$\gamma\gamma \to \pi^0\pi^0$

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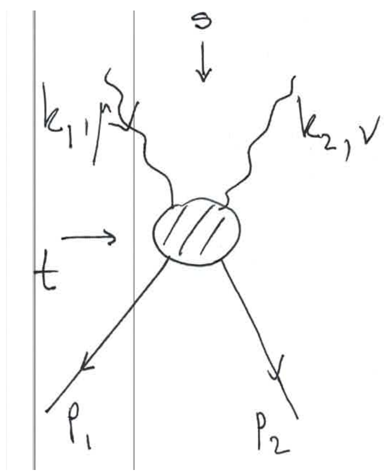
1 The amplitude $\gamma^{(*)}\gamma^{(*)} \to \pi^0\pi^0$



The relevant tensor is:

$$V_{\mu\nu} \equiv= \langle p_1, p_2 \mid T(J_\mu(x)J_\nu(y)) \mid 0\rangle \tag{1}$$

where J_μ is the EM current. Fourier transforming in x and y with momenta k_1 and k_2 respectively, we can write the most general form for $V_{\mu\nu}$ which respects all symmetries:

$$V_{\mu\nu} = \sum_{i=1}^{5} A_i(s,t,u)T^i_{\mu\nu} \tag{2}$$

where $s,\ t,\ u$ are Mandelstam invariants and the tensor basis which respects gauge invariance is:

$$\begin{aligned}
T^1_{\mu\nu} &= k_{1\,\nu}\, k_{2\,\mu} - g_{\mu\nu} k_1\cdot k_2 \\
T^2_{\mu\nu} &= k_{1\,\mu}\, k_{1\,\nu} - g_{\mu\nu} k_1^2 + \frac{1}{k_2\cdot P}(k_{2\,\mu}\, k_1^2 - k_{1\,\mu}\, k_1\cdot k_2) \\
T^3_{\mu\nu} &= k_{2\,\mu}\, k_{2\,\nu} - g_{\mu\nu} k_2^2 + \frac{1}{k_1\cdot P}(k_{1\,\nu}\, k_2^2 - k_{2\,\nu}\, k_1\cdot k_2) \\
T^4_{\mu\nu} &= P_\mu P_\nu - \frac{1}{k_1\cdot k_2}(k_{2\,\mu}\, P_\nu k_1\cdot P + k_{1\,\nu}\, P_\mu k_2\cdot P - g_{\mu\nu} k_1\cdot P k_2\cdot P) \\
T^5_{\mu\nu} &= k_{1\,\mu} k_{2\,\nu} - \frac{1}{k_1\cdot k_2}(k_1^2 k_{2\,\mu}\, k_{2\,\nu} + k_2^2 k_{1\,\mu}\, k_{1\,\nu} - g_{\mu\nu} k_1^2 k_2^2)
\end{aligned} \tag{3}$$

with $P = p_1 - p_2$, we have:

$$\begin{aligned}
k_1\cdot k_2 &= \frac{s}{2} - k_1^2 - k_2^2 \\
k_1\cdot P &= \frac{1}{2}(u - t + p_1^2 - p_2^2) \\
k_2\cdot P &= -\frac{1}{2}(u - t + p_2^2 - p_1^2)
\end{aligned} \tag{4}$$

In the case $p_1^2 = p_2^2$, $k_1\cdot P = -k_2\cdot P = \frac{1}{2}(u-t)$.

Bose symmetry requires that:

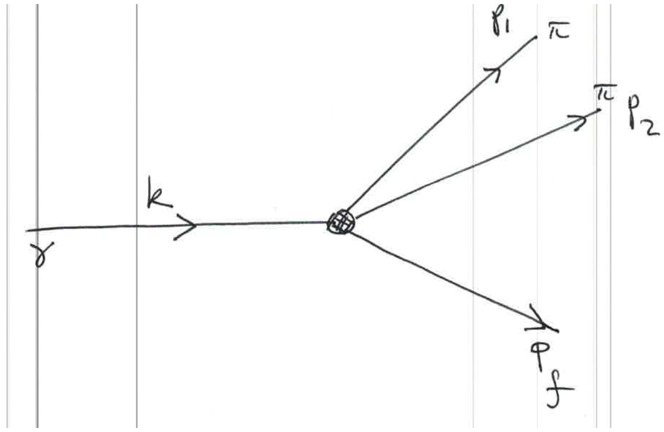
$$\begin{aligned}
T_{\mu\nu}(P, k_1, k_2) &= T_{\mu\nu}(-P, k_1, k_2) \\
&= T_{\nu\mu}(P, k_2, k_1)
\end{aligned} \tag{5}$$

which corresponds also to the exchange $u \leftrightarrow t$. This then implies that:

$$\begin{aligned}
A_2(s,t,u) &= A_3(s,u,t) \\
A_i(s,t,u) &= A_i(s,u,t) \quad i = 1,4,5
\end{aligned} \tag{6}$$

2 $\pi^0\pi^0$ photoproduction

2.1 Kinematics in Lab frame



Definitions:

$$\omega = |\vec{k}|$$

$$\begin{aligned} \vec{p}_\pm &= \vec{p}_1 \pm \vec{p}_2, \quad \mathsf{p}_\pm = |\vec{p}_\pm| \\ \vec{p}_f &= \vec{k} - \vec{p}_+, \quad E_f = \sqrt{\vec{p}_f^2 + M^2} \end{aligned} \tag{7}$$

Spherical coordinates: choose $\vec{k}$ in z direction.

$$\begin{aligned} \vec{p}_\pm &= \mathsf{p}_\pm(\sin\theta_\pm \cos\phi_\pm, \sin\theta_\pm \sin\phi_\pm, \cos\theta_\pm) \\ E_1^2 &= \frac{1}{4}(\mathsf{p}_+^2 + \mathsf{p}_-^2 + 2\mathsf{p}_+ \ \mathsf{p}_- \ \cos\alpha) + \mathsf{M}_\pi^2 \\ E_2^2 &= \frac{1}{4}(\mathsf{p}_+^2 + \mathsf{p}_-^2 - 2\mathsf{p}_+ \ \mathsf{p}_- \ \cos\alpha) + \mathsf{M}_\pi^2 \\ \cos\alpha &= \cos\theta_+ \ \cos\theta_- + \cos(\phi_+ - \phi_-) \sin\theta_+ \ \sin\theta_- \\ \vec{p}_f^{\,2} &= \mathsf{p}_+^2 + \omega^2 - 2\mathsf{p}_+ \ \omega\cos\theta_+ \end{aligned} \tag{8}$$

so that $E_1 + E_2 = \omega + M - E_f$ depends only on p_+ and θ_+.

2.2 Differential cross section

$$\begin{aligned} d\sigma &= \frac{1}{2(4\pi)^5} \frac{|\mathcal{M}|^2}{\omega M E_1 E_2 E_f} \delta(\omega + M - E_1 - E_2 - Ef) \ d^3p_+ d^3p_- \\ &= \frac{1}{2(4\pi)^5} \frac{|\mathcal{M}|^2}{\omega M E_1 E_2 E_f} \delta(\omega + M - E_1 - E_2 - Ef) \ \mathsf{p}_+^2 \mathsf{p}_-^2 d\cos\theta_+ d\cos\theta_- d\phi_+ d\phi_- d\mathsf{p}_+ d\mathsf{p}_- \end{aligned}$$

using that $\mathsf{p}_+\ \mathsf{p}_-\ \cos\alpha = \vec{p}_+\cdot\vec{p}_- = E_1^2 - E_2^2$, we obtain:

$$\delta(\omega + M - E_1 - E_2 - E_f) = 4\frac{E_1 E_2 \mathsf{p}_-}{(E_1+E_2)\mid \mathsf{p}_-^2 - (E_1-E_2)^2\mid}\delta(\mathsf{p}_- - \bar{\mathsf{p}}_-) \tag{9}$$

where

$$\bar{\mathsf{p}}_- = \frac{(E_1+E_2)\sqrt{(E_1+E_2)^2 - \mathsf{p}_+{}^2 - 4M_\pi^2)}}{\sqrt{(E_1+E_2)^2 - \mathsf{p}_+^2\cos^2\alpha}} \tag{10}$$

The diff cross section then becomes:

$$d\sigma = \frac{2}{(4\pi)^5}\ \frac{\mid \mathcal{M}\mid^2}{\omega M E_f (E_1+E_2)\mid \bar{\mathsf{p}}_-^2 - (E_1-E_2)^2\mid}\ \mathsf{p}_+^2\bar{\mathsf{p}}_-^3\ d\cos\theta_+ d\cos\theta_- d\phi_+ d\phi_- d\mathsf{p}_+ \tag{11}$$

where we can use:

$$\begin{aligned} E_1 + E_2 &= \omega + M - E_f \\ (E_1 - E_2)^2 &= (E_1+E_2)^2 - 4E_1E_2 \\ E_1E_2 &= \sqrt{M_\pi^4 + \frac{1}{2}M_\pi^2(\mathsf{p}_+^2 + \mathsf{p}_-^2) + \frac{1}{4}(\mathsf{p}_+^4 + \mathsf{p}_-^4 - \mathsf{p}_+^2\mathsf{p}_-^2\cos(2\alpha))} \end{aligned} \tag{12}$$

It is convenient to express the cross section in terms of the invariant mass squared of the two pion system:

$$W_{\pi\pi} \;=\; (E_1+E_2)^2 - \mathsf{p}_+^2 = 2(\omega^2 + M^2 + \omega M) - 2\omega\mathsf{p}_+\ \cos\theta_+ - 2(\omega+M)E_f \tag{13}$$

where $W_{\pi\pi} > 4M_\pi^2$ and

$$d\mathsf{p}_+ = \frac{E_f}{2(\mathsf{p}_+(\omega+M) - \omega(E_1+E_2)\cos\theta_+)}dW_{\pi\pi} \tag{14}$$

With some work one can replace everywhere p_+ in terms of $W_{\pi\pi}$ using Eq. (13). For this, at a given ω and θ_+, one needs that:

$$W_{\pi\pi}^2 - 4W_{\pi\pi}(M(M+\omega) + \omega^2\sin^2\theta_+) + 4M^2\omega^2 > 0 \tag{15}$$

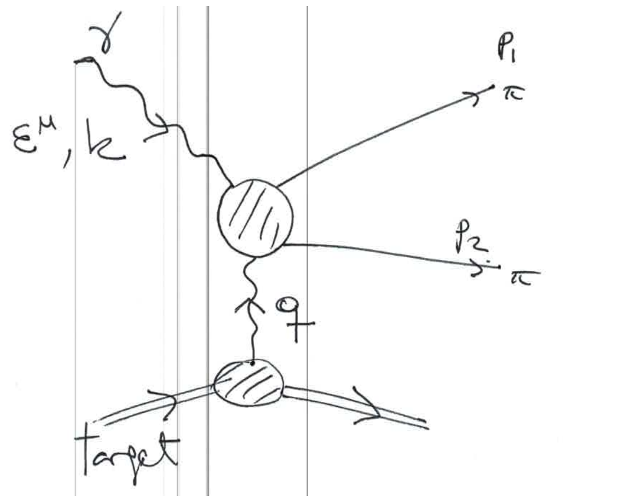
and one gets:

$$\mathsf{p}_+ = \frac{\omega\cos\theta_+(2M\omega + W_{\pi\pi}) \pm (M+\omega)\sqrt{-4M^2\left(W_{\pi\pi} - \omega^2\right) - 4MW_{\pi\pi}\omega + 2W_{\pi\pi}\omega^2\cos 2\theta_+ + W_{\pi\pi}\left(W_{\pi\pi} - 2\omega^2\right)}}{2(M+\omega)^2 - 2\omega^2\cos^2\theta_+} \tag{16}$$

The next step is to determine the physical domain of integration in the angles and $W_{\pi\pi}$. This is being worked out still.

Also, one should find which angular variables are the most convenient to use. This requires that we know in detail the scattering amplitude's angular dependencies in order to make the choice.

3 Primakoff amplitude and cross section



The scattering amplitude is given by the general expression:

$$\mathcal{M} = \epsilon^\mu T_{\mu\nu}(k, q, p_-) \frac{1}{Q^2} J^\nu \tag{17}$$

$T_{\mu\nu}$ is the Compton tensor, $Q^2 = -q^2$, and the target's EM current in the Lab frame we will neglect the spin of the target, and therefore we only care about the its charge:

$$J^\mu = g^{\mu 0} ZeF(Q^2);\ \text{note that we still need to use } q_\nu J^\nu = 0 \tag{18}$$

where $F(Q^2)$ is the charge FF of the target.

Since we are interested in the region of the Primakoff peak, first we approximate the amplitude by using the Compton tensor in the limit of real Compton scattering. This is then directly obtained from the result provided by Bellucci et al. which will be valid for the small $W_{\pi\pi}$ regime. Later I will work out a more detailed analysis where the virtuality Q^2 is also included in the Compton tensor, and we will also need to give the amplitude for intermediate values of $W_{\pi\pi}$ (works of Oller and of Pennington).

So for small Q^2 we have the Compton tensor:

$$\begin{aligned} T_{\mu\nu} &= A(W_{\pi\pi}, t, u)(\frac{1}{2} W_{\pi\pi} g_{\mu\nu} - k_\nu q_\mu) \qquad (19) \\ &+ 2B(W_{\pi\pi}, t, u)((W_{\pi\pi} - q^2) p_{-\mu} p_{-\nu} - 2(k \cdot p_-\ q_\mu p_{-\nu} + q \cdot p_- k_\nu p_{-\mu} - g_{\mu\nu} k \cdot p_-\ q \cdot p_-)) \end{aligned}$$

where $p_- = p_1 - p_2$.

The low energy theorem for Compton scattering gives the following constraints:

$$\begin{aligned} \frac{\alpha}{2M_\pi}(A + 16M_\pi^2 B)|_{W_{\pi\pi}=0, t=M_\pi^2} &= \alpha_\pi \\ -\frac{\alpha}{2M_\pi} A|_{W_{\pi\pi}=0, t=M_\pi^2} &= \beta_\pi \qquad (20) \end{aligned}$$

where α_π β_π are the electric and magnetic polarizabilities respectively.

For the functions A and B there are low energy results in ChPT (Bellucci et al) at two loops. The results are as follows:

$$\begin{aligned} A(s,t,u) &= 4\frac{G_\pi(s)}{sF_\pi^2}(s-M_\pi^2)+U_A+P_A \\ B(s,t,u) &= U_B+P_B \end{aligned} \tag{21}$$

where the functions and polynomials U and P are given in Bellucci's et al., see Appendix:

$$G_\pi(s) = -\frac{1}{(4\pi)^2}\left(1+2\frac{M_\pi^2}{s}\int_0^1\frac{dx}{x}\log(1-\frac{s}{M_\pi^2}x(1-x))\right) \tag{22}$$

Use the integral in terms of dilogarithm functions:

$$\int_0^1\frac{dx}{x}\log(1-Ux(1-x)) = -\mathrm{Li}_2\left(\frac{1}{2}\left(U-\sqrt{U-4}\sqrt{U}\right)\right)-\mathrm{Li}_2\left(\frac{1}{2}\left(U+\sqrt{U-4}\sqrt{U}\right)\right) \tag{23}$$

where in our case U must be taken to have an imaginary part $+i\epsilon$. At low energy $W_{\pi\pi} < (0.4\mathrm{GeV})^2$ the t dependence of the amplitudes A and B is very small and can be neglected. We however should later consider also the effects of $Q^2 > 0$ and check that claim.

3.1 Amplitudes A and B for simulation

We need to have a parametrization which for now gives a sufficiently realistic description for carrying out simulations.

4 Appendix A

4.1 U_A and P_A

$$\begin{aligned} U_A &= \frac{2}{sF_\pi^4}G_\pi(s)\left((s^2-M_\pi^2)J_\pi(s)+C(s)\right)+\frac{\ell_\Delta}{24\pi^2F_\pi^4}(s-M_\pi^2)J_\pi(s) \\ &+ \frac{\ell_2-5/6}{144\pi^2 sF_\pi^4}(s-4M_\pi^2)(H(s)+4(sG_\pi(s)+2M_\pi^2(\tilde{G}_\pi(s)-3\tilde{J}_\pi(s)))d_{00}^2) \\ P_A &= \frac{1}{(4\pi)^2F_\pi^4}(a_1M_\pi^2+a_2s) \end{aligned} \tag{24}$$

where the constants a_1 and a_2 need to be fitted, and:

$$\begin{aligned}
J_\pi(s) &= -\frac{1}{(4\pi)^2}\int_0^1 dx \log(1-\frac{s}{M_\pi^2}x(1-x)) = \frac{2}{(4\pi)^2}\left(1-\frac{\sqrt{4-\frac{s}{{M_\pi}^2}}\tan^{-1}\left(\frac{\sqrt{\frac{s}{{M_\pi}^2}}}{\sqrt{4-\frac{s}{{M_\pi}^2}}}\right)}{\sqrt{\frac{s}{{M_\pi}^2}}}\right) \\
\tilde{J}_\pi(s) &= J_\pi(s) - sJ'_\pi(0) \\
\tilde{G}_\pi(s) &= G_\pi(s) - sG'_\pi(0) \\
H_\pi(s) &= (s-10M_\pi^2)J_\pi(s) + 6M_\pi^2 G_\pi(s)
\end{aligned} \tag{25}$$

and:

$$\begin{aligned}
C(s) &= \frac{1}{48\pi^2}\left(2(\ell_1-\frac{4}{3})(s-2M_\pi^2)^2 + \frac{1}{3}(\ell_2-\frac{5}{6})(4s^2-8sM_\pi^2+16M_\pi^4)\right. \\
&- \left. 3M_\pi^4\ell_3 + 12M_\pi^2(s-M_\pi^2)\ell_4 - 12sM_\pi^2 + 15M_\pi^4\right) \\
d_{00}^2 &= \frac{1}{2}(3\cos^2\theta_{CM}-1)
\end{aligned} \tag{26}$$

where θ_{CM} is the $\gamma\gamma^* \to \pi\pi$ scattering angle in CM, and the low energy constants ℓ_i are known.

4.2 U_B and P_B

$$\begin{aligned}
U_B &= \frac{\ell_2-\frac{5}{6}}{288\pi^2F_\pi^4 s}H_\pi(s) \\
P_B &= \frac{b}{(4\pi F_\pi)^4}
\end{aligned} \tag{27}$$

where b is fitted.