f(x,t) \;=\; f(x,0) + t\frac{df}{dt}(x,0) + \frac{t^2}{2!}\frac{d^2 f}{dt^2}(x,0) + \ldots \tag{164}$$

The first derivative is given by the DGLAP equation itself. Higher derivatives can be obtained by differentiating it:

$$\begin{aligned}
f''(x,t) &= \frac{\alpha_s}{2\pi}\int \frac{dz}{z}P_{qq}(z)\frac{df}{dt}(\frac{x}{z},t) \\
&= \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}P_{qq}(z)\frac{\alpha_s}{2\pi}\int_{\frac{x}{z}}^1 \frac{dz'}{z'}P_{qq}(z)f(\frac{x}{zz'},t) \\
&\;\vdots \\
f^{(h)}(x,t) &= \frac{\alpha_s}{2\pi}\int_x^1 \ldots\ldots \frac{\alpha_s}{2\pi}\int_{x/zz'\ldots z^{(n-1)}}^1 \frac{dz^{(n)}}{z^{(n)}}P_{qq}(z^{(n)})f(\frac{x}{zz'\ldots},t)
\end{aligned} \tag{165}$$

The n-th term in this expansion, proportional to $(\alpha_s\, t)^n$, corresponds to the emission of n gluons (it is just the n-fold iteration of what we did studying the one-gluon emission case).

Parton evolution

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and then

$$\boxed{\mu^2 \frac{df(x, \mu^2)}{d\mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f\left(\frac{x}{z}, \mu^2\right)} \tag{163}$$

This equation is usually called the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equation. As in the case of the resummation of leading logarithms in $R_{e^+e^-}$ induced by the RG invariance constraints, the DGLAP equation – which is the result of RG-invariance – resums a full tower of leading logarithms of Q^2.

Parton evolution

With similar calculations one can include the effect of the other $\mathcal{O}(\alpha_s)$ correction, originating from the splitting into a $q\bar{q}$ pair of a gluon contained in the proton. With the addition of this term, the evolution equation for the density of the ith quark flavour becomes:

$$\boxed{\frac{df_q(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}(z)\, f_i(\frac{x}{z},t) \;+\; P_{qg}(z) f_g(\frac{x}{z},t)\right], \quad \text{with } P_{qg} = \frac{1}{2}\left[z^2 + (1-z)^2\right]} \tag{166}$$

In the case of interactions with a coloured probe (say a gluon) we meet the following corrections, which affect the evolution of the gluon density $f_g(x)$:

$$\frac{df_g(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_{i=q,\bar{q}} f_i\left(\frac{x}{z},t\right) + P_{gg}(z) f_g\left(\frac{x}{z},t\right)\right] \tag{167}$$

with

$$P_{gq}(z) = P_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z} \quad \text{and} \quad P_{gg}(z) = 2C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] \tag{168}$$

Defining the moments of an arbitrary function $g(x)$ as follows:

$$g_n = \int_0^1 \frac{dx}{x}\, x^n\, g(x)$$

it is easy to prove that the evolution equations turn into ordinary linear differential equations:

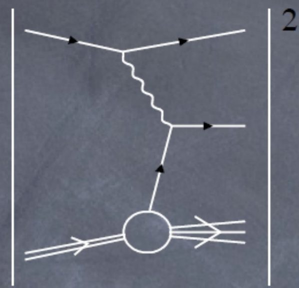
$$\frac{df_i^{(n)}}{dt} = \frac{\alpha_s}{2\pi}[P_{qq}^{(n)} f_i^{(n)} + P_{qg}^{(n)} f_g^{(n)}] \tag{169}$$

$$\frac{df_g^{(n)}}{dt} = \frac{\alpha_s}{2\pi}[P_{gg}^{(n)} f_g + P_{gq}^{(n)} f_i^{(n)}] \tag{170}$$

Parton evolution

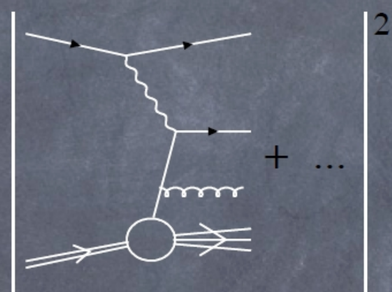
LO, NLO, NNLO...

naive



$$\sigma_q^0 \otimes f_q(x)$$

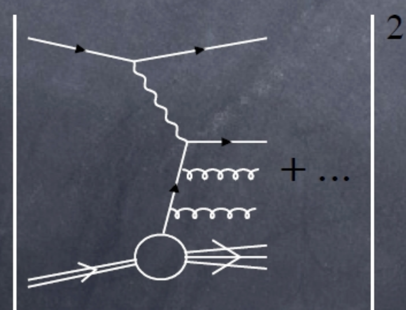
LO



$$\sigma_q^1 \otimes f_q(x) + \sigma_g^1 \otimes f_g(x)$$

$$\sigma_q^0 \otimes f_q^{LO}(x, Q^2)$$

NLO



$$\sigma_q^2 \otimes f_q(x) + \sigma_g^2 \otimes f_g(x)$$

$$\sigma_q^1 \otimes f_q^{NLO}(x, Q^2) + \sigma_g^1 \otimes f_g^{NLO}(x, Q^2)$$

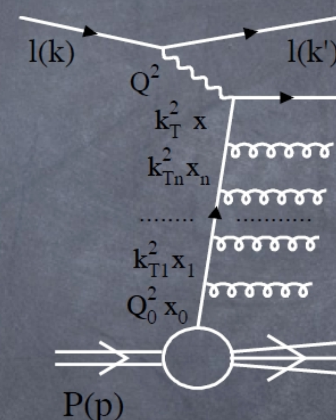
Parton evolution

Why not just a fixed order calculation?

fixed order may miss some contributions: gluon dynamics in DIS at $\mathcal{O}(\alpha_s^1)$

cross sections from the previous order are much simpler!

resummation:

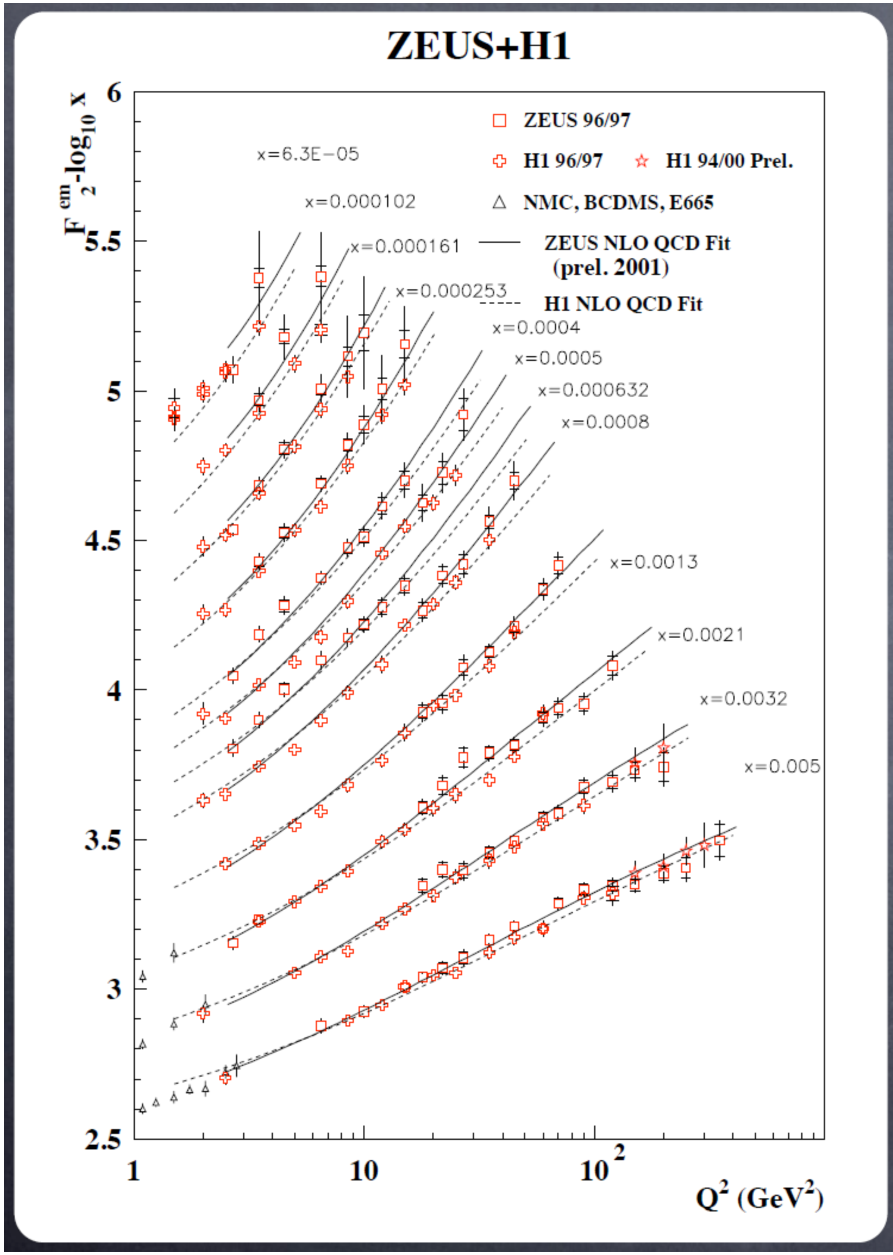


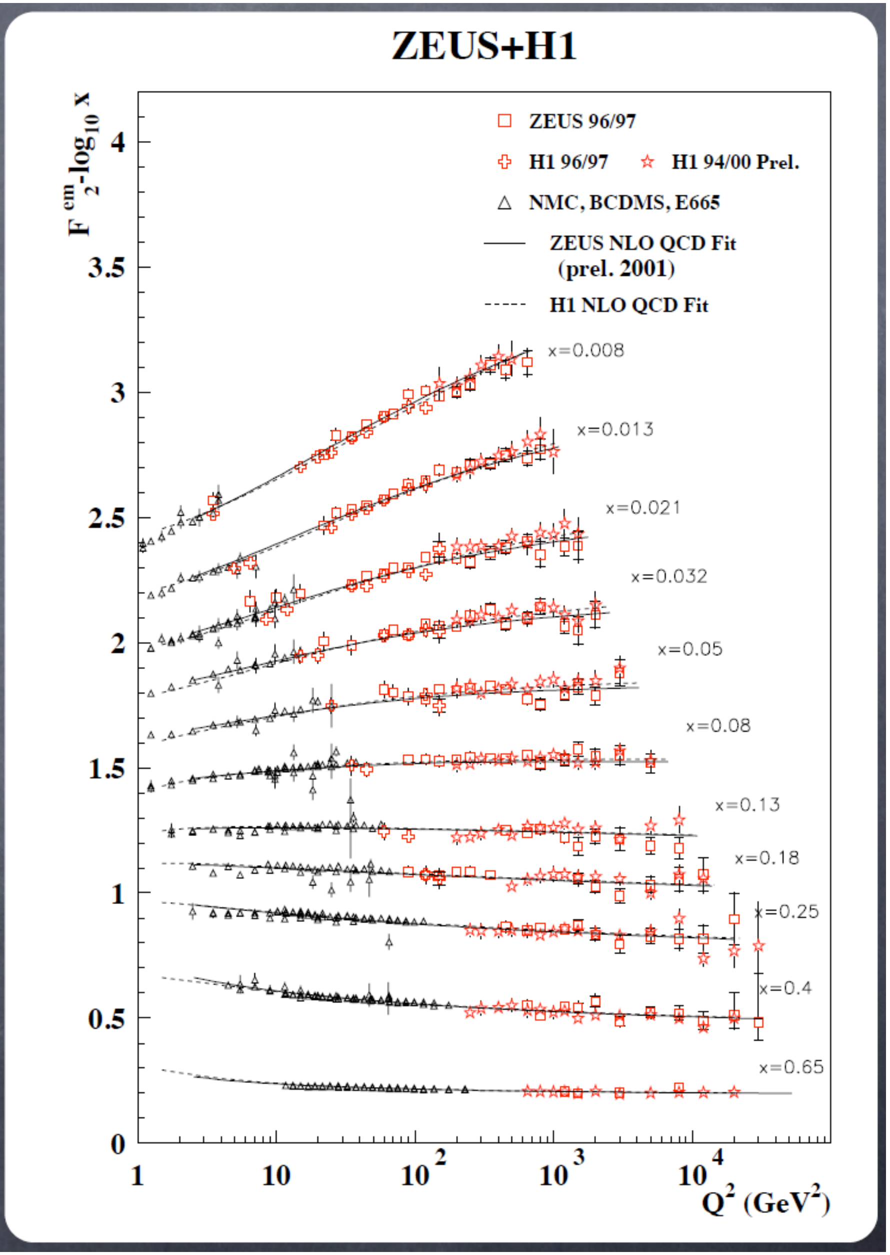
running $\alpha_s^{1-loop}(Q^2)$ + $f_i^{LO}(x, Q^2)$ $\qquad$ $\alpha_s^n(\mu^2)[\log(\frac{Q^2}{\mu^2})]^n$

"leading $\log(Q^2)$"

$\alpha_s^{2-loop}(Q^2)$ + $f_i^{NLO}(x, Q^2)$ $\qquad$ $\alpha_s^n(\mu^2)[\log(\frac{Q^2}{\mu^2})]^n$ + $\alpha_s^n(\mu^2)[\log(\frac{Q^2}{\mu^2})]^{n-1}$

Parton evolution





Giving these lectures
was a tremendous,
titanic effort ...
but it was well worth it
You have been great !

Thank you and good luck for your PhDs !

Summary

This is a worm gear

These are worm gear distribution functions

$$h_{1L}^{q\perp}(x,\mathbf{k}_T^2)$$

$$g_{1T}^{q}(x,\mathbf{k}_T^2)$$

Alessandro Bacchetta

An elephant is not a mouse multiplied by a thousand

Larry Weinstein

Neohipparion (small horse) Mastodon

(a) (b)

This is a pretzel

This is pretzelosity

$$h_{1T}^{q\perp}(x,\mathbf{k}_T^2)$$

Matthias Burkhardt

We are physicists:
we are arrogant,
we are prejudiced, and
we make assumptions
based on our prejudice.
Whatever we say,
we say it with authority.

Paul Reimer