Computing the transport coefficients of the QGP on the lattice

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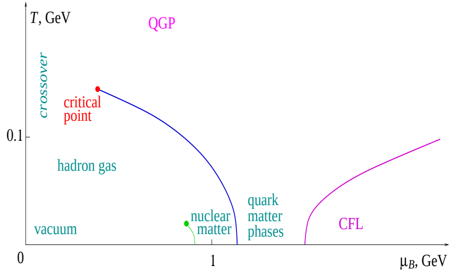
Outline

1. viscosity in heavy ion phenomenology
2. viscosity and the energy-momentum tensor
3. the energy-momentum tensor in hadronic physics
4. conclusions.

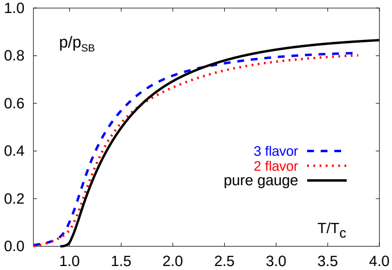
Phase diagram of QCD and equation of state at $\mu_B = 0$

Phase diagram



M. Stephanov, *Lattice* '06

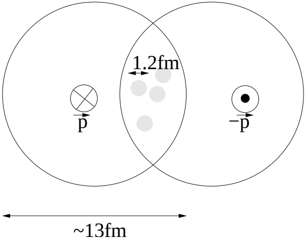
Equation of state



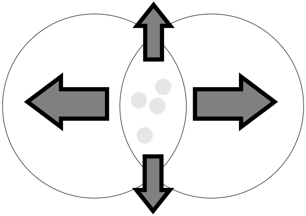
Karsch, *Hard Probes* '06

pressure as a function of temperature on the vertical axis of the phase diagram

Heavy ion collisions and elliptic flow

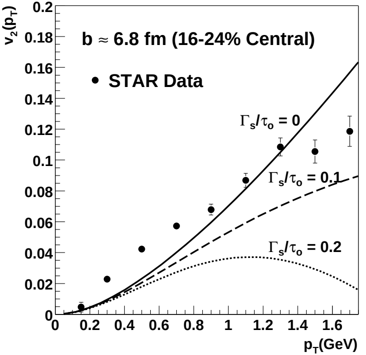


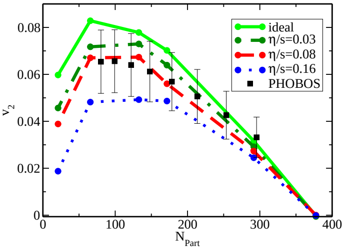
- transv. section of the collision of two gold nuclei
- impact vector $\vec{b}$ determined experimentally by detection of the spectators

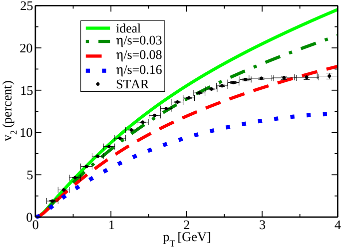


- excess of pressure in the $(\vec{b}, \hat{z})$ plane
- $\Rightarrow$ excess of particles produced in that plane

Hydrodynamics, ideal and viscous



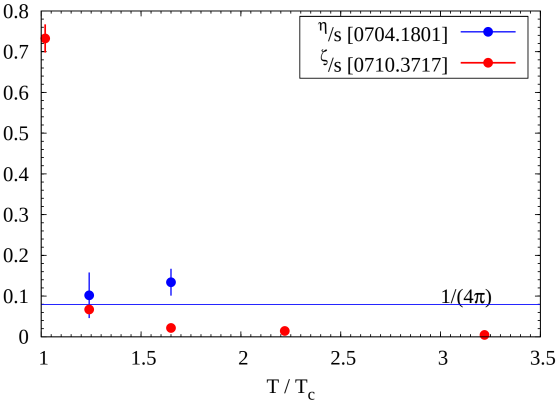




Teaney '03; P. & U. Romatschke, '07;
Heinz, Song '07; Dusling, Teaney '07

anisotropic flow incompatible with $\eta/s \gtrsim 0.2$

The viscosities of gluodynamics from the lattice



HM '07.

$$\eta/s, \zeta/s = \begin{cases} 0.134(33) & 0.008(7) & (T = 1.65\,T_c) \\ 0.102(56) & 0.065(17) & (T = 1.24\,T_c). \end{cases}$$

The viscosity from two-point functions of $T_{\mu\nu}$

- Euclidean formalism, temperature $T = 1/L_0$:

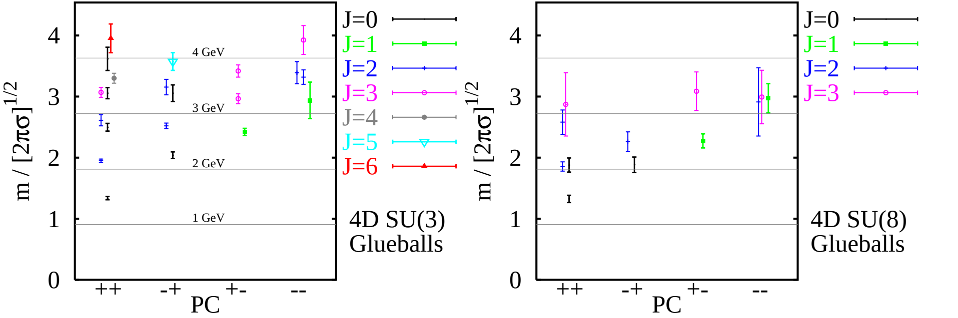
$$C(x_0) = L_0^5 \int d^3\mathbf{x}\, \langle \overline{T}_{12}(0)\overline{T}_{12}(x_0,\mathbf{x})\rangle.$$

- the real-time spectral function $\rho(\omega,\mathbf{0})$ satisfies

$$C(x_0) = L_0^5 \int_0^\infty d\omega \frac{\cosh \omega(\frac{1}{2}L_0 - x_0)}{\sinh \frac{\omega L_0}{2}} \rho(\omega).$$

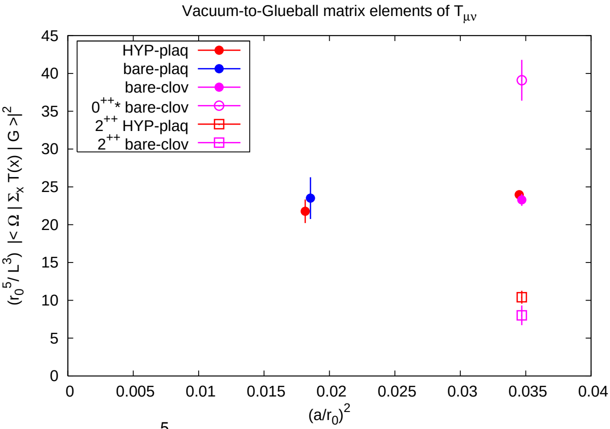
- $\bar{\rho}(\omega) \equiv \rho(\omega)/\tanh \frac{1}{2}\omega L_0$
- $\bar{\rho}(\omega) \geq 0, \quad \bar{\rho}(-\omega) = \bar{\rho}(\omega)$
- the shear viscosity is then given by $\eta(T) = \frac{\pi}{2T}\bar{\rho}(\omega = 0)$
- similar for bulk viscosity ζ with $T_{12} \to \theta$ Kubo formula

Karsch,Wyld '86; Nakamura, Sakai '04



HM, Ph.D. thesis 04

$\langle vac|T_{\mu\nu}|glueball\rangle$ matrix elements prelim.

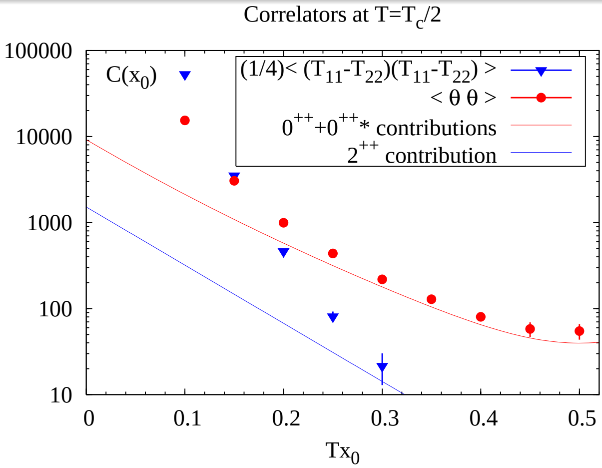


$$\frac{r_0^5}{L^3}|\langle\Omega|\int d^3\mathbf{x}\,\theta(x)\,|0^{++}\rangle|^2 = 23.97(9)$$

$$\frac{r_0^5}{L^3}|\langle\Omega|\int d^3\mathbf{x}\,\theta(x)\,|0^{++*}\rangle|^2 = 39.1(2.7)$$

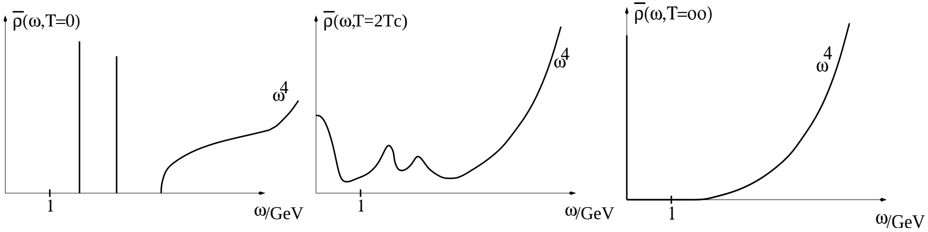
$$\frac{r_0^5}{L^3}|\langle\Omega|\int d^3\mathbf{x}\,\tfrac{1}{2}(T_{11}-T_{22})(x)\,|E^{++}\rangle|^2 = 10.4(9)$$

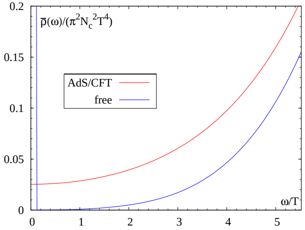
Correlators in the confined phase



- in the scalar channel, the two stable glueballs almost saturate the correlator beyond 0.5fm
- calculation made possible by the multi-level algorithm (HM '03)

Expected form of the spectral function for η

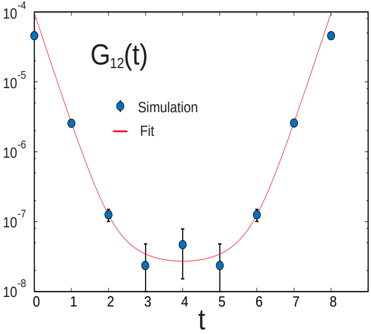




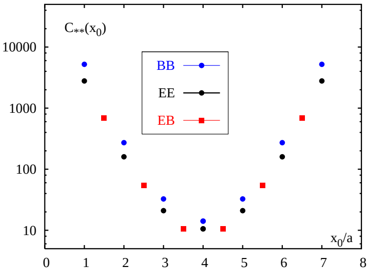
- at $T \gg T_c$, perturbation theory predicts a peak at $\omega = 0$ of width $O(\alpha_s NT)$
- "strong coupling $\leftrightarrow$ smooth spectral function"
  "weak coupling $\leftrightarrow$ spiky spectral function"
- in strongly coupled $\mathcal{N} = 4$ SYM, $\rho(\omega)$ is very smooth
- $\rho(\omega)$ for ζ: no peak at $\omega = 0 \Rightarrow$ more favorable.

A dramatic progress in accuracy ($N_\tau = 8$)

Nakamura,Sakai '04



HM '07



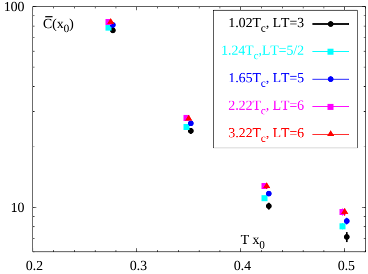
$$C(x_0) = \frac{1}{4}(C_{BB} + C_{EE} + 2C_{EB})$$

- gain ≈ 20 in the error bar on $C(L_0/2)$
- cost formula with multi-level algorithm:

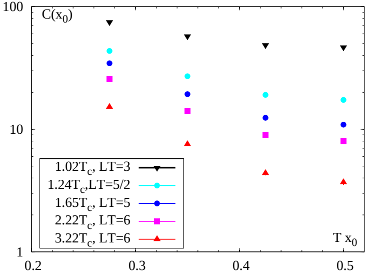
$$\#(PC - days) = 860 \left(\frac{0.051\,fm}{a}\right)^{10} \left(\frac{L}{1.4\,fm}\right)^{3} \left(\frac{1.65\,T_c}{T}\right)^{6} \left(\frac{4\%}{\Delta C(L_0/2)/C(L_0/2)}\right)^{2}$$

The finite T Euclidean correlators ($N_\tau = 8$)

$\langle T_{12} T_{12} \rangle_c$



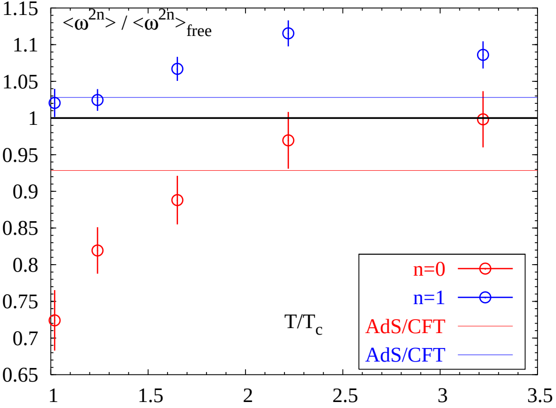
$\langle \theta\ \theta \rangle_c$



- near-conformal behaviour in one channel, large deviations in the other

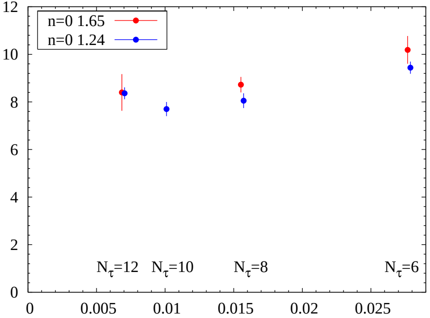
Moments of the tensor spectral function ($N_t = 8$)

$$\langle \omega^{2n} \rangle \equiv \int_0^\infty d\omega \omega^{2n} \frac{\bar{\rho}(\omega)}{\cosh \omega/2T} = T^5 \left. \frac{d^{2n}C}{dx_0^{2n}} \right|_{x_0=L_0/2}$$

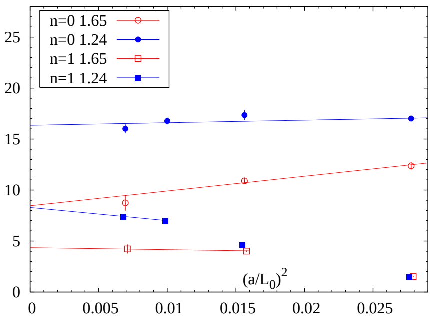


Taking the continuum limit

$\langle T_{12} T_{12} \rangle_c$



$\langle \theta\ \theta \rangle_c$



$$\langle \omega^{2n} \rangle \equiv \int_0^\infty d\omega \omega^{2n} \frac{\rho(\omega)}{\sinh \omega/2T} = T^5 \left. \frac{d^{2n} C}{dx_0^{2n}} \right|_{x_0=L_0/2}$$

- . . . tree-level improvement works well.

The “inverse problem”: solving for $\rho(\omega)$

Given $C_i = \int_0^\infty d\omega \rho(\omega) \frac{\cosh \omega L_0 \tau_i/2}{\sinh \omega L_0/2} \quad (i = 1, \dots N)$, reconstruct $\rho(\omega)$.

- our estimator for ρ: $\widehat{\rho}(\omega) = m(\omega)[1 + \widehat{a}(\omega)]$, $\widehat{a}(\omega) = \sum_{\ell=1}^{N} c_\ell u_\ell(\omega)$
- $m(\omega) > 0$; $m(\omega \gg T)$ has the correct perturbative behavior
- basis u_ℓ determined by singular-value decomposition of $M(x_0, \omega) \stackrel{\text{def}}{=} K(x_0, \omega) m(\omega)$: $\quad M^{\rm t} = U {\rm w} V^{\rm t}, \quad u_\ell(\omega) =$columns of U

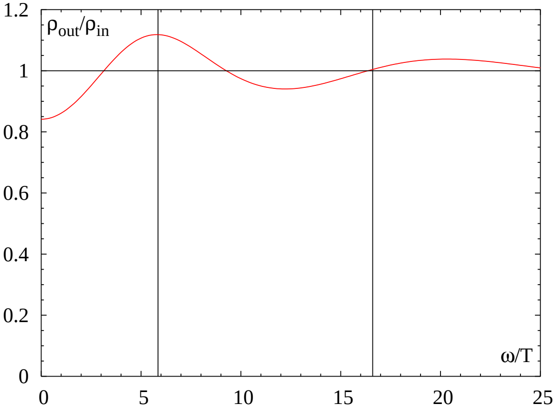
- we are only able to reconstruct a ’fudged’ version of the genuine $a(\omega)$:

$$\widehat{a}(\omega) = \int \widehat{\delta}(\omega, \omega') a(\omega') d\omega' \qquad \widehat{\delta}(\omega, \omega') = \sum_{\ell=1}^{N} u_\ell(\omega) u_\ell(\omega').$$

- $\widehat{\delta}(\omega, \omega')$ is called the resolution function (how complete is the basis?)
- we would like the resolution function to resemble a delta-function.

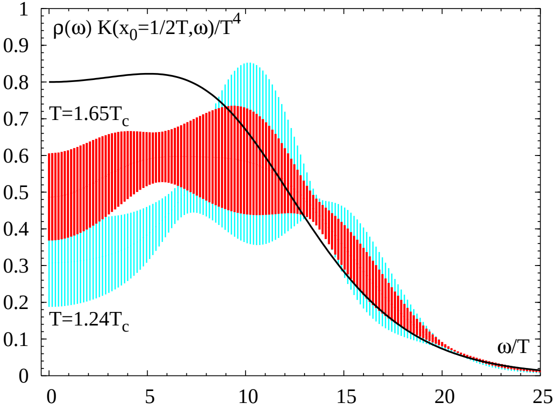
Backus & Gilbert, (geophysics, 1968)

Mock reconstruction of $\mathcal{N} = 4$ SYM spectral function



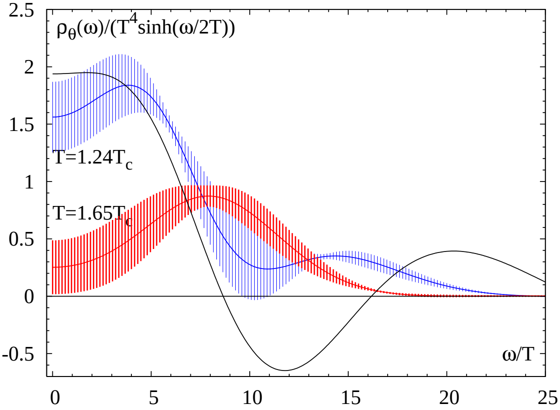
- the method used gives the correct viscosity within $\sim$ 16% for $N_\tau = 8$, assuming the correlator is known exactly
- the reconstructed $\rho(\omega)$ should be regarded as an average within $[\omega - \Delta\omega, \omega + \Delta\omega]$ of the true $\rho(\omega)$.

The reconstructed spectral function for $\langle T_{12} T_{12} \rangle$ $(N_\tau = 8)$



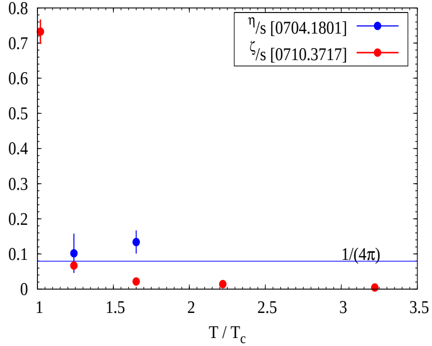
- $\eta/T^3 = \frac{\pi}{2} \times$ *intercept*
- black curve = normalized $\mathcal{N} = 4$ SYM spectral function.

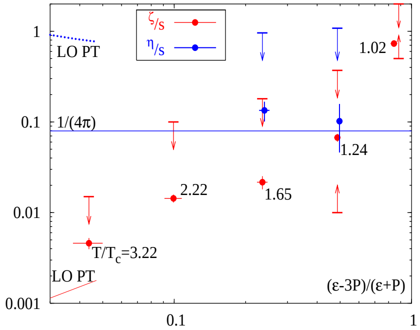
The reconstructed spectral function for $\langle\theta\theta\rangle$ ($N_\tau = 12$)



- $\zeta/T^3 = (\frac{\pi}{18} \times \textit{intercept})$ increasing for $T \to T_c$ cf. (Kharzeev, Tuchin '07)
- black curve $= \hat{\delta}(0, \omega)$

Summary





Perturbative and AdS/CFT calculations:

$$\eta/s,\ \zeta/s = \begin{cases} \dfrac{0.484}{\pi^2\alpha_s^2\log(0.608/\alpha_s)}, & \dfrac{1.25\alpha_s^2}{\pi^2\log(4.06/\alpha_s)} \qquad N_f = 0\ PT \\ 1/(4\pi), & 0 \qquad \mathcal{N} = 4\ SYM. \end{cases}$$

Arnold, Moore, Yaffe '03; Arnold, Dogan, Moore '06;

Policastro, Son, Starinets '01; Kovtun, Son, Starinets '04

The energy-momentum tensor in hadronic physics

The energy-momentum tensor in full QCD

Separating the traceless part $\overline{T}_{\mu\nu}$ from the trace part S for gluons, denoted 'g', and quarks, denoted 'f',

$$
\begin{aligned}
T_{\mu\nu} &\equiv \overline{T}^{\rm g}_{\mu\nu} + \overline{T}^{\rm f}_{\mu\nu} + \tfrac{1}{4}\delta_{\mu\nu}(\theta^{\rm g} + \theta^{\rm f}), \\
\overline{T}^{\rm g}_{\mu\nu} &= \tfrac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma} - F^a_{\mu\alpha}F^a_{\nu\alpha}, \\
\overline{T}^{\rm f}_{\mu\nu} &= \tfrac{1}{4}\textstyle\sum_f \bar\psi_f \overleftrightarrow{D}_\mu \gamma_\nu \psi_f + \bar\psi_f \overleftrightarrow{D}_\nu \gamma_\mu \psi_f - \tfrac{1}{2}\delta_{\mu\nu}\bar\psi_f \overleftrightarrow{D}_\rho \gamma_\rho \psi_f, \\
\theta^{\rm g} &= \beta(g)/(2g)\, F^a_{\rho\sigma}F^a_{\rho\sigma}, \quad \theta^{\rm f} = [1+\gamma_m(g)]\textstyle\sum_f \bar\psi_f m \psi_f
\end{aligned}
$$

- $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$
- $\beta(g)$ is the beta-function
- $\gamma_m(g)$ is the anomalous dimension of the mass operator
- all expressions are written in Euclidean space.

The connection between $T_{\mu\nu}$ and $\langle x \rangle_{\rm f,g}$

- for an on-shell particle with four-momentum $p = (iE_p, \mathbf{p})$, $E_{\mathbf{p}}^2 = M^2 + \mathbf{p}^2$,

$$\begin{aligned}
\langle \Psi, \mathbf{p} | \int d^3\mathbf{z}\, \overline{T}_{00}^{\rm f,g}(z)\, | \Psi, \mathbf{p} \rangle &= [E_{\mathbf{p}} - \tfrac{1}{4} M^2 / E_{\mathbf{p}}]\, \langle x \rangle_{\rm f,g}, \\
\langle \Psi, \mathbf{p} | \int d^3\mathbf{z}\, \theta^{\rm f,g}(z)\, | \Psi, \mathbf{p} \rangle &= (M^2 / E_{\mathbf{p}})\, b_{\rm f,g}, \\
1 = \langle x \rangle_{\rm f} + \langle x \rangle_{\rm g} &= b_{\rm f} + b_{\rm g},
\end{aligned}$$

- states are normalized according to $\langle \mathbf{p} | \mathbf{p} \rangle = 1$.

in the infinite momentum frame:

- $\langle x \rangle_g$ represents the momentum fraction carried by gluons

in the rest frame:

- the gluon contribution to the hadron mass is $\frac{3}{4} M \langle x \rangle_g$
- the contribution of the trace anomaly θ^g to the hadron mass is $\frac{1}{4} b_g M$ [X.D. Ji, '95]

Discretizations of $\overline{T}^{g}_{\mu\nu}$ and θ^{g}

body-centered definition:

$$a^3 \sum_{\mathbf{x}} \overline{T}^{\rm bp}_{00}(x_\odot) = \frac{2Z_{\rm g}(g_0)}{ag_0^2} \sum_{\mathbf{x}}$$

$$\mathrm{Re\,Tr}\left[\sum_k P_{0k}(x) - \sum_{k<l} \tfrac{1}{2}[P_{kl}(x) + P_{kl}(x+a\hat{0})]\right],$$

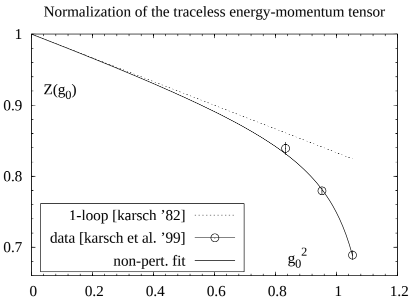
- breaking of continuous translation invariance $\Rightarrow$ finite renormalization, $Z_{\rm g}(g_0) = \frac{\partial\xi_0(\beta,\xi)}{\partial\xi}$
- analog of the axial current normalization factor $Z_A(g_0)$.

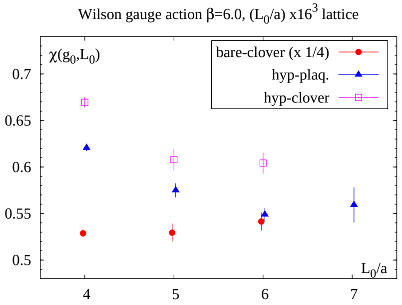
site-centered definition:

$$\overline{T}^{\rm bc}_{00}(x) \equiv \frac{\chi(g_0)Z_{\rm g}(g_0)}{g_0^2} \mathrm{Re\,Tr}\left[\sum_k (\widehat{F}_{0k})^2 - \sum_{k<l} (\widehat{F}_{kl})^2\right]$$

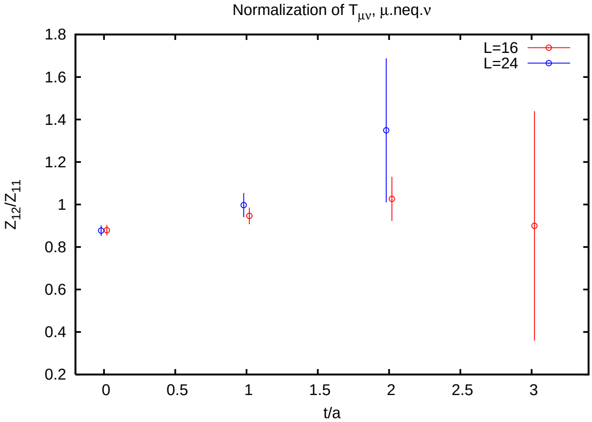
- we calculated the relative normalization factor $\chi(g_0)$
- recently we calculated the normalization of $T_{\mu\neq\nu}$.

Non-perturbative Normalization Factors



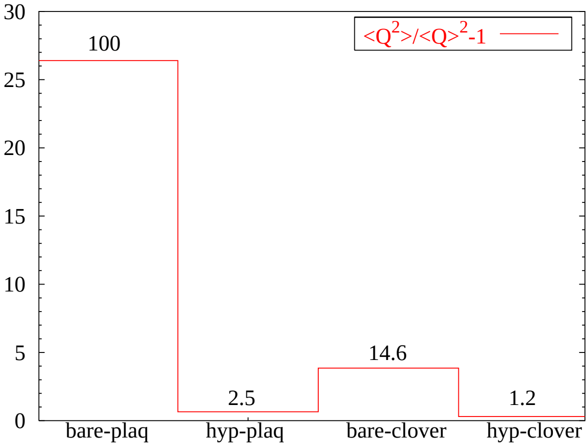


Normalization of $T_{\mu\neq\nu}$ (prelim.)



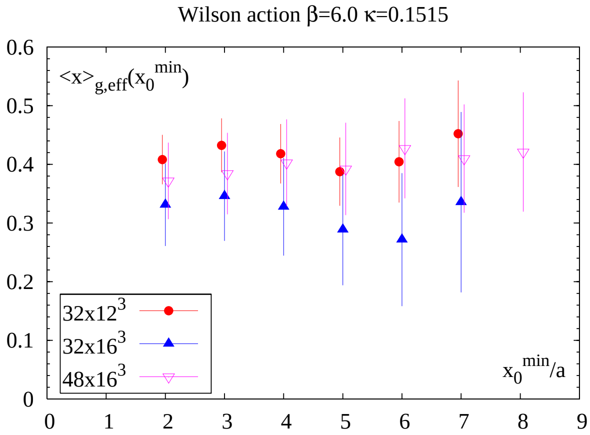
$$\langle vac|T_{12}|T_2^{++}\rangle \stackrel{!}{=} \langle vac|\tfrac{1}{2}(T_{11} - T_{22})|E^{++}\rangle$$

Relative Variance of the Discretizations



- huge differences between operators
- e.g. HYP-smearing the plaquette saves a factor 40 in CPU time

Glue momentum fraction in a 1GeV pion



$$\langle x\rangle_{\rm g}^{(\pi)}(\mu_{\overline{MS}}^2 = 4{\rm GeV}^2) = 0.37(8)_{\rm stat}(12)_{Z_{\rm f}} \qquad (M_\pi = 890{\rm MeV})$$

J.W. Negele, HM ’07, PRD (in press)

Lattice Sum Rule and Normalization of $\langle x\rangle_{\rm g}$

Lattice sum rule: [HM, '06]

$$\frac{3}{4}\Big(E-\frac{1}{3}\textstyle\sum_k \displaystyle\frac{\partial E}{\partial \log L_k}\Big) = \langle\Phi|a^3\sum_{\mathbf{x}} Z_{\rm g}(g_0)\overline{T}_{00}^{\rm g,bare}+Z_{\rm f}(g_0)\overline{T}_{00}^{\rm f,bare}|\Phi\rangle|^{\Phi=\Psi}_{\Phi=\Omega}$$

where

$$a^3\sum_{\mathbf{x}}\overline{T}_{00}^{\rm g,bare}(x_\odot) = \frac{2}{ag_0^2}\sum_{\mathbf{x}}{\rm Re\,Tr}\Big[\textstyle\sum_k P_{0k}(x)-\sum_{k<l}\frac{1}{2}[P_{kl}(x)+P_{kl}(x+a\hat{0})]\Big]$$

$$a^3\sum_{\mathbf{x}}\overline{T}_{00}^{\rm f,bare}(x_\odot) = \frac{3a^3}{4}\sum_{\mathbf{x}}\bar{\psi}(x)[D_0\gamma_0-\frac{1}{2}aD_0^*D_0-\frac{1}{3}(D_k\gamma_k-\frac{1}{2}aD_k^*D_k)]\psi(x)$$

- $Z_{\rm g}(g_0)=\frac{1}{2}\frac{\partial\log(\beta_\sigma/\beta_\tau)(a_\sigma,a_\tau)}{\partial\log a_\tau}$, $Z_{\rm f}(g_0)=-\frac{\partial\log(\kappa_\sigma/\kappa_\tau)(a_\sigma,a_\tau)}{\partial\log a_\tau}$
- this shows, in a particular regularization, that the momentum sum rule $\langle x\rangle_{\rm g}+\langle x\rangle_{\rm f}=1$ holds for appropriate scheme-independent $Z_{\rm f,g}(g_0)$.

Renormalization of $\langle x \rangle_{\rm g}$

- since it is a flavour-singlet operator, $\overline{T}^{\rm g}_{\mu\nu}$ mixes with $\overline{T}^{\rm f}_{\mu\nu}$
- if $\overline{T}^{\rm f,g}_{00}(g_0)$ has been normalized as on previous slide,

$$\begin{bmatrix} \overline{T}^{\rm g}_{00}(\mu) \\ \overline{T}^{\rm f}_{00}(\mu) \end{bmatrix} = \begin{bmatrix} Z_{\rm gg} & 1-Z_{\rm ff} \\ 1-Z_{\rm gg} & Z_{\rm ff} \end{bmatrix} \begin{bmatrix} \overline{T}^{\rm g}_{00}(g_0) \\ \overline{T}^{\rm f}_{00}(g_0) \end{bmatrix},$$

- NB. this form simplifies the calculation of the step-scaling matrix $\Sigma(\bar{g}^2(\mu), a\mu) = \mathbf{Z}(a\mu, g_0)\mathbf{Z}^{-1}(2a\mu, g_0)$ in a finite-volume scheme
- $\Rightarrow$ there are only two independent anomalous dimensions
- $c_{\rm gg,ff}(\bar{g}=0) = \frac{N_{\rm f}}{12\pi^2}, \frac{4}{9\pi^2}$ Gross, Wilczek; Georgi, Politzer '74
- the asymptotic glue momentum fraction is $16/[16+3N_{\rm f}]$ for any hadron.

Result in the Quenched Approximation

- $Z_{\rm gg} = 1$ due to the absence of quark loops $\Rightarrow$

$$\langle x\rangle_{\rm g}(\mu^2) = Z_{\rm g}(g_0)\langle x\rangle_{\rm g} + [Z_{\rm f}(g_0) - Z_{\rm ff}(a\mu, g_0)Z_{\rm f}(g_0)]\ \langle x\rangle_{\rm f}^{\rm bare}$$

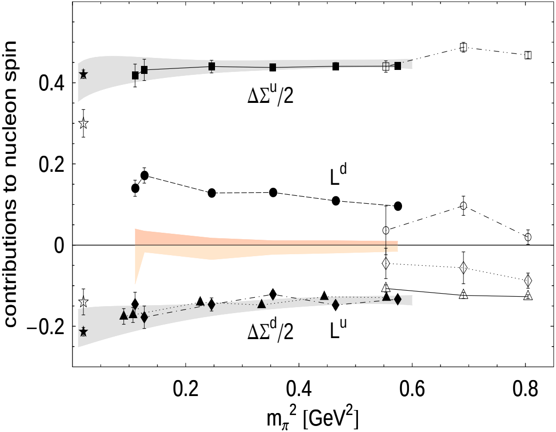
- $Z_{\rm ff}(a\mu, g_0)Z_{\rm f}(g_0) = 0.99(4)$ for the $\overline{MS}$-scheme at $\mu = 2{\rm GeV}$

  [ZeRo Collab, Guagnelli et al. 03; 04]
- $\langle x\rangle^{\rm bare} = 2 \times 0.3080(18)$ (disregarding disconnected diagrams) [ZeRo Collab, Guagnelli et al. 04]
- $Z_{\rm f}(g_0) = 1 + O(g_0^2)$ itself is not known yet beyond treelevel; we take 1.0(2) $\Rightarrow$ with 3000 configurations we obtain

$$\langle x\rangle_{\rm g}^{(\pi)}(\mu^2_{\overline{MS}} = 4{\rm GeV}^2) = 0.37(8)_{\rm stat}(12)_{Z_{\rm f}} \qquad (M_\pi = 890{\rm MeV})$$

- phenomenological estimates: $\langle x\rangle_{\rm g} \approx 0.38(5)$ assuming sea quarks account for 0.1—0.2 of the momentum

Angular momentum contributions to the nucleon



[LHPC 07]

- surprise 1: $\Sigma^d/2 = -L^d$ $\Rightarrow\ J^d = 0$
- surprise 2: $L_u = -L_d \Rightarrow$ $L^{u+d} = 0$
- quark contribution to angular momentum: $J^{u+d} = 0.426(52) \cdot \frac{1}{2}$
- glue contribution: compute form factor of T_{0k}, $q \to 0 \to$ underway.
- a challenge: understand $L_u - L_d < 0$ in the quark model.

Conclusions

1. the observed strong dynamics at RHIC calls for a theoretical understanding of transport coefficients beyond PQCD $\Rightarrow$ compute $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ in lattice QCD

   shear viscosity: $1 < 4\pi\eta/s < 2$ for $1.2T_c < T < 1.7T_c$

   pure glue — assuming $\rho(\omega)$ is smooth for $\omega < 3T$ in that range

   bulk viscosity: small compared to η, but huge rise close to T_c $\Rightarrow$ enhanced entropy production.

2. $T_{\mu\nu}$ also plays a central role in hadronic physics:
   - quark, gluon momentum fractions ($\approx$ 40% for the pion)
   - quark, gluon angular momentum fractions (nucleon)
   - the gluonic renormalization factors are now known (quenched).