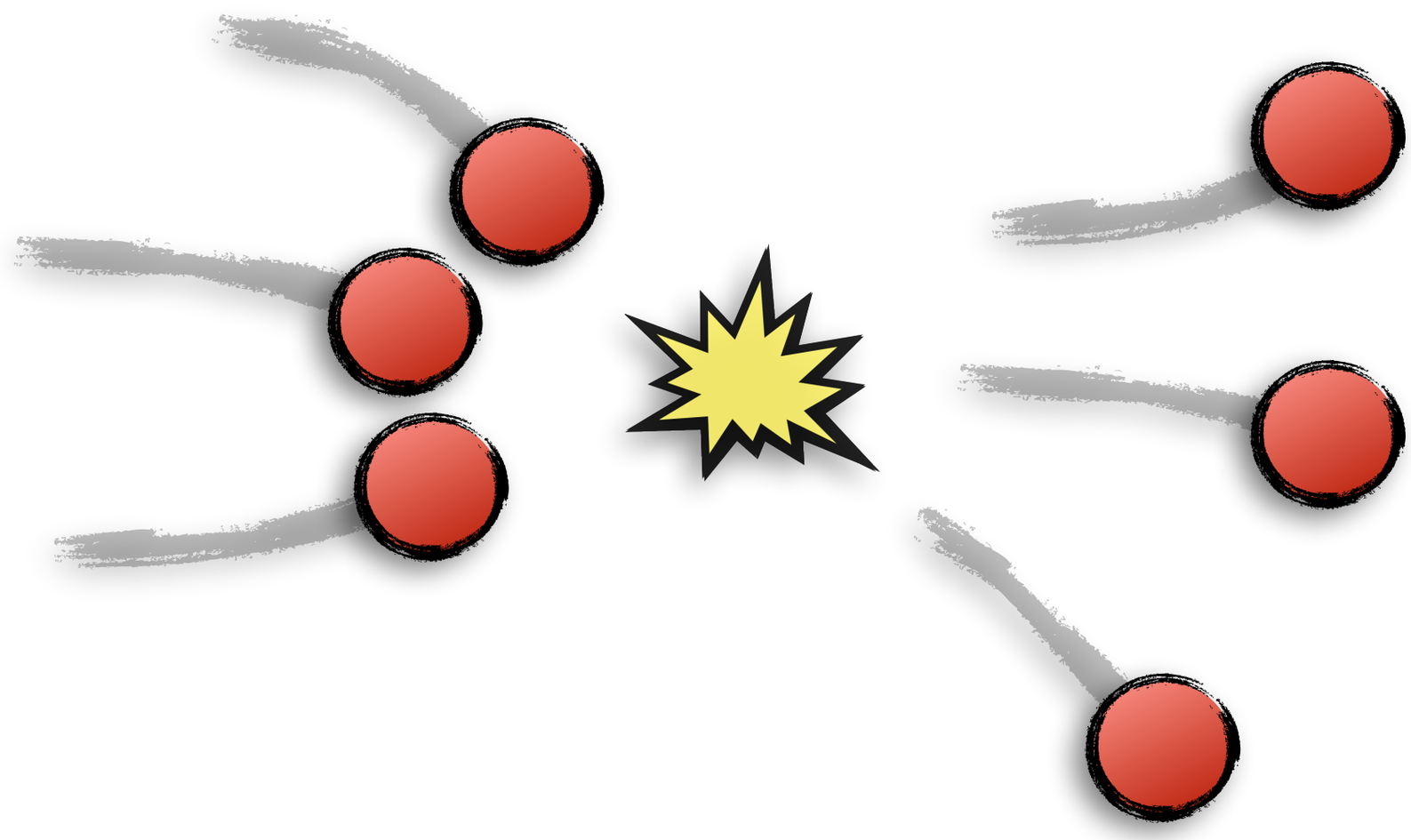
Three-body scattering in the presence of bound states

Sebastian M. Dawid

Thomas Jefferson National Accelerator Facility

"Cake" Seminar

Thursday, September 13th, 2021



Jefferson Lab

Indiana University Bloomington

Previous JLab seminars on three-body physics

Monday, October 15, 2018, 1:00 p.m. Room L102
Maxim Mai (GWU)
3 hadrons in (in-)finite volume

Monday, October 29, 2018, 1:00 p.m. Room L102
Andrew Jackura (Indiana U.)
Phenomenology of 3→3 Scattering

Monday, February 11, 2019, 1:00 p.m. Room L102
Peng Guo (California Bakerfield Uniersity)
Multiple particles interaction in finite volume

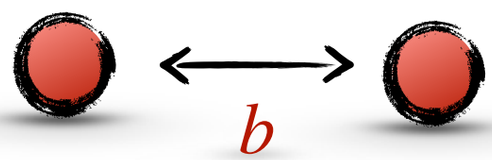
Wednesday, March 13, 2019, 1:00 p.m. Room L102
Raúl Briceño (JLab/ODU)
The three-body problem

Monday, December 14th, 2020, 1:00 p.m.
Fernando Romero López (University of Valencia)
Two- and three-particle scattering amplitudes from lattice QCD

Monday, March 15th, 2021, 1:00 p.m.
Maxwell Hansen (University of Edinburgh)
Three-pion scattering from lattice QCD

V. Efimov, Phys. Lett. B 33 (1970) 563
Braaten, Hammer, Phys. Rept. 428 (2006) 259
Naidon, Endo, Rept. Prog. Phys. 80 (2017) 5, 056001
S. Dawid et al. Phys. Lett. B 777 (2018) 260

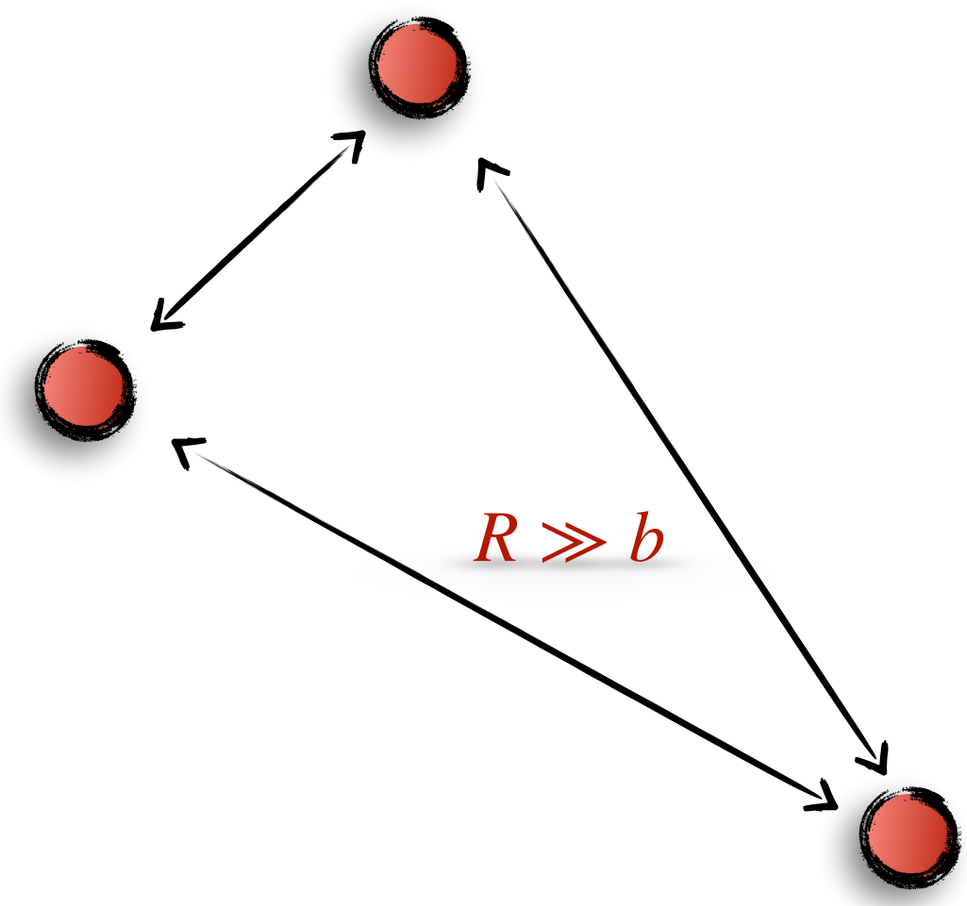
Efimov physics – picture



- Short range, near-resonant interaction (barely supports two-body bound state)

Two bodies

Three bodies

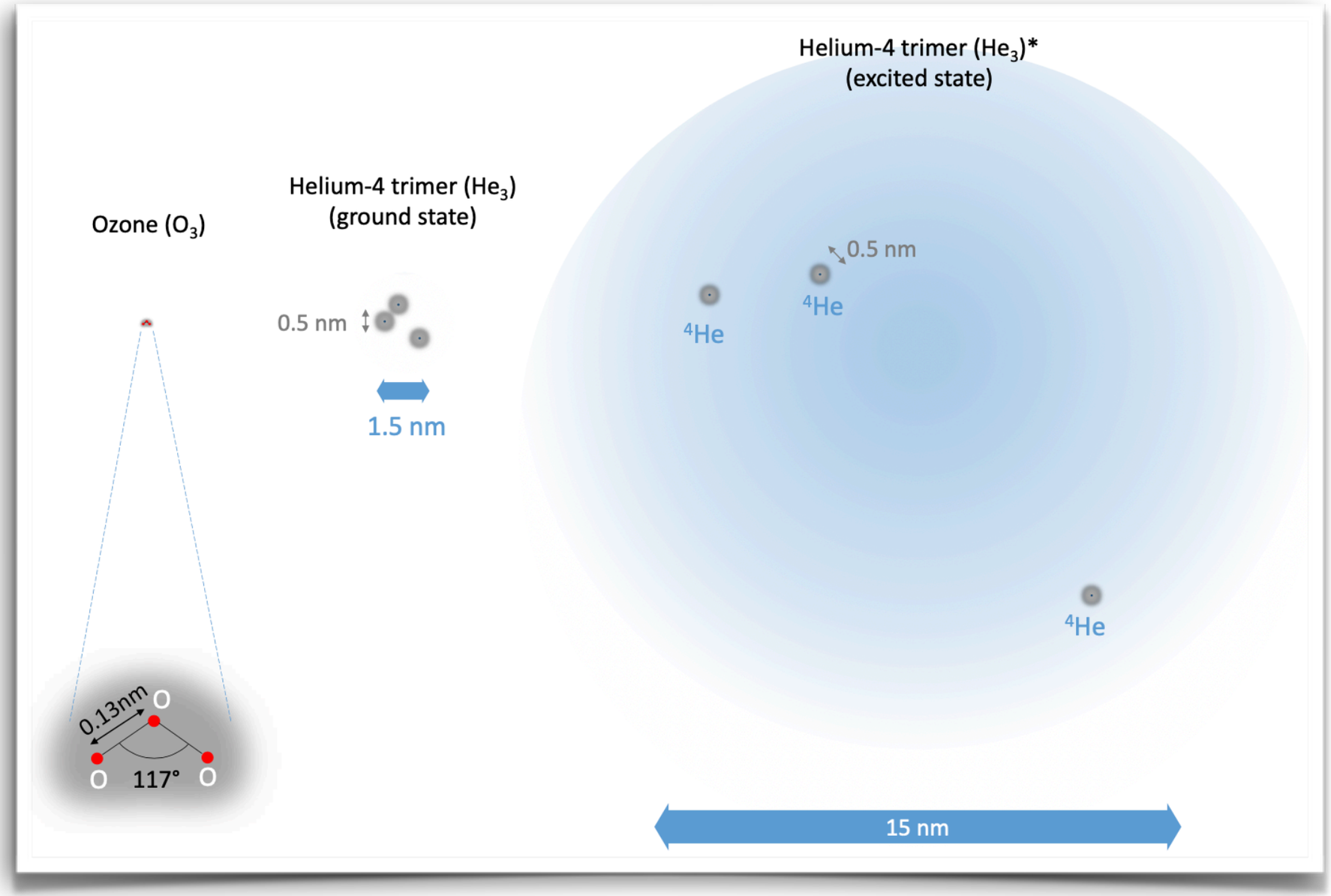


- Induced long-range interaction
- Discrete scale symmetry
- Borromean binding

$$E_n \propto -\left(e^{-\frac{2\pi}{s_0}}\right)^n$$

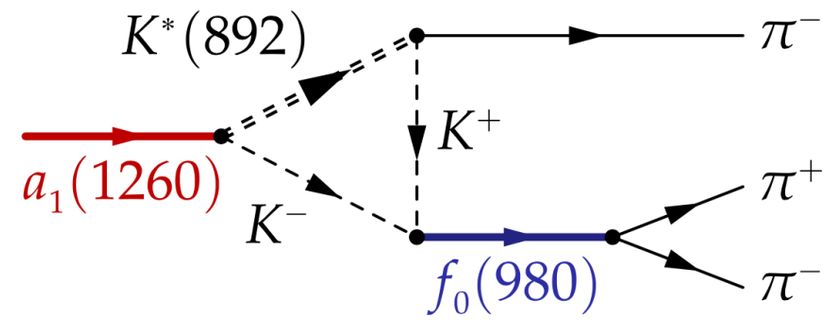
Efimov physics – picture

V. Efimov, Phys. Lett. B 33 (1970) 563
Braaten, Hammer, Phys. Rept. 428 (2006) 259
Naidon, Endo, Rept. Prog. Phys. 80 (2017) 5, 056001
S. Dawid et al. Phys. Lett. B 777 (2018) 260



Three-body processes and hadronic spectrum

- Exotic resonances decay to three-particle final states
  - X(3872), N*(1440), a_1(1260), a_1(1420), ...
- Interpretations
  - molecules,
  - hybrids,
  - diquark-antidiquark,
  - kinematical effects (triangle singularity)



On the nature of a_1(1420)
Mikhasenko, Ketzer, Sarantsev, Phys. Rev. D 91, 094015 (2015)

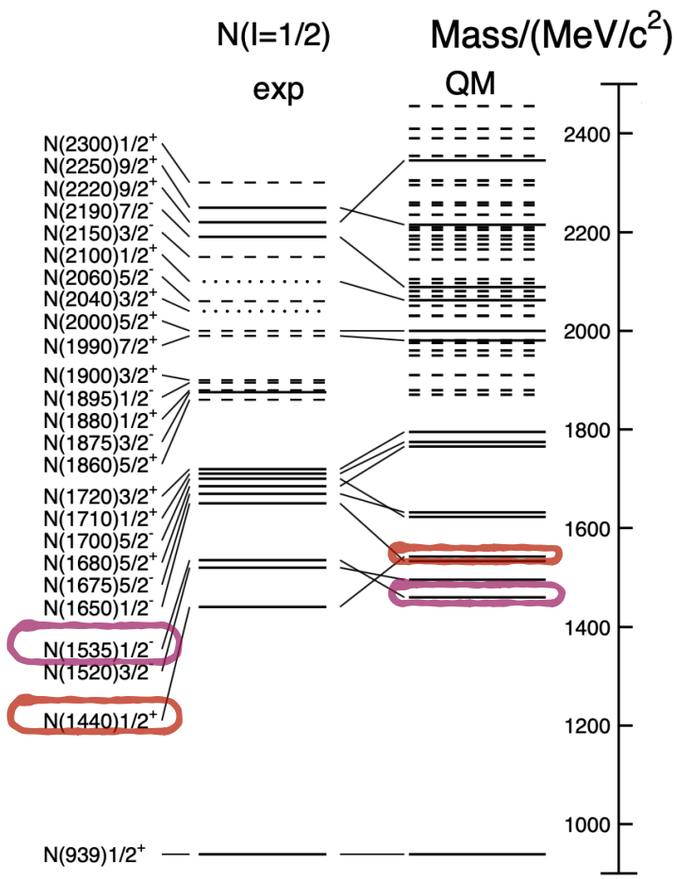
- Goal – to build three-body scattering formalism
  - derive properties of hadrons from the (lattice) QCD
  - having convenient three-body framework for phenomenology

Light-meson spectroscopy with COMPASS
Ketzer, Grube, Ryabchikov, Prog. Part. Nucl. Phys. 113 (2020) 103755

not only a question of computational cost. With decreasing pion mass also the kinematical thresholds for three- and four-hadron channels decrease. In particular highly excited states couple strongly to such multi-hadron final states. However, the current method is not applicable to these channels. A complete finite-volume formalism for three or even more particles would therefore be a major breakthrough for the calculation of masses and decay modes of hadron resonances. Such a formalism is already under development (see Ref. [33] and references therein).

Exotic states

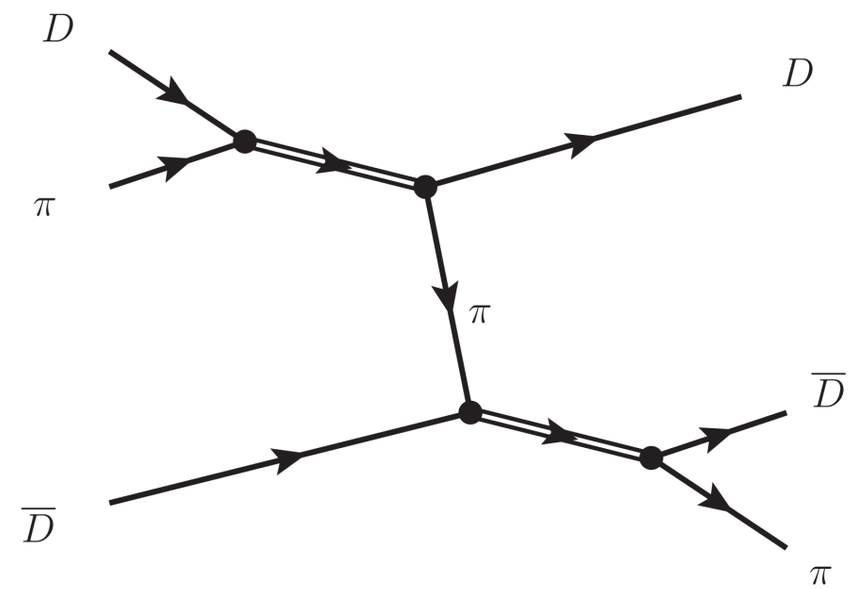
Roper N*(1440)



Pion-nucleon scattering in the Roper channel from lattice QCD,
Lang, Leskovec, Padmanath, Prelovsek, Phys. Rev. D 95 (2017) 1, 014510

Combining experimental $N\pi$ phase shifts with elastic approximation and the Lüscher formalism suggests in the spectrum an additional energy level near the Roper mass $m_R = 1.43$ GeV for our lattice. We do not observe any such additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$.

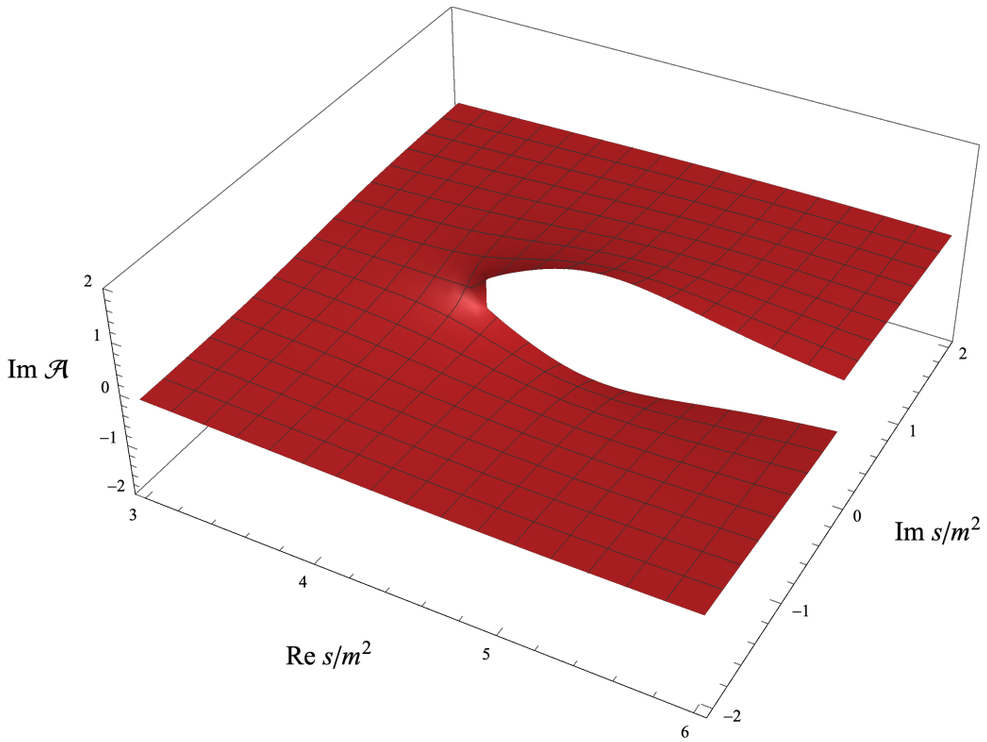
X(3872)



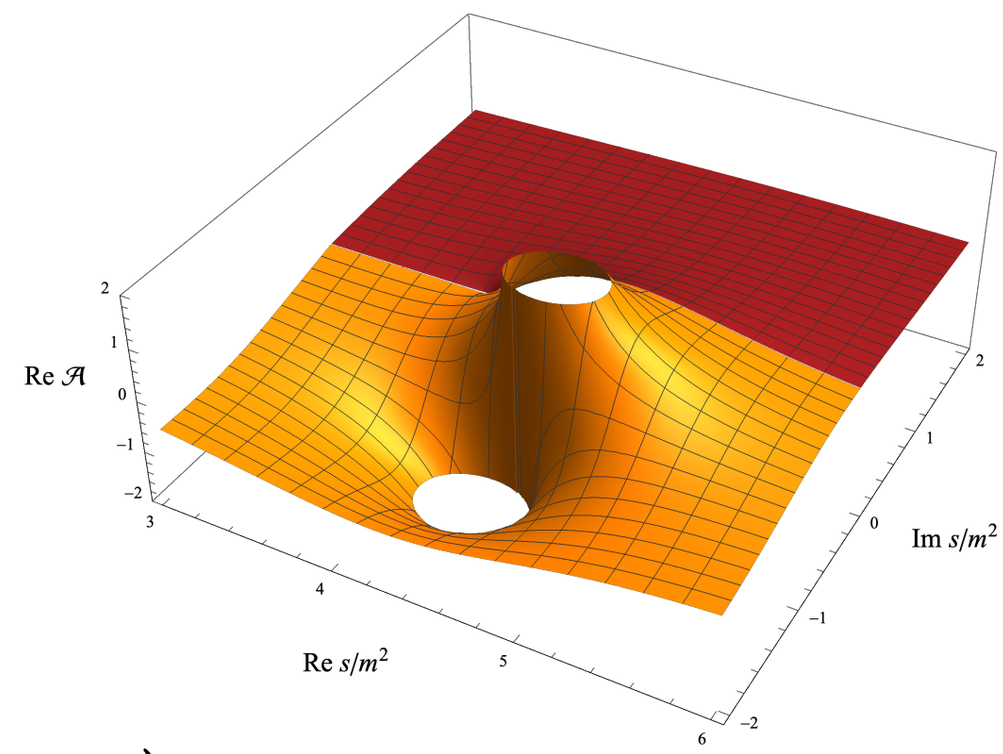
PDG Name	m (MeV)	Γ (MeV)	$I^G(J^{PC})$	Decay	Discovery Year
$\chi_{c1}(3872)$	3871.69 ± 0.17	< 1.2	$0^+(1^{++})$	$\pi^+\pi^- J/\psi$	2003
				$3\pi J/\psi$	
				$D^{*0}\overline{D}^0$	
				$\gamma J/\psi$	
				$\gamma\psi(2S)$	

- About 7 MeV from $D\overline{D}\pi$
- About 0.4 MeV from $D^*\overline{D}$

Fundamentals of the S matrix

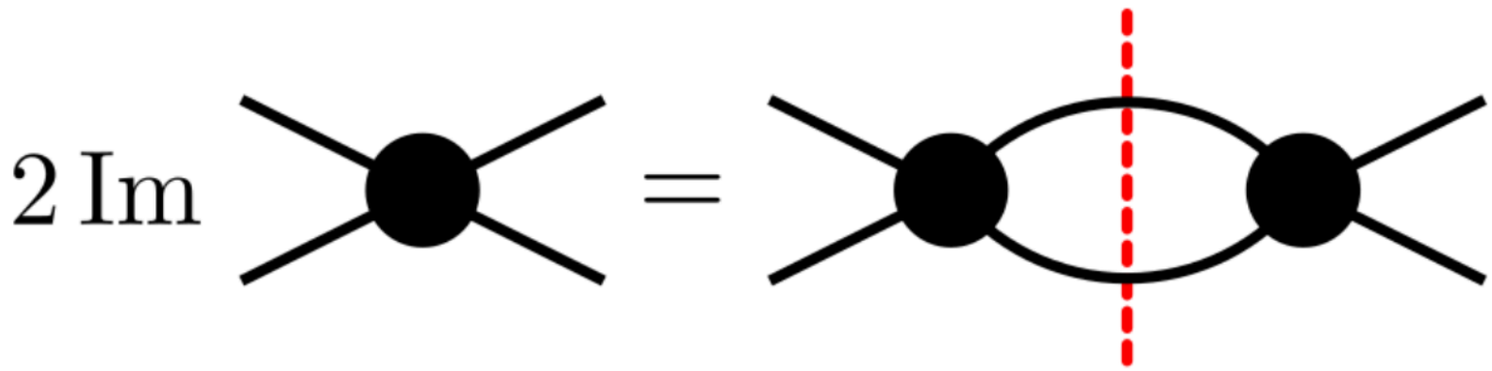


- Analyticity on the first Riemann sheet,
- Bound-states & resonances correspond to poles,
- Branch cuts correspond to open channels



- Principles of the S-matrix theory:
  - Unitarity (probability conservation)
  - Analyticity (causality)
  - Crossing symmetry (particles↔antiparticles)
  - Poincaré symmetry (frame independence)
  - Internal symmetries (charge, isospin, G-parity)

Elastic unitarity in the two-body scattering



- Unitarity relation $SS^\dagger = 1$ implies $\frac{1}{2i}\left(M^\dagger - M\right) = \frac{1}{2i} MM^\dagger$ for $S = 1 + iM$

- After partial wave projection

$$\mathrm{Im}\mathcal{M}_\ell(s) = \rho(s)\,|\mathcal{M}_\ell(s)|^2\,\theta(s-4m^2) \qquad \text{where} \qquad \rho(s) \propto \sqrt{1-\frac{4m^2}{s}}$$

- In consequence

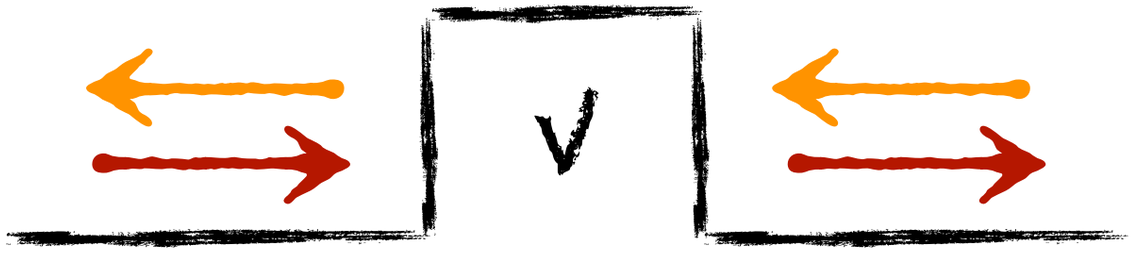
$$\mathscr{M}_\ell^{-1}(s) = \boxed{K_\ell^{-1}(s)} - i\rho(s)$$

- Phase shift

$$K_l^{-1}(s) = \frac{q^\star}{8\pi E^\star}\cot\delta_l(E^\star)$$

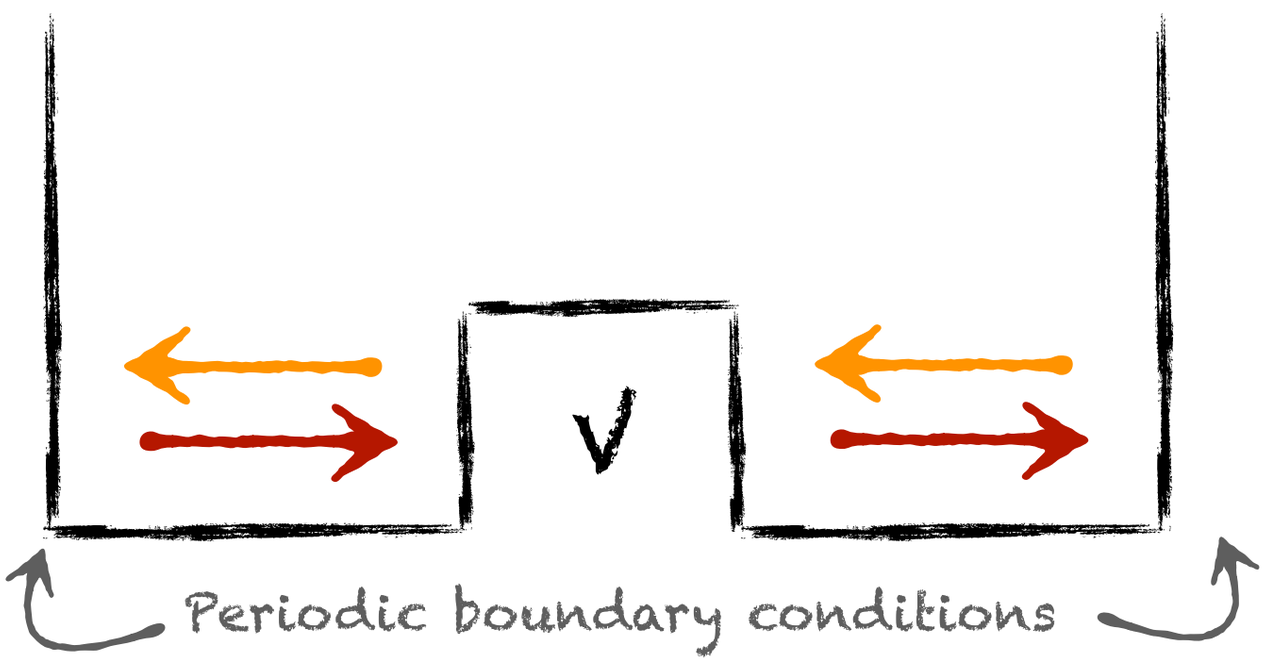
Two-body Lüscher formalism – general idea

- One-dimensional NR problem in infinite volume (IV)



$$\psi(x) \propto \cos(k|x| + \delta(k))$$

- One-dimensional NR problem in finite volume (FV)



- **Functions**

$$\cos\left(\frac{kL}{2} + \delta(k)\right) = \cos\left(\frac{kL}{2} + \delta(k)\right)$$

- **Derivatives**

$$-\sin\left(\frac{kL}{2} + \delta(k)\right) = \sin\left(\frac{kL}{2} + \delta(k)\right)$$

Quantization condition $\quad \dfrac{kL}{2} + \delta(k) = n\pi$

Scattering processes and resonances from lattice QCD
Briceño, Dudek, Young, Rev. Mod. Phys. 90 (2018) 2, 025001

Two-body quantization condition

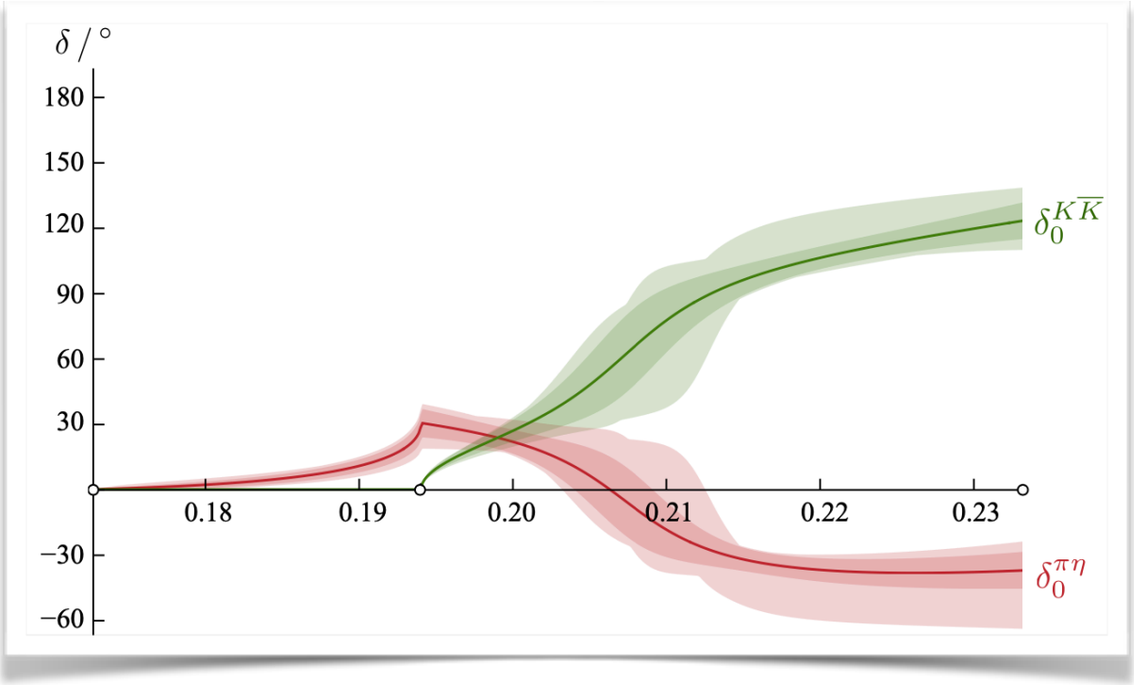
- Relation between IV phase shift and FV spectrum

$$\det\left[\mathscr{M}(s) + F^{-1}(s, \mathbf{P}, L)\right] = 0$$

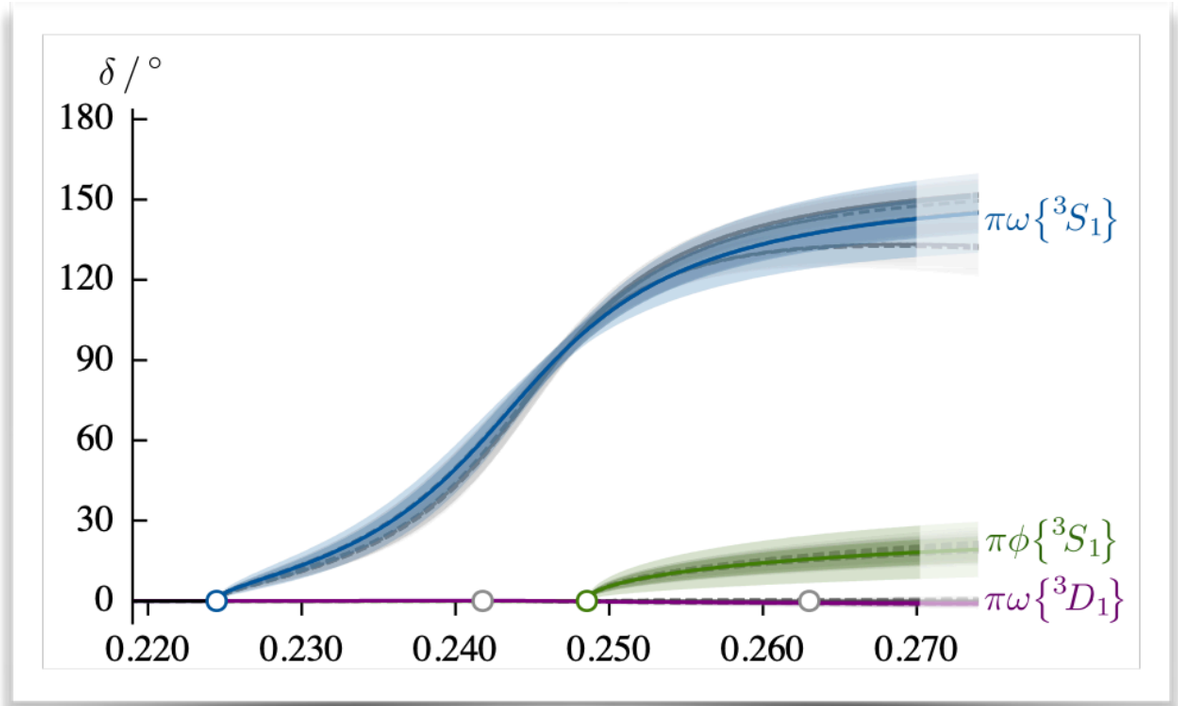
Matrices in the angular momentum space (truncation)

Complicated but known function

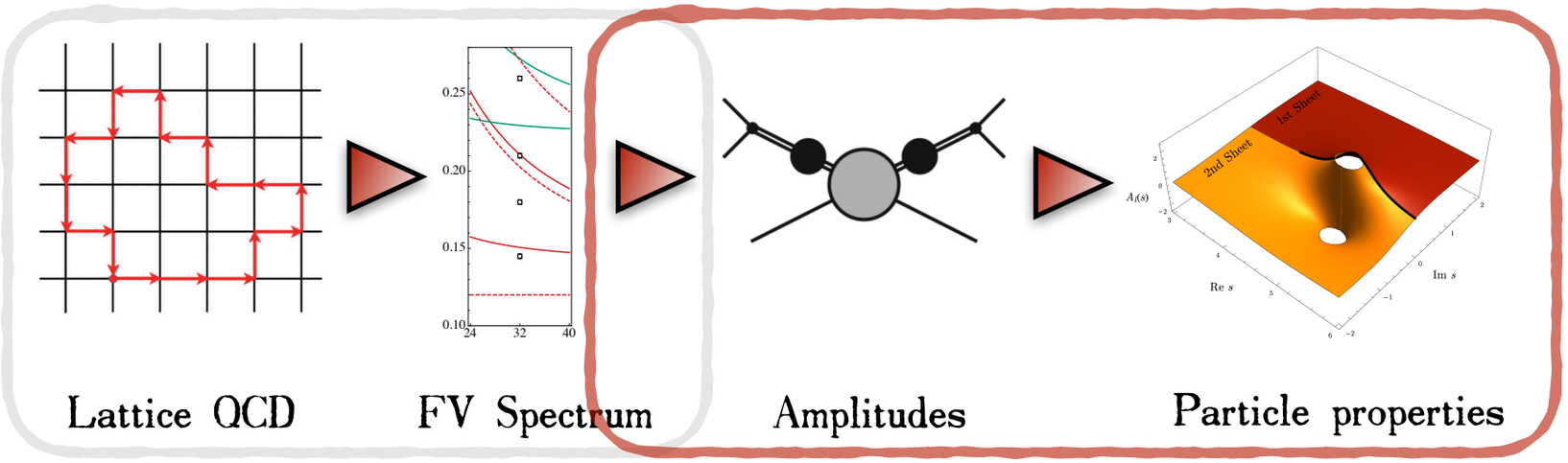
An a₀ resonance in strongly coupled πη,KK scattering from lattice QCD
Dudek, Edwards, Wilson (HadSpec), Phys. Rev. D 100 (2019) 5, 054506



b_1 resonance in coupled $\pi\omega, \pi\phi$ scattering from lattice QCD
Woss et al. (HadSpec), Phys. Rev. D 100 (2019) 5, 054506



The canonical workflow



Our interest

- Finite volume spectrum → Quantization Condition → Three-body K-matrix
- K-matrix + two-body subprocesses → integral equations → 3-body amplitudes
- Final amplitudes analytically continued to the unphysical Riemann sheets

Two main frameworks

- Relativistic Effective Field Theory
  - generic scalar EFT,
  - summation of 2PI, 3PI diagrams,

  (references→next few slides)
- First results – three pions at I=3

  Blanton et al., Phys. Rev. Lett. 124 (2020) 3, 032001
  Hansen et al., Phys. Rev. Lett. 126 (2021) 012001

Jackura et al.
Phys.Rev.D 100 (2019) 3, 034508

Blanton, Sharpe,
Phys.Rev.D 102 (2020) 5, 054515

- Unitarity-based framework
  - parametrization based on the S matrix unitarity,

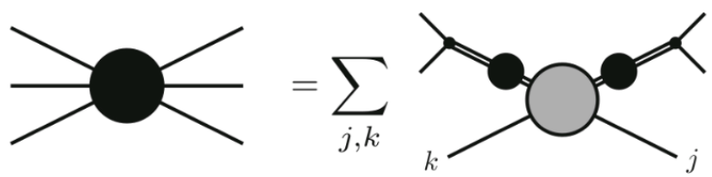
  (references→next few slides)
- First results – three pions at I=3

  Mai, Döring, Phys. Rev. Lett. 122, 062503 (2019)
  Culver et al., Phys. Rev. D 101 (2020) 11, 114507

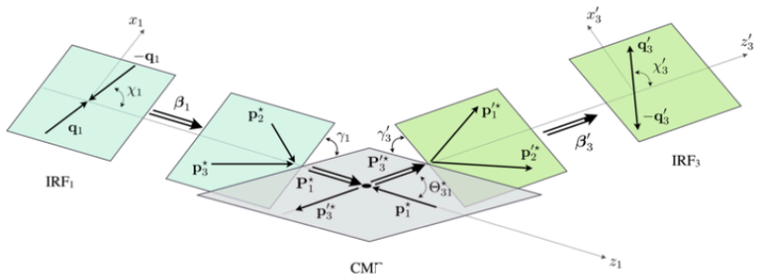
Three-body S-matrix unitarity in one slide

Phenomenology of relativistic 3 → 3 reaction amplitudes within the isobar approximation

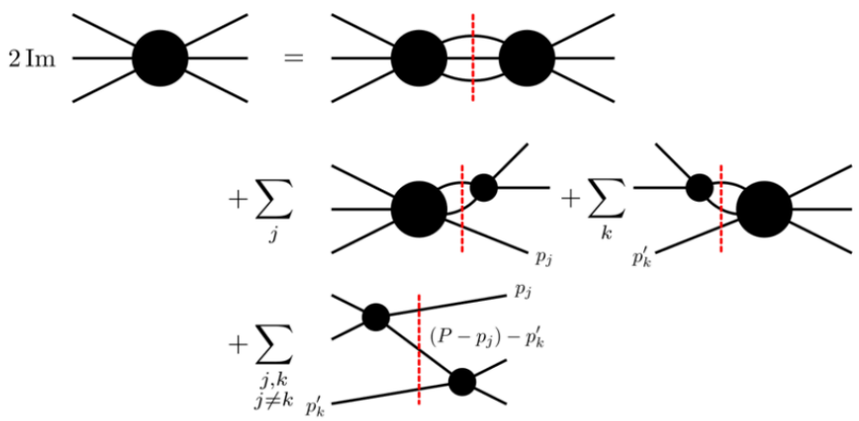
Jackura et al. (JPAC), Eur. Phys. J. C (2019) 1, 56



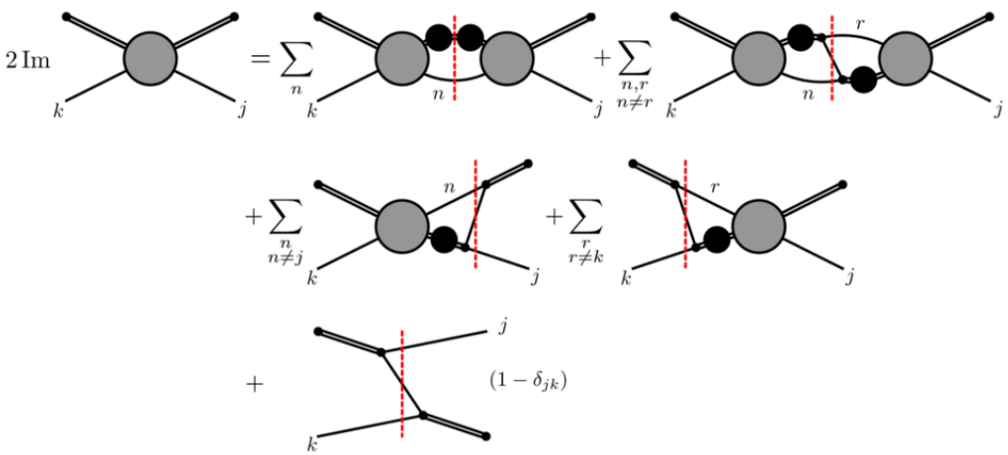
Pair/Isobar and Spectator choice



Partial-wave projection



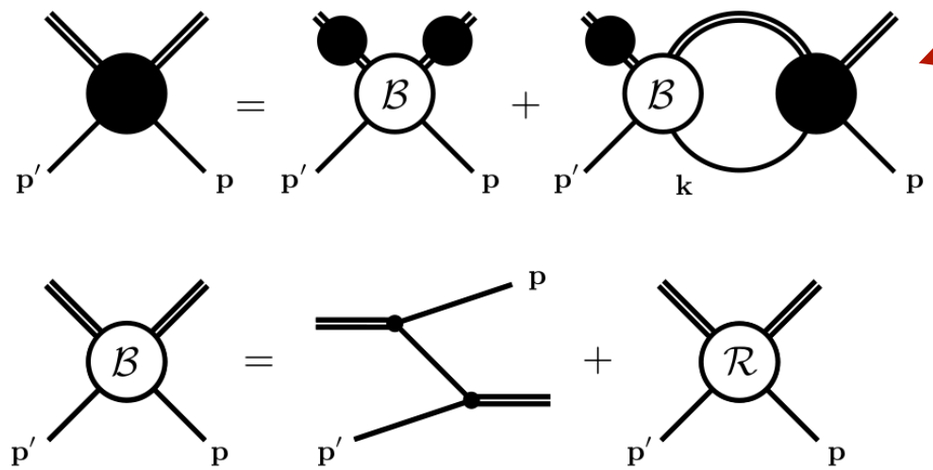
Three-body unitarity relation



Amputation of two-body interactions

The B-matrix approach

- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation



One Particle Exchange Short Range Interactions

Three-body amplitude

$$\mathcal{A}_{\ell m_\ell;\ell' m_\ell}(\sigma', s, \sigma)$$

- pair-spectator,
- partial waves,
- symmetrization,

$$\mathcal{A} = \mathcal{M}_2 \, \mathcal{B} \, \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B} \, \tau \, \mathcal{A}$$

$$\tilde{\mathcal{A}} = \mathcal{B} + \int \mathcal{B} \, \mathcal{M}_2 \, \tau \, \tilde{\mathcal{A}}$$

$$\mathcal{M}_2 = K + K \, i\rho \, \mathcal{M}_2$$

Multi-particle systems on the lattice and chiral extrapolations: a brief review
Mai, Döring, Rusetsky, Eur. Phys. J. ST 230 (2017) 6, 1623

The unitarity-based approach

- **Formalism**

  Relativistic three-body theory with applications to π-N scattering
  Aaron, Amado, Young, Phys. Rev. 174, 2022 (1968)

  Three-body unitarity in finite volume
  Mai, Döring, Eur. Phys. J. A 53 (2017) 12, 240

  Three-body unitarity with Isobars revisited
  Mai et al., Eur. Phys. J. A 53 (2017) 9, 177

  Three-body spectrum in a finite volume: the role of cubic symmetry
  Döring et al., Phys. Rev. D 97 (2018) 11, 114508

  Phenomenology of relativistic 3 → 3 reaction amplitudes within the isobar approximation
  Jackura et al. (JPAC), Eur. Phys. J. C (2019) 1, 56

  Three-body scattering: ladders and resonances
  Mikhasenko et al. (JPAC), JHEP 08 (2019) 1, 080

  Bound states in the B-matrix formalism for the three-body scattering
  Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009

older papers from 60's by Fleming, Holman, Grisaru, Blankenbecler, Cook, Lee, Hwa, Greben, Kok, etc.

- **Lattice spectrum and applications**

  Finite-Volume Spectrum of $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ Systems
  Mai, Döring, Phys. Rev. Lett. 122 (2019) 6, 062503

  Three-body unitarity vs finite-volume $\pi^+\pi^+\pi^+$ spectrum from lattice QCD
  Mai et al., Phys. Rev. D 101 (2020) 5, 054510

  Dalitz plots and lineshape of $a_1(1260)$ from relativistic three-body unitary approach
  Sadasivan et al., Phys. Rev. D 101 (2020) 9, 094018

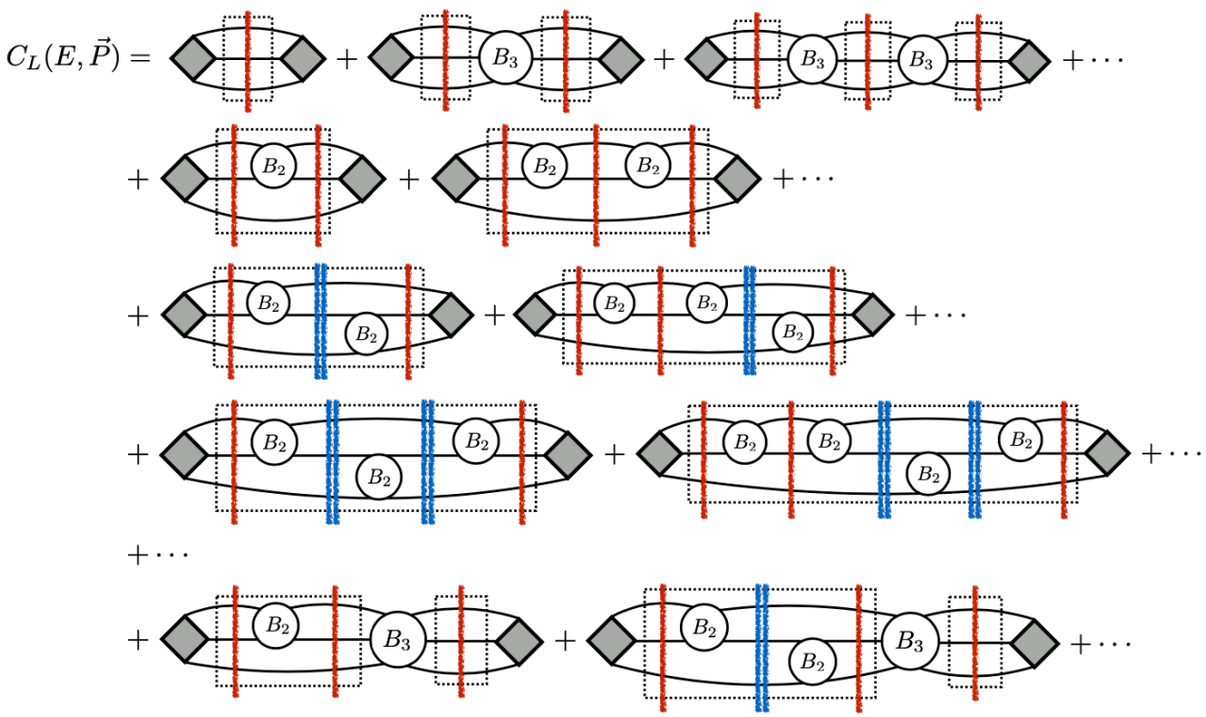
  Three-pion spectrum in the I=3 channel from lattice QCD
  Culver et al., Phys. Rev. D 101 (2020) 11, 114507

  Finite volume energy spectrum of the $K^-K^-K^-$ system
  Alexandru et al., Phys. Rev. D 102 (2020) 11, 114523

  Three-body interactions from the finite-volume spectrum
  Brett et al., Phys. Rev. D 104 (2021) 1, 014501

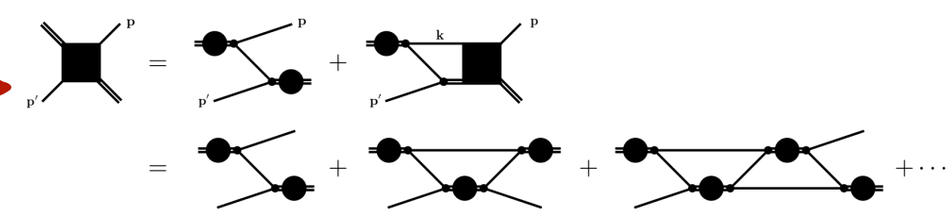
  Three-body dynamics of the $a_1(1260)$ resonance from lattice QCD
  Mai et al. (GWQCD), arxiv:2107.03973 (2021)

REFT three-body formalism

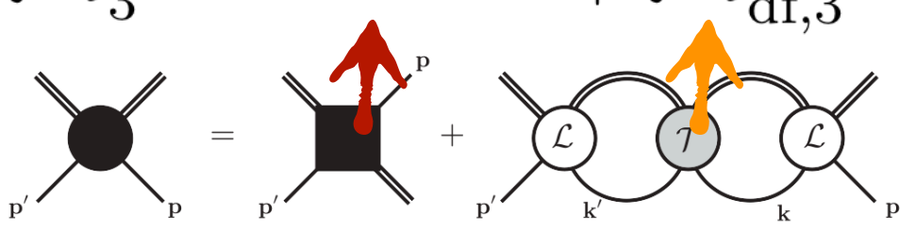


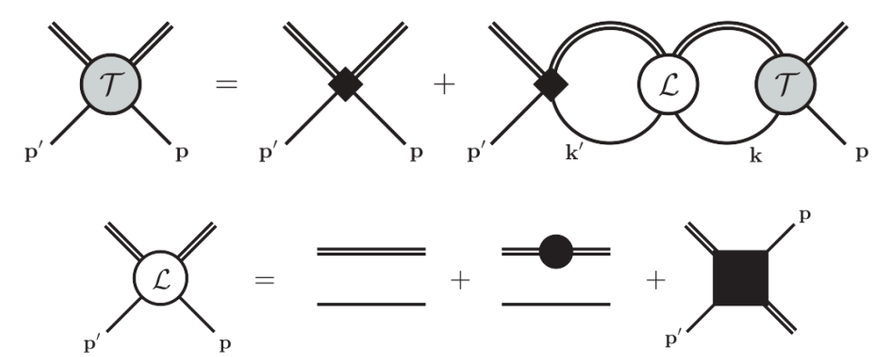
$$\det\left[\mathcal{K}_{\mathrm{df},3}(s) + F_3(s,\boldsymbol{P},L)^{-1}\right] = 0$$

- Infinite volume integral equations:



$$\mathcal{M}_3^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{M}_{\mathrm{df},3}^{(u,u)}$$





REFT approach

- **Formalism**

  Relativistic, model-independent, three-particle quantization condition
  Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

  Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude
  Hansen, Sharpe, Phys. Rev. D (2015) 11, 114509

  Threshold expansion of the three-particle quantization condition
  Hansen, Sharpe, Phys. Rev. D 93 (2016) 9, 096006

  Relating the FV spectrum and the 2-and-3-particle S matrix for relativistic systems of identical scalar particles
  Briceño, Hansen, Sharpe, Phys. Rev. D 95 (2017) 7, 074510

  Three-particle systems with resonant subprocesses in a finite volume
  Briceño, Hansen, Sharpe, Phys. Rev. D 99 (2019) 7, 014516

  Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism
  Briceño et al. Phys. Rev. D 100 (2019) 5, 054508

  Generalizing the relativistic quantization condition to include all three-pion isospin channels
  Hansen, Romero-López, Sharpe, JHEP 07 (2020) 047

  Relativistic three-particle quantization condition for nondegenerate scalars
  Blanton, Sharpe, Phys. Rev. D 103 (2021) 5, 054503

  Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ and related systems
  Blanton, Sharpe, Phys. Rev. D 104 (2021) 3, 034509

- **Lattice spectrum and applications**

  Numerical study of the relativistic three-body quantization condition in the isotropic approximation
  Briceño, Hansen, Sharpe, Phys. Rev. D 98 (2018) 1, 014506

  Implementing the three-particle quantization condition including higher partial waves
  Blanton, Romero-López, Sharpe, JHEP 03 (2019) 106

  I=3 Three-Pion Scattering Amplitude from Lattice QCD
  Blanton, Romero-López, Sharpe, Phys. Rev. Lett. 124 (2020) 3, 032001

  Interactions of two and three mesons including higher partial waves from lattice QCD
  Blanton et al., arxiv:2106.05590 (2021)

  Energy dependent $\pi^+\pi^+\pi^+$ scattering amplitude from lattice QCD
  Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021) 012001

Equivalence of relativistic three-particle quantization conditions
Blanton, Sharpe, Phys.Rev.D 102 (2020) 5, 054515

Equivalence of three-particle scattering formalisms
Jackura et al. (JPAC), Phys.Rev.D 100 (2019) 3, 034508

Equivalence

- Both formalisms equivalent in the FV and IV form

$$\int_{\mathbf{k}} \left[\delta_{\mathbf{p}'\mathbf{k}} + \frac{1}{3} \int_{\mathbf{k}'} \mathcal{U}_{\mathbf{p}'\mathbf{k}'} \mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}} \right] \mathcal{R}_{\mathbf{k}\mathbf{p}} = \int_{\mathbf{k}'} \int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'} \mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}} \mathcal{U}_{\mathbf{k}\mathbf{p}}$$

$$\det \left[\overline{\mathcal{K}}_{2,L}^{-1} + \widetilde{F} + \widetilde{G} - (2\omega L^3)^{-1} \mathcal{R}^{(u,u)} (2\omega L^3)^{-1} \right] = 0$$

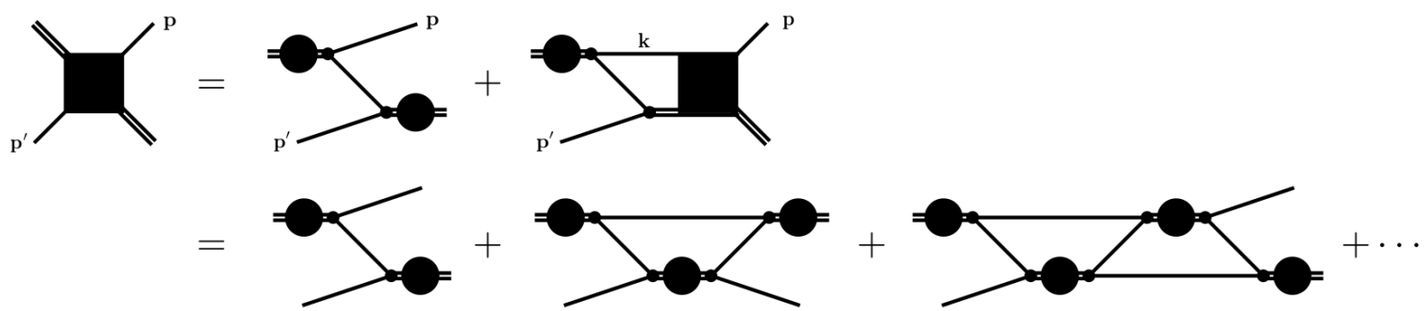
- In practice, differences in the regularization schemes exist in the literature

Short summary

- Three-body physics is not only interesting, but also important for studies of resonances,
  - Lattice QCD,
  - Phenomenology,
- The two-body Lüscher philosophy is well-developed, and guides the three-body research,
- There are two equivalent relativistic formalisms,
  - Unitarity-based,
  - EFT,
- R-matrix or K_{df} are obtained from lattice or modeled; they are related to the physical amplitudes via integral equations,
- Amplitudes must have good analytic properties,

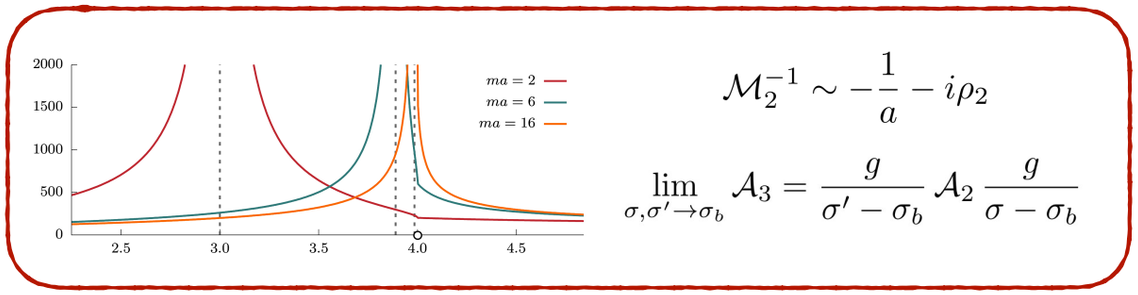
Solving the EFT three-body ladder equation

- Ladder approximation, B = G + (R=0)

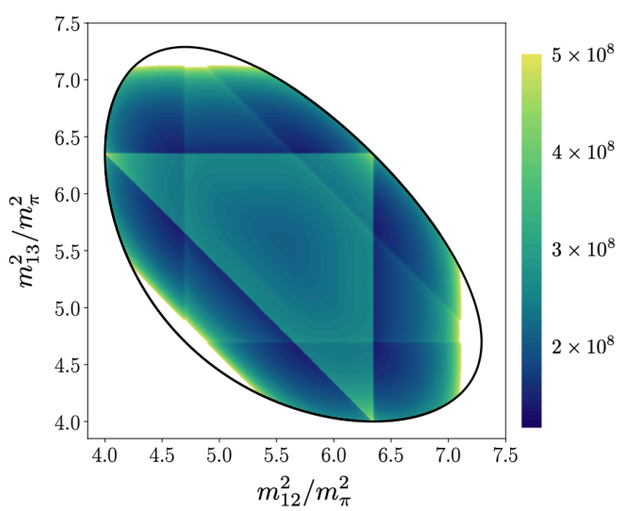


- Numerical solution of the three-body EFT equations

A. Jackura, R. Briceño, S. Dawid, M. H. E Islam, C. McCarty, Phys. Rev. D 104 (2021) 1, 014507



- Similar studies



- **weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$**

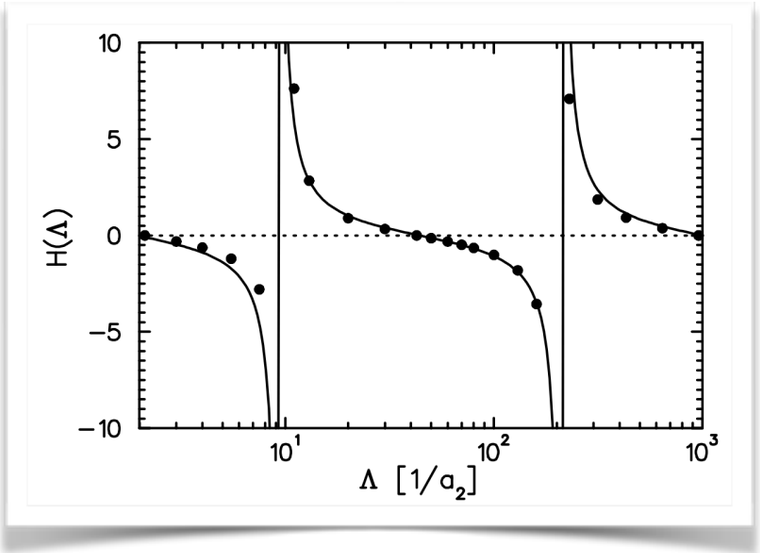
  Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

- **decay** $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$

  Sadasivan et al., Phys.Rev.D 101 (2020) 9, 094018

Nonrelativistic studies

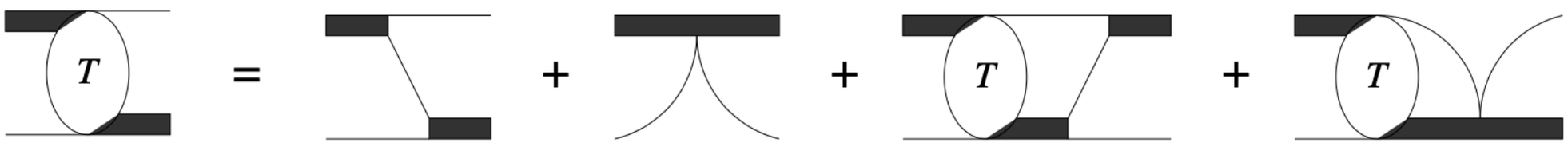
The Three-Boson System with Short-Range Interactions
Bedaque, Hammer, van Kolck, Nucl. Phys. A 646 (1999) 444



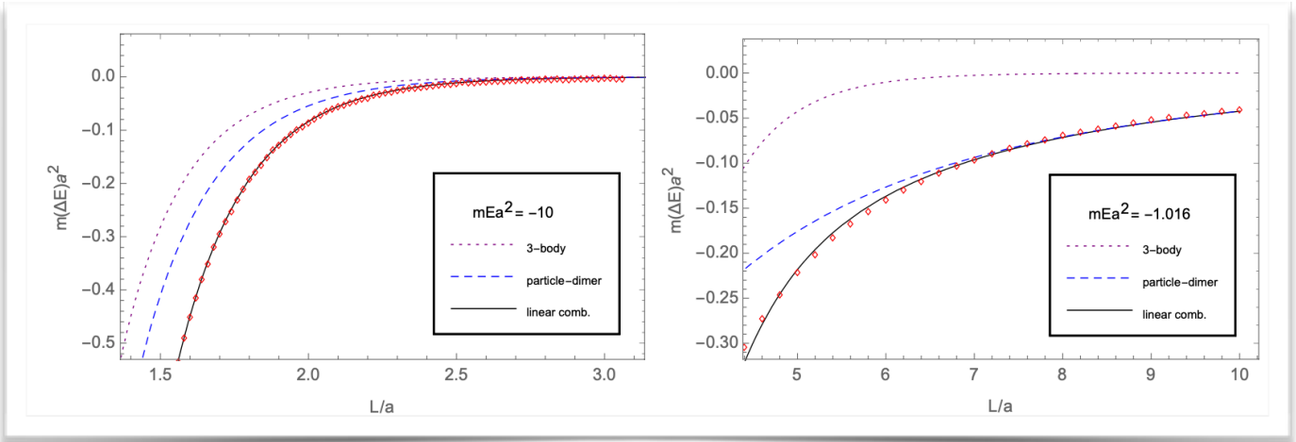
- NREFT approach

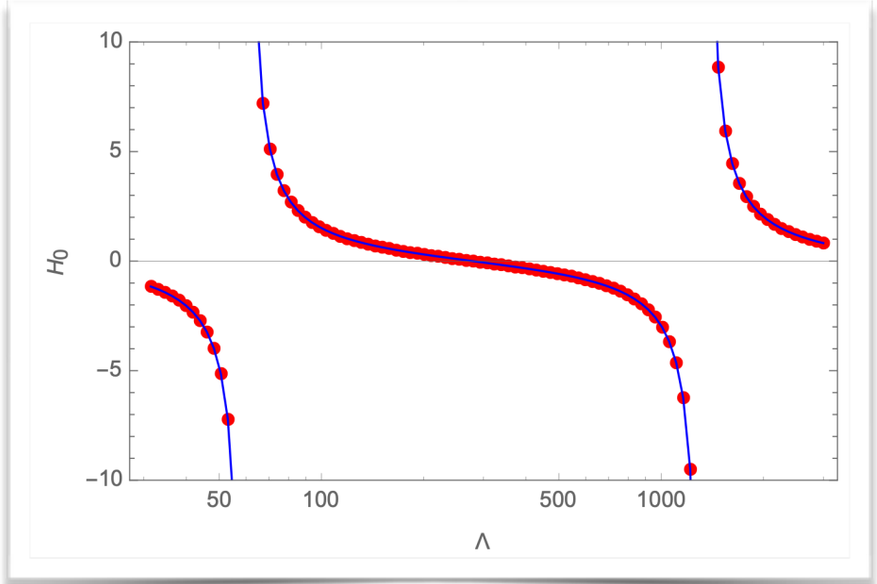
$$\mathcal{L} = \psi^\dagger(i\partial_0 + \frac{\vec{\nabla}^2}{2m})\psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}}(T^\dagger\psi\psi + \text{h.c.}) + hT^\dagger T\psi^\dagger\psi + \ldots$$

- The three-body equation



Three-body spectrum in a finite volume: the role of cubic symmetry
Döring et al., Phys. Rev. D 97 (2018) 11, 114508





Bound-state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

- Start with the quantization condition

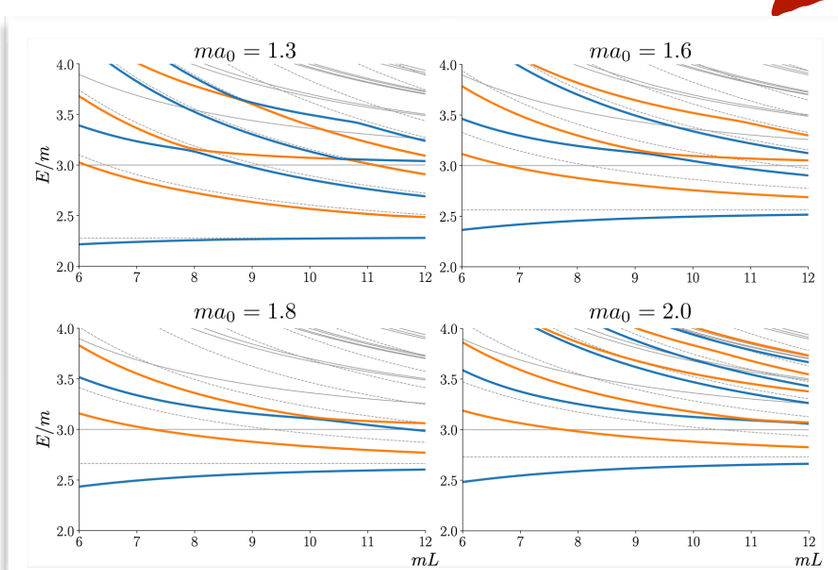
$$\det\left[\mathcal{K}_{\mathrm{df},3}(s) + F_3(s, \boldsymbol{P}, L)^{-1}\right] = 0$$

S-wave, P=0, and K_{df}=0

- Geometric function

$$L^3 F_3^s \equiv \frac{\widetilde{F}^s}{3} - \widetilde{F}^s \frac{1}{1/\widetilde{\mathcal{K}}_2^s + \widetilde{F}^s + \widetilde{G}^s} \widetilde{F}^s$$

Energy levels

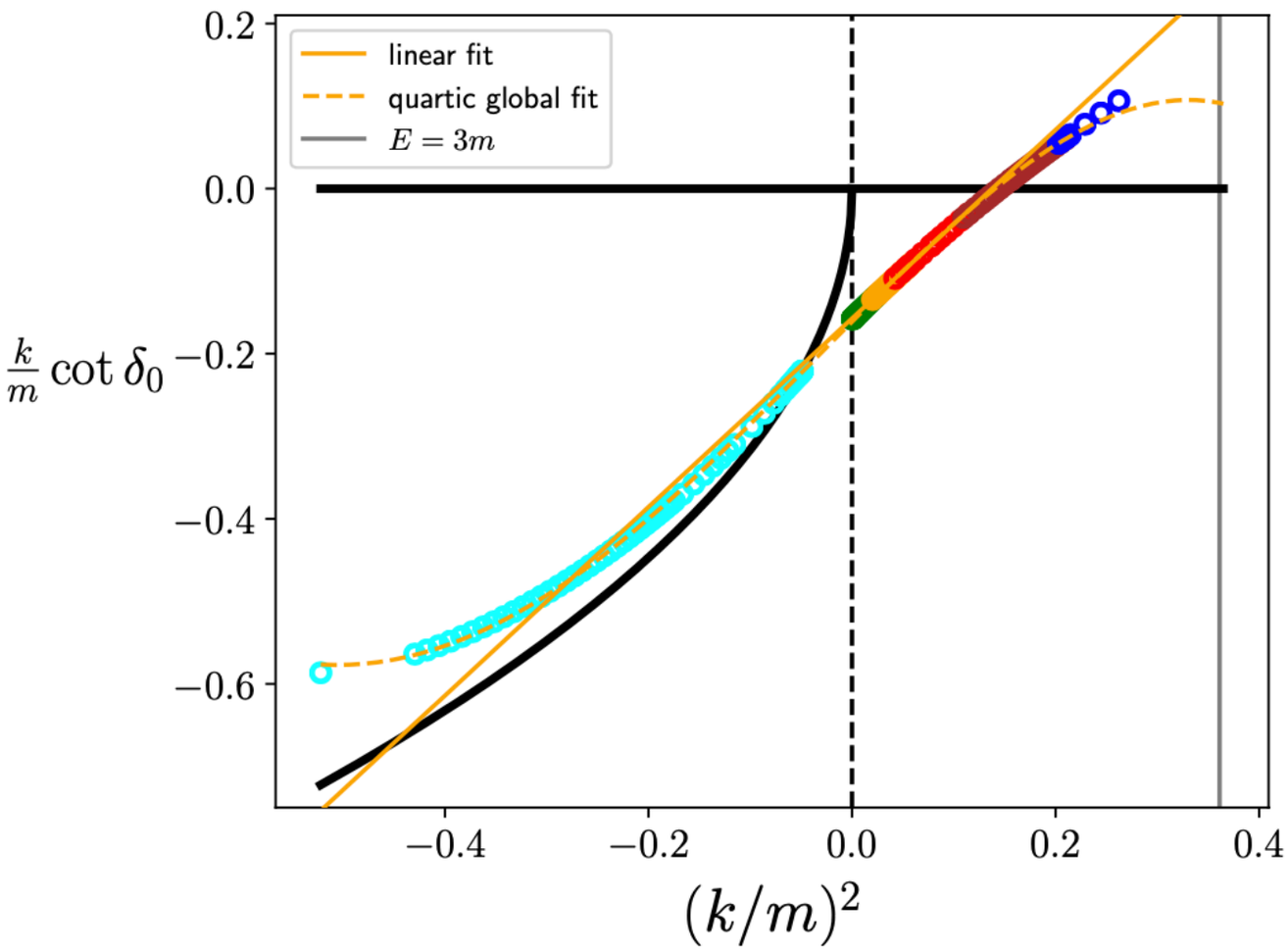


- Noninteracting 3-particle states,
- Noninteracting dimer-particle states,
- Interacting 3-particle, dimer-particle, and trimer states

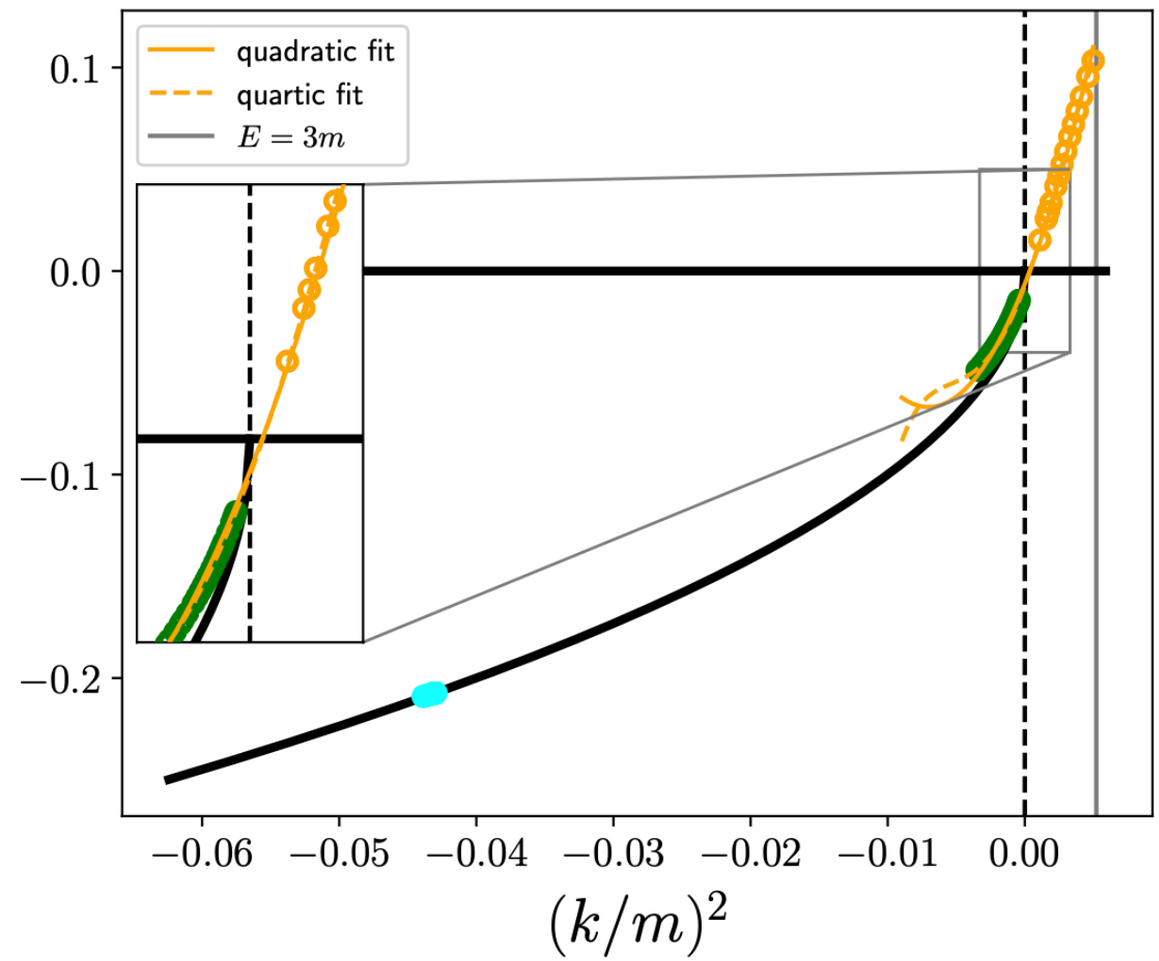
2-body quantization condition

Bound state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007



a=2



a=16

Numerical procedure

- Discretization of the integral equation ⟶ **N linear equations** (Matrix equation)
- Regulation of the bound-state pole via **ϵ-prescription**

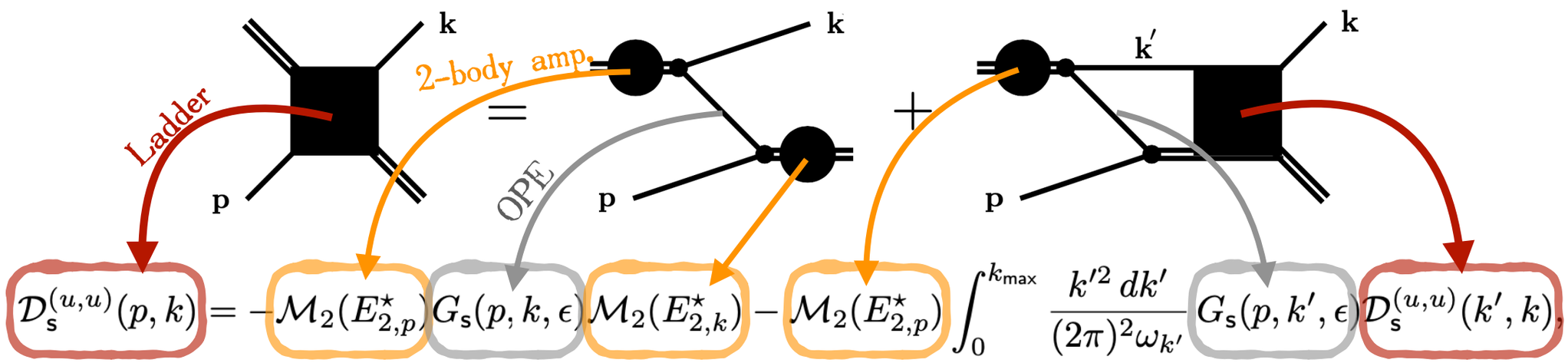
$$\mathcal{A}_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \mathcal{A}_2(s; N, \epsilon)$$

- Systematics
  - **Unitarity test:** $\mathrm{Im}\mathcal{A}_2(s) = \rho_2(s)|\mathcal{A}_2(s)|^2 \longrightarrow \Delta\rho_2 = 100 \times \left| \frac{\mathrm{Im}\,\mathcal{A}_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right|$
  - **Convergence test:** $\Delta_N \mathcal{A}_2 = 2 \times \left| \frac{\mathcal{A}_2(s; N+1, \epsilon) - \mathcal{A}_2(s; N, \epsilon)}{\mathcal{A}_2(s; N+1, \epsilon) + \mathcal{A}_2(s; N, \epsilon)} \right|$

- Methods

"Brute force" | Explicit pole removal | Spline-based quadratures

Brute force method



$$\mathcal{D}_{\mathsf{s}}^{(u,u)}(p,k) = -\mathcal{M}_2(E^\star_{2,p})\, G_{\mathsf{s}}(p,k,\epsilon)\, \mathcal{M}_2(E^\star_{2,k}) - \mathcal{M}_2(E^\star_{2,p}) \int_0^{k_{\max}} \frac{k'^2\, dk'}{(2\pi)^2 \omega_{k'}}\, G_{\mathsf{s}}(p,k',\epsilon)\, \mathcal{D}_{\mathsf{s}}^{(u,u)}(k',k),$$

Discretization

$$\boldsymbol{d}(N,\epsilon) = -\boldsymbol{G}(\epsilon) - \boldsymbol{P}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{d}(N,\epsilon)$$

Amputation

$$\mathcal{D}^{(u,u)}(\mathbf{p},\mathbf{k}) \equiv \mathcal{M}_2(p)\, d^{(u,u)}(\mathbf{p},\mathbf{k})\, \mathcal{M}_2(k)$$

Matrix inversion

$$\boldsymbol{d}(N,\epsilon) = -\left[\mathbb{1} + \boldsymbol{P}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}}\right]^{-1}\cdot\boldsymbol{G}(\epsilon)$$

Limits

$$d^{(u,u)}(p,k) = \lim_{\epsilon\to 0}\lim_{N\to\infty} \boldsymbol{d}(N,\epsilon)$$

Double ordered limit ⟶ single limit

$$\epsilon \propto \eta/N$$

Numerical procedures – continuation

- Interpolation

$$d_S^{(u,u)}(p,k) = -G_S(p,k) - \int_0^\infty \frac{\mathrm{d}k' k'^2}{(2\pi)^2 \omega_{k'}} G_S(p,k') \mathcal{M}_2(k') d_S^{(u,u)}(k',k)$$

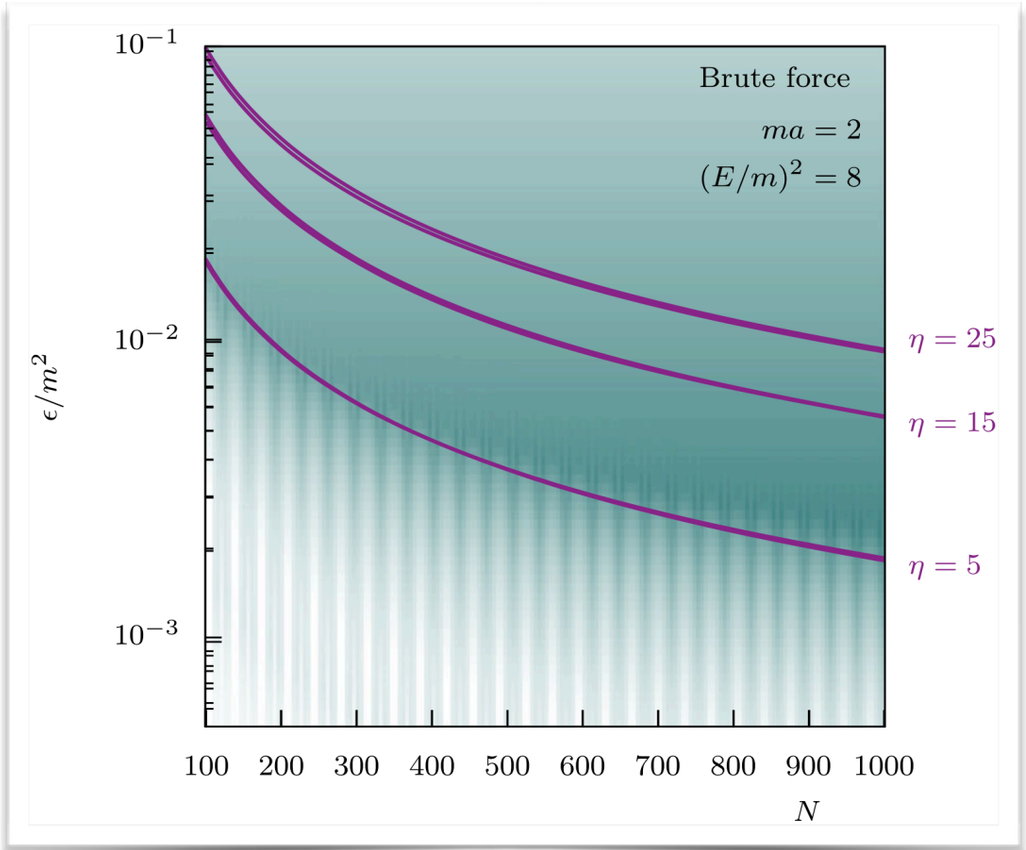
1. Plug the solution
2. Plug the BS momentum
3. Integrate (sum over discrete momenta)

- Limits

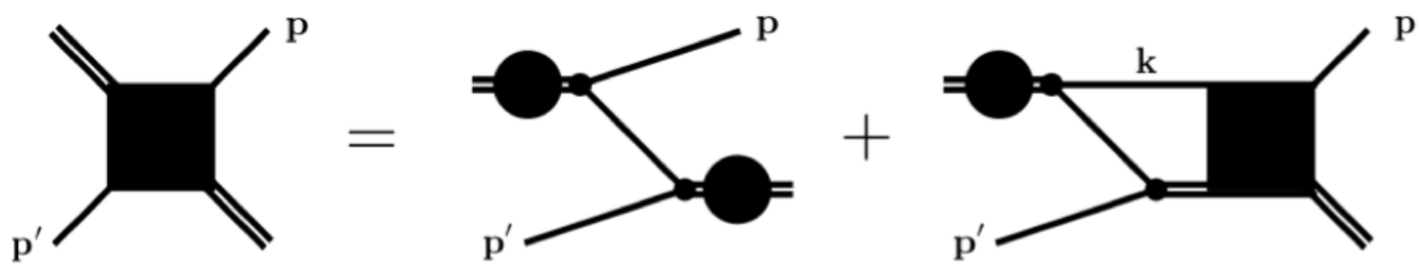
Poisson summation formula ⟶ error

$$\sigma(\epsilon, N) \approx \left| G_s \, \rho_2 \, d^{(u,u)} \right| e^{-\eta}$$

$$\eta = 2\pi\epsilon \, N \times (\text{energy factors})$$



Splines-based method

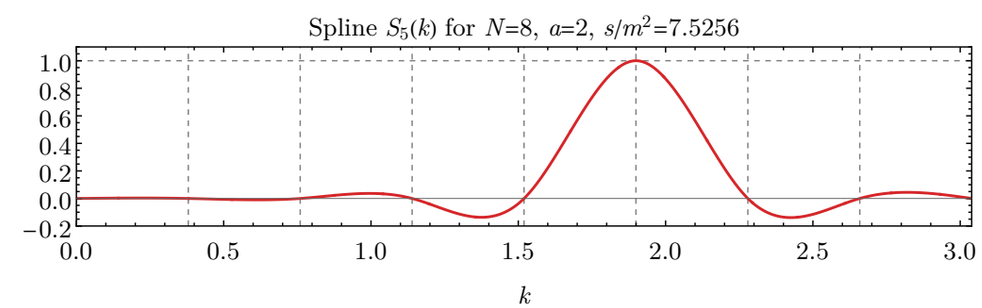


$$d_S^{(u,u)}(p,k) = -G_S(p,k) - \int_0^\infty \frac{\mathrm{d}k' k'^2}{(2\pi)^2 \omega_{k'}} G_S(p,k') \mathcal{M}_2(k') d_S^{(u,u)}(k',k)$$

Quadratures

$$d_S^{(u,u)}(\sigma_k,\sigma_p) = -G_S(\sigma_k,\sigma_p) - \sum_{n=0}^{N_s} \omega_n(\sigma_k,s) d_S^{(u,u)}(\sigma_q,s)$$

$$d_S^{(u,u)}(\sigma_k,\sigma_p) \approx \sum_{n=0}^{N_s} S_n(\sigma_k)\, d_S^{(u,u)}(\sigma_q,\sigma_p)$$

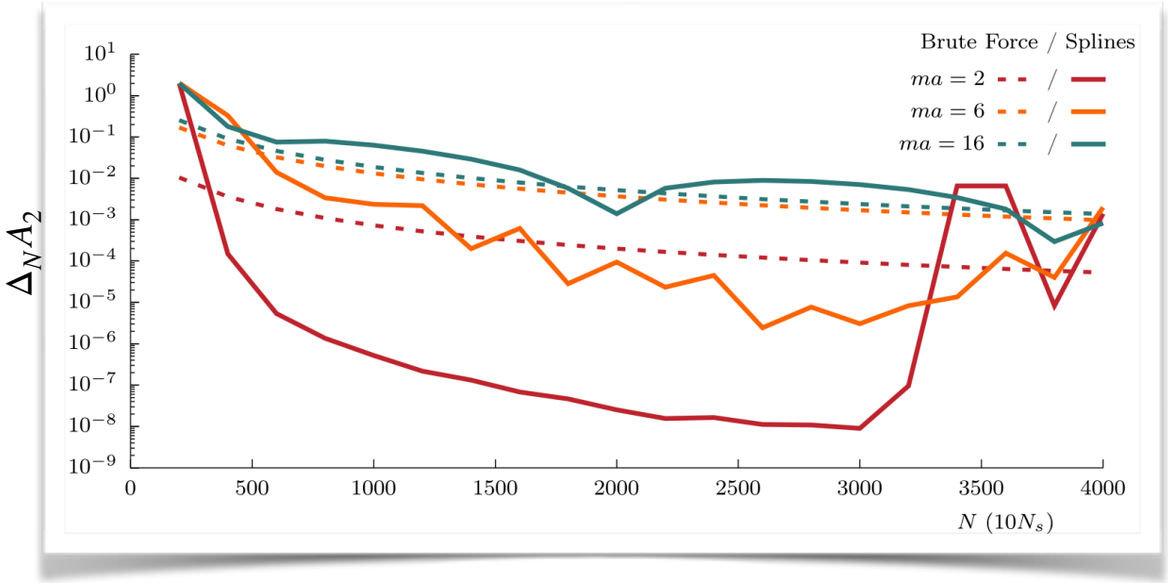


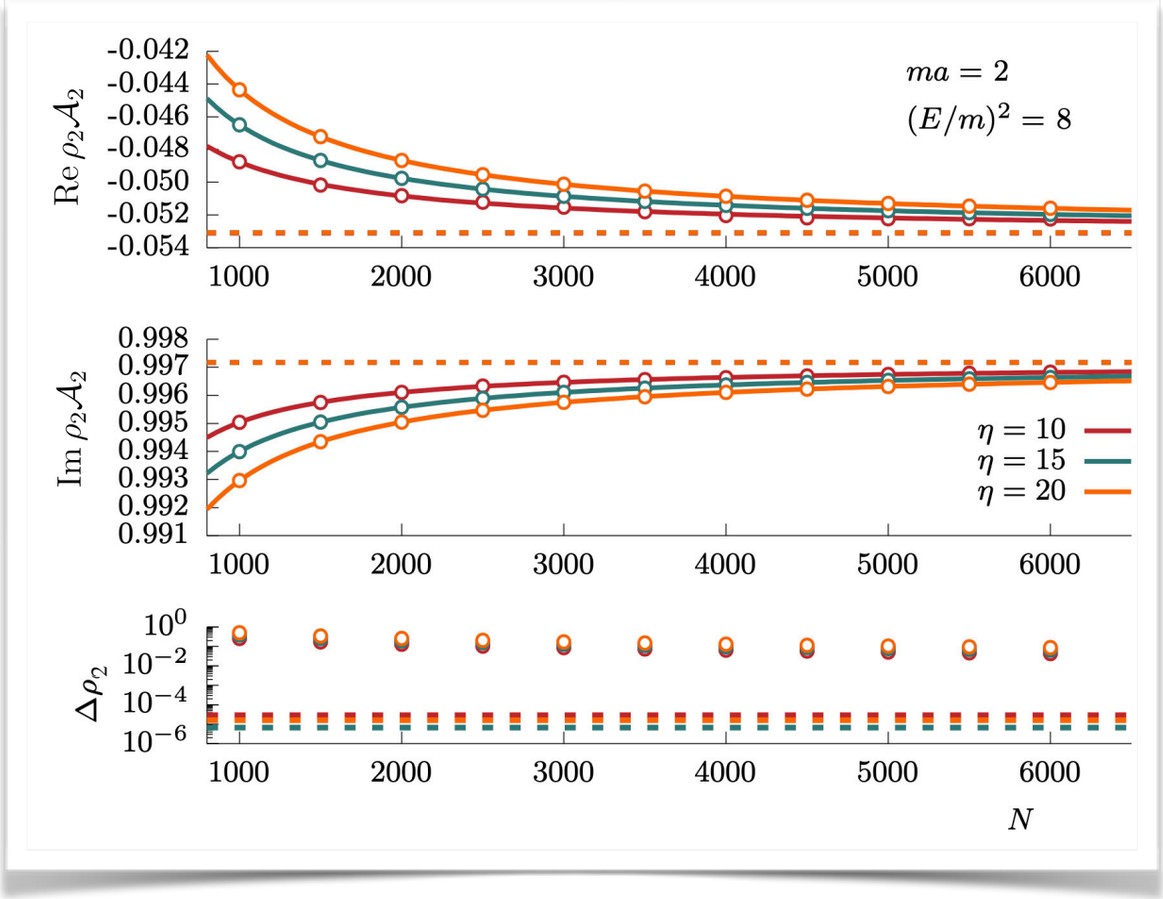
Possible improvement
analytic removal of ϵ

$$\int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{\pi} G_S(\sigma_k,\sigma_q) \frac{\lambda^{1/2}(s,\sigma_q,m^2)}{16\pi s} \mathcal{M}_2(\sigma_q)\, S_i(\sigma_q) = \sum_{n=0}^{N_s} \omega_n(\sigma_k) S_i(\sigma_{q,n})$$

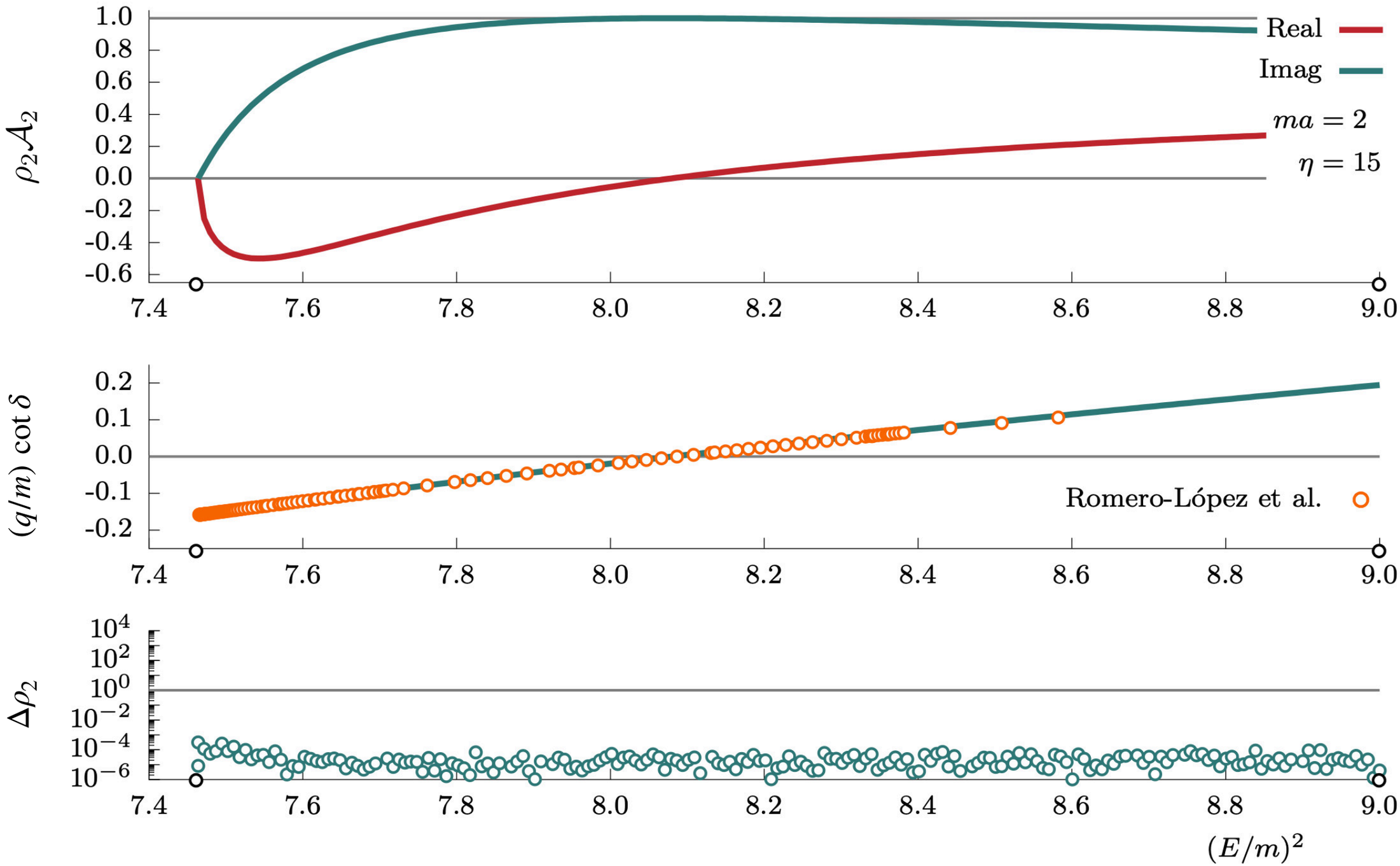
Extrapolations of the results

- Expansion in 1/N
- Epsilon regulator fixed to $\epsilon \propto \eta/N$

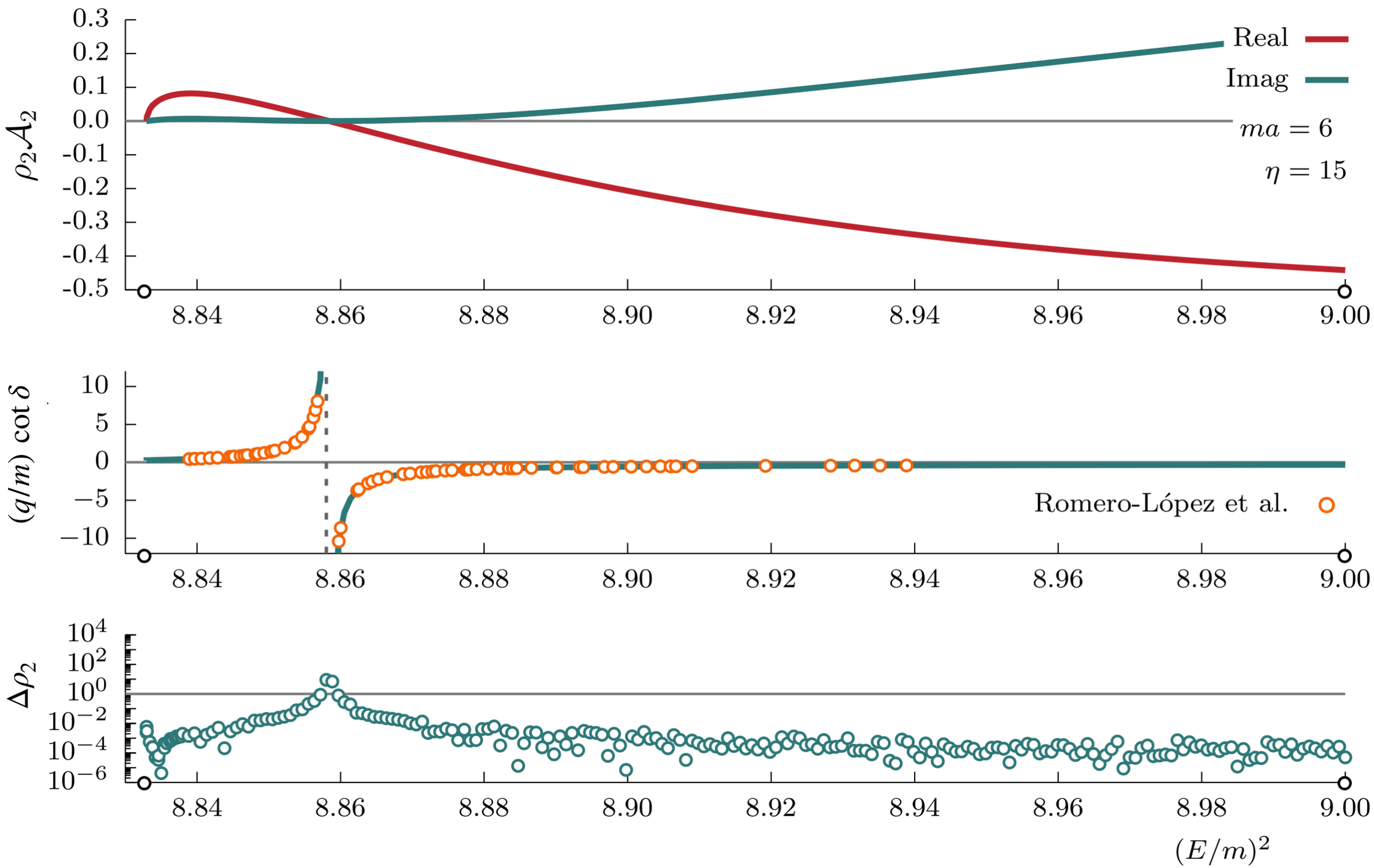




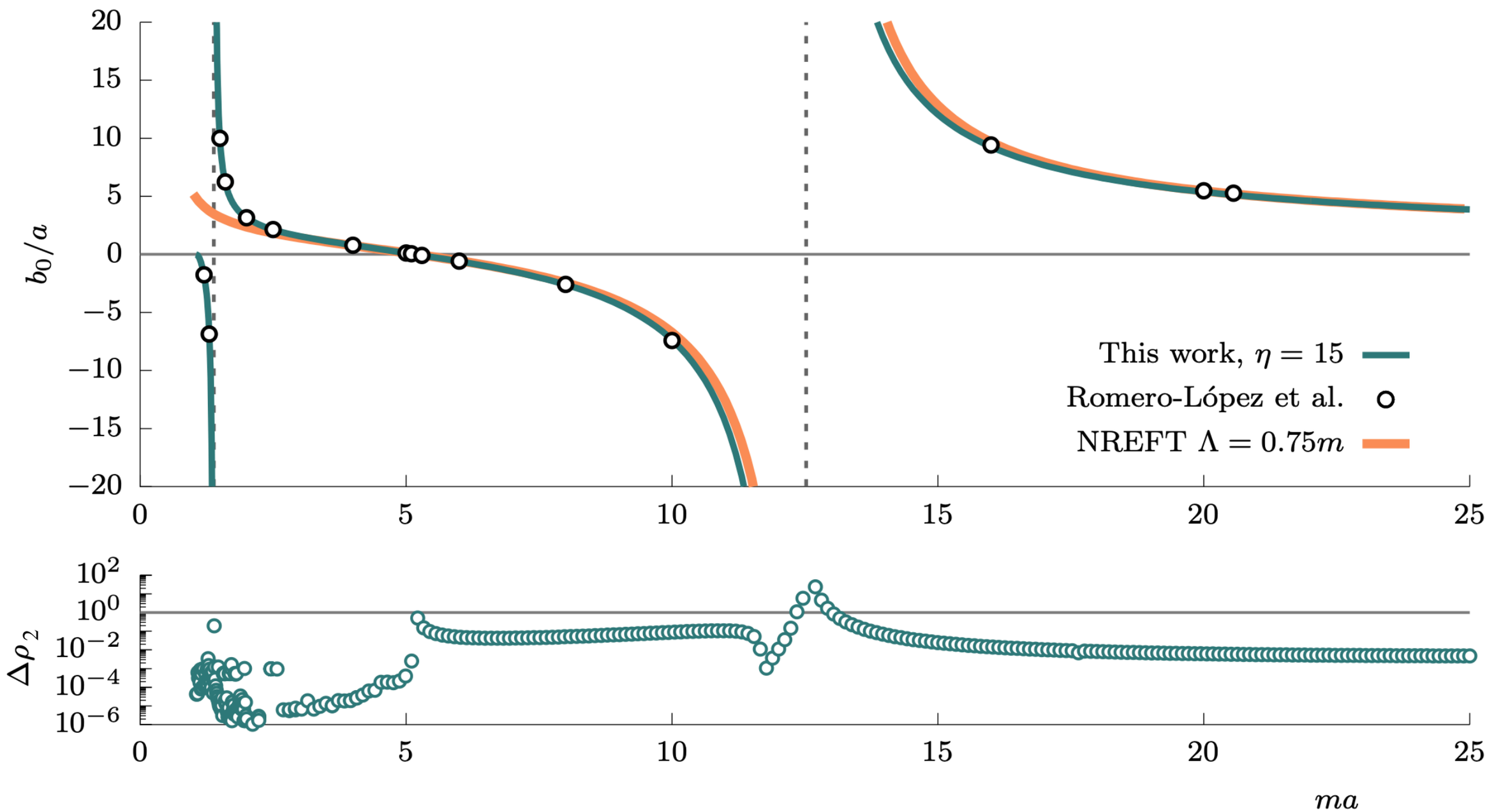
Example results, $M^2 = 3m^2$, i.e. a=2



Example results, $M^2 = 3.89m^2$, i.e. a=6



Example result, three-body scattering length

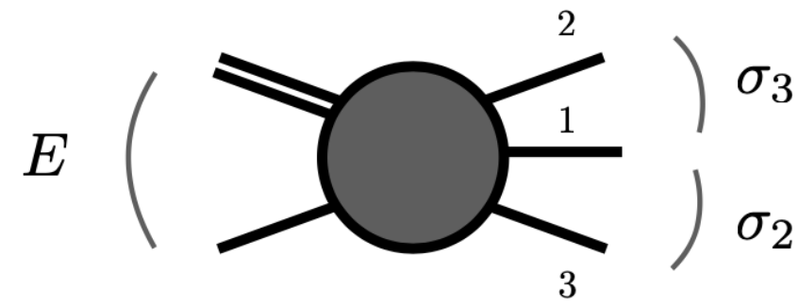


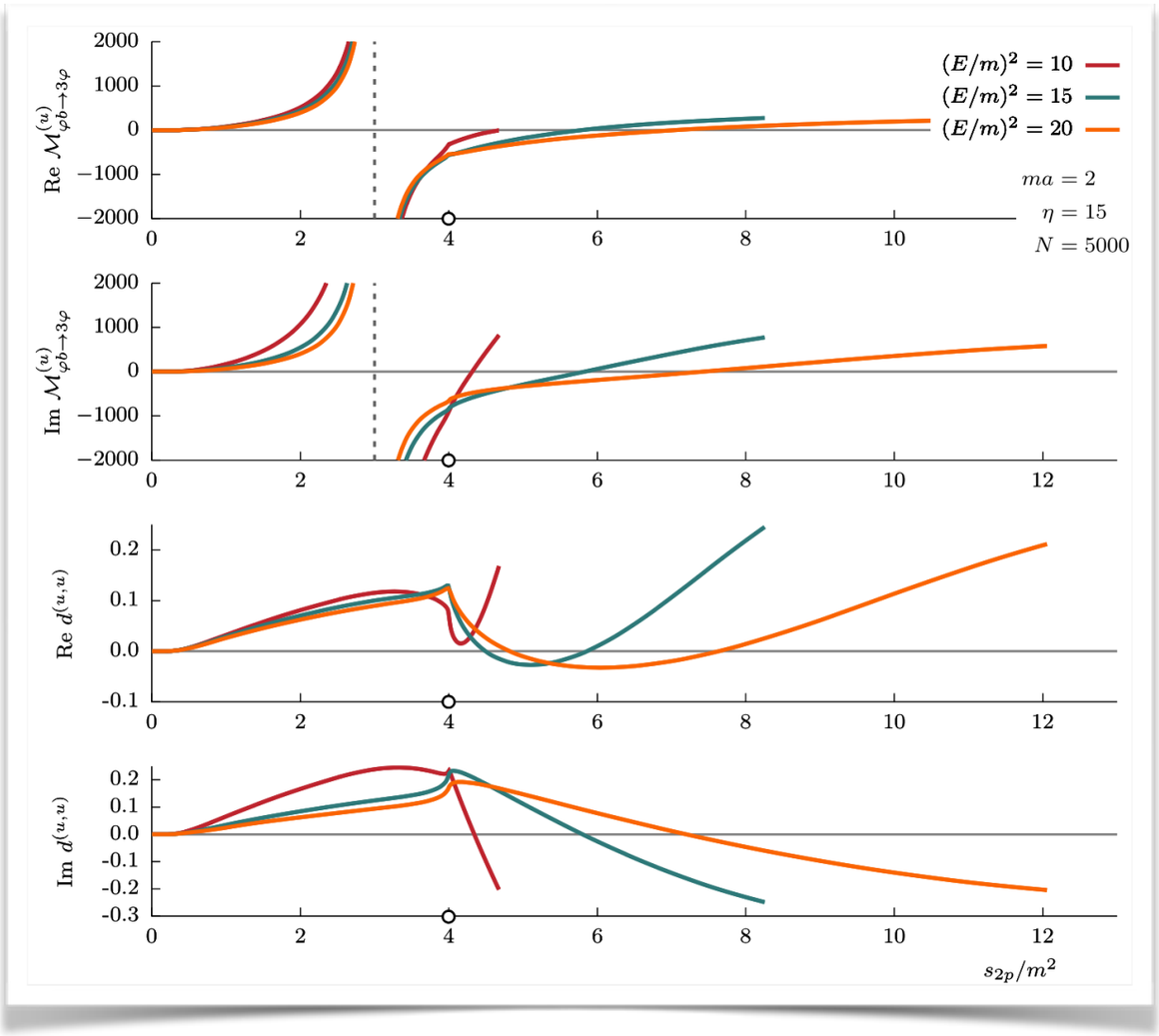
Romero-Lopez et al., JHEP 10 (2019) 007

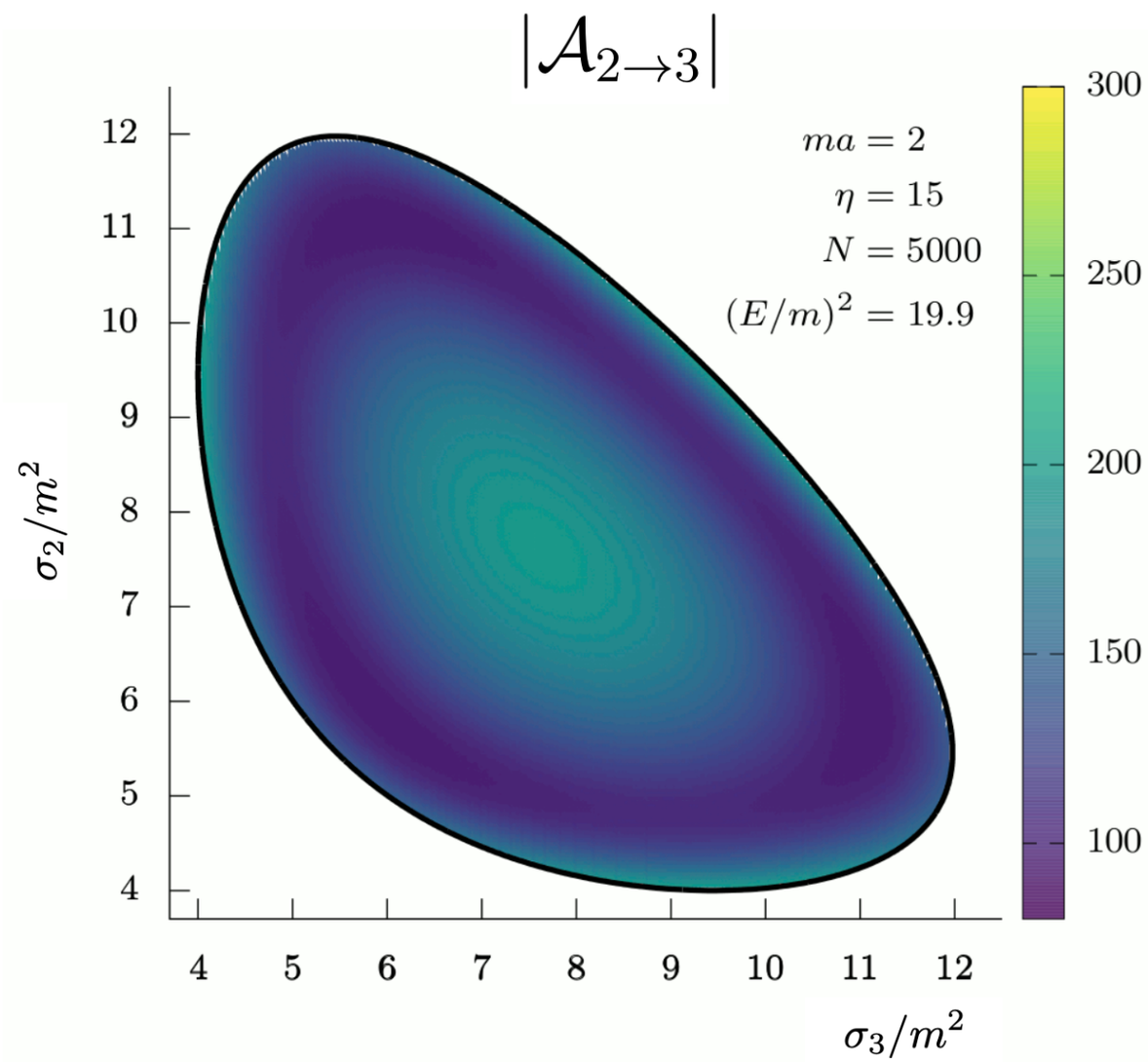
Bedaque et al., Nucl. Phys. A 646 (1999) 444

Example result, 2→3 amplitude

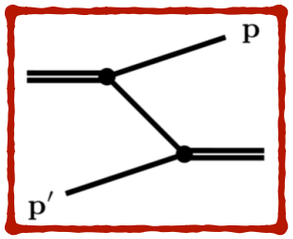
- We are not limited below the three-body threshold,



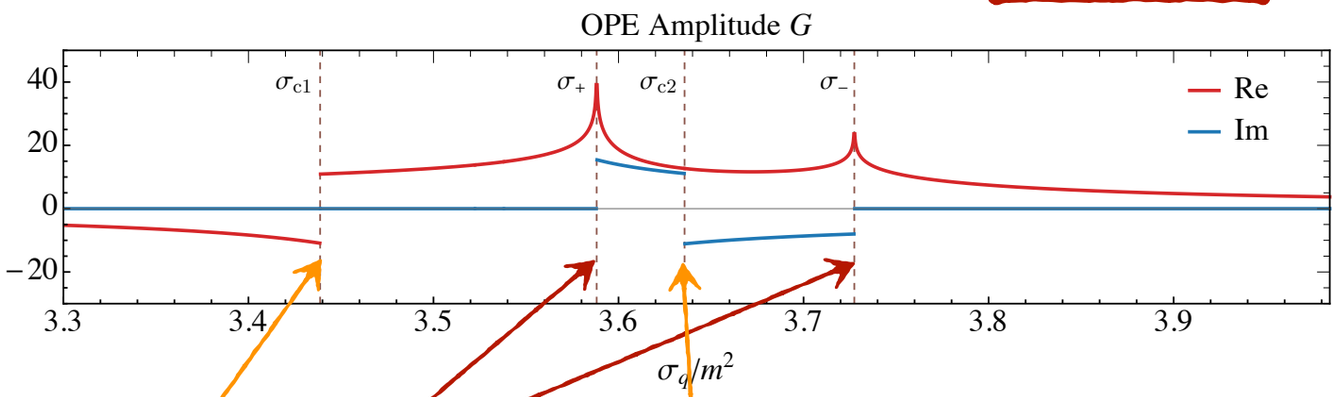




Below the threshold – circular cut



- Analytic structure of the integration kernel
- OPE develops the circular cut



- Real Particle Exchange cut

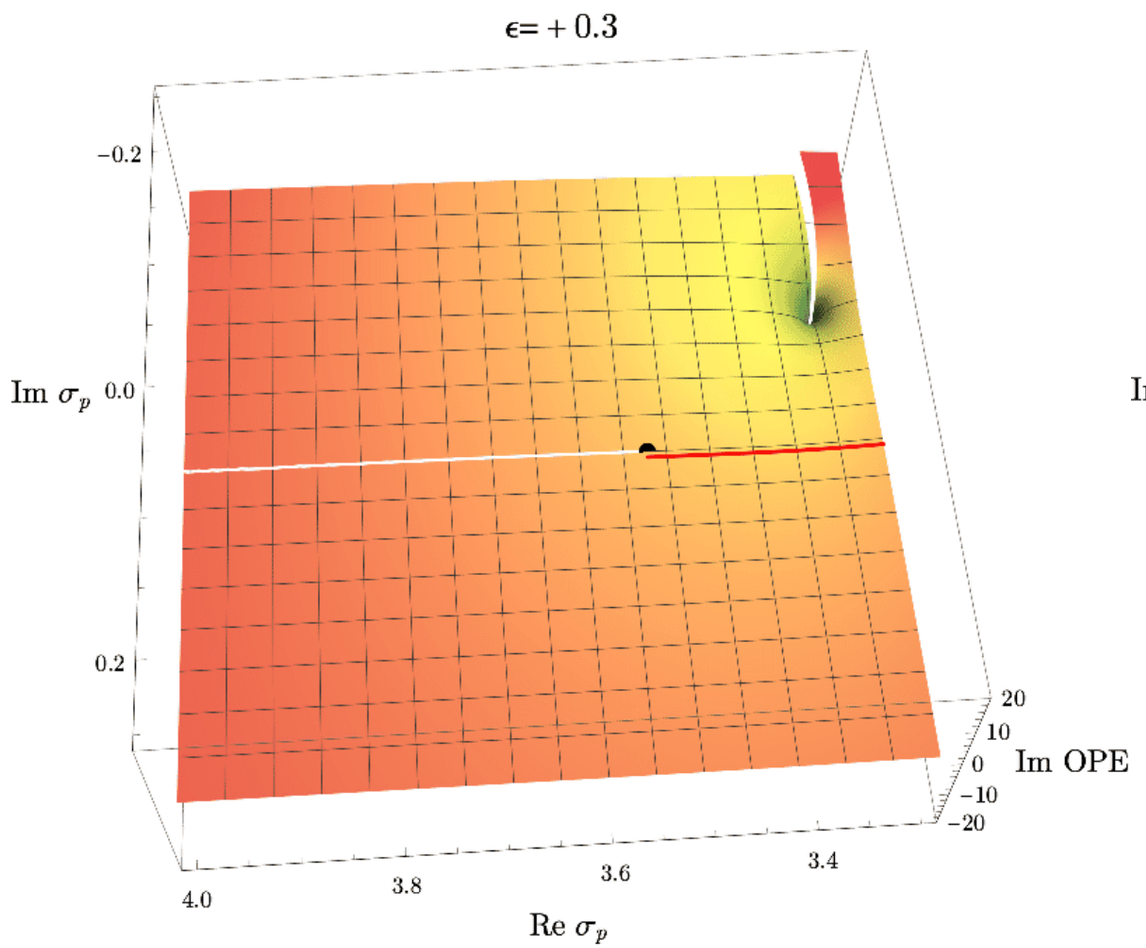
$$\sigma_\pm = \frac{1}{2}(s - \sigma_{p'} + 3m^2) \pm \frac{1}{2\sqrt{\sigma_{p'}}}\sqrt{\sigma_{p'} - 4m^2}\,\lambda^{1/2}(s, \sigma_{p'}, m^2)$$

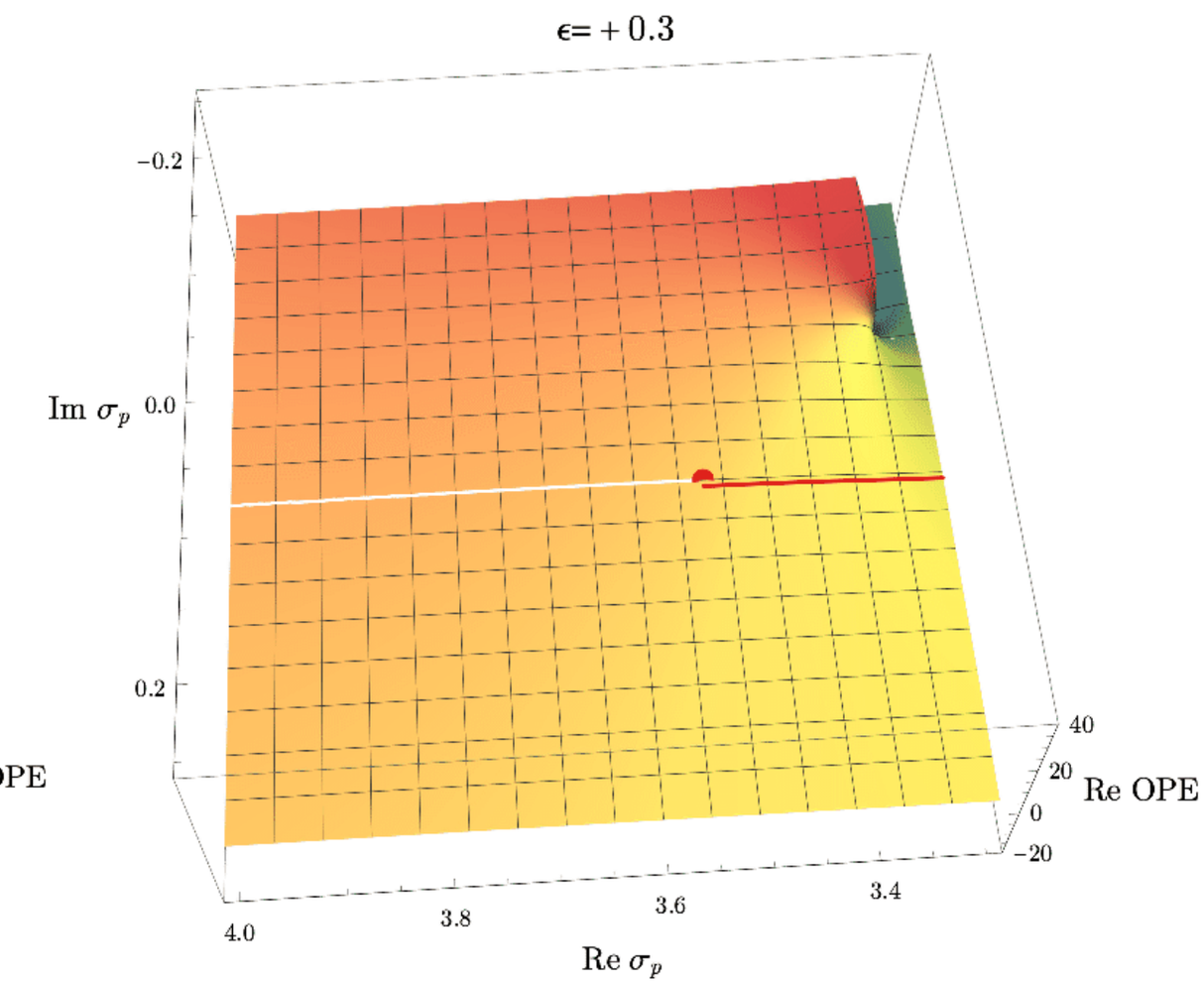
- Crossing the real axis

$$\sigma_{c1} = \frac{(m^2 - s)(m^2 - \sigma_{p'} + s)}{(m^2 - \sigma_{p'} - s)}$$

$$\sigma_{c2} = \frac{(m^2 - s)\,(2m^2 - \sigma_{p'})}{\sigma_{p'}}$$

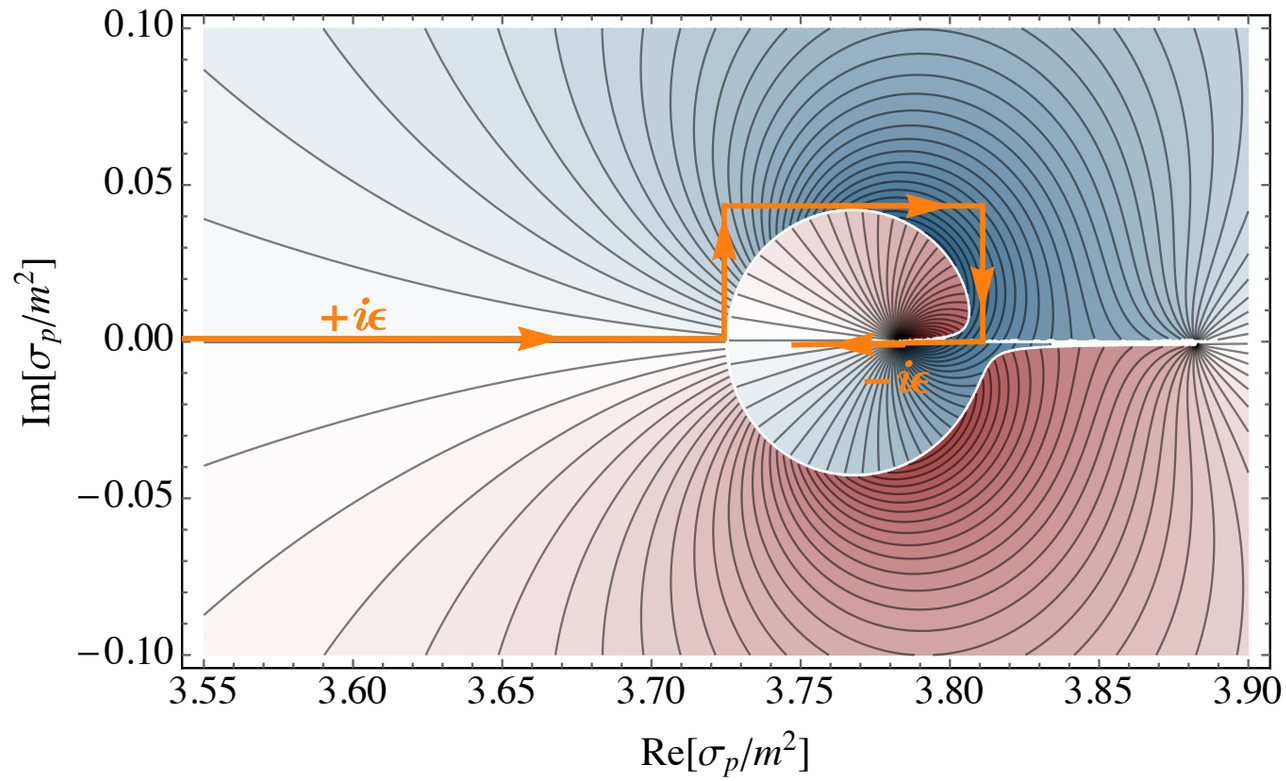
Circular cut in motion picture

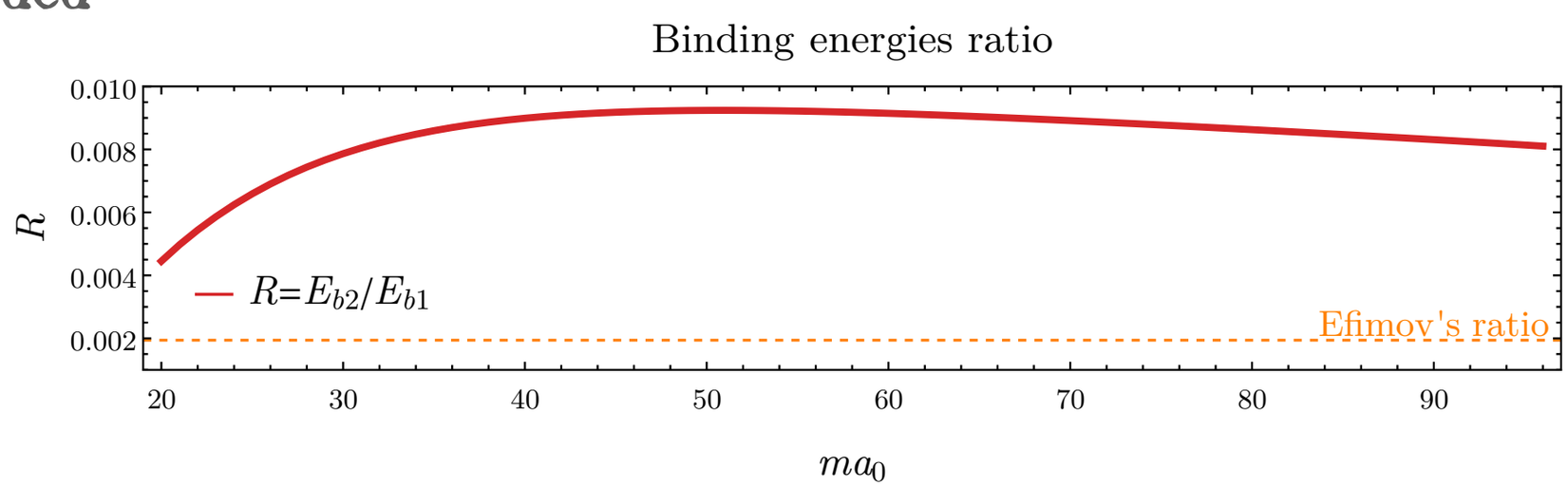




Interpolation and bound-state poles

- Interpolation requires contour deformation
- We need to solve for complex isobar energies lying along the contour
- Three-body poles lie close to the threshold
  - no circular cut
  - no deformation needed

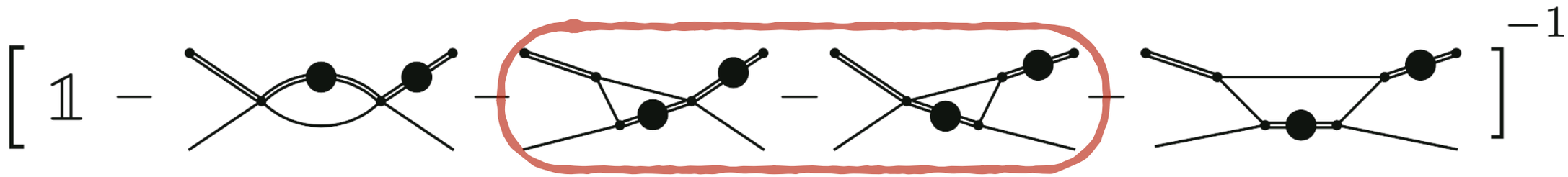


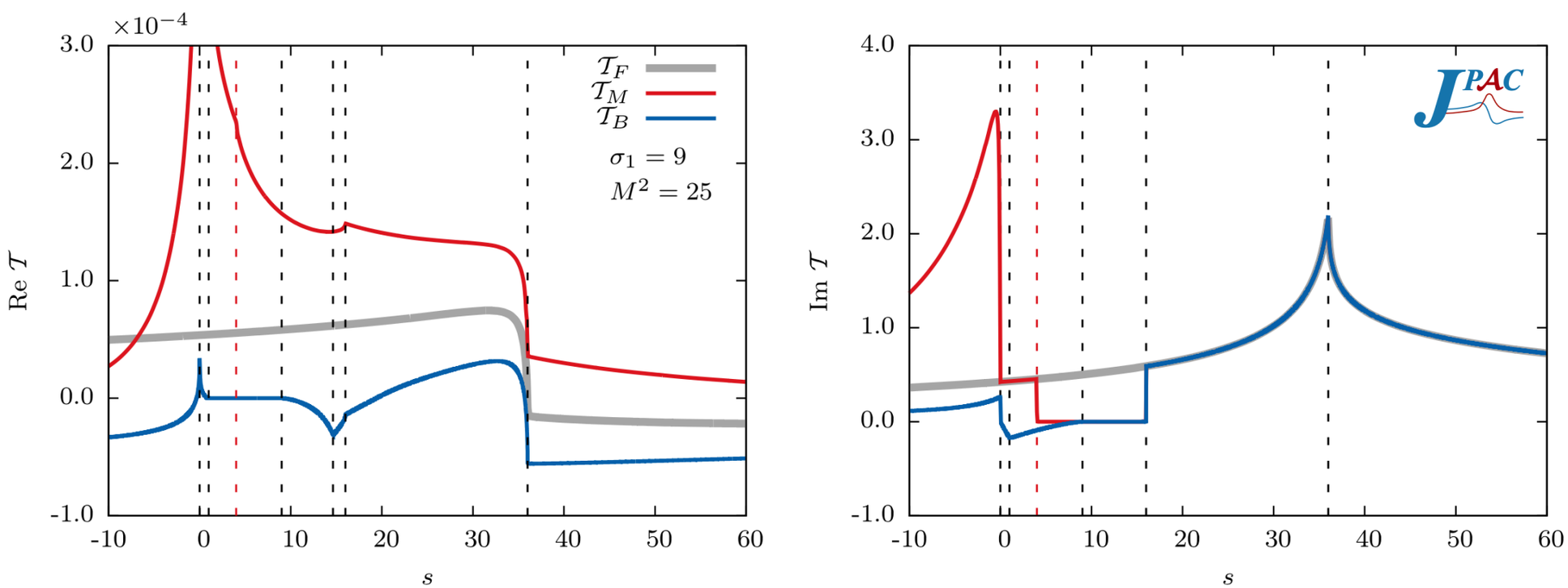


Bound-state–particle scattering

- Model study—formation of the three-body bound states
- Analytic properties of the B-matrix formalism

Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009



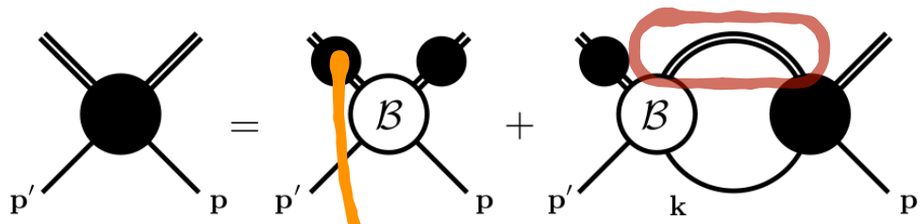


Jackura et al (JPAC), Eur. Phys. J C 79, no. 1, 56 (2019)

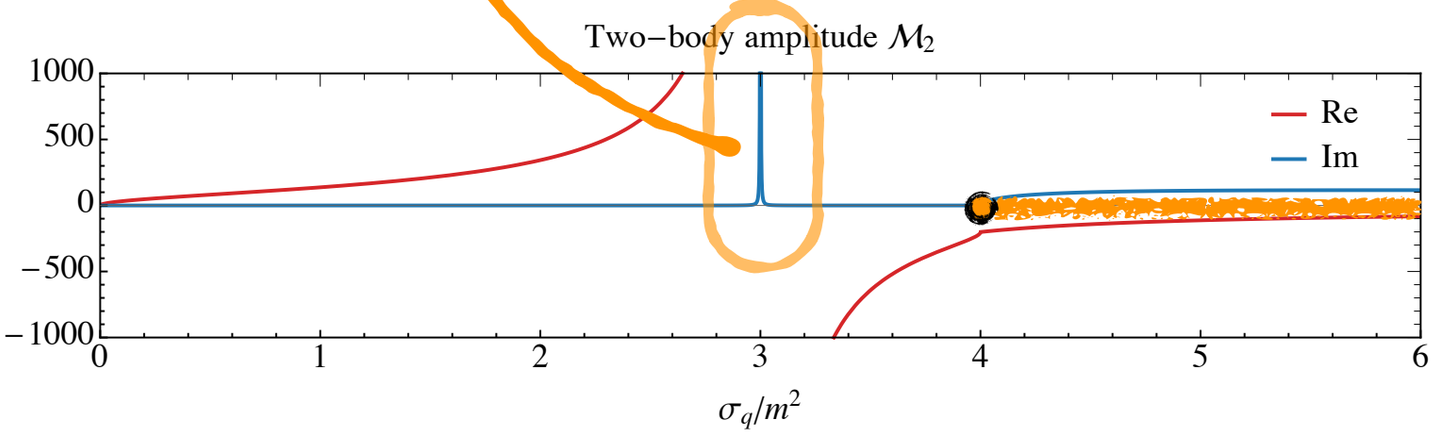
Generalization of the B-matrix formalism

- Amplitude constrained by unitarity in the physical interval

$$\sigma_{\min} = 4m^2$$

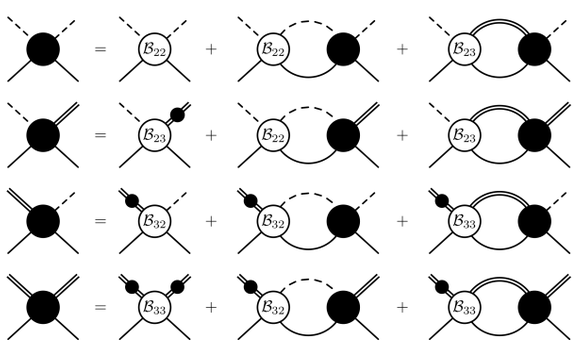


$$\int_{\boldsymbol{q}} \equiv \int \frac{d\Omega_{\boldsymbol{q}}}{4\pi} \int_0^{q_{\max}} \frac{dq\, q^2}{2\pi^2 \omega_q} = \int \frac{d\Omega_{\boldsymbol{q}}}{4\pi} \int_{\sigma_{\min}}^{(\sqrt{s}-m)^2} \frac{d\sigma_{\boldsymbol{q}}}{2\pi} \tau(s, \sigma_{\boldsymbol{q}})$$



- One can not continue below two-body threshold

- Generalization needed to include coupled channels



$$\mathcal{A}_{22} = \mathcal{B}_{22} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{23,\boldsymbol{p}} = \mathcal{B}_{23,\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{qp}},$$

$$\mathcal{A}_{32,\boldsymbol{p}'} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'} \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{33,\boldsymbol{p}'\boldsymbol{p}} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{qp}}\,.$$

Generalization of the B-matrix formalism

- Satisfies unitarity above the three-particle threshold
- New kernels include multi-channel couplings, for example

$$\tilde{\mathcal{A}}_{33} = \mathcal{H}_{33} + \int \mathcal{H}_{33}\, \mathcal{M}_2\, \tilde{\mathcal{A}}_{33}$$

$$\mathcal{H}_{33} = \mathcal{B}_{33} + \frac{\mathcal{B}_{32}\, i\rho_2\, \mathcal{B}_{23}}{1 - i\rho_2\, \mathcal{B}_{22}}$$

- Formal solutions can be found

$$\tilde{\mathcal{A}}_{33} = \frac{1}{\mathbb{1} - \mathcal{H}_{33}\mathcal{M}_2}\mathcal{H}_{33}$$

$$\tilde{\mathcal{A}}_{32} = \frac{1}{\mathbb{1} - \mathcal{H}_{33}\mathcal{M}_2}\mathcal{H}_{32}$$

$$\tilde{\mathcal{A}}_{23} = \frac{1}{\mathbb{1} - \mathcal{H}_{22} i\rho_2}\mathcal{H}_{23}$$

$$\tilde{\mathcal{A}}_{22} = \frac{1}{\mathbb{1} - \mathcal{H}_{22} i\rho_2}\mathcal{H}_{22}$$

$$\mathcal{H}_{32} = \mathcal{B}_{32} + \frac{\mathcal{B}_{32}\, i\rho_2\, \mathcal{B}_{22}}{1 - i\rho_2\, \mathcal{B}_{22}}$$

$$\mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23}\mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33}\, \mathcal{M}_2}\mathcal{B}_{33}$$

$$\mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23}\mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33}\, \mathcal{M}_2}\mathcal{B}_{32}$$

Generalization of the B-matrix formalism

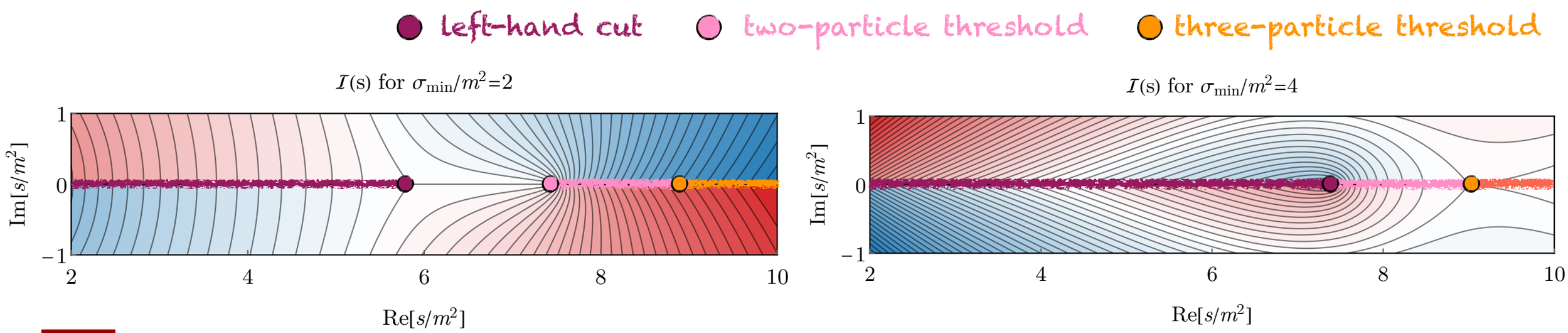
- **Approximation:** all multi-particle interactions are constant and real (couplings g_{ij}),

$$a_{33}(s) = \frac{g_{33}}{1 - g_{22} i\rho_2 - g_{33}\mathcal{I}(s)}, \qquad a_{22}(s) = \frac{g_{22}}{1 - g_{22} i\rho_2 - g_{33}\mathcal{I}(s)}.$$

- Solutions do not satisfy unitarity below the three-body threshold
- Spurious singularities start arbitrarily close to the two-body threshold

Kernel suffering from non-physical left-hand cuts:

$$\mathcal{I}(s) = \int_{\sigma_{\min}}^{(\sqrt{s}-m)^2} \frac{\sigma_q}{2\pi}\, \tau(s,\sigma_q)\, \mathcal{M}_2(\sigma_q)$$

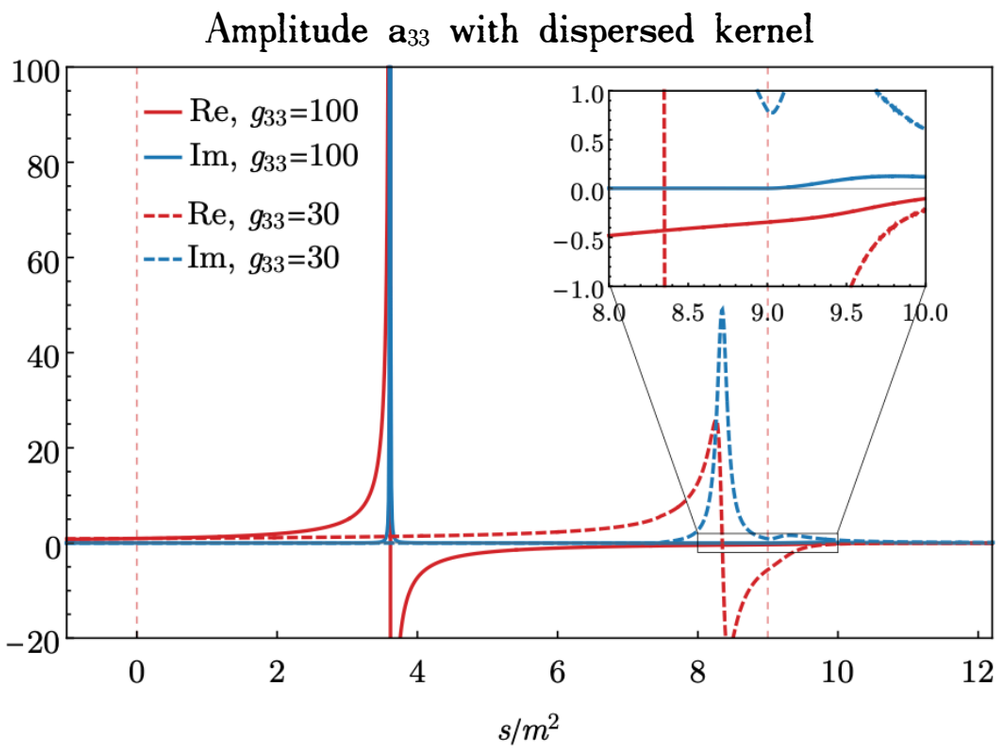


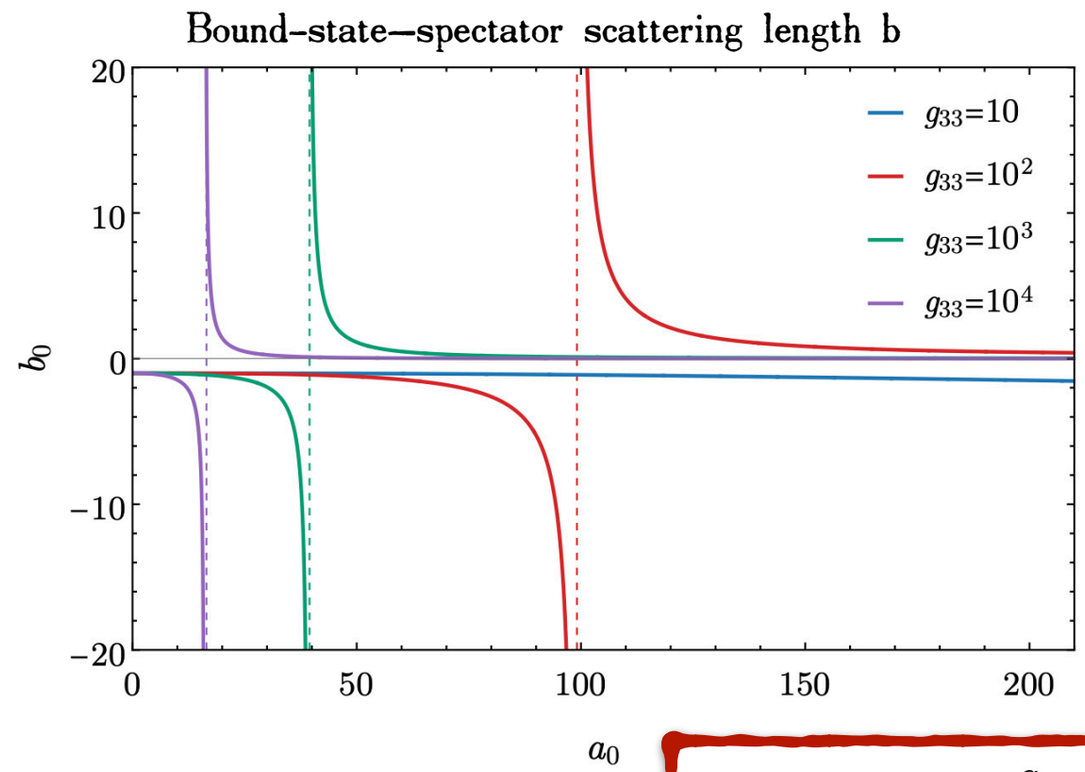
Analyticity of the B-matrix equations

- Dispersion procedure ensures analyticity

Kernel free from those problems:

$$\mathcal{I}_d(s) = \frac{(s-s_s)^2}{\pi}\int_{(3m)^2}^{\infty} ds' \frac{\mathrm{Im}\,\mathcal{I}(s')}{(s'-s-i\epsilon)(s'-s_s-i\epsilon)^2}$$





$$b_0 = -\frac{g_{22}}{\left(1 - g_{33}\,\mathcal{I}_d(s_{\mathrm{th},2})\right)}$$

- Spurious singularities pushed to non-physical Riemann sheets and three-body physics influences the amplitude,
- General dispersion procedure for the three-body unitarity formalism is needed

Including OPE

- How does this generalized B-matrix approach compare with EFT?

- B-matrix formal solution:

$$\widetilde{\mathcal{A}}_{22} = \frac{1}{\mathbb{1} - \mathcal{H}_{22} i\rho_2} \mathcal{H}_{22}$$

where the H matrix is

$$\mathcal{H}_{22} = \cancel{\mathcal{B}_{22}} + \cancel{\mathcal{B}_{23}} \mathcal{M}_2 \frac{1}{1 - \cancel{\mathcal{B}_{33}} \mathcal{M}_2} \cancel{\mathcal{B}_{32}}$$

g_B $\quad -G_S \quad$ g_B

$$\mathcal{H}_{22} = g_B^2 \frac{1}{\mathbb{1} + G_S \mathcal{M}_2} \mathcal{M}_2$$

- REFT ladder formal solution:

$$d = \frac{1}{\mathbb{1} - \mathcal{K} i\rho_2} \mathcal{K}$$

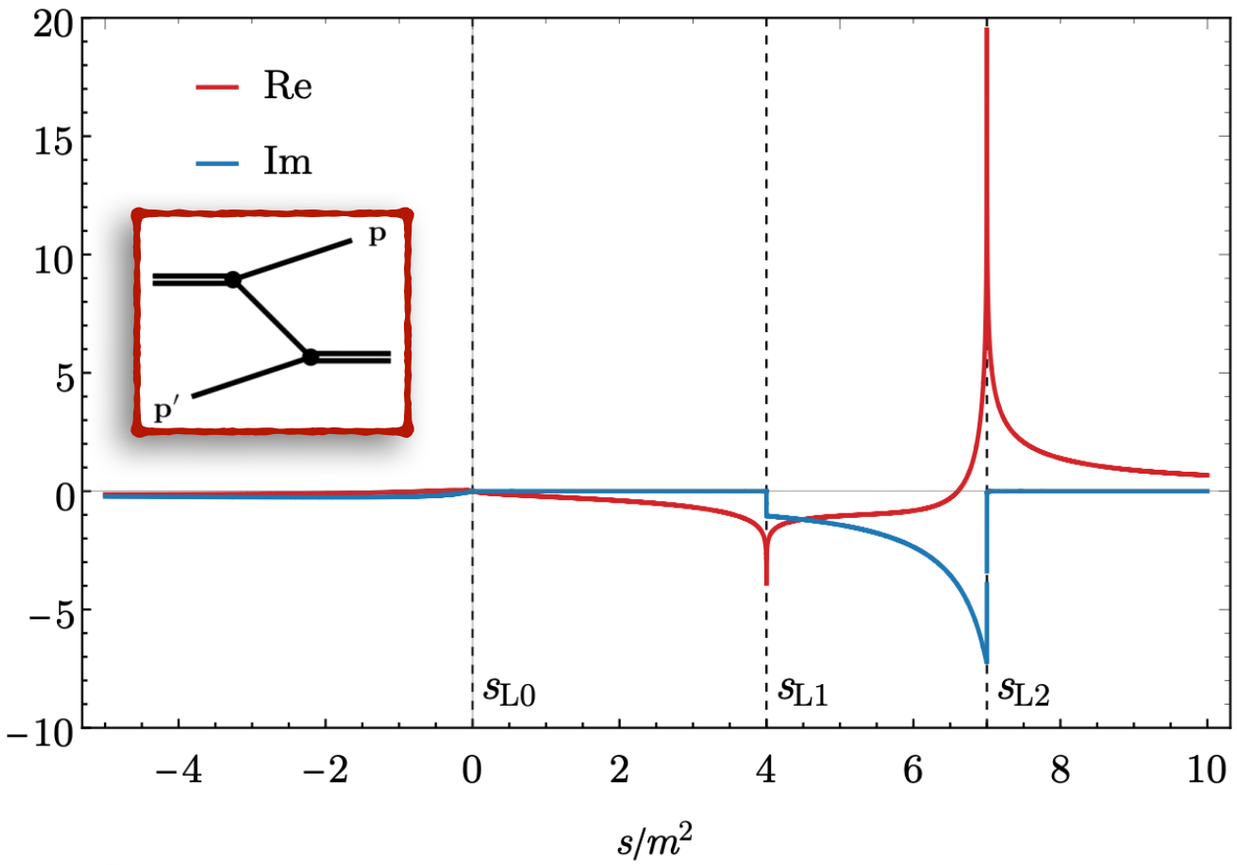
where the K matrix is

$$\mathcal{K} = g^2 \frac{1}{\mathbb{1} + G_S \, \mathrm{P.V.}[\mathcal{M}_2]} (-G_S)$$

Those two are matrices defined in different momentum spaces!

N/D approximation

- Non-perturbative approximation of the ladder solution

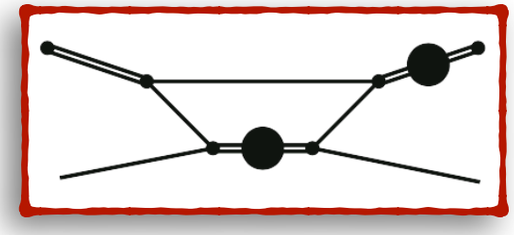


$$\mathcal{A}_2 = \frac{N(s)}{D(s)}$$

- N – contains left hand cuts,
- D – right hand cuts

$$N(s) = \frac{1}{\pi} \int_{-\infty}^{s_{L2}} ds' \, \frac{\mathrm{Im}\,\mathcal{A}_2(s')}{s' - s - i\epsilon} \, D(s') \,,$$

$$D(s) = 1 - \frac{s}{\pi} \int_{(M+m)^2}^{\infty} ds' \, \frac{\rho_2(s')}{s'(s' - s - i\epsilon)} \, N(s')$$



$$D(s) = 1 + \frac{s}{\pi^2} \int_{-\infty}^{s_{2L}} ds' \, g(s, s') \, \mathrm{Im}\,\mathcal{A}_2(s') \, D(s')$$

Conclusions

- Systematic procedure for solving the integral equations
- Agreement with previous studies
- Generalization of the B–matrix formalism
- Study of the analytic properties

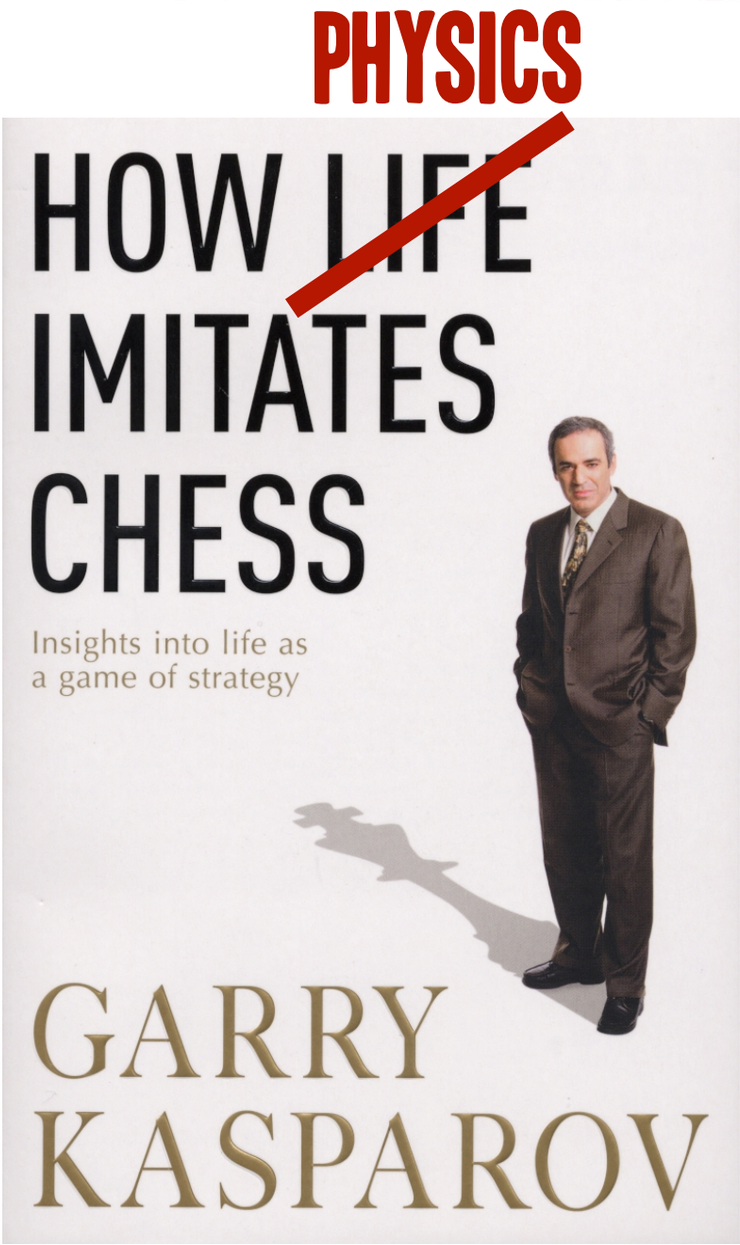
Future prospects

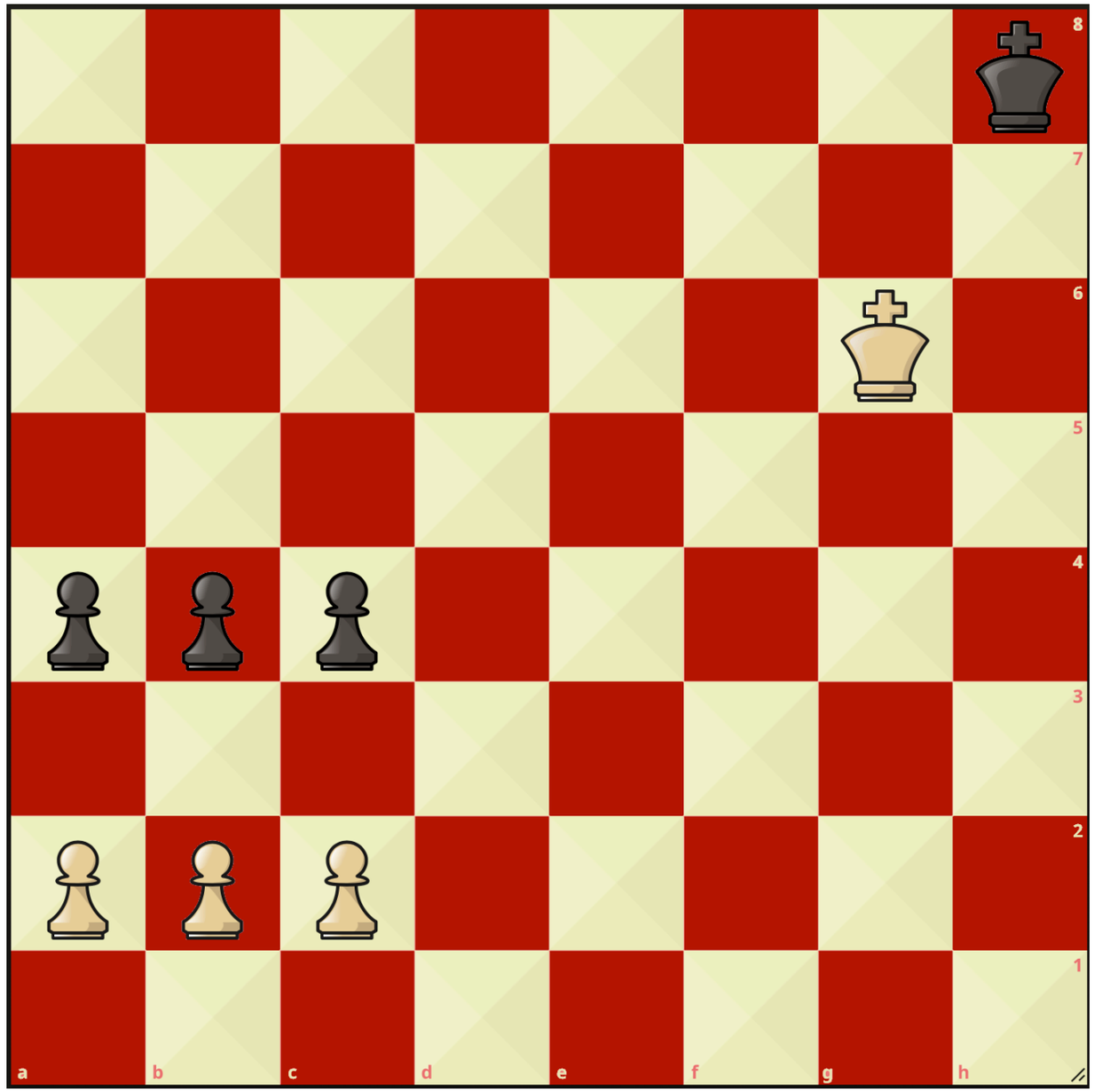
- Continuing below the two–body threshold and to complex energies
- Understanding the Efimov aspects of the model
- Controlling the regularization scheme/ cutoff dependence
- Formulation of the B–matrix satisfying analyticity
- Generalization to arbitrary spins

THANK YOU

I AM BOUND TO PLEASE THEE WITH MY ANSWERS

For three-body experts – black to move





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