CEBAF-TN-0050
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Revised Energy Spread Estimate

In this note the results of CEBAF-TN-0038 are extended to give a more realistic calculation of the energy spread in the beam due to RF phase and amplitude errors. The effects of slow phase errors and the effects of the feedback systems are included in the calculation. If an rms energy spread requirement is to be met, the final formula gives the trade-off between the effects of the slow phase errors and the effects of fast phase errors in the feedback systems.

Brief Description of the RF System

First the ideal system is described. Ideally, the RF system is timed so the bunches arrive synchronized with the RF; the bunch centroid (in phase) should coincide with the RF crest. Mathematically, the requirement is

$$\overline{\Phi}_n = \int \Phi f_n(E,\Phi) dE d\Phi = 0$$

where $f_n(E,\Phi)$ is the single particle longitudinal distribution function of the beam electrons as they enter the nth cavity, E is the kinetic energy, and Φ is the phase with respect to the RF in the nth cavity. Since there is little longitudinal motion within the bunch, in the ideal system the phase of an electron does not change with respect to the crest. Therefore the total energy of an electron emerging from the accelerator, T, is

$$T = E + \sum_{n=1}^{N} \Delta E_n$$

where

$$\Delta E_n = E_0 \cos(\Phi),$$

E_0 is the energy gain per cavity, E is the injection energy, Φ is the injection phase, and N is the total number of cavities. Henceforward $f(E,\Phi)$ denotes the longitudinal distribution function at injection, i.e. $f(E,\Phi)dEd\Phi$ is the probability that an electron initially has energy between E and $E+dE$ and phase between Φ and $d\Phi$.

The beam emerging from the ideal system has a finite energy spread because of the finite energy spread at injection and because of the finite bunch length. The calculations and results are similar to those in Ref. 1.

$$T_{rms}/T = \sqrt{E_{rms}^2/T^2 + \sigma_\Phi^4/2} \tag{1}$$

where E_{rms} is the rms energy spread at injection and σ_Φ is the rms phase spread. For CEBAF the first term in the sum is negligible compared to the second term.

In reality there are voltage errors and (at least) two types of phase errors. An example of the first type of phase error is a phase difference between the voltage in a cavity and the master oscillator phase line. These errors are controlled by dedicated feedback systems which measure the phase of the RF voltage in the cavity, compare it to the phase on the phase line, and shift the phase of the cavity feed voltage until the two phases agree. Such a system can keep the phase of the cavity locked to that of the timing line to a few tenths of a degree[2].

Unfortunately, the phase line itself can introduce a second type of timing error. Such phase line errors, for example time delays due to thermal expansion or contraction of the materials in the phase line or accelerator, are partially corrected by the energy vernier system. The basic idea of the energy vernier system is to set the average energy of the beam as it emerges from the accelerator on the crest of the accelerator as a whole, disregarding the individual errors between the cavities. This is accomplished by having one cavity (or a group of cavities if necessary) powered up or down to set the *average* beam energy and by shifting the injector phase to guarantee that the beam is on the crest of the accelerator as a whole. The phase line errors may be expected to change on a slow time scale (of order greater than minutes).

More generally, the two types of phase errors are distinguished by the time scale over which the error changes. In the former case the fluctuations in the cavity feedback systems are at frequencies beyond the ability of the vernier system to respond. In this paper any phase error at frequencies beyond the vernier system bandwidth is called a high frequency or fast phase error. Mathematically, the high frequency phase errors are characterized by a distribution function which gives the probability that a given fluctuation occurs. Such a distribution is obtained in practice by performing a long term average of the fluctuations.

The thermal expansion phase line errors are examples of errors that change slowly enough that the energy vernier system is able to respond. In this paper such errors are called slow phase errors. They result in slow changes in the energy spread, the energy spread being a function of the slow phase errors. Any phase error within the bandwidth of the vernier system should be treated as a slow phase error below.

With a perfect (i.e. infinite bandwidth) vernier system the results of the calculation show that there is no energy spread from slow phase errors. This behavior arises because the vernier system, given infinite bandwidth, would be able to instantaneously place the bunches on the crest of the accelerator as a whole. In reality, since the vernier system has finite bandwidth and since the fast phase errors of the individual cavities are not corrected, there is a source of energy spread from the fact that the bunches no longer pass through the individual cavities precisely on the crest. The main goal of the following calculation is to quantify the maximum tolerable slow phase error given the fast fluctuation level. Too much slow phase error of the cavities causes the energy spread to deteriorate significantly and can cause a retiming of the whole machine.

Energy Spread Calculation

Fig. 1 gives a schematic diagram of the major error sources in the accelerator. The accelerator consists of N cavities. The energy of an electron passing through the accelerator is

$$T = E + \sum_{n=1}^{N} E_n(1 + A_n)\cos(\phi_n - \Phi_0 + \Phi + \delta_0 + \delta_n)$$

where E_n is the voltage set point of the nth cavity, A_n is the relative voltage error at the nth cavity, ϕ_n is the slow phase error from the master oscillator to the nth cavity, Φ_0 is the injector phase offset, δ_0 is the high frequency phase error in the injector, and δ_n is the fast phase error to the nth cavity. The vernier system sets the injector phase offset; the sign of Φ_0 is chosen for later convenience. The goal is to compute the energy spread as a function of the slow phase errors $\phi_1, ..., \phi_N$.

The *rms* energy spread is evaluated using two suppositions about the cavity phase and voltage errors. An optimistic result is obtained by assuming that the phase errors are not correlated with the voltage errors and that the different cavities are independent. At the other extreme, it is pessimistic to assume that the errors are totally correlated, i.e.

$$A_1 = ... = A_N \quad \text{and} \quad \delta_1 = ... = \delta_N.$$

When the cavity phase and voltage errors are uncorrelated the distribution function is a product of the independent distribution functions

$$F(E, \Phi, \delta_0, \delta_1, A_1, ..., \delta_N, A_N) = f(E, \Phi)\psi_i(\delta_0) \prod_{n=1}^{N} \psi(\delta_n)g(A_n)$$

where $\psi(\delta)$ is the distribution function for the fast cavity phase error and $g(A)$ is the distribution function for the voltage error. Since the phase correction scheme for the injector is somewhat different from that for the cavities one anticipates that the distribution function for the injector phase errors is different from that of the cavities. Consequently, the distribution function for the injector phase errors is denoted by the separate symbol $\psi_i(\delta_0)$. The distribution functions are normalized, meaning that they give probabilities on integration. The function F is used to compute ensemble averages, for example

$$\overline{T} = \int TF(E, \Phi, \delta_0, \delta_1, A_1, ..., \delta_N, A_N)dEd\Phi d\delta_0 d\delta_1 dA_1 ... d\delta_N dA_N.$$

The average energy is

$$\overline{T} = \overline{E} + \sum_{n=1}^{N} E_n I_1(\phi_n, \Phi_0)$$

where

$$I_1(\phi, \Phi_0) = \int \cos(\phi - \Phi_0 + \Phi + \delta_0 + \delta) f(E, \Phi)\psi_i(\delta_0)\psi(\delta) dEd\Phi d\delta_0 d\delta$$

since

$$\int Ag(A)dA = 0.$$

I_1 is reexpressed as

$$I_1(\phi, \Phi_0) = \cos(\phi - \Phi_0)I_c - \sin(\phi - \Phi_0)I_s$$

where

$$I_c = \int \cos(\Phi + \delta_0 + \delta)f(E, \Phi)\psi_i(\delta_0)\psi(\delta)dEd\Phi d\delta_0 d\delta$$

and

$$I_s = \int \sin(\Phi + \delta_0 + \delta)f(E, \Phi)\psi_i(\delta_0)\psi(\delta)dEd\Phi d\delta_0 d\delta.$$

The vernier sets Φ_0 and the cavity excitations so that

$$\overline{T} = \overline{E} + E_f$$

and

$$d\overline{T}/d\Phi_0 = 0,$$

i.e. the energy gain of the accelerator is fixed to E_f and the injector phase offset is set so the bunch is on the crest of the accelerator as a whole. This means

$$\sum_{n=1}^{N} E_n\left[\cos(\phi_n - \Phi_0)I_c - \sin(\phi_n - \Phi_0)I_s\right] = E_f$$

and

$$\sum_{n=1}^{N} E_n\left[\sin(\phi_n - \Phi_0)I_c + \cos(\phi_n - \Phi_0)I_s\right] = 0.$$

It is now possible to solve for the cavity sums:

$$\sum_{n=1}^{N} E_n \cos(\phi_n - \Phi_0) = \frac{I_c E_f}{I_c^2 + I_s^2} \qquad (2a)$$

and

$$\sum_{n=1}^{N} E_n \sin(\phi_n - \Phi_0) = -\frac{I_s E_f}{I_c^2 + I_s^2}. \qquad (2b)$$

Now

$$\overline{T^2} = \overline{E^2} + 2\overline{E}\sum_{n=1}^{N} E_n I_1(\phi_n, \Phi_0)$$

$$+\sum_{n=1}^{N}\sum_{p\neq n}^{N} E_n E_p\left[\cos(\phi_n - \Phi_0)\cos(\phi_p - \Phi_0)I_{cc} + \sin(\phi_n - \Phi_0)\sin(\phi_p - \Phi_0)I_{ss}\right]$$

$$-\sum_{n=1}^{N}\sum_{p\neq n}^{N} E_n E_p\left[\cos(\phi_n - \Phi_0)\sin(\phi_p - \Phi_0)I_{cs} + \sin(\phi_n - \Phi_0)\cos(\phi_p - \Phi_0)I_{sc}\right]$$

$$+(1+I_A)\sum_{n=1}^{N} E_n^2\left[\cos(\phi_n - \Phi_0)\cos(\phi_n - \Phi_0)I_{2cc} + \sin(\phi_n - \Phi_0)\sin(\phi_n - \Phi_0)I_{2ss}\right]$$

$$-(1+I_A)\sum_{n=1}^{N} E_n^2\left[\cos(\phi_n - \Phi_0)\sin(\phi_n - \Phi_0)I_{2cs} + \sin(\phi_n - \Phi_0)\cos(\phi_n - \Phi_0)I_{2sc}\right],$$

where

$$I_{cc} = \int \cos(\Phi + \delta_0 + \delta)\cos(\Phi + \delta_0 + \delta')f(E,\Phi)\psi_i(\delta_0)\psi(\delta)\psi(\delta')dEd\Phi d\delta_0 d\delta d\delta',$$

$$I_{cs} = \int \cos(\Phi + \delta_0 + \delta)\sin(\Phi + \delta_0 + \delta')f(E,\Phi)\psi_i(\delta_0)\psi(\delta)\psi(\delta')dEd\Phi d\delta_0 d\delta d\delta',$$

etc.,

$$I_{2cc} = \int \cos(\Phi + \delta_0 + \delta)\cos(\Phi + \delta_0 + \delta)f(E,\Phi)\psi_i(\delta_0)\psi(\delta)dEd\Phi d\delta_0 d\delta,$$

$$I_{2cs} = \int \cos(\Phi + \delta_0 + \delta)\sin(\Phi + \delta_0 + \delta)f(E,\Phi)\psi_i(\delta_0)\psi(\delta)dEd\Phi d\delta_0 d\delta,$$

etc.,

and

$$I_A = \int A^2 g(A)dA.$$

The relative *rms* energy spread is given by the sum of four terms

$$T_{rms}/\overline{T} = \sqrt{\overline{T^2} - \overline{T}^2}/\overline{T} = \frac{\sqrt{E_{rms}^2 + T_1^2 + T_2^2 + T_3^2}}{\overline{T}}, \tag{3}$$

where E_{rms} is the *rms* energy spread at injection,

$$T_1^2 = \frac{E_f^2}{(I_c^2 + I_s^2)^2}\left[I_c^2(I_{cc} - I_c^2) + I_s^2(I_{ss} - I_s^2) + I_cI_s(I_{cs} - I_cI_s) + I_sI_c(I_{sc} - I_sI_c)\right],$$

$$T_2^2=\sum_{n=1}^{N}E_n^2\big[\cos^2(\phi_n-\Phi_0)(I_{2cc}-I_{cc})+\sin^2(\phi_n-\Phi_0)(I_{2ss}-I_{ss})$$

$$-\cos(\phi_n-\Phi_0)\sin(\phi_n-\Phi_0)(I_{2cs}-I_{cs})-\sin(\phi_n-\Phi_0)\cos(\phi_n-\Phi_0)(I_{2sc}-I_{sc})\big],$$

and

$$T_3^2=I_A\sum_{n=1}^{N}E_n^2\big[\cos^2(\phi_n-\Phi_0)I_{2cc}+\sin^2(\phi_n-\Phi_0)I_{2ss}\big]$$

$$-I_A\sum_{n=1}^{N}E_n^2\big[\cos(\phi_n-\Phi_0)\sin(\phi_n-\Phi_0)I_{2cs}+\sin(\phi_n-\Phi_0)\cos(\phi_n-\Phi_0)I_{2sc}\big].$$

The first term in Eqn. (3) is from the energy spread at injection and is negligible for CEBAF. The second term gives the contributions from the finite bunch length and from the injector phase error. Since these errors are the same at each cavity (i.e. they are correlated errors) the resulting energy spread is independent of the number of cavities. Also the second term is independent of the slow phase errors. The third term is due to the fast phase errors in the individual cavities and due to the fact that the bunches traverse the individual cavities slightly off crest because of the slow phase errors. This contribution vanishes if the fast phase errors vanish since in this case $I_{cc}=I_{2cc}$, etc. For fixed total energy the third term is inversely proportional to the number of cavities. The fourth term in the sum is due to the voltage errors in the system and it is inversely proportional to the number of cavities, the usual result for the effect of uncorrelated errors.

In the totally correlated case the distribution function is

$$F(E,\Phi,\delta_0,\delta_1,A_1,...,\delta_N,A_N)=f(E,\Phi)\psi_i(\delta_0)\psi(\delta_1)g(A_1)\prod_{n=2}^{N}\delta(\delta_1-\delta_n)\delta(A_1-A_n).$$

The result of calculating $\overline{T}$ is the same as before. However,

$$\overline{T^2}=\overline{E^2}+2\overline{E}\sum_{n=1}^{N}E_nI_1(\phi_n,\Phi_0)$$

$$+(1+I_A)\sum_{n=1}^{N}\sum_{p=1}^{N}E_nE_p\big[\cos(\phi_n-\Phi_0)\cos(\phi_p-\Phi_0)I_{2cc}+\sin(\phi_n-\Phi_0)\sin(\phi_p-\Phi_0)I_{2ss}\big]$$

$$-(1+I_A)\sum_{n=1}^{N}\sum_{p=1}^{N}E_nE_p\big[\cos(\phi_n-\Phi_0)\sin(\phi_p-\Phi_0)I_{2cs}+\sin(\phi_n-\Phi_0)\cos(\phi_p-\Phi_0)I_{2sc}\big].$$

Therefore Eqn. (3) still gives the form of the expression for T_{rms} but now

$$T_1^2=\frac{E_f^2}{(I_c^2+I_s^2)^2}\big[I_c^2(I_{2cc}-I_c^2)+I_s^2(I_{2ss}-I_s^2)+I_cI_s(I_{2cs}-I_cI_s)+I_sI_c(I_{2sc}-I_sI_c)\big],$$

$$T_2^2 = 0,$$

and

$$T_3^2 = \frac{I_A E_f^2}{(I_c^2 + I_s^2)^2}\left[I_c^2 I_{2cc} + I_s^2 I_{2ss} + I_c I_s I_{2cs} + I_s I_c I_{2sc}\right].$$

Since the T_2 term is zero, there is no dependence of the result on the slow phase errors ϕ_n. The third term is a factor of N larger than before, as is characteristic in the transition from uncorrelated errors to correlated ones. For 400 cavities this difference can be large.

Example

If

$$f(E,\Phi) = \frac{1}{2\pi\sigma_E\sigma_\Phi}\exp\left[-(E-\overline{E})^2/2\sigma_E^2\right]\exp(-\Phi^2/2\sigma_\Phi^2),$$

$$\psi(\delta) = \frac{1}{\sqrt{2\pi}\sigma_\delta}\exp(-\delta^2/2\sigma_\delta^2),$$

$$\psi_i(\delta_0) = \frac{1}{\sqrt{2\pi}\sigma_i}\exp(-\delta_0^2/2\sigma_i^2),$$

and

$$g(A) = \frac{1}{\sqrt{2\pi}\sigma_A}\exp(-A^2/2\sigma_A^2),$$

then

$$I_c = \exp(-\sigma_\Phi^2/2)\exp(-\sigma_i^2/2)\exp(-\sigma_\delta^2/2),$$

$$I_s = 0,$$

$$I_{cc} = \left[0.5 + 0.5\exp(-2\sigma_\Phi^2)\exp(-2\sigma_i^2)\right]\exp(-\sigma_\delta^2),$$

$$I_{ss} = \left[0.5 - 0.5\exp(-2\sigma_\Phi^2)\exp(-2\sigma_i^2)\right]\exp(-\sigma_\delta^2),$$

$$I_{cs} = I_{sc} = 0,$$

$$I_{2cc} = 0.5 + 0.5\exp(-2\sigma_\Phi^2)\exp(-2\sigma_i^2)\exp(-2\sigma_\delta^2),$$

$$I_{2ss} = 0.5 - 0.5\exp(-2\sigma_\Phi^2)\exp(-2\sigma_i^2)\exp(-2\sigma_\delta^2),$$

$$I_{2cs} = I_{2sc} = 0,$$

and

$$I_A = \sigma_A^2.$$

From the expressions above it is clear that the effects of the finite bunch length and of the injector phase error may be characterized by a single parameter

$$\sigma_I^2 = \sigma_\Phi^2 + \sigma_i^2.$$

This parameter may be regarded as an effective bunch length which depends on the properties of the injector alone. Since the contributions add in quadrature, it is clear that little is gained is by shortening the bunch length (in phase) much less than one can regulate the injector phase or by regulating the injector phase much beyond the bunch length.

To continue it is necessary to specify the vernier system in greater detail. As an example assume that all the cavities except one, the mth cavity, are set so

$$E_n = E_0 = E_f/N \qquad \text{for} \quad n \neq m.$$

This means that Eqns. (2) become

$$E_m \doteq E_0\left[1 + N(\sigma_I^2 + \sigma_\delta^2)/2 + N\Delta\phi^2/2\right]$$

and

$$\Phi_0 \doteq \sum_{n=1}^{N} \phi_n/N$$

to second order in the small quantities $\sigma_I, \ldots,$ and Φ_0. In the above formula

$$\Delta\phi^2 = \sum_{n=1}^{N} (\phi_n - \Phi_0)^2/N$$

is approximately the square of the rms average of the slow phase errors since Φ_0 is approximately equal to the average of the slow phase error. An increase in the voltage of the vernier cavity is needed to compensate for the effect of the errors since they tend to force the bunch particles off the crest of the voltage of the individual cavities.

The two estimates give that the relative *rms* energy spread satisfies

$$\sqrt{D_1} \leq E_{rms}/E \leq \sqrt{D_2} \tag{4}$$

where

$$D_1 \doteq \sigma_I^4/2 + \sigma_\delta^2 \sum_{n=1}^{N} (\phi_n - \Phi_0)^2/N^2 + \sigma_\delta^2(\sigma_\delta^2/2 + \sigma_I^2)/N + \sigma_A^2/N$$

and

$$D_2 \doteq (\sigma_I^2 + \sigma_\delta^2)^2/2 + \sigma_A^2$$

to fourth order in the small quantities. Since the terms neglected are all of sixth order or higher the approximation is very good. The two estimates depend on whether the errors are uncorellated (RH inequality) or corellated (LH inequality). Note that one obtains less energy spread from the slow phase errors, ϕ_n, by regulating the individual cavity feedback systems so that the fast phase errors are reduced (σ_δ decreases).

In the CEBAF recirculating linac it is appropriate to take $N = 400$, i.e. the number of cavities the beam sees on a single pass since the same error occurs during the four passes due to the short recirculation time. After the first pass the errors are correlated with those of the first pass.

By the central limit theorem of probability theory the energy distribution function becomes Gaussian for large numbers of cavities. To obtain a given energy spread, the RF system at minimum must be designed so that the LH inequality is satisfied. Consequently limits on the various terms are calculated. For example, if it is desired to obtain a beam with 10^{-4} FWHM energy spread it is required that

$$\sqrt{D_1} \leq 4.25 \times 10^{-5}$$

assuming the energy distribution is in fact Gaussian. Under optimistic assumptions the injector is capable of producing a beam of full width of about 0.7°. Assuming the phase distribution is parabolic gives $\sigma_\Phi = 2.73 \times 10^{-3}$. It is required that $\sigma_i < 2 \times 10^{-3}(\pm 0.2^\circ)$ (square error distribution) for the injector phase errors to give a contribution to the energy spread no larger than one half the contribution from the bunch length. Under these conditions the effective bunch length is $\sigma_I = 0.0034$.

In Fig. 2 the combinations of σ_δ, $\Delta\phi$, and σ_A which yield 10^{-4} FWHM energy spread are shown. For example if $\sigma_A = 7 \times 10^{-4}$ (corresponding to an amplitude error full width of 2.8×10^{-3}) and if $\sigma_\delta = 1^\circ$ then $\Delta\phi$ must be less than 1.5° to obtain 10^{-4} FWHM energy spread. If $\sigma_A = 7 \times 10^{-4}$ and $\sigma_\delta = .2^\circ$ then $\Delta\phi$ must be less than 7.5°. If $\sigma_A = 7 \times 10^{-4}$ it is not possible to achieve 10^{-4} for $\sigma_\delta > 1.43$. In addition, if $\sigma_A > 8.3 \times 10^{-4}$ no solutions exist with 10^{-4} FWHM energy spread. At the other extreme if $\sigma_A < 0.0002$, then the curve assuming $\sigma_A = 0$ gives a good approximation for the energy spread.

In Fig. 3 the same plot is given for a somewhat larger and more realistic effective bunch length $\sigma_I = 0.0048$. One observes that as a result of the increased bunch length less energy spread may arise from the other terms if the same energy spread is to be achieved. In going to Fig. 4 the effective bunch length is increased to 0.00678. In this case substantial decrease in the sub 10^{-4} energy spread region is observed.

Finally, in Fig. 5 the same plot as Fig. 2 is given but now $\sigma_E = 2.12 \times 10^{-5}$ which corresponds to a case with full width energy spread of 10^{-4}. In this case $\sigma_A < 0.00034$ is needed for a solution to exist. Less spread is available once the term due to the finite bunch length is subtracted, yielding increased sensitivity.

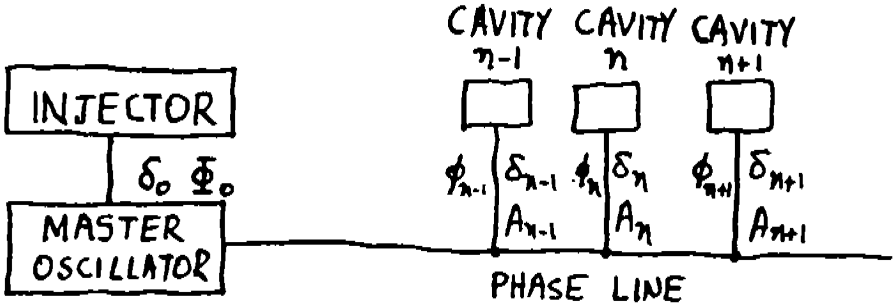
Conclusions:

We have presented a formula that yields the beam energy spread given the injector distribution, the various error distribution functions and a set of slow phase errors. By reducing the size of the fast fluctuations in the individual cavities, the dependence of the energy spread on the slow phase errors is diminished. For this reason it is preferable to have as large a bandwidth as possible for the energy vernier system.

To achieve a given energy spread little is gained by regulating the injector phase to a level much less than the bunch extent in phase. To obtain a FWHM energy spread less than 10^{-4}, slow phase errors up to 6° *rms* are tolerable for a ±0.4° bunch and for ±0.5° fast phase fluctuations. Note that a temperature change of the phase line of only 3° C gives $\Delta\phi = 6.6^\circ$. Finally, smaller bunches help.

References

1. G. Krafft, CEBAF-TN-0038
2. J. Fugitt, CEBAF Design Handbook



δ_i	$i = 0, 1, ..., N$	Fast phase error to the nth cavity
A_i	$i = 1, ..., N$	Cavity amplitude errors
ϕ_i	$i = 1, ..., N$	Slow phase error to the nth cavity
Φ_0		Injector phase set point which keeps the bunches on the crest of accelerator as a whole

Figure 1 Schematic of Possible Errors
In the RF System

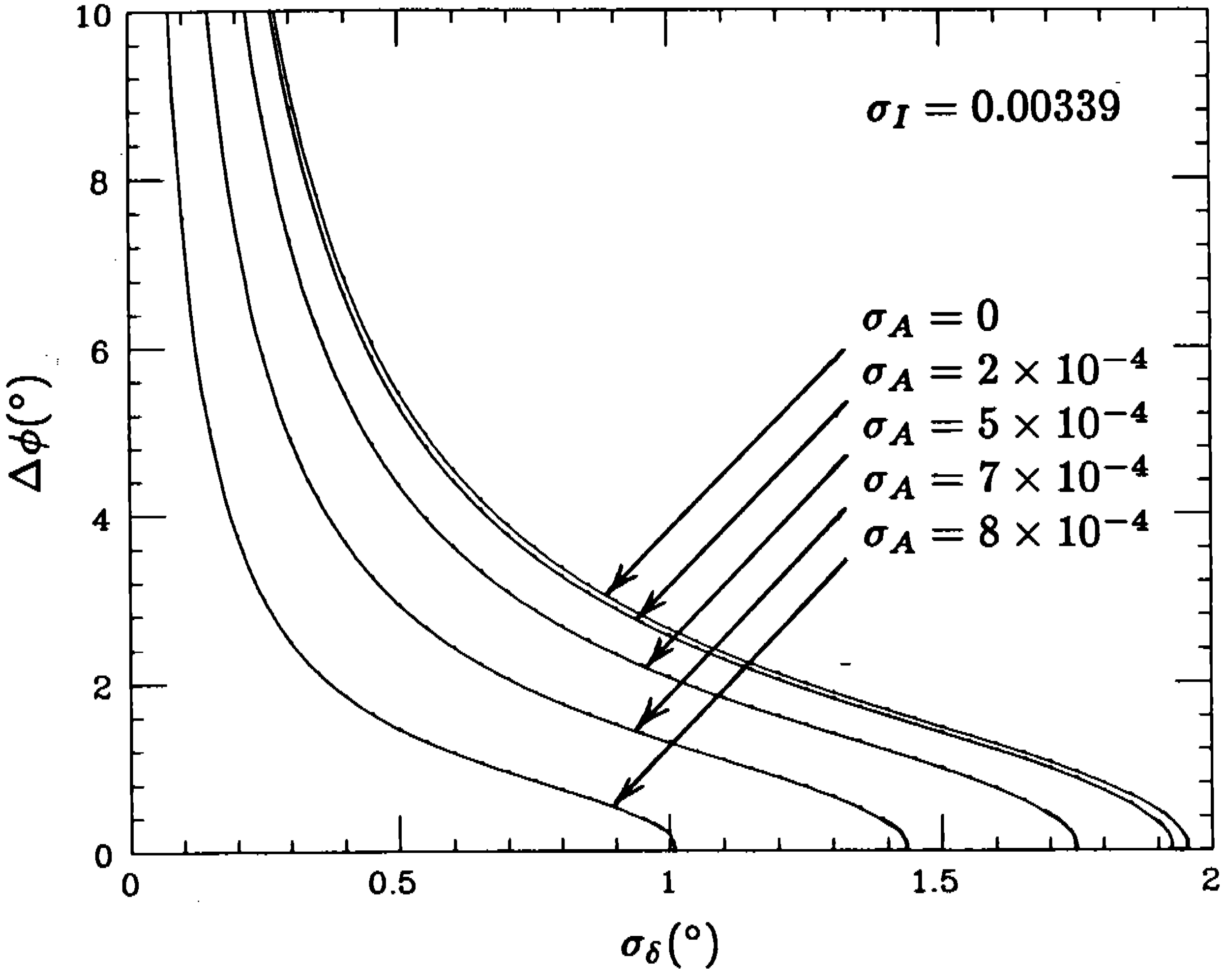


Figure 2 Combinations of σ_δ, $\Delta\phi$, and σ_A
Which Achieve 10^{-4} FWHM Energy Spread When $\sigma_I = 0.00339$

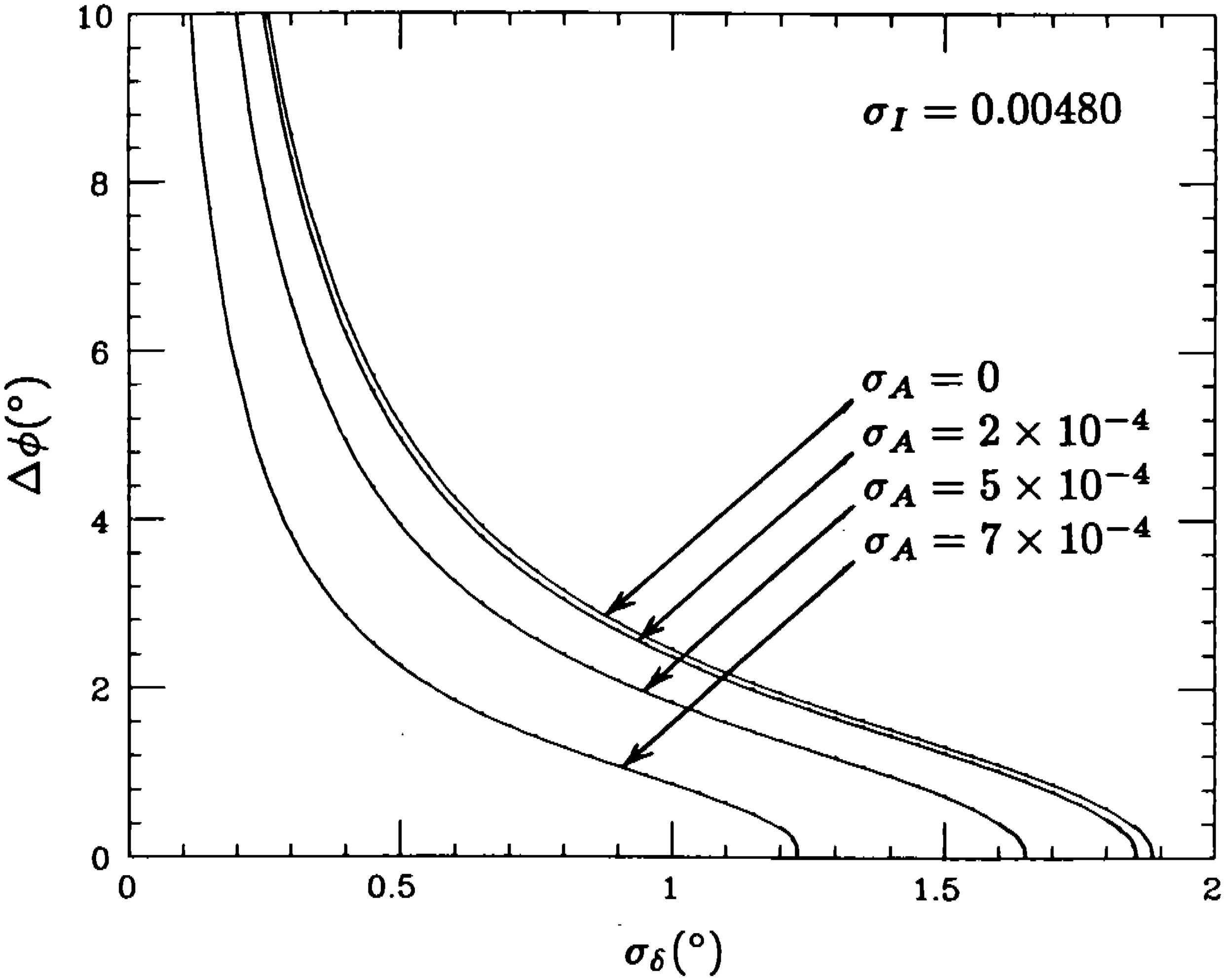


Figure 3 Combinations of σ_δ, $\Delta\phi$, and σ_A
Which Achieve 10^{-4} FWHM Energy Spread When $\sigma_I = 0.00480$

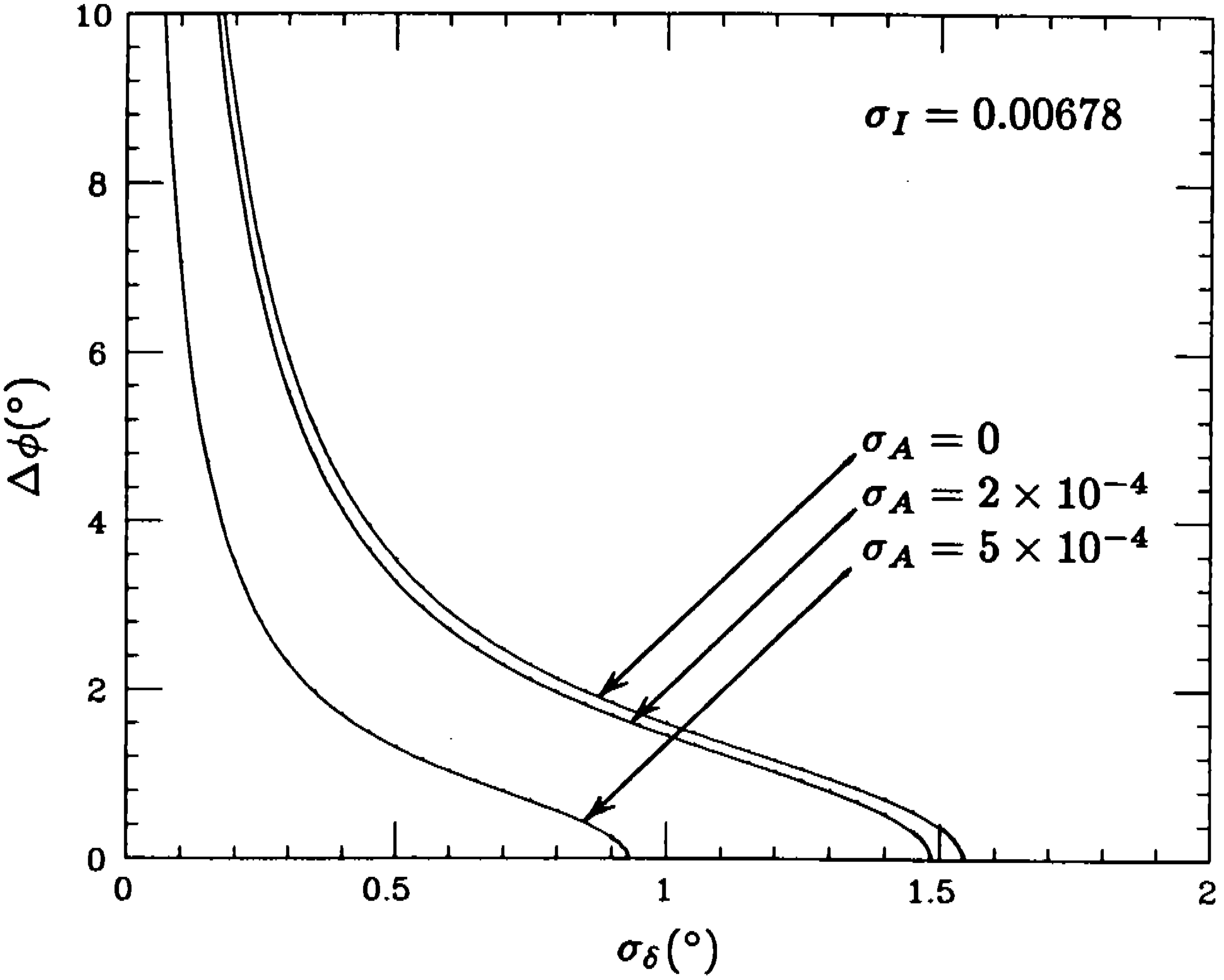


Figure 4 Combinations of σ_δ, $\Delta\phi$, and σ_A
Which Achieve 10^{-4} FWHM Energy Spread When $\sigma_I = 0.00678$

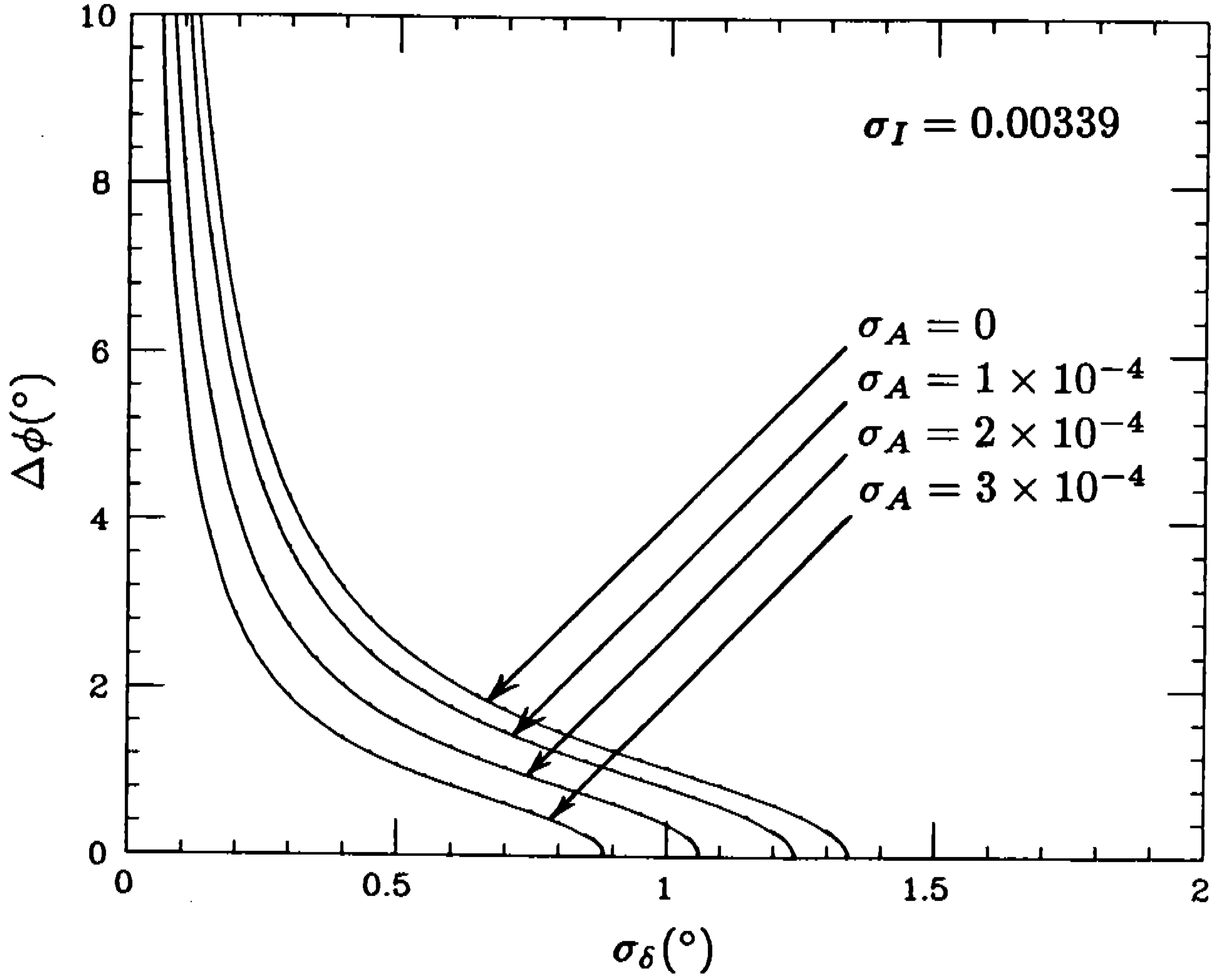


Figure 5 Combinations of σ_δ, $\Delta\phi$, and σ_A
Which Achieve 0.5×10^{-4} FWHM Energy Spread When $\sigma_I = 0.00339$