Kinematic Fitting

Summer Lecture

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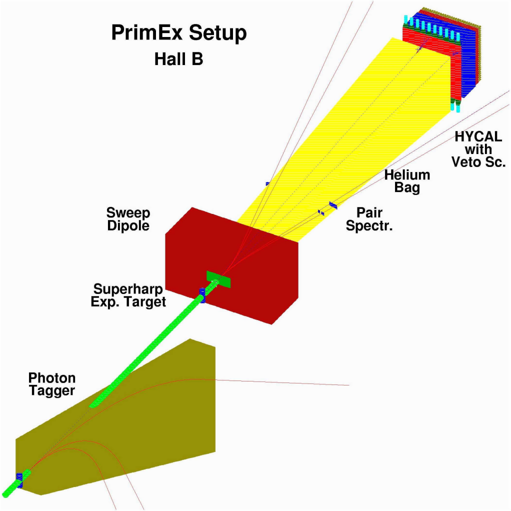
Motivation I

- Particles measured independently
- Often: known relationships between their kinematics
  - energy and momentum conservation in production and decay
    - $\sum_{i=1}^{n} E_{\mathrm{initial},i} = \sum_{i=1}^{n} E_{\mathrm{final},i}$
    - $\sum_{i=1}^{n} \vec{p}_{\mathrm{initial},i} = \sum_{i=1}^{n} \vec{p}_{\mathrm{final},i}$
  - daughters of common parent particle of known mass (Lorentz invariant)
    - $(\sum_{i=1}^{n} E_i)^2 - (\sum_{i=1}^{n} \vec{p}c)^2 = (mc^2)^2$
  - particles from a common "vertex" (spatial correlation)
    - particle trajectories start from single space-time point
- Measured kinematics do not respect these relationships in detail.
  - Exact relationships made approximate by measurement error.

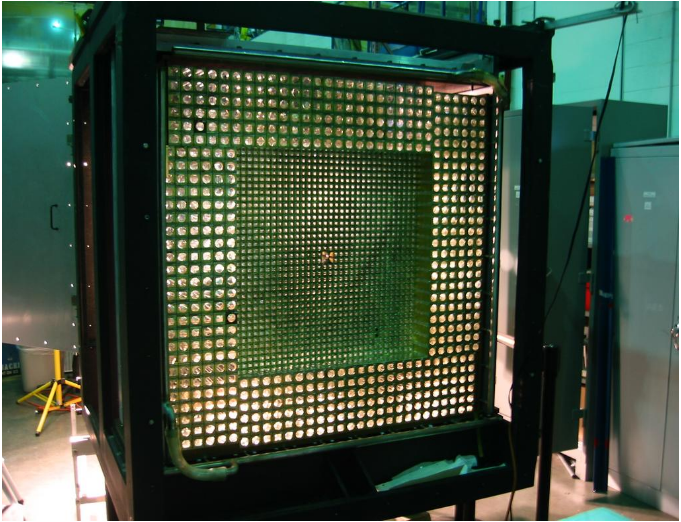
Motivation II

- How to incoporate knowledge of relationships?
  - Kinematic fitting!
- Why bother?
  - better measurements: correlations hint at direction of random measurement errors
  - hypothesis testing: procedure yields statistical assessment of probability of these measurements

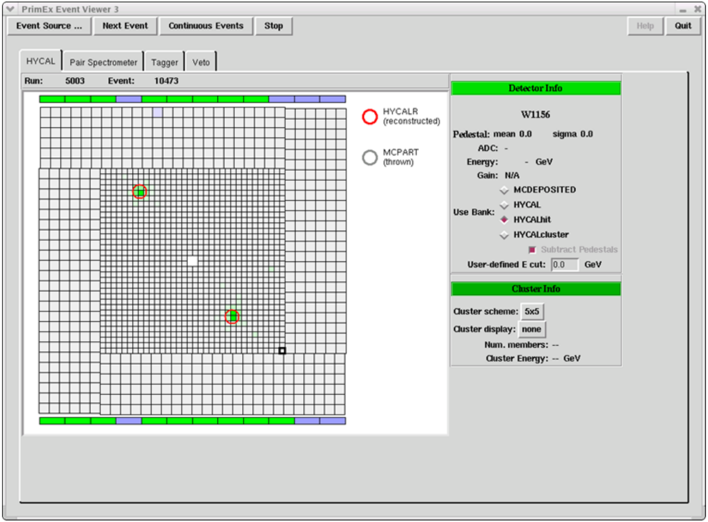
PrimEx Detector



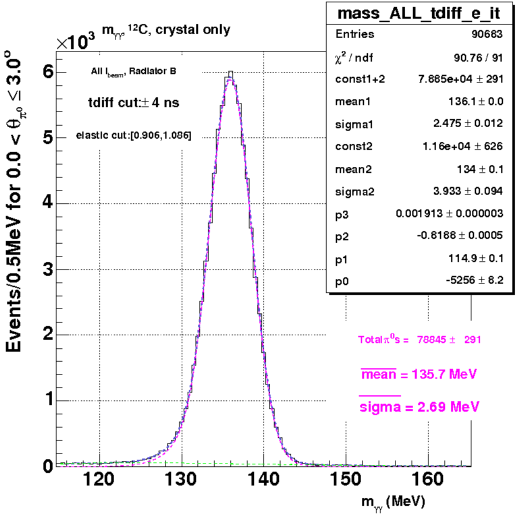
Hybrid Calorimeter



Event Display



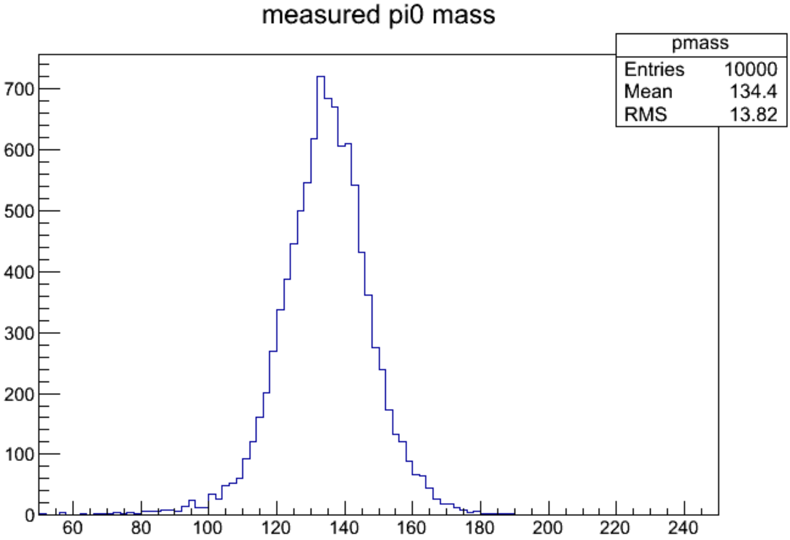
Two-photon Mass Distribution



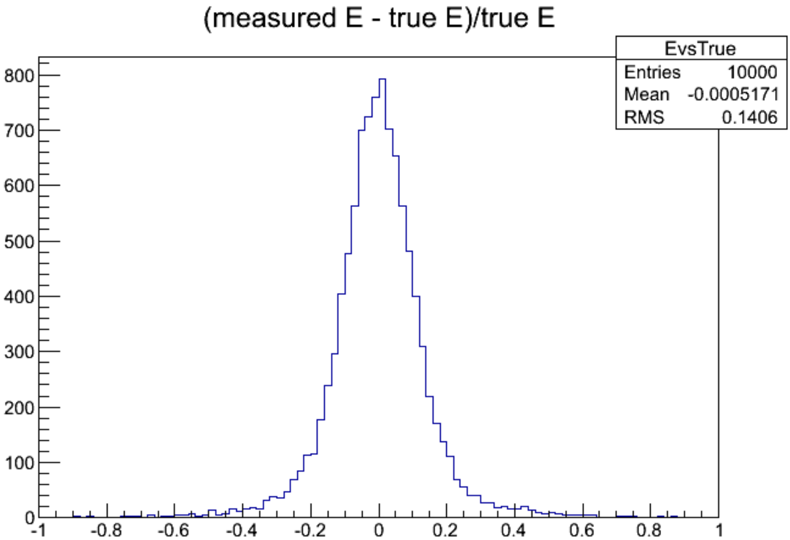
Example: Decay of Neutral Pion into Two Photons

- $\pi^0 \to \gamma\gamma$
- both photons detected
- assume photon directions are known precisely
- energies have relative uncertainty $\sigma_E/E = 5\%/\sqrt{E}$
- for simplicity: look at 500 MeV/c π^0 moving in z-direction
- but, will not assume 500 MeV/c nor z-direction for pion

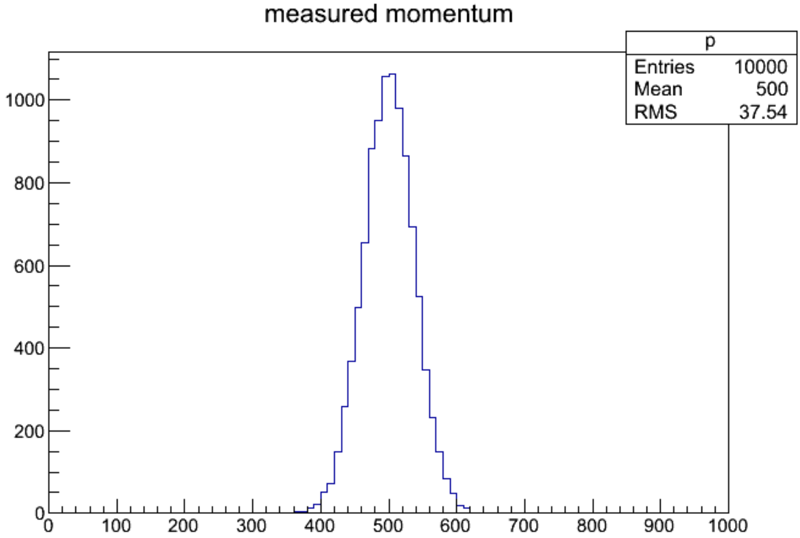
two-photon invariant mass



relative error on single photon energy



two-photon measured momentum



What is the problem?

- want to improve resolution
- assume that photons came from π^0, so mass is known
- can we use this information?
  - could adjust one photon (why just one?)
  - could scale them both (high energy γ: better E measurement)
- could minimize

$$\chi^2 = \left(\frac{E_{1,\mathrm{fit}} - E_{1,\mathrm{meas}}}{\sigma_1}\right)^2 + \left(\frac{E_{2,\mathrm{fit}} - E_{2,\mathrm{meas}}}{\sigma_2}\right)^2$$

- but minimum is clear: $E_{\mathrm{fit}} = E_{\mathrm{meas}}$ (something is missing!)
- must introduce constraint: $(k_1 + k_2)^2 = m_\pi^2$ gives

$$2E_1E_2(1 - \cos\theta) = m_\pi^2$$

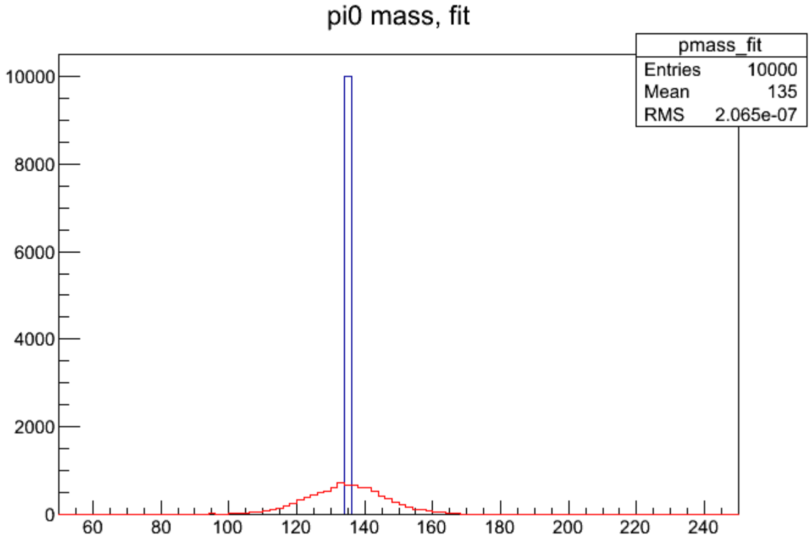
- Problem: minimize χ^2 while simultaneously satisfying constraint

Minimization Strategy

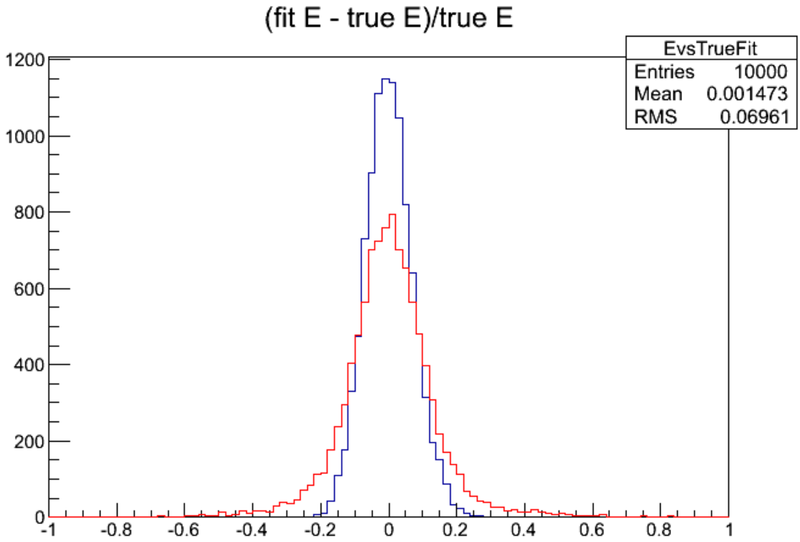
- multi-variable minimization with constraints: Lagrange multipliers
- instead of minimizing over two variables, minimize over three $E_{1,\mathrm{fit}}$, $E_{2,\mathrm{fit}}$, and λ

$$\chi^2 = \left(\frac{E_{1,\mathrm{fit}} - E_{1,\mathrm{meas}}}{\sigma_1}\right)^2 + \left(\frac{E_{2,\mathrm{fit}} - E_{2,\mathrm{meas}}}{\sigma_2}\right)^2$$
$$+2\lambda\left[2E_{1,\mathrm{fit}}E_{2,\mathrm{fit}}(1-\cos\theta) - m_\pi^2\right]$$

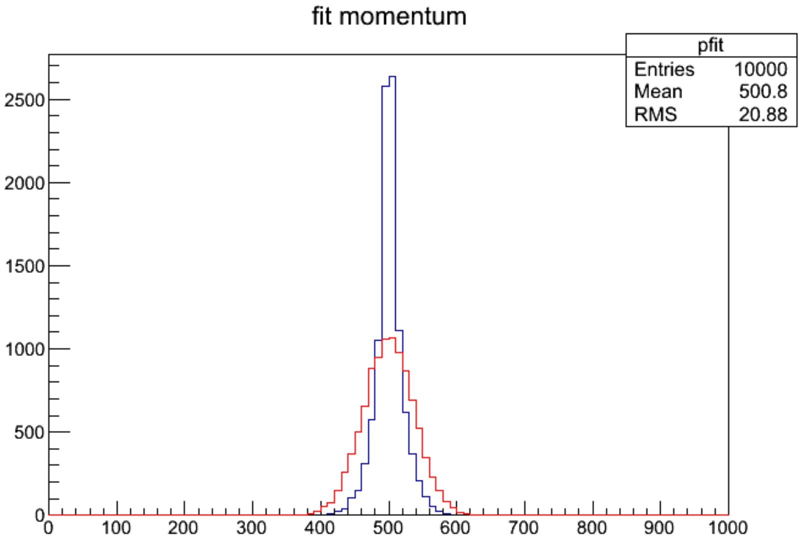
fit two-photon invariant mass



fit relative error on single photon energy



fit two-photon measured momentum



Variable Definitions I

- measured variables

  N number of measured variables (input)
  y vector of measurements, N-dimensional (input)
  V covariance matrix, $N \times N$ (input)
  η vector of fit values of measured variables, N-dimensional

$$\chi^2 = (y - \eta)^T V^{-1} (y - \eta)$$

- unmeasured variables

  J number of unmeasured variables (input)
  ξ vector of unmeasured variable values, J-dimensional

Variable Definitions II

- constraints

K number of constraints (input)
f vector of constraint functions, K dimensional (input)

Each constraint a function of measured and unmeasured variables. When constraint is satisfied

$$f_k(\eta_1, \ldots, \eta_N, \xi_1, \ldots, \xi_J) = 0 \quad \text{for} \quad k = 1, \ldots, K$$

- Lagrange multipliers

λ vector of mulipliers, K-dimensional

Extended χ^2 to be minimized:

$$\chi^2(\eta, \xi, \lambda) = (y - \eta)^T V^{-1}(y - \eta) + 2\lambda^T f(\eta, \xi)$$

Minimization Condition

Set all partial derivatives to zero:

$$\frac{\partial \chi^2}{\partial \eta_n} = \left[-2V^{-1}(y-\eta) + 2F_\eta^T \lambda \right]_n = 0, \quad n = 1, \ldots, N$$

$$\frac{\partial \chi^2}{\partial \xi_j} = \left[2F_\xi^T \lambda \right]_j = 0, \quad j = 1, \ldots, J$$

$$\frac{\partial \chi^2}{\partial \lambda_k} = [2f]_k = 0, \quad k = 1, \ldots, K$$

where

$$(F_\eta)_{kn} = \frac{\partial f_k}{\eta_n} \quad \text{and} \quad (F_\xi)_{kj} = \frac{\partial f_k}{\xi_j}$$

In general, a system of non-linear equations, $N + J + K$ equations with $N + J + K$ unknowns.

Linearize the Constraints; Iterate the Solutions

Go to a linear form of the constraints:

1. Change variables: $\eta = \eta_0 + \delta\eta$, $\xi = \xi_0 + \delta\xi$, and $\lambda = \lambda_0 + \delta\lambda$
2. Take first order in Taylor expansion:
   $f(\delta\eta, \delta\xi, \delta\lambda) = f(\eta_0, \xi_0, \lambda_0) + (F_\eta)(\delta\eta) + (F_\xi)(\delta\xi)$

Now have a linear system of $N + J + K$ equations with $N + J + K$ unknowns, i. e. $\delta\eta$, $\delta\xi$, $\delta\lambda$

- Unique solution!
- But an approximation was made: need to iterate
- Choose new η_0, ξ_0, λ_0 based on $\delta\eta$, $\delta\xi$, $\delta\lambda$.
- After a few iterations, size of $\delta\eta$, $\delta\xi$, $\delta\lambda$ get small, change in χ^2 gets small.
- Also: must initialize variables

Errors

covariance matrices:

- fit variables

$$V_\eta = V\left[I - (G - HUH^T)V\right]$$

- unmeasured variables

$$V_\xi = U$$

- covariances between fit and unmeasured variables

$$V_{\eta,\xi} = -VHU$$

where $G = F_\eta^T S^{-1} F_\eta, \quad H = F_\eta^T S^{-1} F_\xi, \quad U^{-1} = F_\xi^T S^{-1} F_\xi$
and $S = F_\eta^T V F_\eta$

Stretch Functions or "Pulls"

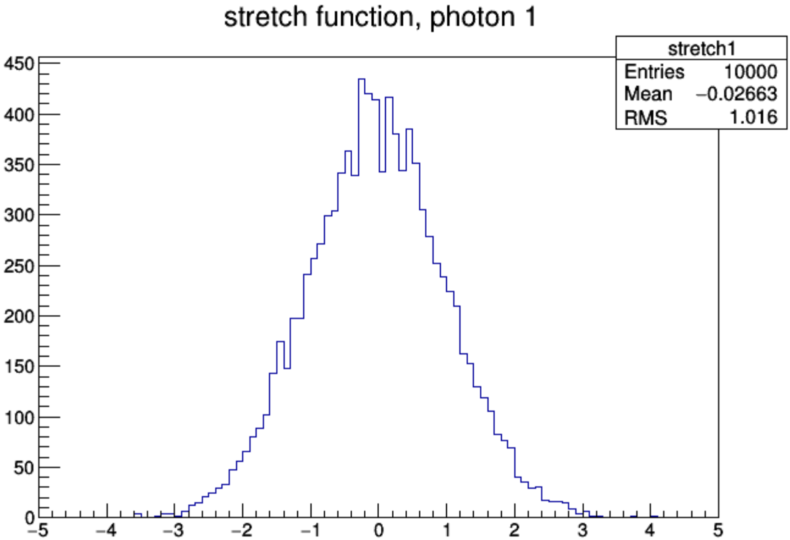
How to tell if the thing is working?

- look at use these N quantities:

$$z_n = \frac{y_n - \eta_n}{\sqrt{\sigma^2(y_n) - \sigma^2(\eta_n)}} \quad n = 1, \ldots, N$$

- Gaussian with mean at 0, σ of 1
- If not there are problems:
  - offset mean: measurements biased
  - wrong width: errors not correct
  - tails: non-Gaussian tails in measurements, background in sample

stretch function



Unmeasured Variables, Number of Constraints

- go back to π^0 decay
- could have introduced unmeasured variables: $p_{\pi,x}$, $p_{\pi,y}$, $p_{\pi,z}$
- but then would have to apply 3-momentum conservation
- now have 4 constraints with 3 unmeasured variables
- used to have 1 constraint with 0 unmeasured variables
- same problem recast: 1-C fit
- $C = K - J$, the number of degrees of freedom

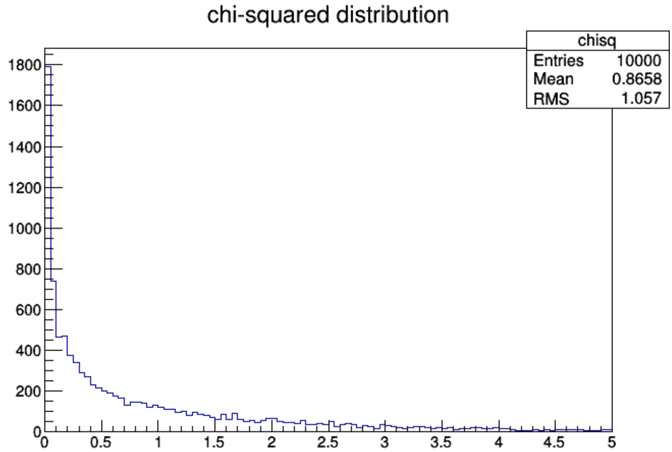
Check χ^2

- χ^2 should have a standard probablility density distribution: $f(\chi^2)$
- Convenient to check χ^2 probability:

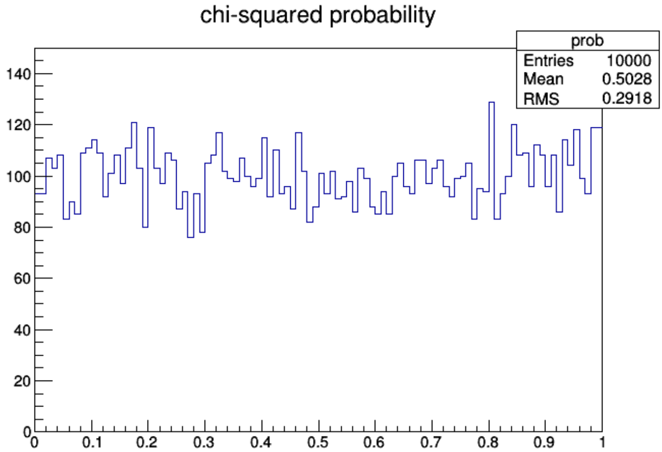
$$P(\chi_0^2) = \int_{\chi_0^2}^{\infty} f(\chi^2) d\chi^2$$

- P runs from 0 to 1
- for nominal χ^2 distribution P will be uniform
- non-uniformity: problem with errors, check the pulls
- often see peaks near 0: bad χ^2, background in sample

χ^2 Distribution



χ^2 Probability Distribution



Summary

- measured variables, with or without statistical correlation, may have physical relationships
- kinematic fit varies values of measured quantities to satisfy relationships
- minimize χ^2 with constraints
- improved measurements
- diagnostics about bias and errors in measurements are generated
- goodness of fit a handle on correctness of physical relationships assumed

Reference: A. G. Frodesen, O. Skjeggestad, H. Tøfte. Probability and Statistics in Particle Physics. Universitetsforlaget, 1979. ISBN 82-00-01906-3