Maximum Likelihood Estimation of Asymmetry and Angular Modulation for Transversity

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Chapter 1

Introduction

1.1 Data Structure of Experiment Transversity

Jefferson Lab Hall A Experiment E06-010 studies neutron single and double spin asymmetry in the semi-inclusive deep inelastic (SIDIS) ${}^3He^{\uparrow}(\overrightarrow{e}, e'\pi/K^{\pm})X$ reactions with polarized electron beam and a transversely polarized ^{3}He target. During production running, there are 4 ${}^3He^{\uparrow}$ target spin directions: vertical up/down (also called $V\pm$) and transverse-in-plane beam right/left (also called $T\pm$). At each so-called target spin state, ${}^3He^{\uparrow}$ spin direction will remain unchanged while beam helicity fast flips at a rate of 30Hz. We tried to take 20min of data at each state, before flipping the spin to an inverse direction and start a new state. There are total roughly 3000 spin states. They are rough averagely distributed between transverse and vertical target direction, as well as positive and negative charged hadrons. Average event count in each state is on the order of amplitude of 100 for $e'pi$ coincidence events and $\leqslant\sim 10$ for $e'K$ coincidence events. From the data, following information has been extracted, which is not concern of this notes.

- total charge and DAQ/electronics live time of each spin state. They could also be counted in helicity states separately.
- target/beam polarization, target density and luminosity
- for each event, following information is known: event physics type (ex. reaction channel), which spin/helicity state it's from and related kinematics variables, ex. ϕ_h, ϕ_S

The total SIDIS yield[3] is simplified as

$$
\begin{aligned}
y(\phi_h, \phi_S) \quad = \quad & \rho \cdot \sigma \cdot a_{T/V\pm}(\phi_h, \phi_S) \left(1 + S_T \sum_j \epsilon_j SSA_j(\phi_h, \phi_S) \right. \\
& \left. + |P_{Beam}| \cdot h \cdot \left(S_T \sum_j \epsilon_j DSA_j(\phi_h, \phi_S) + S_L \cdot \left(\epsilon_{LL} + \epsilon_{LL}^{\cos\phi_h} \cdot \cos(\phi_h) \right) \right) \right)
\end{aligned}
\tag{1.1}
$$

where ρ is longitudinal target density, σ is the cross-sections, $a_{\pm}(\phi_h, \phi_S)$ is the acceptance at transverse or vertical $\pm$ spin state, $SSA/DSA_j(\phi_h, \phi_S)$ is the j^{th} azimuthal single spin (SSA) or double spin (DSA) angular modulation, $\sin(\phi_h + \phi_S)$ for example, $S_{T/L}$ is transverse or longitudinal target polarization in $\boldsymbol{q}$ vector frame, P_{Beam} and h are electron beam polarization and helicity, ϵ_j is the amplitude of each modulation. In this note, longitudinal modulations, ϵ_{LL} and $\epsilon_{LL}^{\cos\phi_h}$, which are contamination to DSA asymmetries, could be extracted from our data or as inputs, base on knowledge from other experiment. Our goal of this notes is develop maximun likelihood based method to extract ϵ_j from Transversity Data.

1.2 Why Maximum likelihood Estimation

Maximum likelihood Estimation (MLE) is a popular statistical method used for fitting a statistical model to data, and providing estimates for the model's parameters. By introducing MLE method into Transversity data analysis, following benefits are expected

- As cross check of existing local-pair angular-binned-fitting method (Blue Team method for short) developed by Blue Team[2]
- High statistic approximation is required for angular-binned-fitting method to be unbiased (in an extreme example, bin fitting method would break down if statistical expectation of count in each bin is less than 1). Therefore, there are practical difficulties for channels with very low statistics (eg. $(e, e'K^-)$). On the other hand, MLE do not have this problem as long as total counts are high (Please refer to section 1.3).
- For angular modulation extraction, part of angle information will be lost during binning process, while MLE would access preserve all the information.
- MLE also offers an alternative method to combine data separated into spin states. Comparing with local-pair method[2], MLE would trade lower statistical uncertainty with higher systematic bias, in the case that local pairs are non-symmetric in sense of effective beam charges. This will be discussed in section 2.2.3.

It's possible to combine use MLE and local-pair method, each for one of two major steps of the asymmetry extraction: combination of information between spin states and extraction of angular modulation. It will be discussed in section 5.1.

1.3 MLE for Combining Yields and Spectrum

As a simple but useful example of MLE method over multiple spin state, this section will demonstrate how to get overall yield from multiple data segments with MLE, under the assumptions that

- *In an experiments, there are multiple data segments (could be runs or spin states), index them by i*
- *In each segment, integrated luminosity ($\widetilde{L}_i = \int L(t)_i dt$) and DAQ/electronics live time (LT_i) is known. We define effective charge $\widetilde{C}_i \equiv \widetilde{L}_i \times LT_i / (Constant)$[1].*

[1]

– Constant here converts units from luminosity to charge.

- *Event count for the channel we study is known (N_i). This could be a specific channel or a bin in a spectrum.*
- *physical yield, y, do not drift (in another word, acceptance and detector efficiency do not drift with time)*
- *As an additional assumption, which is usually true for most cases, each N_i follows Poisson Distribution*[2]

$$\begin{aligned} \Pr(N_i = k) &\equiv f_i(k) \\ &= \text{Poisson}(k,\, y\widetilde{C}_i) \\ &= \frac{(y\widetilde{C}_i)^k}{k!}\exp(-y\widetilde{C}_i) \end{aligned} \tag{1.2}$$

To calculate the maximum likelihood value, we form the log-likelihood function,

$$\begin{aligned} L(y) &\equiv \log \prod_i f_i(N_i) \\ &= \sum_i \left(N_i \log y\widetilde{C}_i - y\widetilde{C}_i \right) + \text{Constant} \end{aligned} \tag{1.3}$$

Take the derivative of L with respect to y and equate it to zero:

$$\begin{aligned} 0 &= dL(y)/dy \\ &= \sum_i \left(N_i y - \widetilde{C}_i \right) \end{aligned} \tag{1.4}$$

So estimator of y is solution to above equation

$$\hat{y}_{MLE} = \sum_i N_i / \sum_i \widetilde{C}_i \tag{1.5}$$

with uncertainty

$$\sigma(\hat{y}_{MLE}) = 1/\sqrt{\sum_i N_i} \tag{1.6}$$

From property of MLE we know $\hat{y}_{MLE}$ is non-biased and efficient at large statistics limit $\sum_i N_i \gg 1$.

Note estimator for y with weighted sum or minimum χ^2 method is

$$\hat{y}_{ws} = \sum_i \frac{N_i}{\widetilde{C}_i} w_i \tag{1.7}$$

$$= \sum_i \widetilde{C}_i / \sum_j \frac{\widetilde{C}_j^2}{N_i} \tag{1.8}$$

2

– This will not be true in special cases, for example, when DAQ prescale> 1. However, if percentage of this channel over all trigger is small, then distribution of N_i will be close to Poisson Distribution.

where

$$w_i \equiv 1/\sigma^2\left(\frac{N_i}{\widetilde{C}_i}\right)/Norm \tag{1.9}$$

$$= \widetilde{C}_i^2/N_i/Norm \tag{1.10}$$

$$= \frac{\widetilde{C}_i^2}{N_i}/\sum_j \frac{\widetilde{C}_j^2}{N_i} \tag{1.11}$$

In case that statistics of each data segment is large, $N_i \gg 1$

$$\hat{y}_{ws} \rightarrow \hat{y}_{MLE} \tag{1.12}$$

However, $\hat{y}_{ws}$ breaks down if N_i is small, especially $N_i = 1$ or 0[3]. $\hat{y}_{MLE}$ remain valid in this case.

1.4 Two Models Describing an Asymmetry Measurement

In this sections, two categories of model are built for extraction of yield asymmetry (non-modulated) between spin states. This is the fundamental process of asymmetry measurements, applicable to both SSA and DSA. They are based on either spin-states or each events. The event based model will be further developed to described angular modulation in section 3.2.1.

Consider the case that, during a stable experiment (no yield drifting), we collected data on multiple target spin states (indexed by i_+ for plus spin states and i_- for minus spin state). For each state, $N_{i_\pm}$ events (independent of each other) are collected. Integrated luminosity and live time[4] *in each spin state are also known to be $\widetilde{L}_{i_\pm} = \int L_{i_\pm} dt$ and $LT_{i_\pm}$. For simplification, we define effective charge*

$$\widetilde{C}_{i_\pm} \equiv \widetilde{L}_{i_\pm} \times LT_{i_\pm}/\left(Units\ Conversion\ Constant\right) \tag{1.13}$$

. Then we would expect

$$\mathrm{E}\left[N_{i_\pm}\right] = y\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon) \tag{1.14}$$

, where $\mathrm{E}\left[\right]$ *is expectation; y is average yield of this experiment; $P_{i_\pm}$ is target polarization; ϵ is physical asymmetry of this process.*

[3] There is a simulation to demonstrate $\hat{y}_{ws}$ is biased in simplified case of $\widetilde{C}_i = Constant$, shown in Figure 3-33 of Yi Qiang's PhD Thesis[4]

[4] *live time here is multiply of both electronics live time and DAQ live time,* ***it includes prescale factor***

For further convenience, we define a series of sum values here:

$$\begin{aligned} N_{\pm} &\equiv \sum_{i_{\pm}} N_{i_{\pm}} \\ N &\equiv N_{+} + N_{-} \\ N_{P,\pm} &\equiv \sum_{i_{\pm}} N_{i_{\pm}} P_{i_{\pm}} \\ N_{P^2,\pm} &\equiv \sum_{i_{\pm}} N_{i_{\pm}} P^2_{i_{\pm}} \\ \widetilde{C}_{\pm} &\equiv \sum_{i_{\pm}} \widetilde{C}_{i_{\pm}} \\ \widetilde{C}_{P,\pm} &\equiv \sum_{i_{\pm}} \widetilde{C}_{i_{\pm}} P_{i_{\pm}} \end{aligned} \tag{1.15}$$

,where $\sum_{i_{\pm}}$ is defined as sum over state index i_{+} or i_{-} over all plus or minus spin states. (Notations like $\widetilde{C}_{T/V\pm}$or $i_{T/V\pm}$ could also be used in later sections, which are defined within transverse or vertical polarized spin states) It's also useful to predefine asymmetries, including

- effective charge asymmetry

$$A_C \equiv \frac{\widetilde{C}_{+} - \widetilde{C}_{-}}{\widetilde{C}_{+} + \widetilde{C}_{-}} \tag{1.16}$$

- polarized effective charge asymmetry

$$A_{CP} \equiv \frac{\widetilde{C}_{P,+} - \widetilde{C}_{P,-}}{\widetilde{C}_{+} + \widetilde{C}_{-}} \tag{1.17}$$

- raw event count asymmetry

$$A_{\text{raw}} \equiv \frac{N_{+} - N_{-}}{N_{+} + N_{-}} \tag{1.18}$$

Finally, we present two statistical models as following:

State-Based Treating each spin state as an independent measurement. Therefore, there are total $N_{\text{state}+} + N_{\text{state}-}$measurements, each of which measures event numbers, $N_{i_{\pm}}$, in spin state $i_{\pm}$. The probability distribution of $N_{i_{\pm}}$ follows Poisson distribution (different for each state) of

$$\begin{aligned} \Pr(N_{i_{\pm}} = k) &= \text{Poisson}(k, y\widetilde{C}_{i_{\pm}}(1 \pm P_{i_{\pm}}\epsilon)) \\ &\equiv \frac{(y\widetilde{C}_{i_{\pm}}(1 \pm P_{i_{\pm}}\epsilon))^k}{k!} \exp(-y\widetilde{C}_{i_{\pm}}(1 \pm P_{i_{\pm}}\epsilon)) \end{aligned} \tag{1.19}$$

Event-Based Treating each event as an independent measurement: Consider the experiment that, although $\widetilde{C}_{i_{\pm}}$ is known, we take out and study each event one by one, randomly, from the data stream (without knowing which spin state it was in before taking it out). Then the spin state ID, $\text{StateID}_{\text{ev}}$, of any event is a discrete random variable. Therefore, upon taking out each event, we make a measurement of independent and identically-distributed (i.i.d.) random variables $\text{StateID}_{\text{ev}}$. The distribution of $\text{StateID}_{\text{ev}}$ is

$$\Pr(\text{StateID}_{\text{ev}} = i_\pm) = \widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon)/Norm \tag{1.20}$$

,where $i_\pm$ stands for the i^{th} state in plus of minors spin state sequence; $Norm$ is normalization factor satisfying

$$\begin{aligned} Norm &= \sum_{\text{states}} \widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon) \\ &= (\widetilde{C}_+ + \widetilde{C}_-)(1 + \epsilon A_{CP}) \end{aligned} \tag{1.21}$$

It will be shown in section 2.2.3 that, despite their different view points, estimators produced from above two statistical model are identical, which suggests they validate each other.

Chapter 2

Estimation of Non-Modulated Asymmetry

Although the final goal of Transversity is to extract angular modulated asymmetries, it is very useful to discuss methods to combine spin states and extract yield asymmetry:

- As data check, it's always useful to check yield asymmetry between spin plus/minus states.
- Angular-binned-fitting method[2] is based on yield asymmetry of each angular bins. It's applicable to both single and double spin asymmetry.
- MLE of angular modulated asymmetry to be discussed in section 3.2 is a further development of this method.

Therefore, in this section, MLE method is discussed to extract of physical asymmetry from data separated into multiple spin states. Statistical model introduced in section 1.4 are directly used here.

2.1 General Formula

2.1.1 State Based Model

To calculate the maximum likelihood value, we form the log-likelihood function,

$$\begin{aligned} L(\epsilon) &= \log \prod_{\text{states}} \frac{(y\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon))^{N_{i_\pm}}}{N_{i_\pm}!} \exp(-y\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon)) \\ &= \sum_{\text{states}} \left(-y\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon) + N_{i_\pm} \log\left(y\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon)\right)\right) + \text{Constant} \end{aligned} \quad (2.1)$$

. We take the derivative of L with respect to parameter ϵ, y respectively and equate them to zero:

$$0 = \frac{dL}{d\epsilon} = -y(\widetilde{C}_{P,+} - \widetilde{C}_{P,-}) + \sum_{\text{states}} \frac{\pm N_{i_\pm} P_{i_\pm}}{1 \pm \epsilon P_{i_\pm}} \quad (2.2)$$

$$0 = \frac{dL}{dy} = -(\widetilde{C}_+ + \widetilde{C}_- + \epsilon(\widetilde{C}_{P,+} - \widetilde{C}_{P,-})) + \frac{N}{y} \quad (2.3)$$

,where $N_{\pm}$ was defined in Eq. 1.15. Now we solve Eq. 2.3 for y:

$$y = \frac{N}{\widetilde{C}_+ + \widetilde{C}_- + \epsilon(\widetilde{C}_{P,+} - \widetilde{C}_{P,-})} \tag{2.4}$$

and send it back to Eq. 2.2 we get equation

$$-\frac{NA_{CP}}{1+\epsilon A_{CP}} + \sum_{i_+}\frac{N_{i_+}P_{i_+}}{1+\epsilon P_{i_+}} - \sum_{i_-}\frac{N_{i_-}P_{i_-}}{1-\epsilon P_{i_-}} = 0 \tag{2.5}$$

. There is no analytical solution to this equation (special cases see section 2.2). An expansion over ϵ simplifies this problem:

$$-NA_{CP}(1-\epsilon A_{CP}) + \sum_{i_+} N_{i_+}P_{i_+}(1-\epsilon P_{i_+}) - \sum_{i_-} N_{i_-}P_{i_-}(1+\epsilon P_{i_-}) + O\left((N_+ - N_-)\epsilon^2\right) = 0 \tag{2.6}$$

. The non-linear residue of expansion is smaller than original form due to cancellation between terms. Solving this equation, we have final estimator for ϵ:

$$\hat{\epsilon} = \frac{N_{P,+} - N_{P,-} - A_{CP}N}{N_{P^2,+} + N_{P^2,-} - A_{CP}^2 N} + O\left(A_{\text{raw}}\epsilon^2\right) \tag{2.7}$$

In Transversity case, all major asymmetries square are much smaller comparing to statistical uncertainty. Therefore, bias $O\left(A_{\text{raw}}\epsilon^2\right)$ could be ignored.

2.1.2 Event Based Model

Similar to last section, we first calculate log-likelihood function with event based model,

$$\begin{aligned} L &= \sum_{\text{ev}} \log\left(\widetilde{C}_{i_\pm}(1 \pm P_{i_\pm}\epsilon)/Norm\right) \\ &= \sum_{\text{ev}} \left(\log\widetilde{C} + \log\left(1 \pm P_{i_\pm}\epsilon\right) - \log\left(Norm\right)\right) \\ &= \sum_{\text{ev}} \log\left(1 \pm P_{i_\pm}\epsilon\right) - N\log\left(1+\epsilon A_{CP}\right) + \text{Constant} \\ &= \sum_{\text{states}} N_{i_\pm}\log\left(1 \pm P_{i_\pm}\epsilon\right) - N\log\left(1+\epsilon A_{CP}\right) + \text{Constant} \end{aligned} \tag{2.8}$$

. Take derivative of L with respect to parameter ϵ and equate them to zero:

$$0 = \frac{dL}{d\epsilon} = -\frac{NA_{CP}}{1+\epsilon A_{CP}} + \sum_{i_+}\frac{N_{i_+}P_{i_+}}{1+\epsilon P_{i_+}} - \sum_{i_-}\frac{N_{i_-}P_{i_-}}{1-\epsilon P_{i_-}} \tag{2.9}$$

, which is identical with Eq. 2.5. Therefore, their results of $\hat{\epsilon}$ are exactly identical.

2.1.3 Uncertainty Estimation,

With the event based model, the information contained in N events about the asymmetry (as defined by Fisher and quoted in [5]) is given by

$$\begin{aligned}
\mathcal{I}(\epsilon) &= -\mathrm{E}\left[\frac{\partial^2}{\partial\epsilon^2}\log L\right] && (2.10)\\
&= -\mathrm{E}\left[\frac{\partial^2}{\partial\epsilon^2}\left(\sum_{\text{states}} N_{i_\pm}\log\left(1\pm P_{i_\pm}\epsilon\right) - N\log\left(1+\epsilon A_{CP}\right)\right)\right] && (2.11)\\
&= \mathrm{E}\left[\sum_{\text{states}} N_{i_\pm}\frac{P_{i_\pm}^2}{\left(1\pm P_{i_\pm}\epsilon\right)^2} - N\frac{A_{CP}^2}{\left(1+\epsilon A_{CP}\right)^2}\right] && (2.12)
\end{aligned}$$

. This expression is difficult to calculate. An expansion over ϵ would make it easier:

$$\mathcal{I}(\epsilon) = \sum_{\text{states}} N_{i_\pm}\left(P_{i_\pm}^2 \mp 2\epsilon P_{i_\pm}^3\right) - N\left(A_{CP}^2 - 2\epsilon A_{CP}^3\right) + O\left(N\epsilon^2\right) \quad (2.13)$$

. The MLE is asymptotically efficient, i.e., it achieves the Cramér-Rao lower bound when the sample size tends to infinity:

$$\begin{aligned}
\sigma^2(\hat{\epsilon}) &= \frac{1}{\mathcal{I}(\epsilon)}\\
&= \frac{1}{N_{P^2,+} + N_{P^2,-} - NA_{CP}^2} + O\left(\frac{\epsilon}{N}\right) && (2.14)
\end{aligned}$$

2.2 Discussion

2.2.1 Special Case : Polarization is identical for all spin states

In case that target polarization is same for all spin states, there are a simple equations of $\hat{\epsilon}$ by solving Eq. 2.5 exactly (without expansion on ϵ).

Here, Eq. 2.5 become

$$-\frac{NA_{CP}}{1+\epsilon A_{CP}} + P\left(\frac{\sum_{i_+} N_{i_+}}{1+\epsilon P} - \frac{\sum_{i_-} N_{i_-}}{1-\epsilon P_{i_-}}\right) = 0 \quad (2.15)$$

, which is analytical solvable. Then we conclude that

$$\begin{aligned}
\hat{\epsilon}_{\text{ConstP}} &= \frac{1}{P}\frac{A_{\text{raw}} - A_C}{1 - A_{\text{raw}}A_C} && (2.16)\\
&= \frac{1}{P}\frac{\frac{N_+}{\widetilde{C}_+} - \frac{N_-}{\widetilde{C}_-}}{\frac{N_+}{\widetilde{C}_+} + \frac{N_-}{\widetilde{C}_-}} && (2.17)
\end{aligned}$$

Eq. 2.17 is same as getting MLE yield from spin ± states separately (as section 1.3) then form an asymmetry, which is also reasonable.

2.2.2 Special Case : There are only one spin state pair

In case that there are only two spin state and target polarization is same for both states, Eq. 2.16 could be further simplified to

$$\hat{\epsilon}_{\text{ConstP}} = \frac{1}{P}\frac{A_{\text{raw}} - A_C}{1 - A_{\text{raw}}A_C} = \frac{1}{P} \times \frac{\frac{N_+}{\widetilde{C}_+} - \frac{N_-}{\widetilde{C}_-}}{\frac{N_+}{\widetilde{C}_+} + \frac{N_-}{\widetilde{C}_-}} \tag{2.18}$$

. As we expected, this expression is identical to that of Blue Team's note[2].

2.2.3 Compared with Local-Pair Method : Statistics for Systematics

So, what's the difference between MLE result Eq. 2.7 and local pair method developed in [2]? I would argue that MLE gains lower statistic uncertainty by having risk of higher systematics, which is related with time dependent efficiency drift. Size of this trade-off is proportional to local charge asymmetry$\left(\widetilde{C}_{i_+} - \widetilde{C}_{i_-}\right) / \left(\widetilde{C}_{i_+} + \widetilde{C}_{i_-}\right)$. To help better illustrate this point, a simple experiment is constructed:

Consider an experiment with 4 spin states: $1+, 1-, 2+, 2-$. *We set physical asymmetry is* 0*, so that any asymmetry given estimator is its bias. But yield is assumed to be drifting with time, but remains constant in each state, say* $y_{i_\pm}$*. To simplify, target polarization is constant* 100%*.* Just to be clear, estimators from both methods are expressed here:

$$\hat{\epsilon}_{\text{MLE}} = \frac{\frac{N_{1+}-N_{1-}+N_{2+}-N_{2-}}{N_{1+}+N_{1-}+N_{2+}+N_{2-}} - \frac{\widetilde{C}_{1+}-\widetilde{C}_{1-}+\widetilde{C}_{2+}-\widetilde{C}_{2-}}{\widetilde{C}_{1+}+\widetilde{C}_{1-}+\widetilde{C}_{2+}+\widetilde{C}_{2-}}}{1 - \frac{N_{1+}-N_{1-}+N_{2+}-N_{2-}}{N_{1+}+N_{1-}+N_{2+}+N_{2-}} \times \frac{\widetilde{C}_{1+}-\widetilde{C}_{1-}+\widetilde{C}_{2+}-\widetilde{C}_{2-}}{\widetilde{C}_{1+}+\widetilde{C}_{1-}+\widetilde{C}_{2+}+\widetilde{C}_{2-}}} \tag{2.19}$$

$$\hat{\epsilon}_{\text{Local Pair}} = \frac{a_1\left(\frac{N_{1+}}{\widetilde{C}_{1+}} - \frac{N_{1-}}{\widetilde{C}_{1-}}\right) + a_2\left(\frac{N_{2+}}{\widetilde{C}_{2+}} - \frac{N_{2-}}{\widetilde{C}_{2-}}\right)}{a_1\left(\frac{N_{1+}}{\widetilde{C}_{1+}} + \frac{N_{1-}}{\widetilde{C}_{1-}}\right) + a_2\left(\frac{N_{2+}}{\widetilde{C}_{2+}} + \frac{N_{2-}}{\widetilde{C}_{2-}}\right)} \tag{2.20}$$

where $a_1/a_2 = (\widetilde{C}_{1+} + \widetilde{C}_{1-})/(\widetilde{C}_{2+} + \widetilde{C}_{2-})$. In large event number estimation, $N_{i_\pm} \to y_{i_\pm}\widetilde{C}_{i_\pm}$, we study following cases:

- $\widetilde{C}_{1+}/\widetilde{C}_{1-} = \widetilde{C}_{2+}/\widetilde{C}_{2-}$, or local charge asymmetry is equal to that of global. Then $\hat{\epsilon}_{\text{MLE}} = \hat{\epsilon}_{\text{Local Pair}}$.
- To help better illustrate the effect of local charge asymmetry , we exaggeratedly assume $\widetilde{C}_{1+} = 100\widetilde{C}_{1-} = 100\widetilde{C}_{2+} = \widetilde{C}_{2-}$ and $y_{i_\pm}$ is constant. Then $\sigma\left(\hat{\epsilon}_{\text{MLE}}\right) \approx \frac{\sqrt{2}}{10}\sigma\left(\hat{\epsilon}_{\text{Local Pair}}\right)$.
- At same huge local charge asymmetry, $\widetilde{C}_{1+} = 100\widetilde{C}_{1-} = 100\widetilde{C}_{2+} = \widetilde{C}_{2-}$, we further assume $y_{1+} = y_{1-} \neq y_{2+} = y_{2-}$. Then $\hat{\epsilon}_{\text{MLE}} \approx \left(y_{1+} - y_{2-}\right) / \left(y_{1+} + y_{2-}\right)$ gets biased by yield drifts, while $\hat{\epsilon}_{\text{Local Pair}}$=0 remain unbiased.

Therefore, when there is local charge asymmetry fluctuation, MLE achieves lower statistical uncertainty by best (statistically) matching ± states beyond local pairs, taking the risk that yield could drift. It's possible to suppress this risk by applying time dependent yield correction on effective charge $\widetilde{C}_{i_\pm}$, if relative yield drift is well known. Possible ways include

- Average yield fitting of a stable period of runs. Blue Team has made this study. However, for such period, total effective charge asymmetry is small, which mean, effectively, MLE matches spin state within this period. Therefore, the total amount of this correction is limited.

- Estimate relative coincidence yield drift by studying single channels, which have higher statistics. However, coincidence yield drift will not be equal to multiply of drift of two single channels due to acceptance/distribution difference. Extensive study is necessary to get more precise prediction.

Chapter 3

Estimation of Angular Modulation for SSA

In the case that electron beam is not polarized, SIDIS azimuthal yield, Eq. 1.1, can be simplified as

$$y(\phi_h, \phi_S) = \rho \cdot \sigma \cdot a_{\pm}(\phi_h, \phi_S)(1 + P \sum_j \epsilon_j A_j(\phi_h, \phi_S)), \tag{3.1}$$

, where ρ is longitudinal target density, σ is the unpolarized cross-section, $a_{\pm}(\phi_h, \phi_S)$ is the acceptance at spin state $\pm$, $A_j(\phi_h, \phi_S)$ is the j^{th} azimuthal angular modulation, $\sin(\phi_h + \phi_S)$ for example, P is target polarization[1], and ϵ_j as the amplitude of each modulation. Our goal is to get estimators of $\hat{\epsilon}_j$ as well as their uncertainties and correlations, which takes the form of covariance matrix. Further pointing out, acceptance for plus and minus spin states are different. However, due to our exact spin direction flip, we have

$$a_+(\phi_h, \phi_S) = a_-(\phi_h, \phi_S + \pi) \tag{3.2}$$

In section 3.1, a special case (no live time and charge asymmetry) will be studied with both MLE and weighted sum method. It will be demonstrated that MLE give identical result as weighted sum, given asymmetry is small. Then MLE will be expanded to process data in real experiment situations (including live time, charge asymmetry and target polarization).

3.1 As a start : Single Data Section with Symmetric Acceptance

3.1.1 Introduction

As a start, we introduce to a complete solution to a simpler ideal case:

In a Transversity-like experiment with multiple target spin states, there is no overall charge asymmetry, lift time asymmetry and time dependent yield drift. Target polarization is constant (since it's easy to process by transform $\epsilon \to \epsilon/P$, $\sigma(\epsilon) \to \sigma(\epsilon)/P$, we simply set it to be 100%)

[1] Since in transversity configuration, at SSA is most sensitive to data taken with vertical target direction, during which target spin direction is mostly transverse to $\mathbf{q}$. In this case $S_T \to P_{\text{Target}}$ as in Eq. 1.1.

In this special case, we can further expand event based model of section 1.4 to describe this experiment. Here is an important step: by summing over all spin states, for each incoming electron, it has equal chance to interact with a nuclei of spin plus or minus. Therefore, we can use a single probability function to describe (ϕ_h, ϕ_S) distribution of each event with an effective acceptance, $a(\phi_h, \phi_S) = (a_+(\phi_h, \phi_S) + a_-(\phi_h, \phi_S))/2$. Now we can translate the yield (Eq. 3.1) into probability, a normalization factor $Norm(\{\epsilon_j\})$ needs to be applied due to the undetermined ϵ_j

$$f(\phi_h, \phi_S) = \rho \cdot \sigma \cdot a(\phi_h, \phi_S)(1 + P \sum_j \epsilon_j A_j(\phi_h, \phi_S))/Norm(\epsilon_i) \tag{3.3}$$

where

$$Norm(\epsilon_i) = (\int \int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S)(1 + \sum_j \epsilon_j A_j(\phi_h, \phi_S))d\phi_h d\phi_S \tag{3.4}$$

Consider the fact that $a(\phi_h, \phi_S) = a(\phi_h, \phi_S + \pi)$, an interesting fact with this kind of acceptance is that, when $m + n =$ odd we have

$$\int_0^{2\pi} a(\phi_S) \cos^m(\phi_S) \sin^n(\phi_S) d\phi_S = 0. \tag{3.5}$$

In the case of single spin asymmetries on a transversely polarized target, we have at most five azimuthal terms [3],

- Three in leading twist: $\sin(\phi_h - \phi_S)$, $\sin(\phi_h + \phi_S)$ and $\sin(3\phi_h - \phi_S)$;
- Two in higher twist: $\sin(\phi_S)$ and $\sin(2\phi_h - \phi_S)$.

And all of these terms have *odd* total order of $\sin(\phi_S)$ and $\cos(\phi_S)$, or

$$A_i(\phi_h, \phi_S) = -A_i(\phi_h, \phi_S + \pi) \tag{3.6}$$

. Now let's go back to see the normalization factor in Eq. 3.4. Because of the nature of the asymmetry components, the normalization become a constant:

$$Norm(\{\epsilon_j\}) = \int_0^{2\pi} \int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S)(1 + d\phi_h d\phi_S) = const. \tag{3.7}$$

3.1.2 Maximum Likelihood

Therefore the log-likelihood function of all the event sample is

$$L = \log \prod_{ev} f(\phi_h, \phi_S). \tag{3.8}$$

And the best sets of ϵ_is maximizes the total likelihood with

$$\frac{\partial \log L}{\partial \epsilon_i} = 0. \tag{3.9}$$

With N as a constant, we can rewrite Equation 3.9 as

$$\begin{aligned}\frac{\partial L}{\partial \epsilon_i} &= \sum_{ev}(\frac{\partial \log(\sigma N \cdot a(\phi_h, \phi_S))}{\partial \epsilon_i} + \frac{\partial \log(1 + \sum_j \epsilon_j A_j(\phi_h, \phi_S))}{\partial \epsilon_i}) \\ &= \sum_{ev} \frac{\partial \log(1 + \sum_j \epsilon_j A_j(\phi_h, \phi_S))}{\partial \epsilon_i} \\ &= \sum_{ev} \frac{A_i(\phi_h, \phi_S)}{1 + \sum_j \epsilon_j A_j(\phi_h, \phi_S)} = 0 \end{aligned} \quad (3.10)$$

And in case that the asymmetries are small, the denominator can be expanded into polynomials

$$\sum_{ev} \frac{A_i(\phi_h, \phi_S)}{1 + \sum_j \epsilon_j A_j(\phi_h, \phi_S)} = \sum_{ev} A_i(\phi_h, \phi_S)(1 - \sum_j \epsilon_j A_j(\phi_h, \phi_S) + O(A^2)) = 0 \quad (3.11)$$

With $A_j \ll 1$, we can ignore higher order terms to get a set of linear solutions for ϵ_j as written in matrix format:

$$\begin{pmatrix} \sum\limits_{ev} A_1 A_1 & \sum\limits_{ev} A_1 A_2 & \dots \\ \sum\limits_{ev} A_2 A_1 & \sum\limits_{ev} A_2 A_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum\limits_{ev} A_1 \\ \sum\limits_{ev} A_1 \\ \vdots \end{pmatrix} \quad (3.12)$$

or

$$\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum\limits_{ev} A_1 A_1 & \sum\limits_{ev} A_1 A_2 & \dots \\ \sum\limits_{ev} A_2 A_1 & \sum\limits_{ev} A_2 A_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{-1} \begin{pmatrix} \sum\limits_{ev} A_1 \\ \sum\limits_{ev} A_1 \\ \vdots \end{pmatrix} \quad (3.13)$$

3.1.3 Weighted Sum

The method of weighted sum [5] was successful used in various analysis of azimuthal asymmetries with only one angular variable and symmetric acceptance, and now let's see whether we can adopted in the SIDIS case with two variables.

With Equation (**??**) and (3.5) and that fact that all five modulations, A_i, have *odd* total order of ϕ_S in SIDIS, we have the following relations.

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} y(\phi_h, \phi_S) A_i d\phi_h d\phi_S &= \int_0^{2\pi}\int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S)(1 + \sum_j \epsilon_j A_j) A_i d\phi_h d\phi_S \\ &= \sum_j \epsilon_j \int_0^{2\pi}\int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S) A_j A_i d\phi_h d\phi_S \end{aligned} \quad (3.14)$$

and

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} y(\phi_h, \phi_S) A_i A_j d\phi_h d\phi_S &= \int_0^{2\pi}\int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S)(1 + \sum_k \epsilon_k A_k) A_i A_j d\phi_h d\phi_S \\ &= \int_0^{2\pi}\int_0^{2\pi} \sigma \cdot a(\phi_h, \phi_S) A_i A_j d\phi_h d\phi_S \end{aligned} \quad (3.15)$$

Combine Equation (3.14) and (3.15), we get

$$\int_0^{2\pi}\int_0^{2\pi} y(\phi_h,\phi_S)A_i d\phi_h d\phi_S = \sum_j \epsilon_j \int_0^{2\pi}\int_0^{2\pi} y(\phi_h,\phi_S)A_i A_j d\phi_h d\phi_S \tag{3.16}$$

Since $y(\phi_h,\phi_S)$ is the yield, doing the integral $\int_0^{2\pi}\int_0^{2\pi} y(\phi_h,\phi_S)x(\phi_h,\phi_S)d\phi_h d\phi_S$ is effectively doing the sum over all obtained events with weight $x(\phi_h,\phi_S)$, $\sum_{ev} x(\phi_h,\phi_S)$, and the overall scaling factor can be ignored since it's applied on both sides of the equation. Therefore we can rewrite Equation 3.16 with the sum and result is exactly same as Equation (3.12)

$$\begin{pmatrix} \sum\limits_{ev} A_1A_1 & \sum\limits_{ev} A_1A_2 & \dots \\ \sum\limits_{ev} A_2A_1 & \sum\limits_{ev} A_2A_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum\limits_{ev} A_1 \\ \sum\limits_{ev} A_1 \\ \vdots \end{pmatrix}.$$

3.1.4 Uncertainty Estimation

The uncertainty estimation procedure is similar to that of section 2.1.3. Therefore we just quote the conclusion from reference[5]: the covariance matrix of $\hat{\epsilon}_i$ is

$$\mathbf{V}\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum\limits_{ev} A_1A_1 & \sum\limits_{ev} A_1A_2 & \dots \\ \sum\limits_{ev} A_2A_1 & \sum\limits_{ev} A_2A_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{-1} \tag{3.17}$$

3.2 Angular Modulation Extraction with Spin States

Following last section, we will discuss angular modulation extraction in close-to-reality case that

- *During an experiment, there are multiple spin states. We categorize them into Target Transversely Polarized Plus/Minus States, indexed by $i_{T\pm}$, and Vertically Polarized Polarized Plus/Minus States, indexed by $i_{V\pm}$*
- *there are charge/live time asymmetry between spin states*
- *each spin state could carry a different target polarization $P_{i_{T/V\pm}}$. However, all event in same spin state share same polarization.*
- *yield is stable*[2]

[2]

– time dependent drift is ignored in this section. It will introduce a systematic bias to estimator of this study

3.2.1 Building the statistical Model

Back to event based model as discussed in section 1.4, for each event, we have 3 observables $(\mathrm{StateID}, \phi_h, \phi_S)_{\mathrm{ev}}$. Considering equation 3.1, the probability distribution of each event is

$$\begin{aligned} \Pr(\mathrm{StateID} = i_{T/V\pm}, \phi_h, \phi_S) &\equiv f_{i_{T/V\pm}}(\phi_h, \phi_S) & (3.18) \\ &= \frac{\widetilde{C}_{i_{T/V\pm}} a_{T/V\pm}(\phi_h, \phi_S)}{Norm(\{\epsilon_i\})} \left(1 + P_{i_{T/V\pm}} \sum_j \epsilon_j A_j(\phi_h, \phi_S)\right) & (3.19) \end{aligned}$$

, where normalization function

$$Norm(\{\epsilon_i\}) = \sum_{i_{T/V\pm}} \iint d\phi_h d\phi_S \widetilde{C}_{i_{T/V\pm}} a_{T/V\pm}(\phi_h, \phi_S) \left(1 + P_{i_\pm} \sum_j \epsilon_j A_j(\phi_h, \phi_S)\right) \quad (3.20)$$

. Here $\sum_{i_{T/V\pm}}$ is defined as sum over all spin states with state index $i_{T/V\pm}$, while $\sum_j$ is sum over all angular modulation terms $A_j(\phi_h, \phi_S)$. Quoting Eq. 3.6 and Eq. 3.2, normalization function could be further simplify to

$$\begin{aligned} Norm(\{\epsilon_i\}) &= \sum_{i_{T/V+}} \iint d\phi_h d\phi_S \widetilde{C}_{i_{T/V+}} a_{T/V+}(\phi_h, \phi_S)(1 + P_{i_{T/V+}} \sum_j \epsilon_j A_j(\phi_h, \phi_S) \\ &\quad + \sum_{i_{T/V-}} \iint d\phi_h d\phi_S \widetilde{C}_{i_{T/V-}} a_{T/V-}(\phi_h, \phi_S)(1 + P_{i_{T/V-}} \sum_j \epsilon_j A_j(\phi_h, \phi_S) \\ &= \sum_{i_{T/V+}} \iint d\phi_h d\phi_S \widetilde{C}_{i_{T/V+}} a_{T/V+}(\phi_h, \phi_S)(1 + P_{i_{T/V+}} \sum_j \epsilon_j A_j(\phi_h, \phi_S) \\ &\quad + \sum_{i_{T/V-}} \iint d\phi_h d\phi_S \widetilde{C}_{i_{T/V-}} a_{T/V+}(\phi_h, \phi_S + \pi) \\ &\quad \times \left(1 - P_{i_{T/V-}} \sum_j \epsilon_j A_j(\phi_h, \phi_S + \pi)\right) \\ &= \tilde{a}_T \left(\widetilde{C}_{T+} + \widetilde{C}_{T-}\right) + \tilde{a}_V \left(\widetilde{C}_{V+} + \widetilde{C}_{V-}\right) \\ &\quad + \left(\widetilde{C}_{P,T+} - \widetilde{C}_{P,T-}\right) \sum_j \epsilon_j \tilde{a}_{T,j} + \left(\widetilde{C}_{P,V+} - \widetilde{C}_{P,V-}\right) \sum_j \epsilon_j \tilde{a}_{V,j} \\ &= \left(\tilde{a}_T \left(\widetilde{C}_{T+} + \widetilde{C}_{T-}\right) + \tilde{a}_V \left(\widetilde{C}_{V+} + \widetilde{C}_{V-}\right)\right) \left(1 + \sum_j \epsilon_j A_{CP,j}\right) & (3.21) \end{aligned}$$

, where

$$\tilde{a}_{T/V} \equiv \iint a_{T/V+}(\phi_h, \phi_S) d\phi_h d\phi_S \tag{3.22}$$

$$\tilde{a}_{T/V,\,i} \equiv \iint a_{T/V+}(\phi_h, \phi_S) A_i(\phi_h, \phi_S) d\phi_h d\phi_S \tag{3.23}$$

$$A_{CP,\,i} \equiv \frac{\tilde{a}_{T,\,i}\left(\widetilde{C}_{P,\,T+} - \widetilde{C}_{P,\,T-}\right) + \tilde{a}_{V,\,i}\left(\widetilde{C}_{P,\,V+} - \widetilde{C}_{P,\,V-}\right)}{\tilde{a}_{T}\left(\widetilde{C}_{T+} + \widetilde{C}_{T-}\right) + \tilde{a}_{V}\left(\widetilde{C}_{V+} + \widetilde{C}_{V-}\right)} \tag{3.24}$$

in another word, for transverse or vertical polarized separately, $\tilde{a}_{T/V}$ is integrated acceptance and $\tilde{a}_{T/V,\,i}$ is normalized A_i modulated acceptance, and $A_{CP,\,i}$ is polarization and acceptance weighted effective charge asymmetry[3]. In ideal Transversity setup that instrument acceptance and efficiency is symmetric relative to central horizontal plane, there will be $\tilde{a}_{T,\,i} \to 0$ or the affect of transverse charge asymmetry is much smaller than that of vertical[4].

3.2.2 MLE estimator and uncertainty

To calculate the maximum likelihood value, we form the log-likelihood function,

$$\begin{aligned} L\{\epsilon_i\} &= \log\left(\prod_{\text{ev}} f_{i_\pm}(\phi_h, \phi_S)\right) \\ &= \sum_{\text{ev}} \log\left(1 + P_{i_{T/V\pm}} \sum_j \epsilon_j A_j(\phi_h, \phi_S)\right) - N \log Norm(\{\epsilon_i\}) + \text{Constant} \end{aligned} \tag{3.25}$$

where $\sum_{\text{ev}}$ is summing over all events in both transverse and vertical target states, while $\sum_j$ remains summing over all angular modulations. Take derivative of L with respect to parameter ϵ_k and equate them to zero:

$$\begin{aligned} 0 &= \frac{dL}{d\epsilon_k} \\ &= \sum_{\text{ev}} \frac{P_{i_{T/V\pm}} A_k(\phi_h, \phi_S)}{1 + P_{i_{T/V\pm}} \sum_j \epsilon_j A_j(\phi_h, \phi_S)} - N \frac{A_{CP,\,k}}{1 + \sum_j \epsilon_j A_{CP,\,j}} \end{aligned} \tag{3.26}$$

$$\begin{aligned} &= \sum_{\text{ev}} \left(P_{i_{T/V\pm}} A_k(\phi_h, \phi_S) \left(1 - P_{i_{T/V\pm}} \sum_j \epsilon_j A_j(\phi_h, \phi_S)\right)\right) \\ &\quad - N A_{CP,\,k} \left(1 - \sum_j \epsilon_j A_{CP,\,j}\right) + O\left((N_+ - N_-) \sum_{mn} \epsilon_m \epsilon_n\right) \end{aligned} \tag{3.27}$$

[3] Section 5.2 is dedicated to discuss property of acceptance and $A_{CP,\,i}$, while its estimation is shown in section 5.2.3.1.

[4] Real Transversity data suggests for Seivers modulation, $\tilde{a}_T \sim -3\%$ while $\tilde{a}_V \sim -93\%$, which validates this discussion

To further simplify this expression, we define following event sums:

$$\sum [PA_k] \equiv \sum_{\text{ev}} P_{i_{T/V\pm}} A_k(\phi_h, \phi_S) \tag{3.28}$$

$$\sum \left[P^2 A_j A_k\right] \equiv \sum_{\text{ev}} P^2_{i_{T/V\pm}} A_j(\phi_h, \phi_S) A_k(\phi_h, \phi_S) \tag{3.29}$$

So

$$0 = \sum [PA_k] - \sum_j \epsilon_j \sum \left[P^2 A_j A_k\right] - NA_{CP,\,k} + NA_{CP,\,k} \sum_j \epsilon_j A_{CP,\,j} + O\left((N_+ - N_-) \sum_{mn} \epsilon_m \epsilon_n\right) \tag{3.30}$$

Therefore, we have multiple equations with index k

$$\sum_j \epsilon_j \left(\sum \left[P^2 A_j A_k\right] - NA_{CP\,,j} A_{CP,\,k}\right) = \sum [PA_k] - NA_{CP,\,k} + O\left((N_+ - N_-) \sum_{mn} \epsilon_m \epsilon_n\right) \tag{3.31}$$

We define matrix

$$\mathbf{F} \equiv \begin{pmatrix} \sum \left[P^2 A_1 A_1\right] - NA^2_{CP,\,1} & \sum \left[P^2 A_1 A_2\right] - NA_{CP,\,1} A_{CP,\,2} & \cdots \\ \sum \left[P^2 A_2 A_1\right] - NA_{CP,\,2} A_{CP,\,1} & \sum \left[P^2 A_2 A_2\right] - NA^2_{CP,\,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{3.32}$$

$$\boldsymbol{B} \equiv \begin{pmatrix} \sum [PA_1] - NA_{CP,\,1} \\ \sum [PA_2] - NA_{CP,\,2} \\ \vdots \end{pmatrix} \tag{3.33}$$

$$\boldsymbol{\epsilon} \equiv \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \end{pmatrix} \tag{3.34}$$

Then equation for estimators could be expressed in matrix

$$\mathbf{F}\boldsymbol{\epsilon} = \boldsymbol{B} + O\left(\frac{N_+ - N_-}{N} \sum_{mn} \epsilon_m \epsilon_n\right) \tag{3.35}$$

The estimator are solved by

$$\hat{\boldsymbol{\epsilon}} = \mathbf{F}^{-1}\boldsymbol{B} + O\left(\frac{N_+ - N_-}{N} \sum_{mn} \epsilon_m \epsilon_n\right) \tag{3.36}$$

To estimate its uncertainty, again we build Fisher information matrix for log likelihood formula 3.25:

$$
\begin{aligned}
[\mathcal{I}(\boldsymbol{\epsilon})]_{jk} &= -\mathrm{E}\left[\frac{\partial^2 L}{\partial\epsilon_j\partial\epsilon_k}\right] \\
&= -\mathrm{E}\left[\frac{\partial^2}{\partial\epsilon_j\partial\epsilon_k}\left(\sum_{\mathrm{ev}}\log\left(1+P_{i_{T/V\pm}}\sum_i \epsilon_i A_i(\phi_h,\phi_S)\right)-N\log\left(1+\sum_i \epsilon_i A_{CP,\,i}\right)\right)\right] \\
&= -\mathrm{E}\left[\frac{\partial}{\partial\epsilon_j}\left(\sum_{\mathrm{ev}}\frac{P_{i_{T/V\pm}}A_k(\phi_h,\phi_S)}{1+P_{i_{T/V\pm}}\sum_i\epsilon_i A_i(\phi_h,\phi_S)}-N\frac{A_{CP,\,k}}{1+\sum_i\epsilon_i\tilde{a}_i}\right)\right] \\
&= \mathrm{E}\left[\sum_{\mathrm{ev}}\frac{P^2_{i_{T/V\pm}}A_k(\phi_h,\phi_S)A_j(\phi_h,\phi_S)}{\left(1+P_{i_{T/V\pm}}\sum_i\epsilon_i A_i(\phi_h,\phi_S)\right)^2}-N\frac{A_{CP,\,k}A_{CP,\,j}}{(1+A_{CP}\sum_i\epsilon_i\tilde{a}_i)^2}\right] \\
&= \sum\left[P^2 A_j A_k\right]-NA_{CP\,,j}A_{CP,\,k}+O\left(N\sum_{ijk}A_{CP,\,i}A_{CP,\,j}A_{CP,\,k}\sum_i\epsilon_i\right) \qquad (3.37)
\end{aligned}
$$

At large statistic approximation, covariance matrix is given by

$$
\begin{aligned}
\mathbf{V}\begin{pmatrix}\hat{\epsilon}_1\\ \hat{\epsilon}_2\\ \vdots\end{pmatrix} &= \begin{pmatrix}\sum\left[P^2A_1A_1\right]-NA^2_{CP,\,1} & \sum\left[P^2A_1A_2\right]-NA_{CP,\,1}A_{CP,\,2} & \cdots\\ \sum\left[P^2A_2A_1\right]-NA_{CP,\,2}A_{CP,\,1} & \sum\left[P^2A_2A_2\right]-NA^2_{CP,\,2} & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix}^{-1} \\
&\quad +O\left(\frac{1}{N}\sum_{ijk}A_{CP,\,i}A_{CP,\,j}A_{CP,\,k}\sum_i\epsilon_i\right) \qquad (3.38)
\end{aligned}
$$

or

$$
\mathbf{V}(\hat{\boldsymbol{\epsilon}})=\mathbf{F}^{-1}+O\left(\frac{\sum_{ijk}A_{CPi}A_{CPj}A_{CPk}\sum_i\epsilon_i}{N}\right) \qquad (3.39)
$$

3.2.3 Discussion

- $\sum_T[PA_k]$ and $\tilde{a}_{T,\,i}$ are small relative to those from vertical spin states. Therefore, $\{\hat{\epsilon}_i\}$ is less depending on transverse data. It is as we expected, since transverse data do not carry single spin asymmetry to first order.
- If events distribution are correlated for two angular modulations A_i and A_j, then interference term, $\sum\left[P^2A_jA_k\right]$, will be non-zero. Following Eq. 3.39, uncertainty of $\hat{\epsilon}_i$ will increase. Besides, correlation elements will show up in the covariance matrix $\mathbf{V}$.

Chapter 4

Estimation of Angular Modulated for DSA

4.1 Build the statistical model

As a related physics goal with SSA, DSA concerns beam helicity related asymmetry, or DSA_j in Eq. 1.1. From analysis point of view it features,

- In general we treat target spin = ±1 states in two separate series. With in each series, an angular modulation on helicity asymmetry could be extracted. Results between two series are expected to be consistent.
- Charge asymmetry between helicity = ±1 states are very small (online monitoring controlled to below 0.1%). Therefore, SSA asymmetry terms would largely canceled out during the process of forming helicity asymmetry.
- Also because charge asymmetry between helicity is small, bias of MLE introduced by yield drift will be suppressed, as shown in section 2.2.3.
- Back to first bullet, in real case there are 2 target spin direction: Transverse and Vertical. To combine their information and form full statistics angular modulation, it's best to perform a global MLE analysis on all states. However, it's possible to check consistency between the following two series, whose results are largely statistically independent of each other: series 1= $V_{\text{Spin}=+} + T_{\text{Spin}=+} + V_{\text{Spin}=-}$, series 2= $V_{\text{Spin}=+} + T_{\text{Spin}=-} + V_{\text{Spin}=-}$

In this section we study DSA under following assumptions:

- *During an experiment, there are multiple spin states. We categorize them into Target Transversely Polarized Plus/Minus ($T_\pm$) States, indexed by $i_{T\pm}$, and Vertically Polarized Polarized Plus/Minus ($V_\pm$) States, indexed by $i_{V\pm}$. Each spin state contains data from two beam helicity, a second index is added to spin state index to specify helicity, eg. $N_{+,i_{V-}}$ for event count at i^{th} minus spin state with helicity$= +1$.*
- *There could be charge/live time asymmetry between spin states. Charge/live time is known for each spin. S*imilar as Eq. *1.13, we define effective charge* $\widetilde{C}_{h,i_{T/V\pm}} \equiv \widetilde{L}_{h,i_{T/V\pm}} \times LT_{h,i_{T/V\pm}} /$ (Average Target Density)

- *each spin state could carry a different target/beam polarization. However, all event in same spin state share same polarization. Transverse and longitudinal target polarization could be expressed as* [1]

$$S_T = P_{Target} \sin\theta_S \quad (4.1)$$
$$S_L = P_{Target} \cos\theta_S \quad (4.2)$$

And for further simplification, we define composite polarization

$$P_{i_{T/V\pm}} \equiv (P_{Beam} * P_{Target})_{i_{T/V\pm}} \quad (4.3)$$

- *yield is stable (time dependent drift is ignored in this section; later it will show up as systematic bias)*
- *longitudinal modulations, ϵ_{LL} and $\epsilon_{LL}^{\cos\phi_h}$, are assumed as inputs, base on knowledge from other experiment. It's also possible to fit them from data as discussed in section* **??***.*

Reviewing Eq. 1.1, DSA cross sections could be simplified as

$$y(\phi_h, \phi_S) = \rho \cdot \sigma \cdot a_{T/V\pm}(\phi_h, \phi_S, \theta_S) \Big(1 + h \cdot P_{i_{T/V\pm}} \times \left(\sin\theta_S \cdot \sum_j \epsilon_j A_j(\phi_h, \phi_S) + \cos\theta_S \cdot \left(\epsilon_{LL} + \epsilon_{LL}^{\cos(\phi_h)} \cos(\phi_h)\right)\right)\Big) \quad (4.4)$$

where $a_{T/V\pm}(\phi_h, \phi_S, \theta_S)$ is 3D acceptance on $(\phi_h, \phi_S, \theta_S)$. Because of spin direction if flipped exactly 90 degree between $\pm$ spin states,

$$a_{T/V+}(\phi_h, \phi_S, \theta_S) = a_{T/V-}(\phi_h, \phi_S + \pi, \pi - \theta_S) \quad (4.5)$$

$A_j(\phi_h, \phi_S)$ is double spin asymmetry angular modulation terms[3]:

- One in leading twist: $\cos(\phi_h - \phi_S)$
- Two in higher twist: $\cos(\phi_S)$ and $\cos(2\phi_h - \phi_S)$

They are also following Eq. 3.6 as in single spin asymmetry case.

Further expanding event based model in section 1.4, each event from data stream is treated as a measurement of a group of 5 random variables (helicity, StateID, ϕ_h, ϕ_S, θ_S). Joint probability distribution is

$$\begin{aligned} &\Pr(\text{helicity} = h, \text{ StateID} = i_{T/V\pm}, \phi_h, \phi_S, \theta_S) \\ &\equiv f_{h, i_{T/V\pm}}(\phi_h, \phi_S, \theta_S) \\ &= \frac{\widetilde{C}_{h, i_{T/V\pm}} a_{T/V\pm}(\phi_h, \phi_S, \theta_S)}{Norm(\{\epsilon_i\})} \quad (4.6) \\ &\times \left(1 + h \cdot P_{i_{T/V\pm}} \sin\theta_S \sum_j \epsilon_j A_j(\phi_h, \phi_S) + h \cdot P_{i_{T/V\pm}} \cos\theta_S \left(\epsilon_{LL} + \epsilon_{LL}^{\cos(\phi_h)} \cdot \cos(\phi_h)\right)\right) \quad (4.7) \end{aligned}$$

[1] *Notice the S_L sign convention (spin along momentum transfer is +) is inverse to* ref [3]

, where normalization function

$$\begin{aligned}
Norm(\{\epsilon_i\}) &= \sum_{i_{T/V\pm}} \sum_{h=\pm1} \iiint d\phi_h d\phi_S d\theta_S \widetilde{C}_{h,i_{T/V\pm}} a_{T/V\pm}(\phi_h, \phi_S) \\
&\times \left(1 + h \cdot P_{i_{T/V\pm}} \sin\theta_S \sum_j \epsilon_j A_j(\phi_h, \phi_S) + h \cdot P_{i_{T/V\pm}} \cos\theta_S \left(\epsilon_{LL} + \epsilon_{LL}^{\cos(\phi_h)} \cdot \cos(\phi_h)\right)\right) \\
&= \sum_{i_{T/V\pm}} \sum_{h=\pm1} \widetilde{C}_{h,i_{T/V\pm}} \\
&\times \left(\tilde{a}_{T/V} + h \cdot P_{i_{T/V\pm}} \left(\sum_j \epsilon_j \tilde{a}^{\sin\theta_S}_{T/V\pm,j} + \epsilon_{LL} \tilde{a}^{\cos\theta_S}_{T/V\pm} + \epsilon_{LL}^{\cos(\phi_h)} \tilde{a}^{\cos\theta_S \cos\phi_h}_{T/V\pm}\right)\right) \qquad (4.8)
\end{aligned}$$

, where $\sum_{i_{T/V\pm}}$ is sum over all spin states with StateID variable $i_{T/V\pm}$, $\sum_{h=\pm1}$ is summing over both helicity states with helicity variable h. Also, acceptance modulations are defined below (also considering symmetry of Eq. 4.5 after spin flip $\phi_S \to \phi_S + \pi$, $\theta_S \to \pi - \theta_S$)

$$\tilde{a}_{T/V} \equiv \iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) d\phi_h d\phi_S d\theta_S \qquad (4.9)$$

$$= \iiint a_{T/V-}(\phi_h, \phi_S, \theta_S) d\phi_h d\phi_S d\theta_S \qquad (4.10)$$

$$\tilde{a}^{\sin\theta_S}_{T/V,i} \equiv \iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) \sin\theta_S A_i(\phi_h, \phi_S) d\phi_h d\phi_S d\theta_S \qquad (4.11)$$

$$= -\iiint a_{T/V-}(\phi_h, \phi_S, \theta_S) \sin\theta_S A_i(\phi_h, \phi_S) d\phi_h d\phi_S d\theta_S \qquad (4.12)$$

$$\tilde{a}^{\cos\theta_S}_{T/V} \equiv \iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) \cos\theta_S d\phi_h d\phi_S d\theta_S \qquad (4.13)$$

$$= -\iiint a_{T/V-}(\phi_h, \phi_S, \theta_S) \cos\theta_S d\phi_h d\phi_S d\theta_S \qquad (4.14)$$

$$\tilde{a}^{\cos\theta_S \cos\phi_h}_{T/V} \equiv \iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) \cos\theta_S \cos\phi_h d\phi_h d\phi_S d\theta_S \qquad (4.15)$$

$$= -\iiint a_{T/V-}(\phi_h, \phi_S, \theta_S) \cos\theta_S \cos\phi_h d\phi_h d\phi_S d\theta_S \qquad (4.16)$$

Here $\tilde{a}_{T/V}^{\cos\theta_S}$ and $\tilde{a}_{T/V}^{\cos\theta_S\cos\phi_h}$ are suppressed compared with $\tilde{a}_{T/V,i}^{\sin\theta_S}$, because $a_{T/V\pm}(\phi_h, \phi_S, \theta_S)$ concentrate on $\theta_S \sim 90°$ region. Then we redefine

$$\widetilde{C}_{h,T/V\pm} \equiv \sum_{i_{T/V\pm}} \widetilde{C}_{h,i_{T/V\pm}} \tag{4.17}$$

$$\widetilde{C}_{P,h,T/V\pm} \equiv \sum_{i_{T/V\pm}} \widetilde{C}_{h,i_{T/V\pm}} P_{i_{T/V\pm}} \tag{4.18}$$

$$A_{CP,j} \equiv \frac{\sum_{T/V\pm} \pm\tilde{a}_{T/V,j}^{\sin\theta_S}\left(\widetilde{C}_{P,+,T/V\pm} - \widetilde{C}_{P,-,T/V\pm}\right)}{\sum_{T/V\pm} \tilde{a}_{T/V}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)} \tag{4.19}$$

$$A_{CP}^{\cos\theta_S} \equiv \frac{\sum_{T/V\pm} \pm\tilde{a}_{T/V}^{\cos\theta_S}\left(\widetilde{C}_{P,+,T/V\pm} - \widetilde{C}_{P,-,T/V\pm}\right)}{\sum_{T/V\pm} \tilde{a}_{T/V}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)} \tag{4.20}$$

$$A_{CP}^{\cos\theta_S\cos\phi_h} \equiv \frac{\sum_{T/V\pm} \pm\tilde{a}_{T/V}^{\cos\theta_S\cos\phi_h}\left(\widetilde{C}_{P,+,T/V\pm} - \widetilde{C}_{P,-,T/V\pm}\right)}{\sum_{T/V\pm} \tilde{a}_{T/V}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)} \tag{4.21}$$

where, $\sum_{T/V\pm}$ is sum over 4 spin direction configurations. Estimation of above asymmetries are discussed in section 5.2.3.2. With new variables, normalization function is further simplified to

$$\begin{aligned}
Norm(\{\epsilon_i\}) &= \sum_{\text{All States}} \left(\left(\widetilde{C}_{+,i_{T/V\pm}} + \widetilde{C}_{-,i_{T/V\pm}}\right)\tilde{a}_{T/V}\right. \\
&\quad \left. + P_{i_{T/V\pm}}\left(\widetilde{C}_{+,i_{T/V\pm}} - \widetilde{C}_{-,i_{T/V\pm}}\right)\left(\sum_j \epsilon_j \tilde{a}_{T/V\pm,j}^{\sin\theta_S} + \epsilon_{LL}\tilde{a}_{T/V\pm}^{\cos\theta_S} + \epsilon_{LL}^{\cos(\phi_h)}\tilde{a}_{T/V\pm}^{\cos\theta_S\cos\phi_h}\right)\right) \\
&= \left(\sum_{T/V\pm}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)\right) \\
&\quad \times \left(1 + \sum_j \epsilon_j A_{CP,j} + \epsilon_{LL} A_{CP}^{\cos\theta_S} + \epsilon_{LL}^{\cos(\phi_h)} A_{CP}^{\cos\theta_S\cos\phi_h}\right)
\end{aligned} \tag{4.22}$$

Here $A_{CP}^{\cos\theta_S}$ and $A_{CP}^{\cos\theta_S\cos\phi_h}$ are much suppressed compared with $A_{CP,j}$ due to similar suppression on modulated acceptance as discussed above. And general A_{CP} is small, because helicity related charge asymmetry was controlled to be small ($<\sim 0.1\%$) during data taking as well as asymmetry of live time, $LT_\pm$, was shown to be small too.

4.2 Extraction of Modulated Asymmetry

To calculate the maximum likelihood value, we form the log-likelihood function,

$$\begin{aligned} L\{\epsilon_i\} &= \log\left(\prod_{\rm ev} f_{h,i_{T/V\pm}}(\phi_h,\phi_S,\theta_S)\right) \\ &= \sum_{\rm ev}\log\left(1+h\cdot P_{i_{T/V\pm}}\sin\theta_S\sum_j \epsilon_j A_j(\phi_h,\phi_S)+h\cdot P_{i_{T/V\pm}}\cos\theta_S\left(\epsilon_{LL}+\epsilon_{LL}^{\cos(\phi_h)}\cdot\cos(\phi_h)\right)\right) \\ &\quad -N\log\left(Norm(\{\epsilon_i\})\right)+\text{Constant} \end{aligned} \tag{4.23}$$

where, N is total event counts, $\sum_{\rm ev}$ is summing over all events in both transverse and vertical target states, while $\sum_j$ is summing over all angular modulations. Take derivative of L with respect to parameter ϵ_k and equate them to zero:

$$\begin{aligned} 0 &= \frac{dL}{d\epsilon_k} \\ &= \sum_{\rm ev}\frac{h\cdot P_{i_{T/V\pm}}\sin\theta_S A_k(\phi_h,\phi_S)}{1+h\cdot P_{i_{T/V\pm}}\sin\theta_S\sum_j\epsilon_j A_j(\phi_h,\phi_S)+h\cdot P_{i_{T/V\pm}}\cos\theta_S\left(\epsilon_{LL}+\epsilon_{LL}^{\cos(\phi_h)}\cdot\cos(\phi_h)\right)} \\ &\quad -N\frac{A_{CP,k}}{1+\sum_j\epsilon_j A_{CP,j}+\epsilon_{LL}A_{CP}^{\cos\theta_S}+\epsilon_{LL}^{\cos(\phi_h)}A_{CP}^{\cos\theta_S\cos\phi_h}} \end{aligned}$$

In the case that modulations are small, above equations could be expanded with ϵ_j, ϵ_{LL} and $\epsilon_{LL}^{\cos(\phi_h)}$:

$$\begin{aligned} 0 &= \sum_{\rm ev} h\cdot P_{i_{T/V\pm}}\sin\theta_S A_k(\phi_h,\phi_S) \\ &\quad \times\left(1-h\cdot P_{i_{T/V\pm}}\sin\theta_S\sum_j\epsilon_j A_j(\phi_h,\phi_S)-h\cdot P_{i_{T/V\pm}}\cos\theta_S\left(\epsilon_{LL}+\epsilon_{LL}^{\cos(\phi_h)}\cdot\cos(\phi_h)\right)\right) \\ &\quad -N\times A_{CP,k}\left(1-\sum_j\epsilon_j A_{CP,j}-\epsilon_{LL}A_{CP}^{\cos\theta_S}-\epsilon_{LL}^{\cos(\phi_h)}A_{CP}^{\cos\theta_S\cos\phi_h}\right) \\ &\quad +O\left((N_+-N_-)\sum_{mn}\epsilon_m\epsilon_n\right) \end{aligned} \tag{4.24}$$

Define event sums

$$\sum[hPA_k] \equiv \sum_{\rm ev} h\cdot P_{i_{T/V\pm}}\sin\theta_S A_k(\phi_h,\phi_S) \tag{4.25}$$

$$\sum\left[P^2A_jA_k\right] \equiv \sum_{\rm ev} P^2_{i_{T/V\pm}}\sin^2\theta_S A_j(\phi_h,\phi_S)A_k(\phi_h,\phi_S) \tag{4.26}$$

$$\sum\left[P^2\cos\theta_S A_k\right] \equiv \sum_{\rm ev} P^2_{i_{T/V\pm}}\sin\theta_S A_k(\phi_h,\phi_S)\cos\theta_S \tag{4.27}$$

$$\sum\left[P^2\cos\theta_S\cos\phi_h A_k\right] \equiv \sum_{\rm ev} P^2_{i_{T/V\pm}}\sin\theta_S A_k(\phi_h,\phi_S)\cos\theta_S\cos\phi_h \tag{4.28}$$

Eq. 4.24 could be expressed as

$$\sum_j \epsilon_j \left(\sum \left[P^2 A_j A_k\right] - N \times A_{CP,\,j} \times A_{CP,\,k} \right)$$
$$= \sum \left[hPA_k\right] - N \times A_{CP,\,k} \tag{4.29}$$
$$-\epsilon_{LL} \left(\sum \left[P^2 \cos\theta_S A_k\right] - N \times A_{CP,\,k} A_{CP}^{\cos\theta_S} \right)$$
$$-\epsilon_{LL}^{\cos(\phi_h)} \left(\sum \left[P^2 \cos\theta_S \cos\phi_h A_k\right] - N \times A_{CP,k} A_{CP}^{\cos\theta_S \cos\phi_h} \right)$$
$$+O\left((N_+ - N_-) \sum_{mn} \epsilon_m \epsilon_n \right) \tag{4.30}$$

Solution of which, also expressed in matrix form, are estimators

$$\begin{aligned}
\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \vdots \end{pmatrix} &= \begin{pmatrix} \sum\left[P^2 A_1 A_1\right] - N A_{CP,\,1}^2 & \sum\left[P^2 A_1 A_2\right] - N A_{CP,\,1} A_{CP,\,2} & \cdots \\ \sum\left[P^2 A_2 A_1\right] - N A_{CP,\,2} A_{CP,\,1} & \sum\left[P^2 A_2 A_2\right] - N A_{CP,\,2}^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^{-1} \\
&\times \left(\begin{pmatrix} \sum\left[hPA_1\right] - N A_{CP,\,1} \\ \sum\left[hPA_2\right] - N A_{CP,\,2} \\ \vdots \end{pmatrix} \right. \\
&-\epsilon_{LL} \begin{pmatrix} \sum\left[P^2 \cos\theta_S A_1\right] - N \times A_{CP}^{\cos\theta_S} A_{CP,1} \\ \sum\left[P^2 \cos\theta_S A_2\right] - N \times A_{CP}^{\cos\theta_S} A_{CP,2} \\ \vdots \end{pmatrix} \\
&\left. -\epsilon_{LL}^{\cos\phi_h} \begin{pmatrix} \sum\left[P^2 \cos\theta_S \cos\phi_h A_1\right] - N \times A_{CP}^{\cos\theta_S \cos\phi_h} A_{CP,1} \\ \sum\left[P^2 \cos\theta_S \cos\phi_h A_2\right] - N \times A_{CP}^{\cos\theta_S \cos\phi_h} A_{CP,2} \\ \vdots \end{pmatrix} \right) \qquad (4.31) \\
&\equiv \mathbf{F}^{-1} \left(\boldsymbol{B} - \epsilon_{LL} \mathbf{F}_{LL} - \epsilon_{LL}^{\cos\phi_h} \mathbf{F}_{LL}^{\cos\phi_h} \right)
\end{aligned}$$

where

$$\mathbf{F}_{i,\,j} \equiv \sum \left[P^2 A_i A_j\right] - N A_{CP,\,i} A_{CP,\,j} \tag{4.32}$$
$$\boldsymbol{B}_i \equiv \sum \left[hPA_i\right] - N A_{CP,\,i} \tag{4.33}$$
$$\mathbf{F}_{LL,\,i} \equiv \sum \left[P^2 \cos\theta_S A_i\right] - N \times A_{CP}^{\cos\theta_S} A_{CP,\,i} \tag{4.34}$$
$$\mathbf{F}_{LL,\,i}^{\cos\phi_h} \equiv \sum \left[P^2 \cos\theta_S \cos\phi_h A_i\right] - N \times A_{CP}^{\cos\theta_S \cos\phi_h} A_{CP,\,i} \tag{4.35}$$

It's covariance matrix is

$$\begin{aligned}
\mathbf{V}\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \vdots \end{pmatrix} &= \begin{pmatrix} \sum\left[P^2 A_1 A_1\right] - N A_{CP,\,1}^2 & \sum\left[P^2 A_1 A_2\right] - N A_{CP,\,1} A_{CP,\,2} & \cdots \\ \sum\left[P^2 A_2 A_1\right] - N A_{CP,\,2} A_{CP,\,1} & \sum\left[P^2 A_2 A_2\right] - N A_{CP,\,2}^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^{-1} \\
&\equiv \mathbf{F}^{-1} \qquad (4.36)
\end{aligned}$$

4.3 Discussion

4.3.1 Longitudinal asymmetries

4.3.1.1 Size of Impact on DSA measurements

Longitudinal asymmetries, ϵ_{LL} and $\epsilon_{LL}^{\cos\phi_h}$, show up as two correction terms in Eq. 4.31. Size of this correction is proportional to $\mathbf{F}_{LL,i} \sim \sum[\cos\theta_S]$. Therefore, in a perfect situation, that target spin direction perfectly transverse relative to $\vec{q}$ or θ_S, longitudinal terms will not affect DSA results. In experiment Transversity, θ_S take an average of $90 \pm 7°$, which suggests the suppression is around 12%.

4.3.1.2 Extracting Longitudinal Asymmetries from Transversity Data

It's also possible to treat longitudinal asymmetries as unknown variables. Following similar procedure as last section, treating longitudinal term as additional modulation other than n known DSA modulations

$$\widetilde{A}_{1\ldots n}(\phi_h, \phi_S, \theta_S) \equiv \sin\theta_S A_{1\ldots n}(\phi_h, \phi_S) \tag{4.37}$$

$$\widetilde{A}_{n+1}(\phi_h, \phi_S, \theta_S) \equiv \cos\theta_S \tag{4.38}$$

$$\widetilde{A}_{n+2}(\phi_h, \phi_S, \theta_S) \equiv \cos\theta_S \cos\phi_h \tag{4.39}$$

Defining

$$\sum\left[hP\widetilde{A}_i\right] \equiv \sum_{\text{ev}} h \cdot P_{i_{T/V\pm}} \widetilde{A}_i(\phi_h, \phi_S, \theta_S) \tag{4.40}$$

$$\sum\left[P^2\widetilde{A}_i\widetilde{A}_j\right] \equiv \sum_{\text{ev}} P^2_{i_{T/V\pm}} \sin^2\theta_S \widetilde{A}_i(\phi_h, \phi_S, \theta_S)\widetilde{A}_j(\phi_h, \phi_S, \theta_S) \tag{4.41}$$

$$\mathbf{F}^{(n+2)}_{i,j} \equiv \sum\left[P^2\widetilde{A}_i\widetilde{A}_j\right] - NA_{CP,i}A_{CP,j} \tag{4.42}$$

$$\boldsymbol{B}^{(n+2)}_{i} \equiv \sum\left[hP\widetilde{A}_i\right] - NA_{CP,i} \tag{4.43}$$

it's easy to show the estimator are solution of following equation

$$\mathbf{F}^{(n+2)} \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \\ \epsilon_{LL} \\ \epsilon_{LL}^{\cos\phi_h} \end{pmatrix} = \boldsymbol{B}^{(n+2)} \tag{4.44}$$

or

$$\begin{pmatrix} \hat{\epsilon}_1 \\ \vdots \\ \hat{\epsilon}_n \\ \hat{\epsilon}_{LL} \\ \hat{\epsilon}_{LL}^{\cos\phi_h} \end{pmatrix} = \left(\mathbf{F}^{(n+2)}\right)^{-1} \boldsymbol{B}^{(n+2)} \tag{4.45}$$

with covariance matrix of

$$\mathbf{V}\begin{pmatrix}\hat{\epsilon}_1\\ \vdots\\ \hat{\epsilon}_n\\ \hat{\epsilon}_{LL}\\ \hat{\epsilon}_{LL}^{\cos\phi_h}\end{pmatrix} = \left(\mathbf{F}^{(n+2)}\right)^{-1} \tag{4.46}$$

In case that ϵ_{LL} and $\epsilon_{LL}^{\cos\phi_h}$ are known from other experiment, which is assumption of last section, only the first n equation of $n+2$ equation set 4.44 is useful. Therefore, it reduces to a n(equation) $\times$ n(variable) format, with index $i = 1 \cdots n$:

$$\sum_{j=1}^{n}\mathbf{F}_{i,j}^{(n+2)}\epsilon_j + \epsilon_{LL}\mathbf{F}_{i,n+1}^{(n+2)} + \epsilon_{LL}^{\cos\phi_h}\mathbf{F}_{i,n+2}^{(n+2)} = \boldsymbol{B}_i^{(n+2)} \tag{4.47}$$

Solution of this equation is exactly Eq. 4.31, as we expected.

4.3.1.3 Rough Estimation of Longitudinal Modulation from World Data

It's more likely that transversity data would not provide a better information on longitudinal modulations than world data. Therefore, it's important to translate world data into ϵ_{LL} and $\epsilon_{LL}^{\cos\phi_h}$ as input to this study. By comparing equation 4.4 with Eq. 2.7 as of ref [3], we have

$$\epsilon_{LL} = -\frac{\sqrt{1-\varepsilon^2}F_{LL}}{F_{UU,T}+\varepsilon F_{UU,L}} \tag{4.48}$$

$$\epsilon_{LL}^{\cos\phi_h} = -\frac{\sqrt{2\varepsilon\,(1-\varepsilon)}F_{LL}^{\cos\phi_h}}{F_{UU,T}+\varepsilon F_{UU,L}} \tag{4.49}$$

where to leading order

$$F_{UU,T} = \mathcal{C}\left[f_1 D_1\right] \tag{4.50}$$

$$F_{UU,L} = 0 \tag{4.51}$$

$$F_{LL} = \mathcal{C}\left[g_{1L} D_1\right] \tag{4.52}$$

$$F_{LL}^{\cos\phi_h} = \frac{2M}{Q}\mathcal{C}\left[\ldots\right] \tag{4.53}$$

$F_{LL}^{\cos\phi_h}$ is higher twist term. $\mathcal{C}\left[f\,D\right]$ is sum over quark species and convolution over transverse $\boldsymbol{p}_T$ and $\boldsymbol{k}_T$

$$\mathcal{C}\left[f\,D\right] \equiv x\sum_{\alpha} e_\alpha^2 \int d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T \delta^2\left(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{p}_{h\perp}/z\right) f^\alpha(x,\boldsymbol{p}_T^2) D^\alpha(z,\boldsymbol{k}_T^2) \tag{4.54}$$

Therefore, $\epsilon_{LL}^{\cos\phi_h}$ is assumed to be 0 for this section. Besides, consider the leading contribution of $D_1^{\pi^\pm}$ is valance u/d quark in transversity, we ignore other quark species. Further by assuming a Gaussian like $\boldsymbol{p}_T/\boldsymbol{k}_T$ distribution

$$D_1(z,\boldsymbol{k}_T^2) = D_1(z)\frac{1}{\pi\mu_D^2}\exp\left(-\frac{\boldsymbol{k}_T^2}{\mu_D^2}\right) \tag{4.55}$$

$$f_1(x,\boldsymbol{p}_T^2) = f_1(x)\frac{1}{\pi\mu_0^2}\exp\left(-\frac{\boldsymbol{p}_T^2}{\mu_0^2}\right) \tag{4.56}$$

$$g_1(x,\boldsymbol{p}_T^2) = g_1(x)\frac{1}{\pi\mu_1^2}\exp\left(-\frac{\boldsymbol{p}_T^2}{\mu_1^2}\right) \tag{4.57}$$

and assumption of

$$\mu_D^2 \approx \mu_0^2 \approx \mu_1^2 \tag{4.58}$$

we have

$$\begin{aligned}
\epsilon_{LL} &= -\sqrt{1-\varepsilon^2}\frac{\mathcal{C}\left[g_{1L}D_1\right]}{\mathcal{C}\left[f_1D_1\right]} \\
&= -\sqrt{1-\varepsilon^2}\frac{\sum_\alpha e_\alpha^2 g_1^\alpha(x)D^\alpha(z)\int d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\delta^2\left(\boldsymbol{p}_T-\boldsymbol{k}_T-\boldsymbol{p}_{h\perp}/z\right)\frac{1}{\pi\mu_D^2}\exp\left(-\frac{\boldsymbol{k}_T^2}{\mu_D^2}\right)\frac{1}{\pi\mu_1^2}\exp\left(-\frac{\boldsymbol{p}_T^2}{\mu_1^2}\right)}{\sum_\alpha e_\alpha^2 f_1^\alpha(x)D^\alpha(z)\int d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\delta^2\left(\boldsymbol{p}_T-\boldsymbol{k}_T-\boldsymbol{p}_{h\perp}/z\right)\frac{1}{\pi\mu_D^2}\exp\left(-\frac{\boldsymbol{k}_T^2}{\mu_D^2}\right)\frac{1}{\pi\mu_0^2}\exp\left(-\frac{\boldsymbol{p}_T^2}{\mu_0^2}\right)} \\
&\approx -\sqrt{1-\varepsilon^2}\frac{\Delta u}{u}(\text{or }\frac{\Delta d}{d}) \qquad (4.59)\\
\epsilon_{LL}^{\cos\phi_h} &= O\left(\frac{M}{Q}\right)\sim 0 \qquad (4.60)
\end{aligned}$$

where $\frac{\Delta u}{u}$ and $\frac{\Delta d}{d}$ could be extracted from fitting of world data[6].

4.3.2 Asymmetry of Effective Charge

Asymmetry of effective charge is generally small between helicity states. This bring two effects

Correction on final modulation due to effective charge asymmetry ($A_{CP,\,i}$) is small. Real data show overall $A_{CP,\,i} \lesssim PPM$

Extra Bias Due to Yield Drift is tiny in DSA case. Referring to discussion on section 2.2.3, extra yield drift dependence on MLE than local pair method is proportional to effective charge asymmetry. Upper limit of the systematic bias could be estimated by

$$\begin{aligned}
\text{bias} &\leq (\text{Max Yield Drift})\times(\text{Local Charge Asymmetry}) &(4.61)\\
&\sim 10\%\times 1000\text{PPM} &(4.62)\\
&\sim 100\text{PPM} &(4.63)
\end{aligned}$$

which is far negligible relative to uncertainties of coincidence asymmetry.

4.3.3 Higher Twist $F_{LU}^{\sin\phi_h}$ term

There is an other helicity dependent term $F_{LU}^{\sin\phi_h}$ as of Eq. 2.7 in ref [3]. Although it's a higher twist term, it do not flip sign of asymmetry when spin flips (all other helicity dependent term flips). Therefore, we have another clean channel to separate this term.

Chapter 5

Discussion and Conclusions

5.1 Application and Comparison

There are two major steps in Transversity asymmetry extraction:

1. combining data from spin states
2. extract angular modulation

It's possible to use either MLE or Blue Team's method[2] of each of the steps. Therefore useful combinations besides Blue Team's notes are listed below.

5.1.1 MLE for both steps

Directly use Eq. 3.36 for estimator and Eq. 3.39 for uncertainty estimation. There are potential higher systematics due to detector efficiency drift as discussed in section 2.2.3. It's probably best method for Kaon case, whose statistics are very limited. Also for MLE is also great for DSA analysis, since MLE became unbiased to yield drift due to small charge asymmetries.

5.1.2 MLE for step 1, Fitting for step 2

First divide data into 2D angular bins, then for each bin, use Eq. 2.7 and 2.14 to get asymmetry for each bin by summing over spin states. Then perform a 2D angular fitting to extract angular modulations. It's doable for pion case, could serve as a check/alternative to Blue Team's method.

5.1.3 MLE for step 2 then perform local pair sum

For each super local spin pairs, extract it's angular modulation with Eq. 3.36 and 3.39 (set spin state $=1\pm$). Then do χ^2 fit modulations over all pairs. There is potential danger of low count in some pair, leaving a point with very large and unreliable error bar.

5.1.4 possible local pair based MLE

There is possibility to form a local pair based MLE method, which similar to 5.1.1, but preserve the structure of local pair. This method is expected to be as robust as blue team local pair method, as well as features MLE reliability at low statistics. One naive way to construct is as following:

Reevaluating Eq. 3.35, and applying it to each local pair. Then there will be N_{pair} equations with matrix form, call them

$$\mathbf{F}_i \epsilon = \mathbf{B}_i \tag{5.1}$$

Then add them together with coefficient

$$w_i = \frac{1}{1/\tilde{C}_{i_+} + 1/\tilde{C}_{i_-}} \tag{5.2}$$

and we have one matrix equation

$$\left(\sum_i w_i \mathbf{F}_i \right) \epsilon = \sum_i w_i \mathbf{B}_i \tag{5.3}$$

or

$$\hat{\epsilon} = \left(\sum_i w_i \mathbf{F}_i \right)^{-1} \left(\sum_i w_i \mathbf{B}_i \right) \tag{5.4}$$

Similar to the one on Blue Team's method[2], this equation features same resistance to yield drift[1], while remain free of angular binning. However, a mathematical study is necessary to perfect this idea.

5.2 Acceptance

5.2.1 Why Acceptance is in MLE estimators?

Acceptance is shown in final modulation results of both SSA (Eq. 3.36) and DSA (Eq. 4.31). It appear in asymmetry correction terms (Eq. 3.24 and Eq. 4.19), which take a general form of

$$A_{CP} \sim \frac{\iiint a_{T/V\pm}(\phi_h, \phi_S, \theta_S) A_i(\phi_h, \phi_S) d\phi_h d\phi_S d\theta_S}{\iiint a_{T/V\pm}(\phi_h, \phi_S, \theta_S) d\phi_h d\phi_S d\theta_S} \times \frac{\widetilde{C}_+ - \widetilde{C}_-}{\widetilde{C}_+ + \widetilde{C}_-} \tag{5.5}$$

where $A_i(\phi_h, \phi_S)$ is SIDIS angular modulation and $\frac{\widetilde{C}_+ - \widetilde{C}_-}{\widetilde{C}_+ + \widetilde{C}_-}$ is asymmetry of effective charge. It suggests

- In case that there is no asymmetry of effective charge between $\pm$ states, acceptance will not affect our result.
- In case that acceptance is symmetric, $A_{CP} = 0$ independent of asymmetry of effective charge. In another word, angular modulation is not extracted by comparing between spin states, rather than comparing data between opposite parts of acceptance.

Following is an naive explanation, why acceptance is in the MLE estimator, while local pair method is free of it:

[1] To be more precise, bias of this method is 0 if yield remain constant with in each spin pair (although it is allowed to drift from pair to pair).

- if the charge asymmetry = 0, or there are comparable amount of data from both spin direction, the MLE would compare them, form asymmetry and give an modulation on the asymmetry.
- In case there are difference charge between spin states, local pair method, would scale down weight of larger state and form an asymmetry between ± then get an modulation without worrying about acceptance. However, effectively, MLE splits larger state in two parts: one of them paired with smaller state and do the same as local pair method. For rest part, MLE compare it's yield with acceptance, which is suppose by MLE to be well known parameter. Final MLE result are combination this two parts.

5.2.2 How to Get Acceptances, to What Precision?

Since our effective charge asymmetry is small ($< 1\%$ for SSA and PPM level for DSA), precision required on acceptance is also suppress by same ratio. Therefore, the requirement is loose, considering we are expecting order of magnitude of ~1% level statistics precision.

To extract acceptance, we can bin data into angular bins and sum over equal amount (effective charge) of ± states. As shown in Eq. 1.1, the distribution of the sum is equivalent to

$$sum = \langle y_+(\phi_h, \phi_S) + y_-(\phi_h, \phi_S + \pi)\rangle_{helicity} \propto a_{T/V+}(\phi_h, \phi_S) \tag{5.6}$$

Besides, since only relative angular modulated terms are shown in expression, it's also possible to extract the ratio using event sums,

$$\frac{\iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) A_i(\phi_h, \phi_S) d\phi_h d\phi_S d\theta_S}{\iiint a_{T/V+}(\phi_h, \phi_S, \theta_S) d\phi_h d\phi_S d\theta_S} \sim \frac{\sum_{ev} A_i(\phi_h, \phi_S)}{\sum_{ev} N} \tag{5.7}$$

which will be further explained in following sections. Uncertainty in this study is comparable to extraction of yield, which means usually much more precise than asymmetries.

5.2.3 Acceptance Estimation with Event Sum

As suggested by last section, it's possible to estimate relative angular modulation from event sum method. Advantage of this method is that it's using similar method and data structure as those in MLE, which simplifies analysis procedures. In this section we will discuss how to estimate SSA acceptance related term $A_{CP,i}$ as of Eq. 3.24 and DSA acceptance related term $A_{CP,j}$, $A_{CP}^{\cos\theta_S}$ and $A_{CP}^{\cos\theta_S \cos\phi_h}$ as of Eq. 4.19.

5.2.3.1 SSA Acceptance Estimation

Starting with Eq. 3.24, although acceptance $a_{T+}(\phi_h, \phi_S)$ is no necessarily equal to $a_{V+}(\phi_h, \phi_S + \pi/2)$ (due to virtual photon is not exactly transverse to $\vec{q}$), it's fair to assume that

$$\iint a_{V+}(\phi_h, \phi_S) d\phi_h d\phi_S = \iint a_{T+}(\phi_h, \phi_S) d\phi_h d\phi_S \tag{5.8}$$

or

$$\tilde{a} \equiv \tilde{a}_T = \tilde{a}_V \tag{5.9}$$

As a result, Eq. 3.24 is simplified to

$$A_{CP,i} \equiv \frac{\left(\widetilde{C}_{P,T+} - \widetilde{C}_{P,T-}\right)(\tilde{a}_{T,i}/\tilde{a}) + \left(\widetilde{C}_{P,V+} - \widetilde{C}_{P,V-}\right)(\tilde{a}_{V,i}/\tilde{a})}{\left(\widetilde{C}_{T+} + \widetilde{C}_{T-}\right) + \left(\widetilde{C}_{V+} + \widetilde{C}_{V-}\right)} \tag{5.10}$$

Then by checking Eq. 3.1, relative angular modulation $\tilde{a}_{T/V,i}/\tilde{a}$ is estimated by

$$\begin{aligned}
\frac{\tilde{a}_{T/V,i}}{\tilde{a}} &\equiv \frac{\iint a_{T/V+}(\phi_h,\phi_S)A_i(\phi_h,\phi_S)d\phi_h d\phi_S}{\iint a_{T/V+}(\phi_h,\phi_S)d\phi_h d\phi_S} \\
&= \frac{\iint \left(y_{T/V+}(\phi_h,\phi_S) + y_{T/V-}(\phi_h,\phi_S+\pi)\right) A_i(\phi_h,\phi_S)d\phi_h d\phi_S}{\iint \left(y_{T/V+}(\phi_h,\phi_S) + y_{T/V-}(\phi_h,\phi_S+\pi)\right) d\phi_h d\phi_S} \\
&= \frac{\iint y_+(\phi_h,\phi_S)A_i(\phi_h,\phi_S)d\phi_h d\phi_S - \iint y_-(\phi_h,\phi_S)A_i(\phi_h,\phi_S)d\phi_h d\phi_S}{\iint y_+(\phi_h,\phi_S)d\phi_h d\phi_S + \iint y_-(\phi_h,\phi_S)d\phi_h d\phi_S} \\
\text{Estimate}\left[\frac{\tilde{a}_{T/Vi}}{\tilde{a}}\right] &= \frac{\sum_{\text{ev},T/V+} A_i(\phi_h,\phi_S)/\widetilde{C}_{T/V+} - \sum_{\text{ev},T/V-} A_i(\phi_h,\phi_S)/\widetilde{C}_{T/V-}}{N_{T/V+}/\widetilde{C}_{T/V+} + N_{T/V-}/\widetilde{C}_{T/V-}}
\end{aligned} \tag{5.11}$$

Where $N/\widetilde{C}_{T/V+}$ are sums of event numbers or effective charges as defined in Eq. 1.15, $\sum_{\text{ev},T/V\pm}$ is defined as sum over event in a specific spin direction ($T/V\pm$).

5.2.3.2 DSA Acceptance Estimation

Similar as SSA case, we follow Eq. 4.19 and assume $\tilde{a} \equiv \tilde{a}_V = \tilde{a}_T$, which is reasonable due to spin rotations and flips. Therefore,

$$\begin{aligned}
A_{CP,j} &\equiv \frac{\sum_{T/V\pm} \pm\tilde{a}^{\sin\theta_S}_{T/V,j}\left(\widetilde{C}_{P,+,T/V\pm} - \widetilde{C}_{P,-,T/V\pm}\right)}{\sum_{T/V\pm} \tilde{a}_{T/V}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)} \\
&= \frac{\sum_{T/V\pm} \pm\left(\widetilde{C}_{P,+,T/V\pm} - \widetilde{C}_{P,-,T/V\pm}\right)\left(\tilde{a}^{\sin\theta_S}_{T/V,j}/\tilde{a}\right)}{\sum_{T/V\pm}\left(\widetilde{C}_{+,T/V\pm} + \widetilde{C}_{-,T/V\pm}\right)}
\end{aligned} \tag{5.12}$$

where $\tilde{a}^{\sin\theta_S}_{T/V,j}/\tilde{a}$ could be calculated with Eq. 5.11.

5.3 To Be Further Studied

- Yield drift correction as discussed in section 2.2.3
- Removing angular dependent background
- Simulations to evaluate effect of efficiency drift on bias
- Weighted sum with charge correction

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