Analytic Evolution of Singular Distribution Amplitudes in QCD

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Outline

Quantum Chromodynamics

- Fundamental theory of strong interactions
- Non-Abelian gauge theory with gauge group SU(3)
- Force carriers: Colored gluons
- $L_{QCD} = -\frac{1}{4} F^{(a)}_{\mu\nu} F^{(a)\mu\nu} + i \sum_q \bar{\psi}^i_q \left[\gamma\mu (D_\mu)_{ij} + i\, m\right] \psi^j_q$
  - $F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$

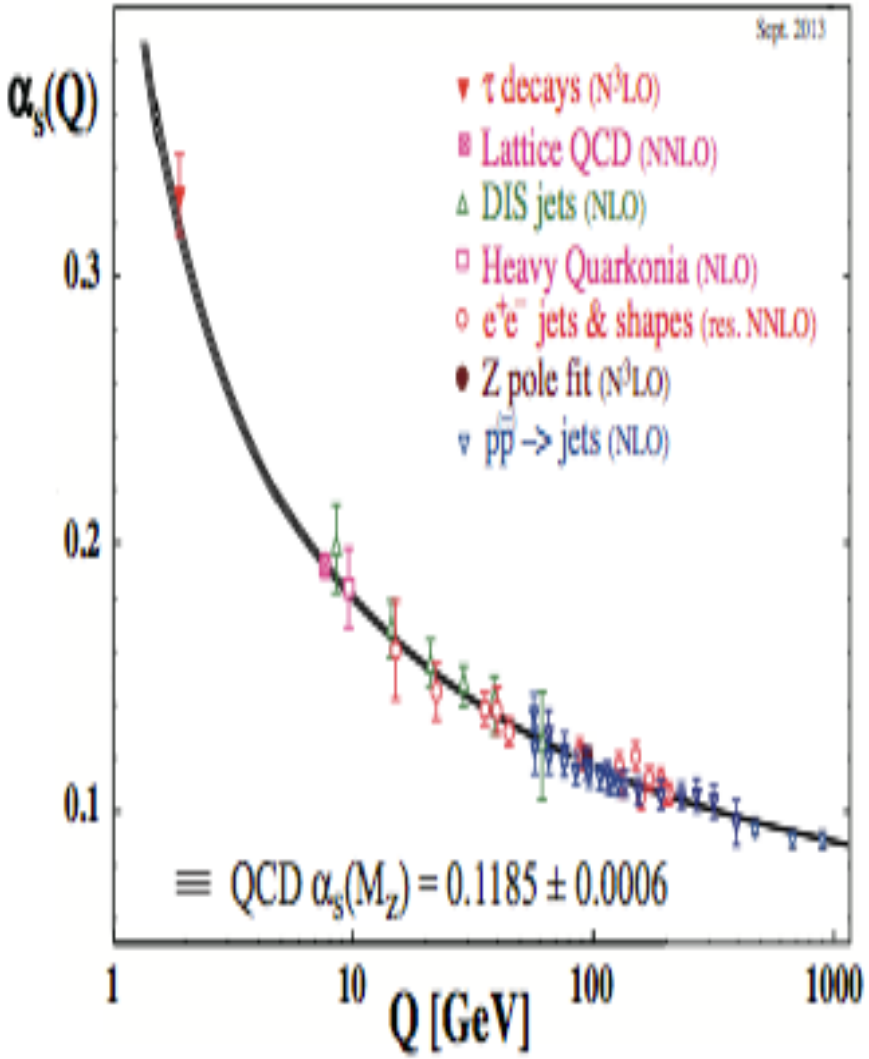
    (Field strength tensor)
  - $(D_\mu)_{ij} = \delta_{ij} \partial_\mu + i g_s \sum_a \frac{\lambda^a_{ij}}{2} A^a_\mu$

    (Covariant derivative)

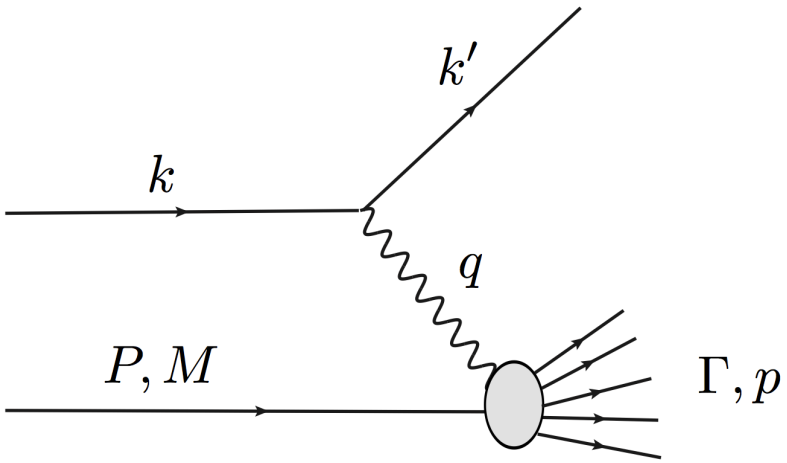
Running Coupling Constant

- Coupling constant α_s depends on Q^2
- A renormalization scale μ is introduced within perturbation theory
- $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+(\beta_0\alpha_s(\mu^2)/4\pi)\ln(Q^2/\mu^2)}$$ with

$$\beta_0 = 11 - \frac{2}{3}n_f \,,$$



Deep Inelastic Scattering



- Test of perturbative QCD
- Deep $\Rightarrow$ Virtual photon can probe very small distances($Q^2 \gg M_p^2$)
- Inelastic $\Rightarrow$ Virtual photon destruct the target hadron($W^2 \gg M_p^2$)
- General form of the scattering $\Rightarrow l + h \rightarrow l' + X$
- l and l' are leptons, h is hadron target and X is the sum of all possible hadron states.

Factorization

- Separation of long and short distances
- Short distances $\Rightarrow$ Perturbatively calculable
- $\underbrace{\text{Long distances}}_{\text{PDF+DA}}$ $\Rightarrow$ Experimentally measurable
  - Parton Distribution Functions (PDFs): Probability densities to find a parton carrying a momentum fraction x
  - Distribution Amplitudes (DAs): Describe how the longitudinal momentum of a fast-moving hadron is shared among constituents belonging to a particular Fock component

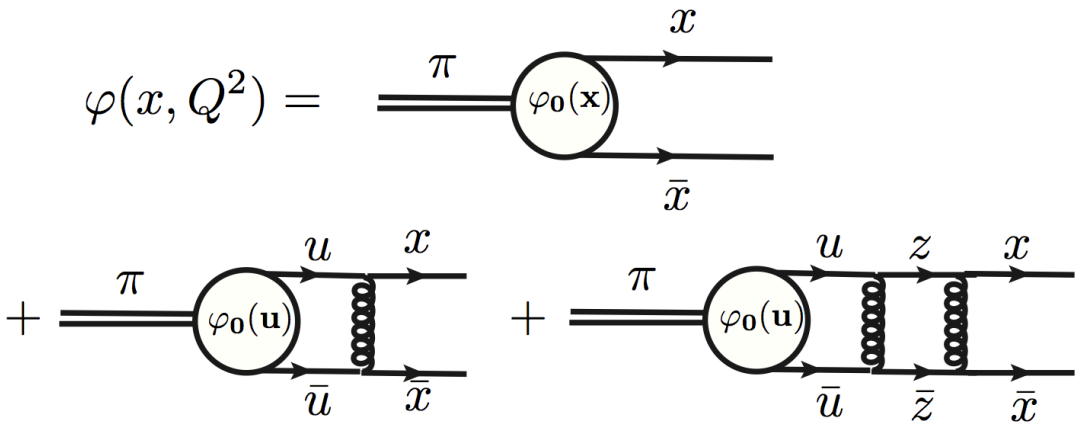
Illustration of Factorization

- Pion form factor $F_\pi(Q^2)$ at Q^2

$$F_\pi(Q^2) = \int_0^1 dx \int_0^1 dy \varphi^\dagger(y, Q^2)\, T(x, y, Q^2)\, \varphi(x, Q^2)$$

  - $T_{Born}(x, y, Q^2) = \frac{4}{3} \frac{16\pi}{Q^2(1-x)(1-y)}$
  - $\varphi(x, Q^2)$ is pion DA

$\varphi(x, Q^2) =$



$+ \cdots$

Evolution

- Evolution equations predict the parton distributions for any scale μ' using the measured parton distribution at a scale μ.
- In order to describe this dependence of the parton distributions on the renormalization scale, μ and μ' should be large enough so that running coupling constants $\alpha_s(\mu)$ and $\alpha_s(\mu')$ are small.

Main Points of the Talk

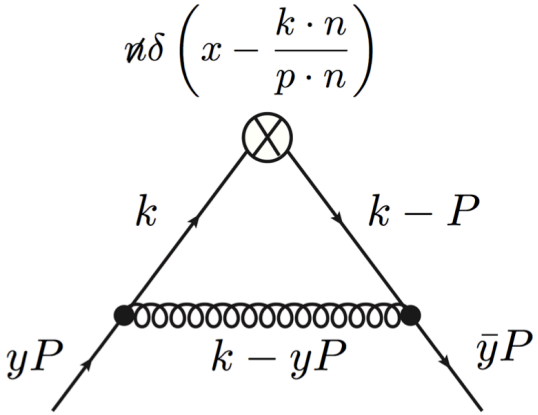
- Developing an Analytic Method for Evolution of Singular DAs
- Application of the Method to Two-Photon Generalized Distribution Amplitude (GDA)
- Application of the Method to DGLAP Evolution
- Application of the Method to Evolution of Double Distributions

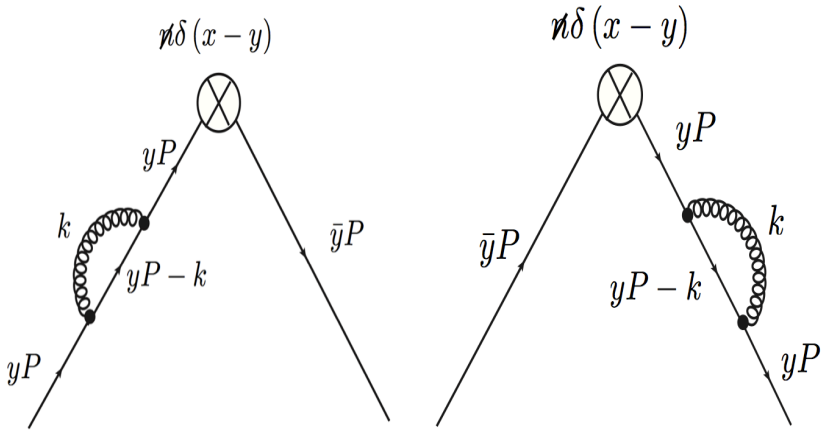
Motivations

- Why to Study Evolution of Singular DAs?
  Modeling Generalized Parton Distribtions (GPDs)
  - Non-analyticity of GPDs at the border $x = \pm\xi$ can cause cusps, jumps and delta functions
  - Inefficiency of well established standard way for non-singular DAs

  $\Rightarrow$ Need for analytic method. Standard way is not efficient for singular DAs
- Why to apply the method to two-photon GDA?
  - Two-photon GDAs include some singular functions.
- Goal
  - Instead of relying on the numerical calculations for evolution of singular DAs we will have understanding about the evolution because of the analytic solutions

Diagramatic Expansions of the LO ERBL Kernel





$$\rightarrow V_1(x,y) = \left(\frac{x}{y}\left(1+\frac{1}{y-x}\right)\right)\theta(x<y) \rightarrow V_2(x,y) = -\delta(y-x)\int_0^1 dz V_1(z,y)$$
$$+(x\rightarrow\overline{x},\ \ y\rightarrow\overline{y})$$

$V_1(x,y) - \delta(y-x)\int_0^1 dz V_1(z,y) \rightarrow$ “+"-Prescription

The non-forward Evolution Kernel and Evolution Equation

$$V(x,y) = \left(\frac{x}{y}\left(1+\frac{1}{y-x}\right)\right)_+ \theta(x<y) + \left(\frac{\overline{x}}{\overline{y}}\left(1+\frac{1}{x-y}\right)\right)_+ \theta(y<x)$$

where $\overline{x} \equiv 1-x$ and $V(x,y)_+ = V(x,y) - \delta(y-x)\int_0^1 V(z,y)dz$

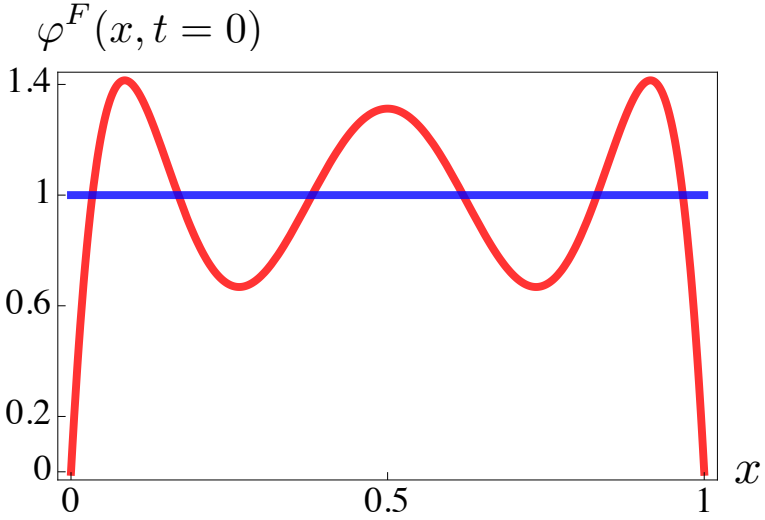
Explicit form of ERBL evolution equation:

$$\begin{aligned}\frac{\partial\varphi(x,t)}{\partial t} &= \int_0^1 \left[V(x,y)\varphi(y,t) - V(y,x)\,\varphi(x,t)\right]\,dy \\ &= \int_0^1 V(x,y)[\varphi(y,t)-\varphi(x,t)]dy + \varphi(x,t)\int_0^1 [V(x,y)-V(y,x)]dy\end{aligned}$$

where $t = 2\ln\ln(\mu^2/\Lambda^2)/b_0$

Calculation of Evolution Equation

- Standard Way: Gegenbauer polynomial expansion
  - $x(1-x)\,C_n^{3/2}(1-2x)$ are eigenfunctions of evolution kernel
  - Oscillating functions: inefficient for expansion of singular DAs
  - Illustration of Gegenbauer expansion of a flat function



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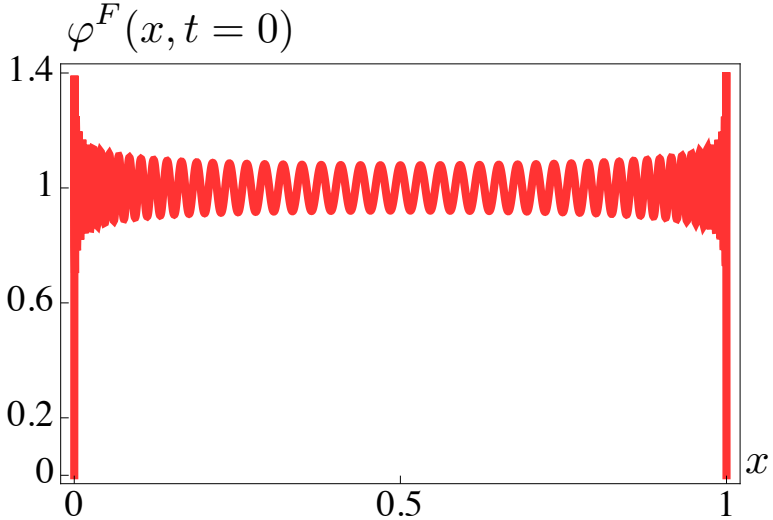


50 Terms

Calculation of Evolution Equation

- Alternative way: Iteration

$$\varphi(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi_n(x)$$

with the functions $\varphi_n(x)$ satisfying the recurrence relation

$$\begin{aligned}\varphi_{n+1}(x) &= \int_0^1 [V(x,y)]_+ \varphi_n(y)\, dy \\ &= \int_0^1 \left[V(x,y)\varphi_n(y) - V(y,x)\,\varphi_n(x)\right]\, dy\end{aligned}$$

Evolution of Flat DA

Rearranged evolution equation

$$\frac{\partial \varphi(x,t)}{\partial t} = \int_0^1 V(x,y)[\varphi(y,t) - \varphi(x,t)]dy + \varphi(x,t) \underbrace{\int_0^1 [V(x,y) - V(y,x)]dy}_{=3/2+x\ln\bar{x}+\bar{x}\ln x \equiv v(x)}$$

$\Rightarrow$ The ansatz

$$\varphi(x,t) = e^{tv(x)}\Phi(x,t)$$

Equation for $\Phi(x,t)$

$$\frac{\partial \Phi(x,t)}{\partial t} = \int_0^1 V(x,y) \times \left[e^{t[v(y)-v(x)]}\Phi(y,t) - \Phi(x,t)\right] dy$$

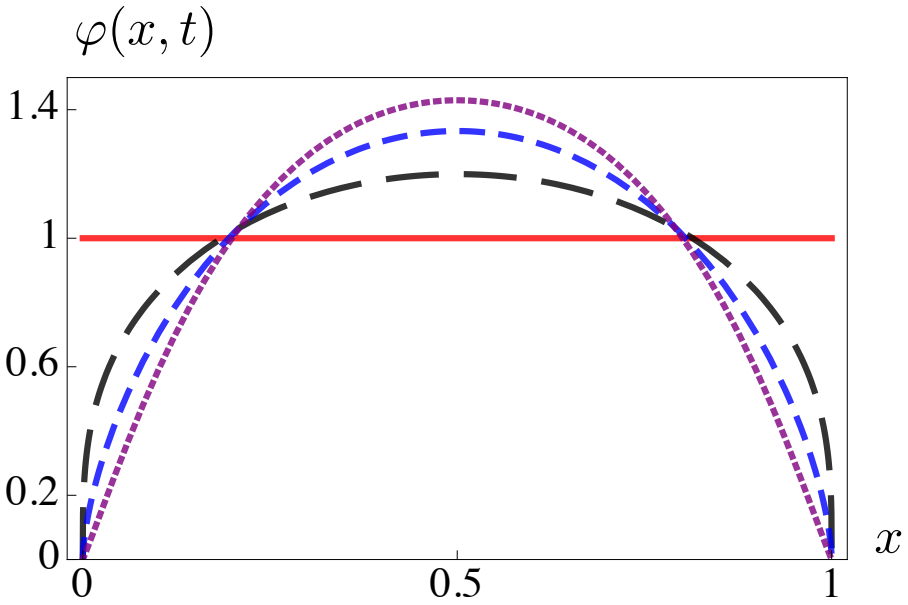
- $\Phi(x,t)$ in series of t

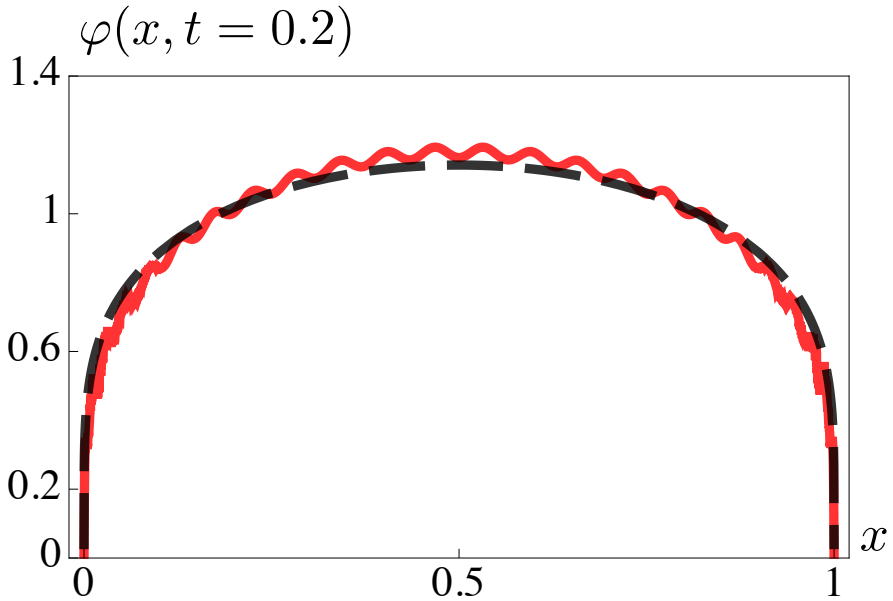
$$\Phi(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Phi_n(x)$$

- Recurrence relation for $\Phi_n(x)$

$$\Phi_{n+1}(x) = \int_0^1 V(x,y) \left[\sum_{l=0}^{n} \frac{n!}{(n-l)!\,l!}\Phi_l(y)\left[v(y)-v(x)\right]^{n-l} - \Phi_n(x)\right] dy$$

Evolution of Flat DA





50 Terms

From red to purple: $t = 0, t = 0.3, t = 0.6, t = 1.0$

(A.T, Radyushkin,2011)

Evolution of Jumps: Simple Case

DA has a jump in $0 < x < 1$ interval

- Simplest example

$$\varphi_0^J(x) = \begin{cases} 1 & 0 < x \leqslant 1/2 \\ -1 & 1/2 < x < 1 \end{cases}$$

- Contribution from the singular part of the kernel

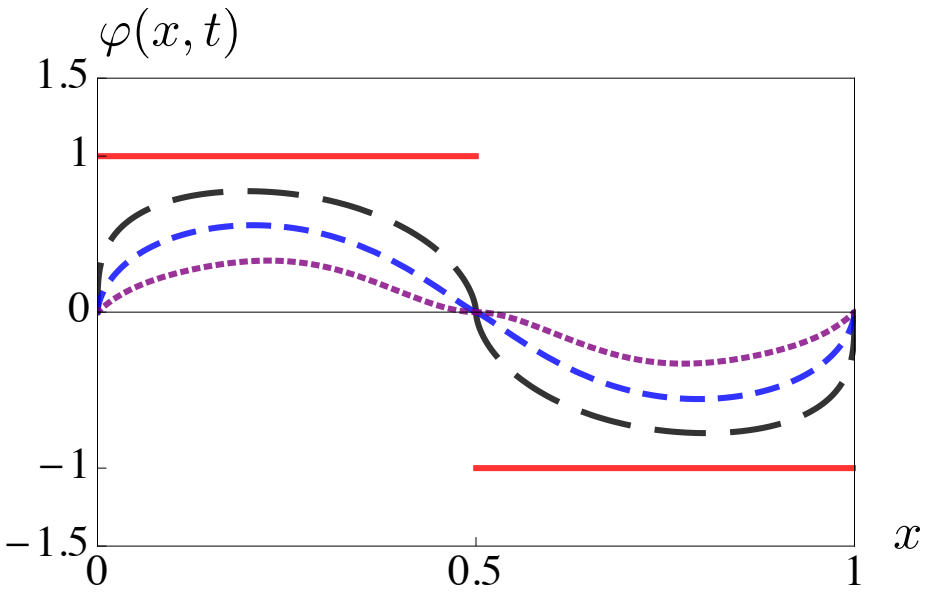
$$\Phi_1^{J\,\mathrm{sing}}(x) = \begin{cases} 2\ln\left[\frac{1-2x}{\bar{x}}\right] \\ -2\ln\left[\frac{2x-1}{x}\right] \end{cases}$$

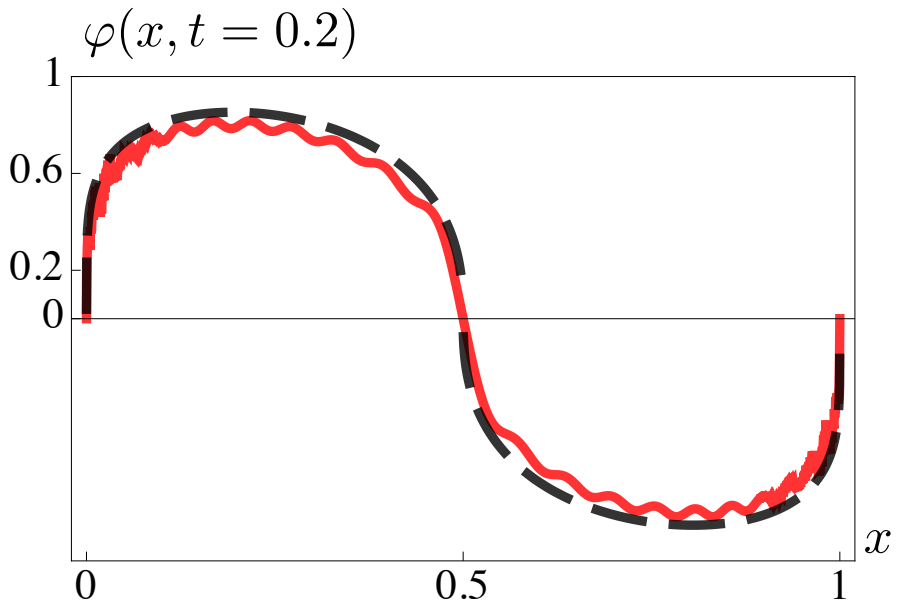
- Ansatz

$$\underbrace{\varphi(x,t) = e^{2t}(x\bar{x})^t|1-2x|^{2t}\Phi(x,t)}_{\Downarrow}$$

$$\Phi_1(x) = -2\ln\bar{x}\;\theta(0 < x \leqslant 1/2) - \{x \to \bar{x}\}$$

Evolution of Step Function





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Evolution of Jumps: General Case

- More general jump structure

$$\varphi_0^J(x,\zeta;a,b) = \begin{cases} a & 0 < x \leqslant \zeta \\ b & \zeta < x < 1 \end{cases}$$

- Contribution from the singular part of the kernel

$$\Phi_1^{J\,\mathrm{sing}}(x,\zeta;a,b) = \begin{cases} (a-b)\ln\left[\frac{\zeta-x}{(1-x)\zeta}\right] \\ -(a-b)\ln\left[\frac{x-\zeta}{(1-\zeta)x}\right] \end{cases}$$

- For jumping part

$$\varphi^J(x,\zeta;a,b) = \frac{a-b}{2}e^{2t}(x\bar{x})^t\left[\left(\frac{1-x/\zeta}{1-x}\right)^{2t}\theta(x<\zeta) + \left(\frac{1-\bar{x}/\bar{\zeta}}{1-\bar{x}}\right)^{2t}\theta(x>\zeta)\right]\Phi(x,\zeta) + \Psi(x,\zeta;t)$$

Evolution of Jumps: More General Case

- Function with antisymmetric jumps at $x = \zeta_i \Rightarrow$ Ansatz

$$\varphi(x,t) = \underbrace{\Phi(x,t)}_{e^{t[v(x)+w(x)]}\Phi_0(x)} + \underbrace{\Psi(x,t)}_{\text{regular function vanishing at } t=0}$$

  - $\left.\frac{\partial \Psi(x,t)}{\partial t}\right|_{t=0} \equiv \chi(x) \to \chi(x)$ is continuous
  - Corrections to this approximation can be found by

$$\Psi(x,t) = \sum_{n=1} \Psi_n(x,t), \quad \Psi_1(x,t) \equiv t\,\chi(x) + \delta\Psi_1(x,t)$$

- Final ansatz $\varphi(x,t) = \Phi(x,t) + \underbrace{\Psi_\chi(x,t)}_{\text{smooth part}} + \underbrace{\delta\Psi(x,t)}_{\text{remainder}}$

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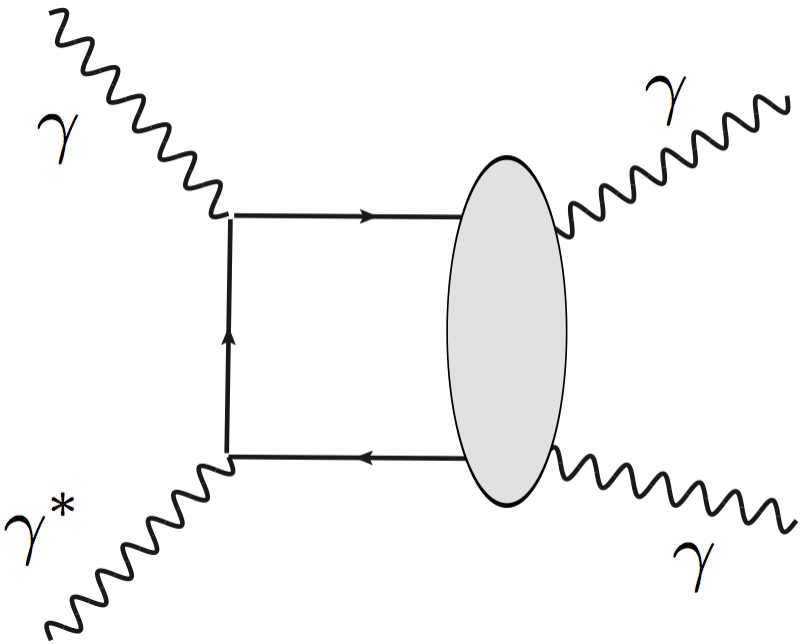
Two-Photon Generalized Distribution Amplitude (GDA)

- GDA: describes the transition of a quark-antiquark or a gluon-gluon pair into a photon pair

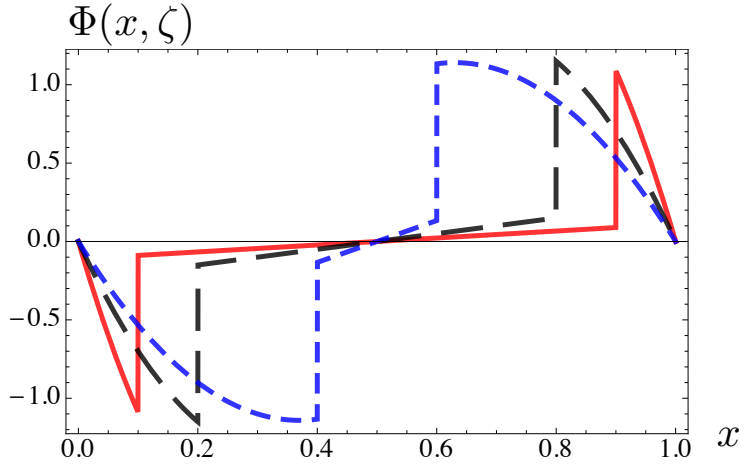


- In lowest order, non-singlet vector part of two-photon GDA

$$\psi^q(x,\zeta,Q^2) = \frac{N_C\, e_q^2}{2\pi^2} \log \frac{Q^2}{m^2} \varphi(x,\zeta)$$

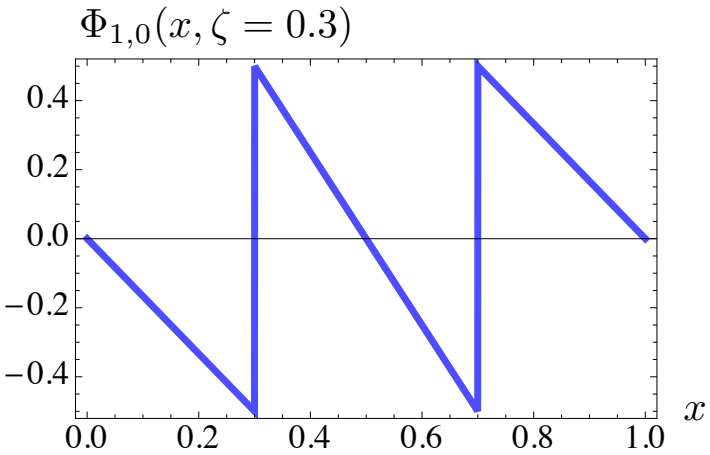
$$\Rightarrow \varphi(x,\zeta) = \frac{\bar{x}(2x-\zeta)}{\bar{\zeta}}\theta(x-\zeta) + \frac{\bar{x}(2x-\bar{\zeta})}{\zeta}\theta(x-\bar{\zeta}) - \frac{x(2\bar{x}-\bar{\zeta})}{\zeta}\theta(\zeta-x) - \frac{x(2\bar{x}-\zeta)}{\bar{\zeta}}\theta(\bar{\zeta}-x)$$

Two-Photon Generalized Distribution Amplitude (GDA)

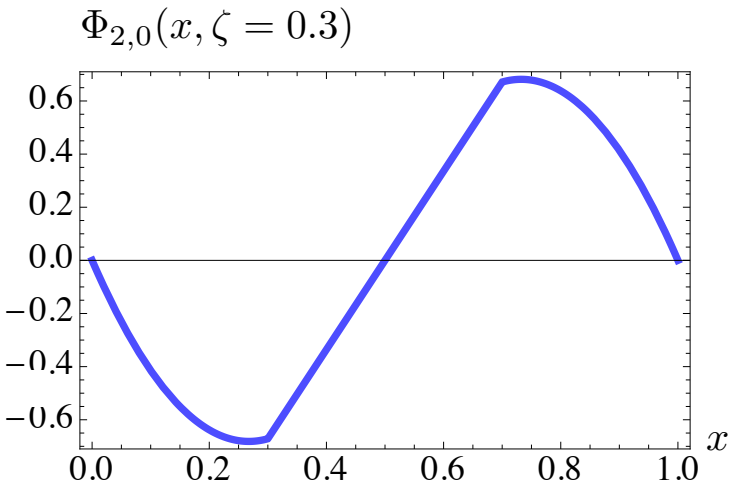


Jump Part

Cusp Part



a

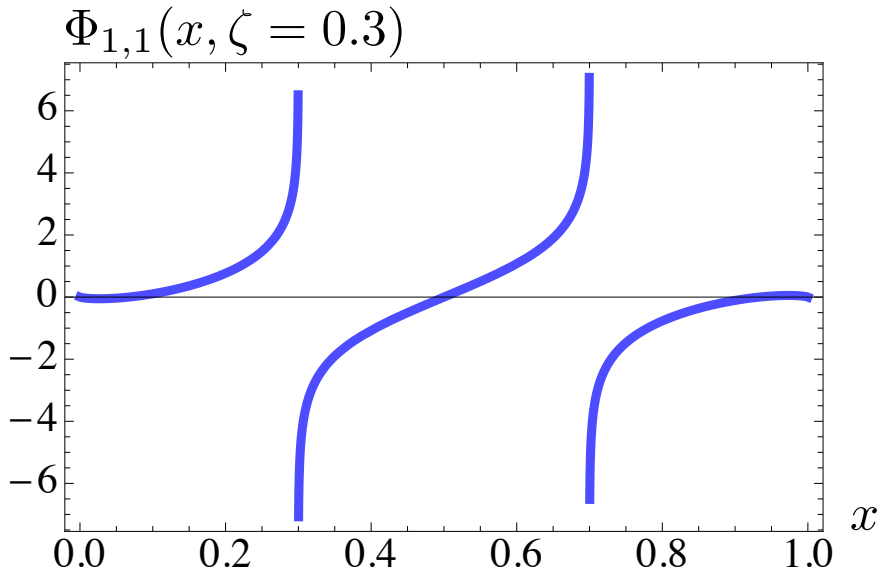


b

Jump part of GDA: $\Phi_1(x, \zeta; t)$

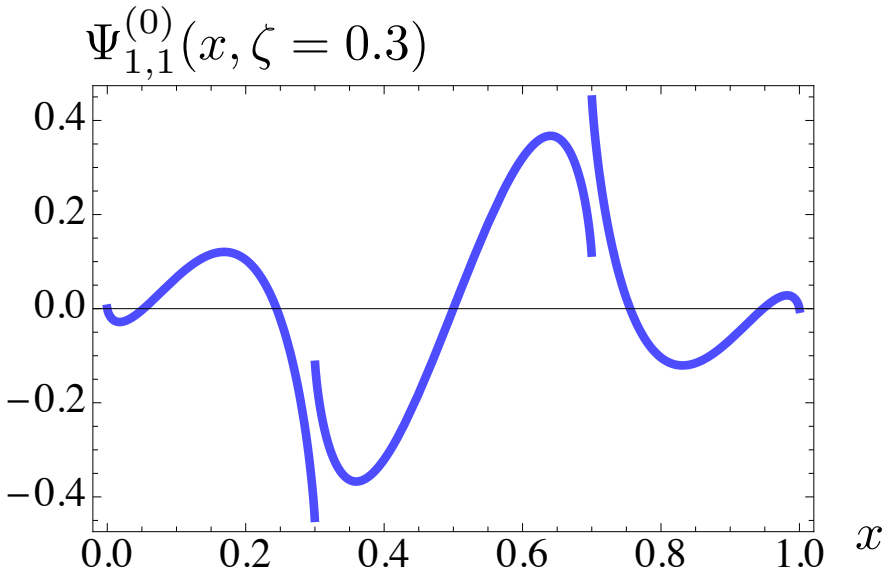
$$\Phi_{1,0}(x,\zeta) = -\frac{x}{2\zeta}\theta(0 < x < \zeta) + \frac{\bar{x}}{2\zeta}\theta(\bar{\zeta} < x < 1) + \frac{1-2x}{2(1-2\zeta)}\theta(\zeta < x < \bar{\zeta})$$

- Straightforward first iteration



Jump part of GDA: $\Phi_1(x, \zeta; t)$

Summation of the divergent terms: $2(\ln|x-\zeta| + \ln|x-\bar{\zeta}|)\Phi_1(x,\zeta)$
$e^{t[v(x)+w(x)]}\Phi_0(x) \Rightarrow w_0(x,\zeta) = 4 + 2\ln|x-\zeta| + 2\ln|x-\bar{\zeta}|$



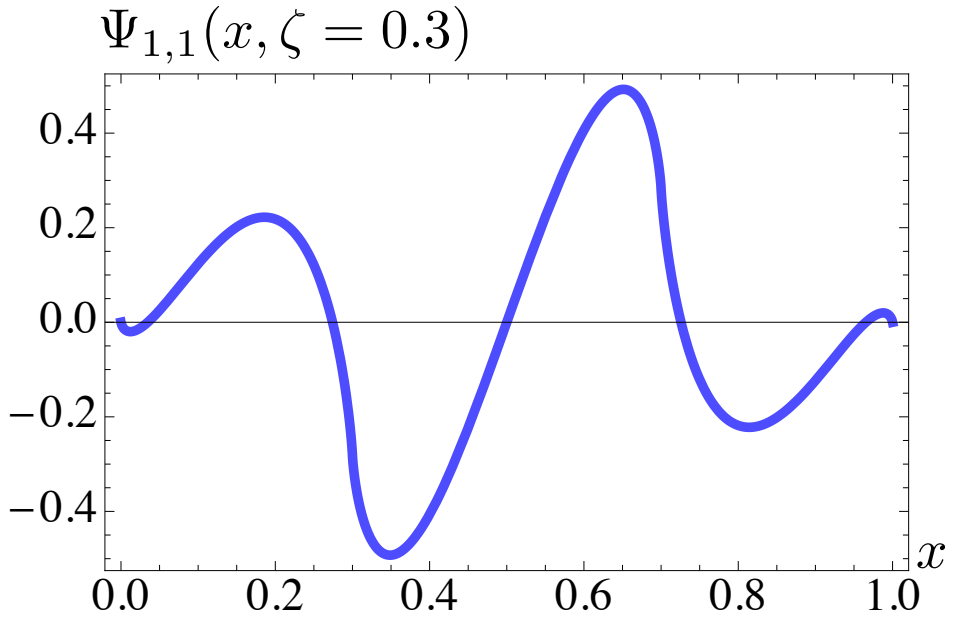
FINITE JUMPS

Jump part of GDA: $\Phi_1(x,\zeta;t)$

- Explicit calculation gives

$$\Psi^{(0)}_{1,1}(\zeta_+,\zeta) - \Psi^{(0)}_{1,1}(\zeta_-,\zeta) = 4 + 2\ln(|1-2\zeta|) + (2-\zeta)\ln\zeta + (2-\bar{\zeta})\ln\bar{\zeta} \equiv -w_1(\zeta)$$

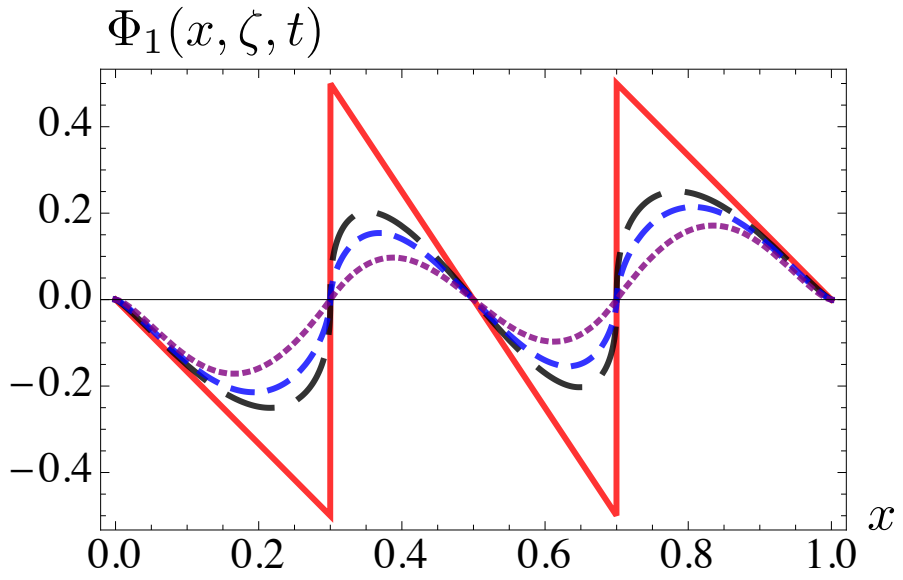
- Adding $w_1(\zeta)\Phi_{1,0}(x,\zeta)$ to $\Psi^{(0)}_{1,1}(x,\zeta)$, $\Rightarrow$ $\underbrace{\chi(x,\zeta)}_{\text{continuous}}$



Jump part of GDA: $\Phi_1(x, \zeta; t)$

Φ-part of the Ansatz $\varphi(x, \zeta, t) = \Phi(x, \zeta, t) + \Psi(x, \zeta, t)$

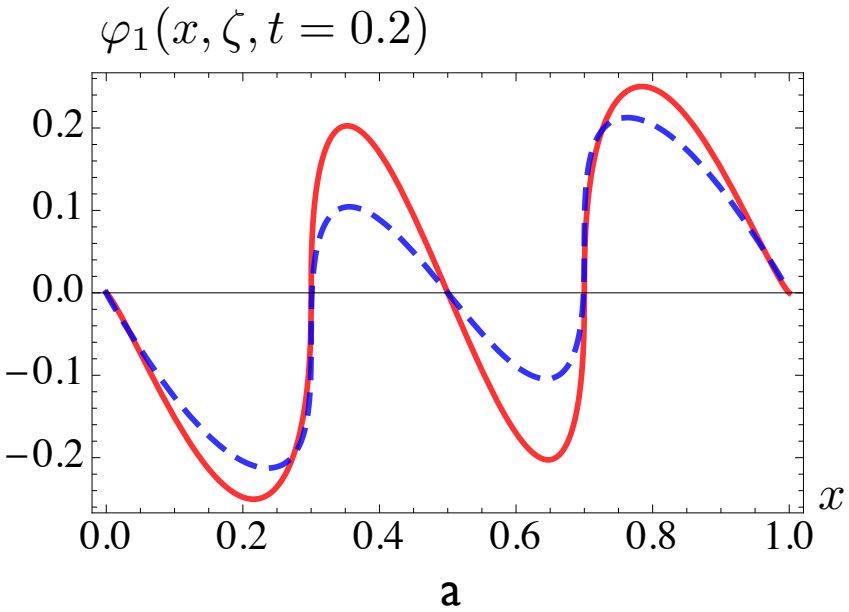
$$\Phi_1(x, \zeta, t) = e^{tv(x)} \left(\frac{|1 - x/\zeta||1 - x/\bar{\zeta}|}{|1 - 2\zeta|} \right)^{2t} \zeta^{t\zeta} \bar{\zeta}^{t\bar{\zeta}} \Phi_{1,0}(x, \zeta)$$

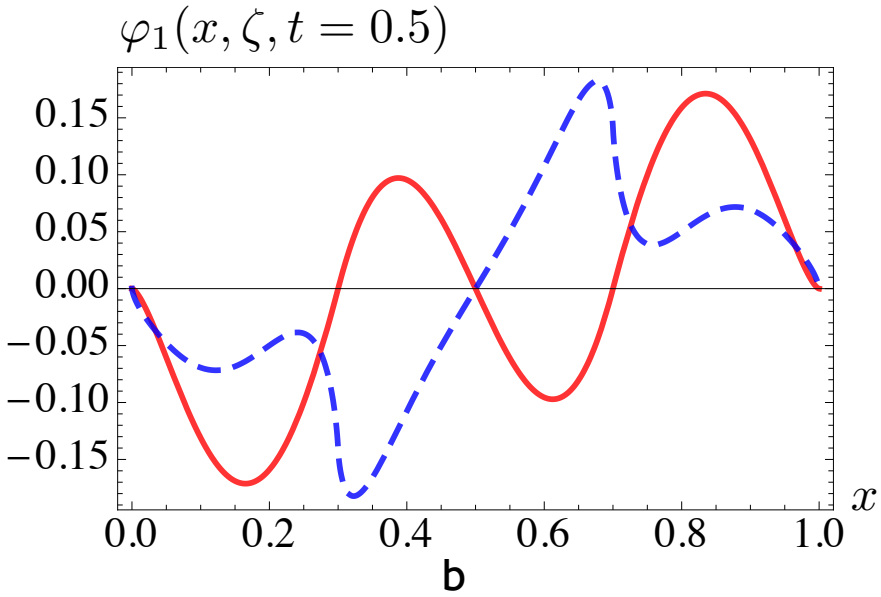


From red to purple: $t = 0, t = 0.2, t = 0.3, t = 0.5$

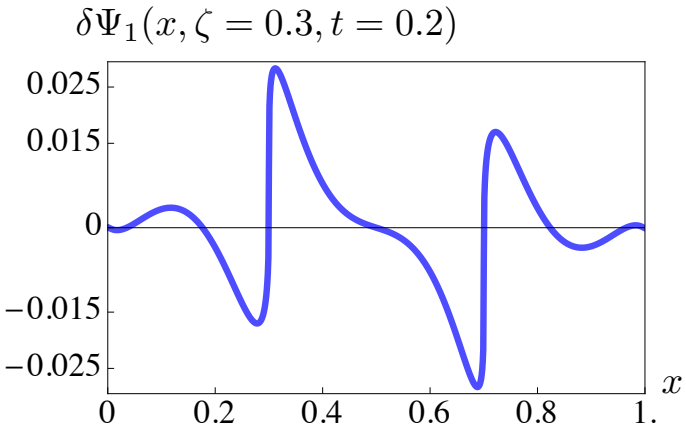
Jump part of GDA: $\Phi_1(x, \zeta; t)$

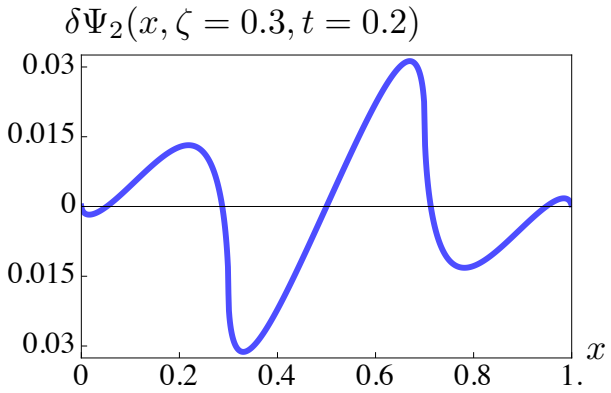
Correction due to the Ψ term from $\varphi(x, \zeta, t) = \Phi(x, \zeta, t) + \Psi(x, \zeta, t)$

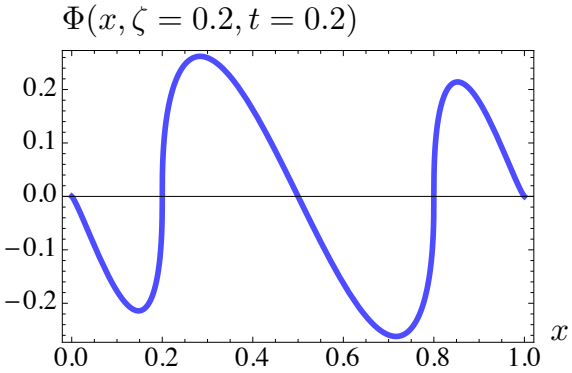


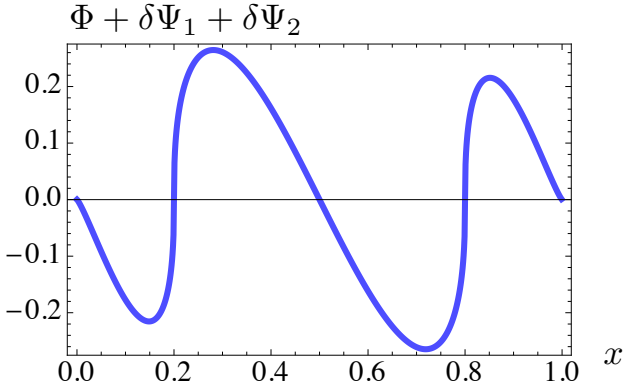


Jump part of GDA: $\Phi_1(x, \zeta; t)$





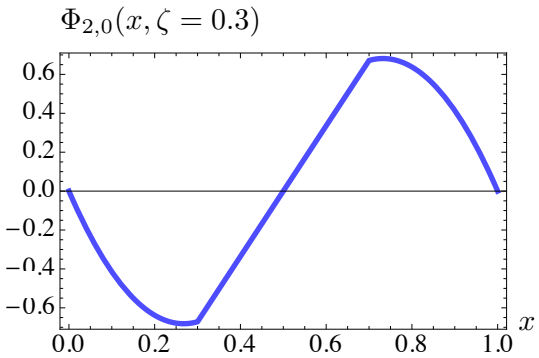


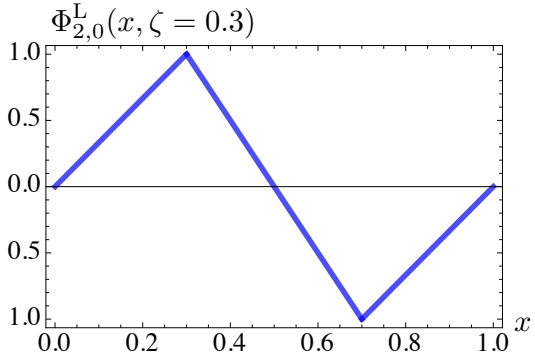


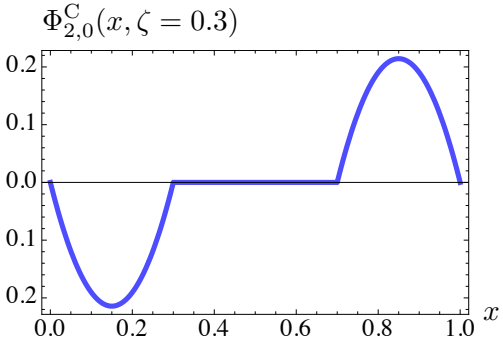
$$\Psi(x, \zeta, t) \approx t\chi(x, \zeta) \ \surd$$

Cusp Part of GDA:$\Phi_2(x,\zeta;t)$

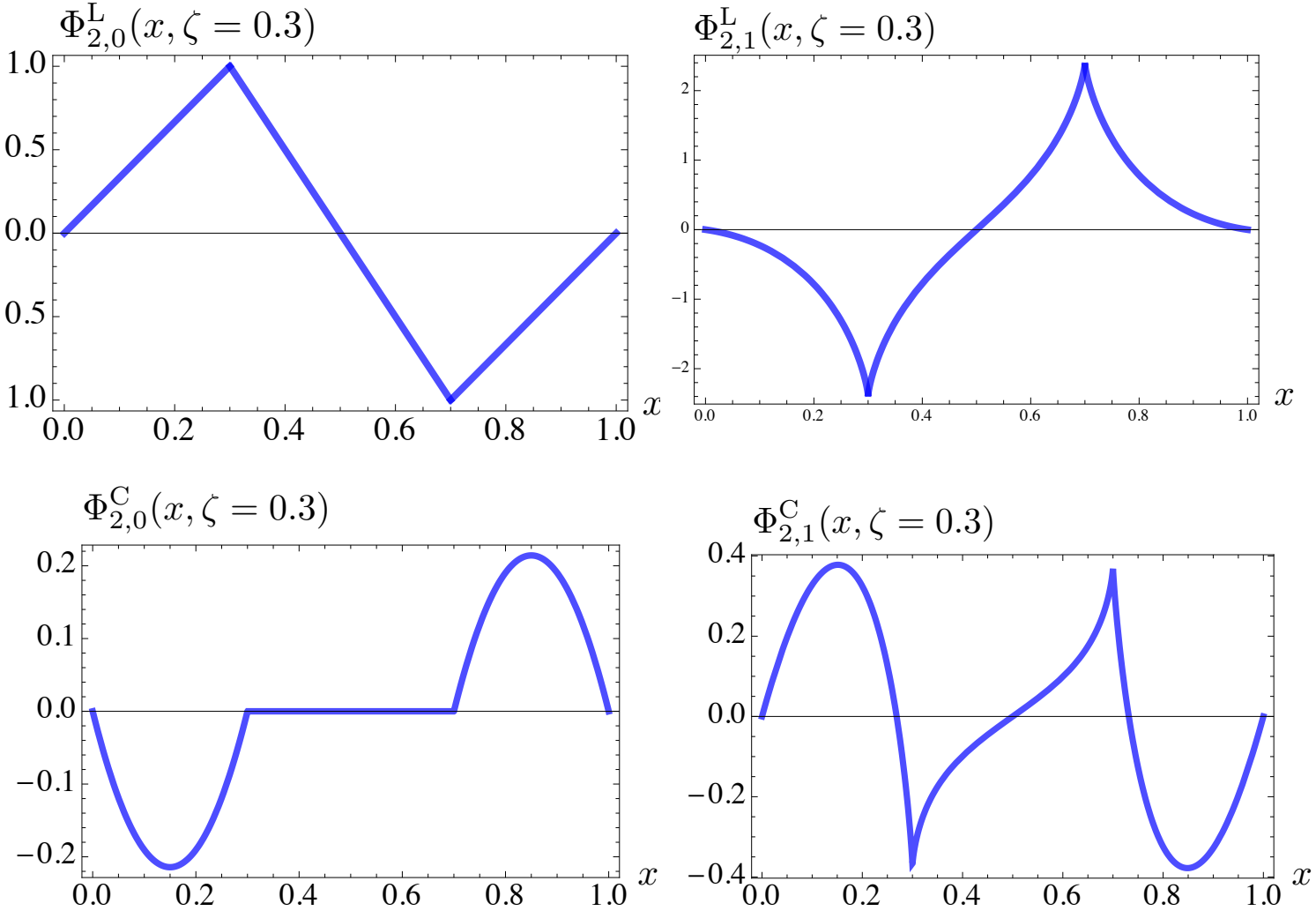
$$\Phi_{2,0}(x,\zeta) = \frac{-x(1-4x+5\zeta-4\zeta^2)}{2\bar{\zeta}\zeta}\theta(\zeta-x) - \frac{\bar{x}(3-4x-5\zeta+4\zeta^2)}{2\bar{\zeta}\zeta}\theta(x-\bar{\zeta}) - \frac{(1-2x)(1+\zeta-4\zeta^2)}{2\bar{\zeta}(1-2\zeta)}\theta(x-\zeta)\theta(\bar{\zeta}-x)$$



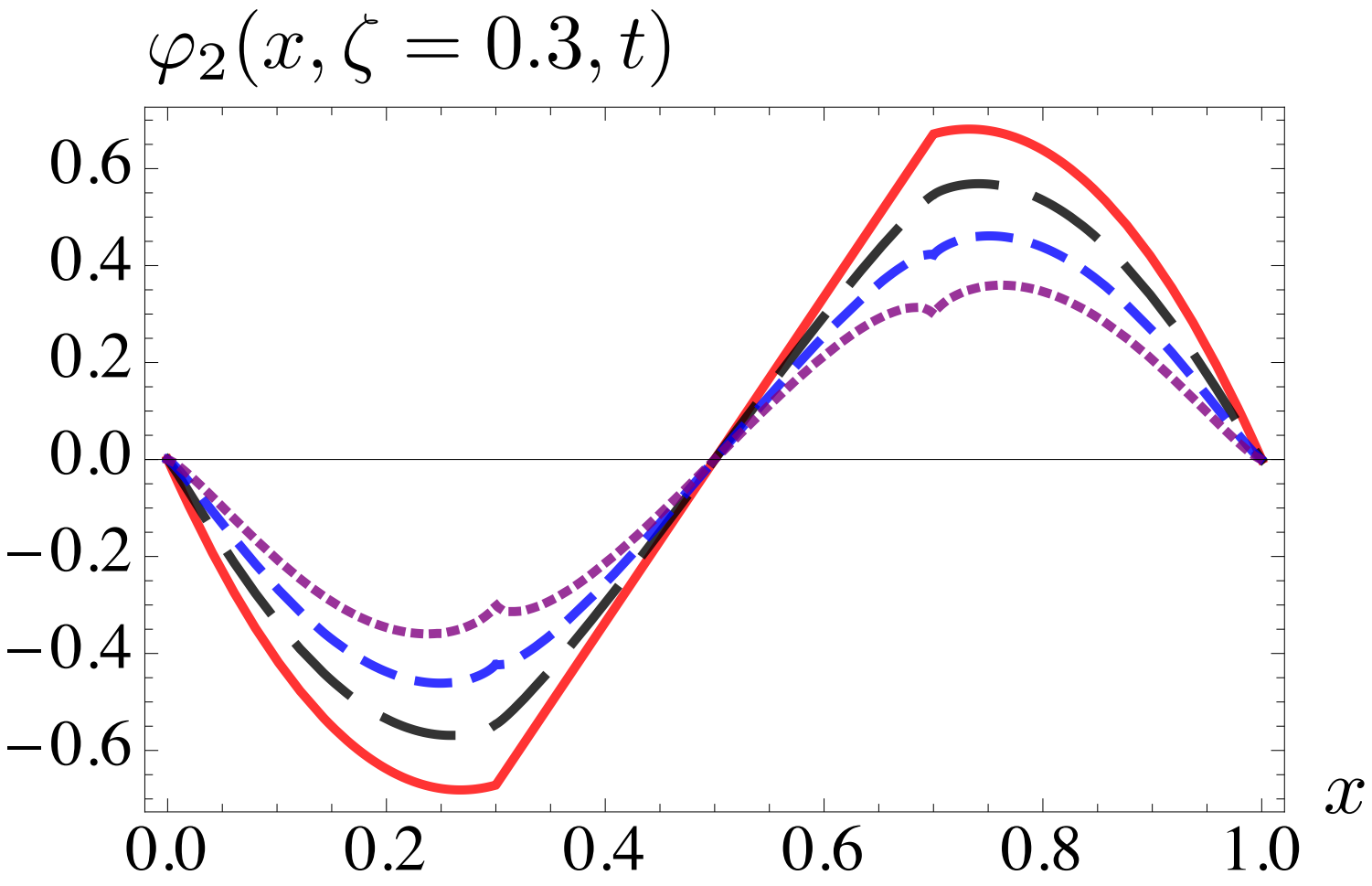




Cusp Part of GDA:$\Phi_2(x, \zeta; t)$



Cusp Part of GDA:$\Phi_2(x, \zeta; t)$

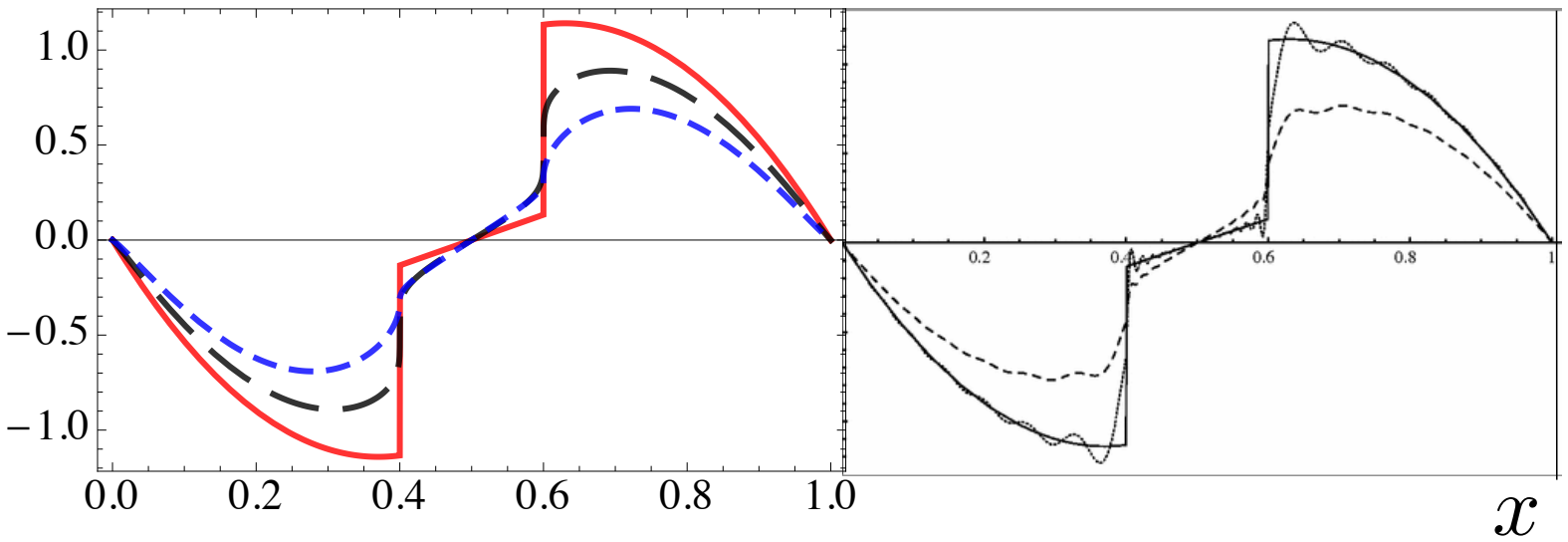


From red to purple: $t = 0, t = 0.2, t = 0.3, t = 0.5$

Result for Two-Photon GDA

$$\Psi^q(x,\zeta;t) = \Psi_1(x,\zeta;t) + \Psi_2(x,\zeta;t)$$

$$\varphi(x,\zeta=0.4,t)$$



(A. Radyushkin, A.T., 2014)

(Beiyad, Pire, Szymanowski, 2008)

From red to blue: $t=0, t=0.1, t=0.2$

Main Points of the Talk

- Developing an Analytic Method for Evolution of Singular DAs $\surd$
- Application of the Method to Two-Photon Generalized Distribution Amplitude $\surd$
- Application of the Method to DGLAP Evolution
- Application of the Method to Evolution of Double Distributions

DGLAP Evolution

- LO quark-quark DGLAP kernel: A quark with momentum fraction x could have come from a parent quark with a larger momentum fraction y. Usual way to write evolution equation in DGLAP region is

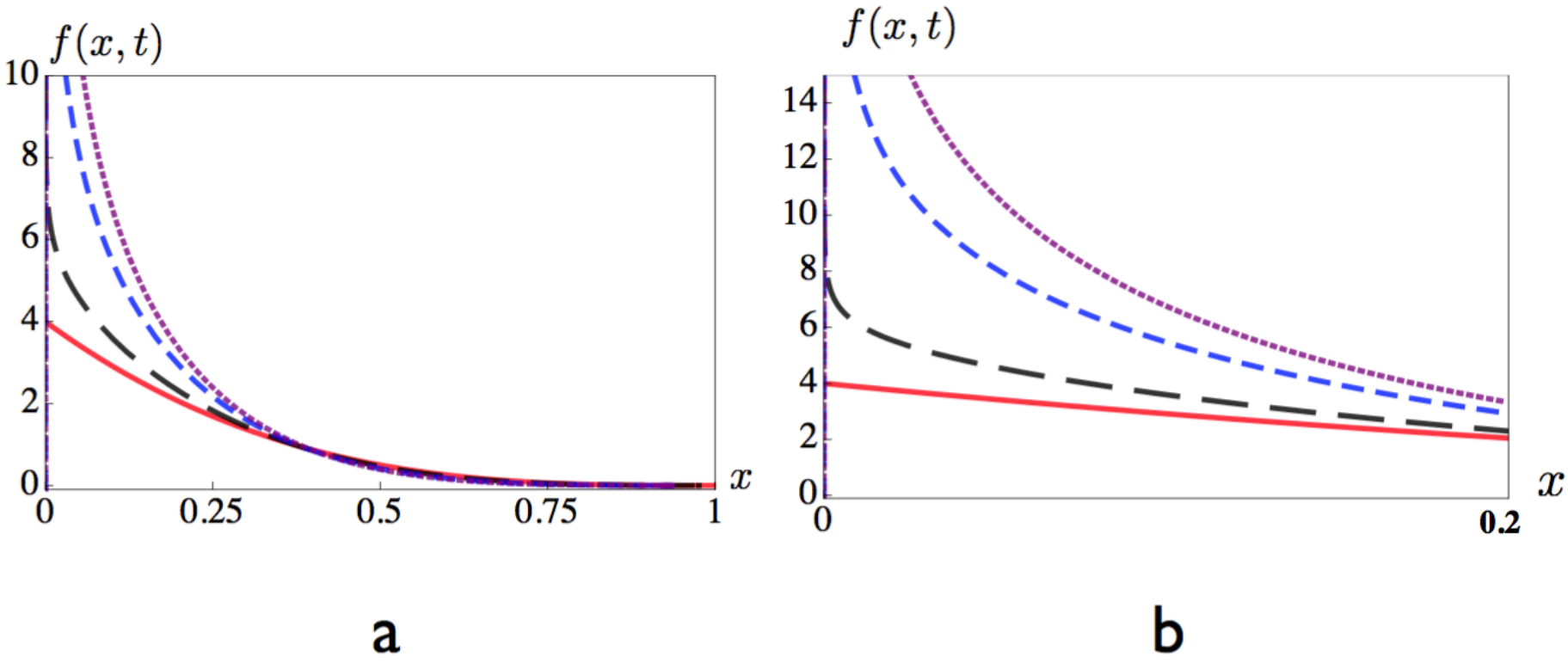
$$\frac{df(x,t)}{dt} = \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) f(y)$$

- Rewrite the evolution equation ($P(x/y) = P(x,y)$)

$$\frac{df(x,t)}{dt} = \int_0^1 \frac{dy}{y} P(x,y)[f(y) - f(x)] + f(x) \int_0^1 \left[\frac{P(x,z)}{z} - \frac{P(z,x)}{x}\right] dz$$

Evolution of Typical PDF $(1-x)^3$

- Ansatz $f(x,t) = e^{(x+1/2)t}\bar{x}^{2t}x^{-t}F(x,t)$



From red to purple: $t=0, t=0.2, t=0.4, t=0.5$

Main Points of the Talk

- Developing an Analytic Method for Evolution of Singular DAs $\surd$
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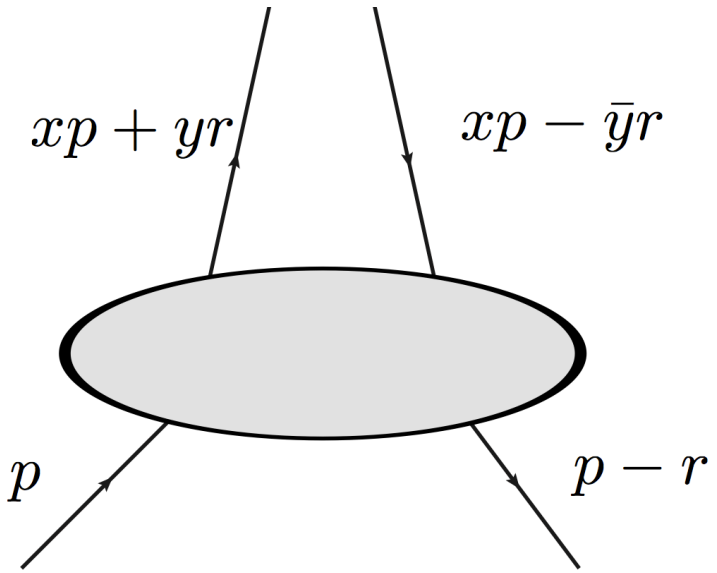
Evolution of Double Distributions

- Nonforward matrix element $\langle p-r \,|\, \mathcal{O}(0,z) \,|\, p\rangle \,|_{\,z^2=0}$ can be parametrized by DDs



- Integration over $y \Rightarrow$ PDFs
  $\int_0^{1-x} R^{ab}(x,y;\xi,\eta)\, dy = \frac{1}{\xi}\, P^{ab}(x/\xi)$
- Integration over $x \Rightarrow$ DAs
  $\int_0^{1-y} R^{QQ}(x,y;\xi,\eta) dx = V^{QQ}(y,\eta)$

Evolution of Double Distributions

- Double Distribution kernel

$$R^{QQ}(x,y;\xi,\eta) = \frac{\alpha_s}{\pi} C_F \frac{1}{\xi} \Bigg\{ \theta(0 \le x/\xi \le \min\{y/\eta, \overline{y}/\overline{\eta}\}$$

$$+ \quad \frac{\theta(0 \le x/\xi \le 1) x/\xi}{(1 - x/\xi)} \left[\frac{1}{\eta} \delta(x/\xi - y/\eta) + \frac{1}{\overline{\eta}} \delta(x/\xi - \overline{y}/\overline{\eta}) \right]$$

$$- \quad \delta(1 - x/\xi) \delta(y - \eta) \left[\frac{1}{2} + 2 \int_0^1 \frac{z}{1-z} dz \right] \Bigg\}$$

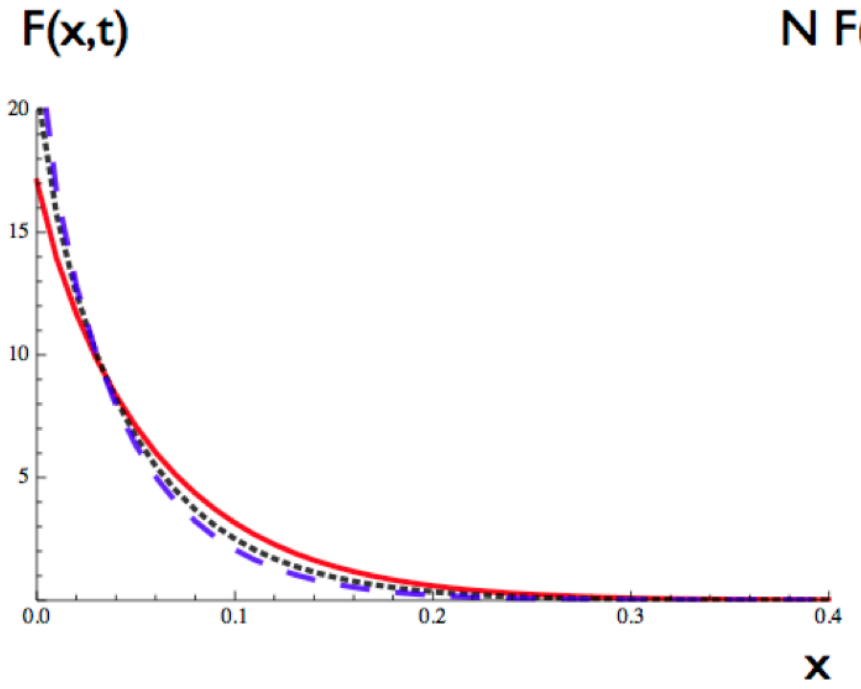
- Some algebra + rearranging

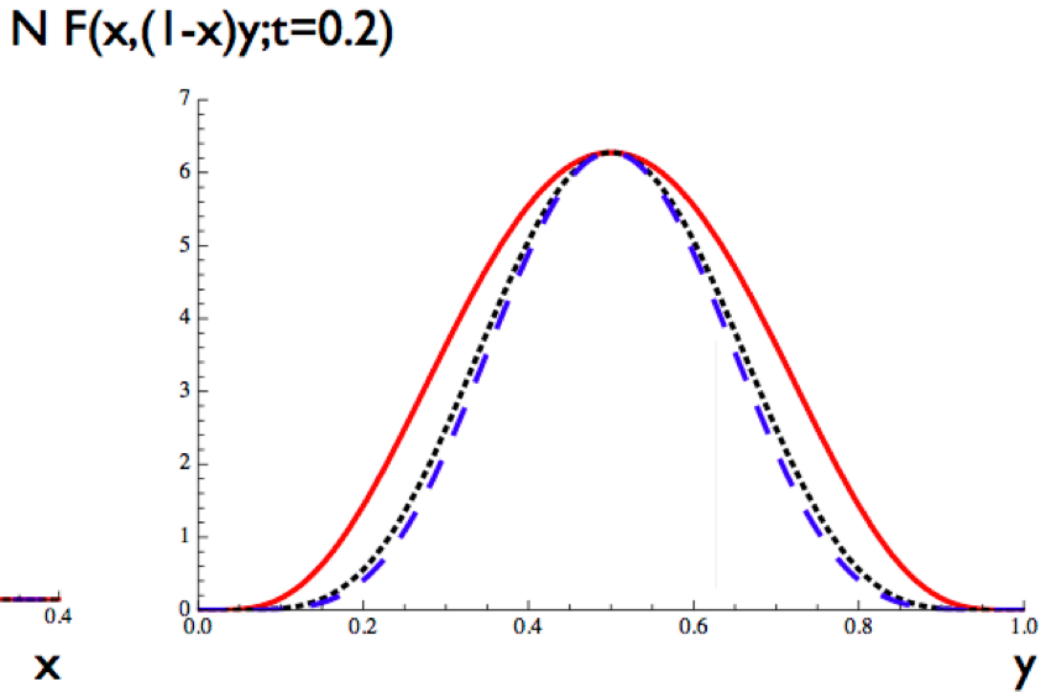
$$\frac{\partial F(x,y;t)}{\partial t} = \int_y^{\frac{y}{y+x}} \frac{d\eta}{\eta - y} \left[F\left(\frac{x\eta}{y}, \eta; t \right) - F(x,y;t) \right]$$

$$+ \int_0^y \frac{d\eta}{y - \eta} \left[F\left(\frac{x\overline{\eta}}{\overline{y}}, \eta; t \right) - F(x,y;t) \right]$$

$$+ F(x,y;t) \left[2 + \ln\left(\frac{y(1-x-y)}{\overline{y}(x+y)} \right) \right)$$

Evolution of $[y(1-x-y)]^{10}$





Summary and Conclusion

- We described an analytic method for evolution of singular DAs which appear in models for GPDs
  - Logaritmic singularities are summed from the start, which produces a continuous curve.
  - Unlike the standard method which needs an infinite amount of terms for singular DAs, our method requires only one or two iterations for precise results.
- We applied our method to GDAs
  - With just one or two iterations, we obtained more accurate results than the existing numerical method for evolution of two-photon GDA
- We illustrated application of the method to DGLAP evolution and evolution of DDs

Results are presented in the following publications:

- A. V. Radyushkin and A. Tandogan. Analytic Evolution of Singular Distribution Amplitudes in QCD. Phys. Rev. D, 89:074003, Mar 2014
- A. Tandogan and A. V. Radyushkin. Studies of analytic evolution of two- photon generalized distribution amplitude in QCD. Int.J.Mod.Phys.Conf.Ser., 25:1460037, 2014
- A. Tandogan and A. V. Radyushkin. Method of Analytic Evolution of Flat Distribution Amplitudes in QCD. Int.J.Mod.Phys. Conf.Ser., 04:227-238, 2011