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TN38
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Energy Spread From RF Phase and Amplitude Errors

In this note the energy spread due to RF phase and amplitude errors is estimated.

The fundamental inputs to the theory are the distribution function for the phase error $f(\delta)$ and the distribution function for the amplitude error $g(A)$. The distribution functions are normalized, meaning that the probability of the phase error lying between δ and $\delta + d\delta$ is $f(\delta)d\delta$ and the probability of the amplitude error lying between A and $A+dA$ is $g(A)dA$.

For simplicity assume that the accelerator consists of N indentical cavities. The energy gain of a particle passing through the nth cavity is

$$\Delta E_n = E_0(1 + A_n)\cos(\delta_n + \delta_0)$$

where E_0 is the energy gain per cavity, A_n is the amplitude error at the nth cavity, δ_n is the phase error at the nth cavity, and δ_0 is the injection phase of the particle. The total energy is

$$E = \sum_{n=1}^{N} \Delta E_n.$$

The *rms* energy spread is evaluated using two suppositions about the errors. An optimistic result is obtained by assuming that the phase errors are not correlated with the amplitude errors and that the different cavities are independent. At the other extreme, it is pessimistic to assume that the errors are totally correlated, i.e.

$$A_1 = ... = A_n \quad \text{and} \quad \delta_1 = ... = \delta_n.$$

When the errors are uncorrelated the distribution function is a product of the independent distribution functions

$$F(\delta_1, A_1, ..., \delta_N, A_N) = \prod_{n=1}^{N} f(\delta_n)g(A_n)$$

where the function F is used to compute ensemble averages, for example

$$\overline{E} = \int EF(\delta_1, A_1, ..., \delta_N, A_N)d\delta_1 dA_1 ... d\delta_N dA_N.$$

The average energy is

$$\overline{E} = NE_0 I_1$$

where

$$I_1(\delta_0) = \int_{-\infty}^{\infty} \cos(\delta + \delta_0) f(\delta) d\delta$$

since

$$\int_{-\infty}^{\infty} A g(A) dA = 0.$$

Because

$$E^2 = \sum_{m=1}^{N} \sum_{n=1}^{N} \Delta E_m \Delta E_n$$

one obtains

$$\overline{E^2} = N(N-1) E_0^2 I_1^2 + N E_0^2 I_2 + N E_0^2 I_2 I_3$$

where

$$I_2(\delta_0) = \int_{-\infty}^{\infty} \cos^2(\delta + \delta_0) f(\delta) d\delta$$

and

$$I_3(\delta_0) = \int_{-\infty}^{\infty} A^2 g(A) dA.$$

The relative *rms* energy spread is thus

$$E_{rms}/E = \sqrt{\overline{E^2} - \overline{E}^2}/E = \frac{\sqrt{I_2 + I_2 I_3 - I_1^2}}{\sqrt{N}}. \tag{1}$$

In the totally correlated case the distribution function is

$$F(\delta_1, A_1, ..., \delta_N, A_N) = f(\delta_1) g(A_1) \prod_{n=2}^{N} \delta(\delta_1 - \delta_n) \delta(A_1 - A_n).$$

The result of performing the averages is

$$\overline{E} = N E_0 I_1$$

and

$$E_{rms}/E = \sqrt{I_2 + I_2 I_3 - I_1^2}.$$

For 400 cavities this is a factor of 20 worse than the optimistic result.

Example

If

$$f(\delta) = \frac{1}{\sqrt{2\pi}\sigma_\delta} \exp(-\delta^2/2\sigma_\delta^2)$$

and

$$g(A) = \frac{1}{\sqrt{2\pi}\sigma_A} \exp(-A^2/2\sigma_A^2)$$

then

$$I_1 = \exp(-\sigma_\delta^2/2)\cos(\delta_0)$$

$$I_2 = 0.5 + 0.5\exp(-2\sigma_\delta^2)\cos(2\delta_0)$$

and

$$I_3 = \sigma_A^2.$$

In either case the average energy is lower than the maximum energy by

$$NE_0 - \overline{E} \doteq \sigma_\delta^2/2.$$

The relative *rms* energy spread satisfies

$$\frac{\sqrt{\sigma_\delta^4/2 + \sigma_\delta^2\delta_0^2 + \sigma_A^2}}{\sqrt{N}} \le E_{rms}/E \le \sqrt{\sigma_\delta^4/2 + \sigma_\delta^2\delta_0^2 + \sigma_A^2}. \tag{2}$$

In the CEBAF recirculating linac it is appropriate to take $N = 400$, i.e. the number of cavities the beam sees on a single pass, since the same error occurs during the four passes due to the short recirculation time. After the first pass the errors are correlated with those of the previous pass. Using $N = 400$, $\sigma_\delta = 0.017$ ($\pm 1°$), $\delta_0 = 0.017$, and $\sigma_A = 2 \times 10^{-4}$, the limits are

$$2 \times 10^{-5} < E_{rms}/E < 4 \times 10^{-4}.$$

When more is known about the possible errors, this result should be sharpened.