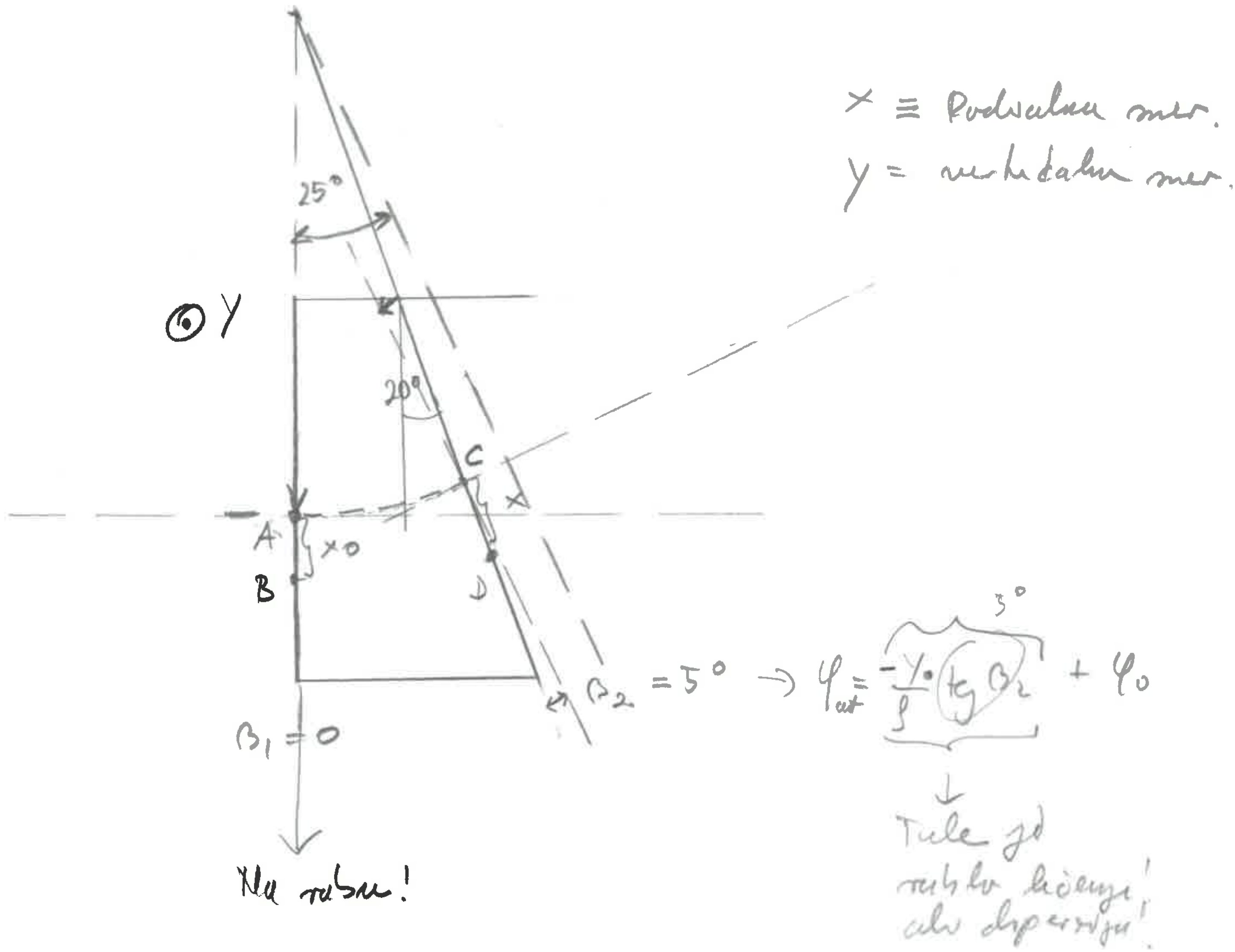
0.1412



$$y = 1 \cdot y_0 + 0$$

$$\varphi = \underbrace{-\frac{y_0}{f} \cdot tg(\beta_1)}_{\Delta\varphi} + \varphi_0 = \varphi_0$$

The thin lens approximation is usually very good, as may be seen by expanding the second of Eqs. (8) or (9) in powers of kL. The result is

$$t_i = t_e = \frac{L}{2}\left(1 + \frac{1}{12}k^2L^2 + \cdots\right)$$

for a converging lens, and

$$t_i = t_e = \frac{L}{2}\left(1 - \frac{1}{12}k^2L^2 + \cdots\right)$$

for a diverging lens. Thus the thin lens approximation holds when $k^2L^2 \ll 1$. Expanding the focal lengths given by Eqs. (8) and (9) in powers of kL we find

$$\frac{1}{f_C} = k^2L\left(1 - \frac{1}{6}k^2L^2 + \cdots\right), \quad (11)$$

and

$$\frac{1}{f_C} = -k^2L\left(1 + \frac{1}{6}k^2L^2 + \cdots\right).$$

Thus in thin lens approximation

$$f_C = -f_D = 1/k^2L. \quad (12)$$

Quadrupole Pairs

In order to obtain a system which focuses in both the xz and yz planes it is necessary to combine two (or more) quadrupole lenses. By suitable choice of the focal lengths f_1 and f_2 of two lenses and their separation, we can obtain a system which has the same focal point in both planes. However, as will be shown, the focal lengths cannot simultaneously be made equal, because the principal planes are not coincident. Conversely, we can obtain equal focal lengths in both planes by having the focal points noncoincident. Thus quadrupole systems are inherently astigmatic.

For simplicity we consider a pair of quadrupoles which are individually thin lenses, with focal lengths f_1 and f_2. The center-to-center separation is t. Then the transformation through the system is

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix},$$

or

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{bmatrix} 1 - t/f_1 & t \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{t}{f_1 f_2} & 1 - t/f_2 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}.$$

Comparison of this matrix with the right-hand side of Eq. (7) shows that the pair of lenses is equivalent to a single thick lens. The focal length F_x of the system in the xz plane is given by

$$\frac{1}{F_x} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}. \quad (13)$$

The entrance principal plane is a distance

$$t_{ix} = F_x t / f_2 \quad (14)$$

after the center of the first lens, and the exit principal plane is a distance

$$t_{ex} = F_x t / f_1 \quad (15)$$

before the center of the second lens. In thin lens approximation, the individual lens focal lengths in the yz plane are just the negatives of the xz plane values. Then the system focal length in the yz plane is given by

$$\frac{1}{F_y} = -\frac{1}{f_1} - \frac{1}{f_2} - \frac{t}{f_1 f_2}, \quad (16)$$

where f_1 and f_2 are the focal lengths of the individual lenses *in the xz plane*. From Eqs. (13) and (16) we see that F_x and F_y can simultaneously be positive. In particular $F_x = F_y$ if $f_2 = -f_1$. However, if F_x and F_y do have the same sign, the principal planes for the xz and yz planes are not coincident. From Eq. (14)

$$t_{ix} - t_{iy} = (t/f_2)(F_x + F_y).$$

A pair of quadrupoles cannot be regarded as a thin lens because from Eqs. (14) and (15)

$$t_{ix} + t_{ex} = F_x t\left(\frac{1}{f_1} + \frac{1}{f_2}\right). \quad (17)$$

For a thin lens $t_{ix} + t_{ex} = t$, which according to Eqs. (13) and (17) requires that $t/f_1 f_2 = 0$. The distance between principal planes is

$$t - (t_{ix} + t_{ex}) = -t^2 F_x / f_1 f_2,$$

which is often not negligible in practice.

III. UNIFORM FIELD BENDING MAGNETS

The geometry of a magnet which deflects the central trajectory through angle α is shown in Fig. 2. We will refer to the plane in which the bending occurs as the radial plane. The central trajectory is defined to be the trajectory which has radial plane coordinates $x=0$ and $\theta=0$ always, and which has the central momentum $p = eB\rho$. The angles β_1 and β_2 are chosen to be positive in the direction which provides positive vertical (i.e., perpendicular to radial plane) focusing, as shown in Fig. 2.

Radial Orbits

We will first calculate the coordinates (x, θ) at the magnet exit of a particle which enters the magnet with coordinates (x_0, θ_0) and momentum $p' = p + \Delta p$, for the case $\beta_1 = \beta_2 = 0$. In this case the central trajectory is a circular arc of radius ρ and length $\rho\alpha$ and the trajectory of the particle with momentum p' is an arc of radius $\rho + \Delta\rho$, where

$$\Delta p / p = \Delta\rho / \rho. \quad (18)$$

This case is illustrated in Fig. 3. In Fig. 3, AC is the arc of

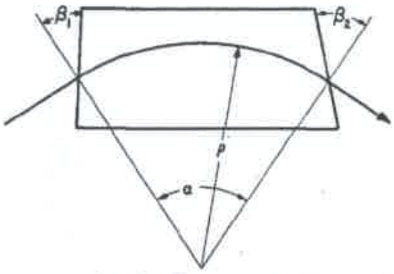


FIG. 2. Geometry of a bending magnet. The angles β_1 and β_2 between radii which intersect the pole edges at the central trajectory, and the pole edges, are chosen positive in the directions which provide positive vertical focusing.

the central trajectory, with center of curvature at 0. The entrance and exit faces of the magnet are OB and OD, and

$$OA=OC=\rho. \tag{19}$$

BD is the trajectory of the particle with momentum p', so that BD is a circular arc with its center of curvature at P;

$$BP=DP=\rho+\Delta\rho. \tag{20}$$

The initial coordinates of the latter trajectory are $x_0(=AB)$ and θ_0. The final coordinates are $x(=CD)$ and θ. We construct OQ perpendicular to OA, PQ parallel to OA, and RP perpendicular to OC. Then since $OC+CD=OR+RD$, and $RD=PD\cos\theta$, we obtain from Eqs. (19) and (20), and $x=CD$

$$x=OR+(\rho+\Delta\rho)\cos\theta-\rho. \tag{21}$$

However,

$$OR=OQ\sin\alpha+PQ\cos\alpha. \tag{22}$$

In Fig. 3 we see that $BP\cos\theta_0+PQ=OA+AB$. Using Eqs. (19) and (20), and $x_0=AB$ we have

$$PQ=\rho+x_0-(\rho+\Delta\rho)\cos\theta_0.$$

Substituting this result in Eq. (22) and using the fact that

$$OQ=BP\sin\theta_0=(\rho+\Delta\rho)\sin\theta_0,$$

we obtain

$$OR=(\rho+\Delta\rho)\sin\theta_0+[\rho+x_0-(\rho+\Delta\rho)\cos\theta_0]\cos\alpha. \tag{23}$$

Substitution of Eq. (23) in Eq. (21) yields an *exact* expression for x.

To obtain a first-order solution for x we use the inequalities $\theta_0\ll1$, $x_0\ll\rho$, $\Delta\rho\ll\rho$, and $\theta\ll1$. Dropping all terms in Eqs. (21) and (23) which are second order or higher in x_0, θ_0, $\Delta\rho$, θ, or their products, we obtain

$$x=x_0\cos\alpha+\theta_0\rho\sin\alpha+\Delta\rho(1-\cos\alpha). \tag{24}$$

The coordinate θ may be obtained by noting in Fig. 3 that $\sin\theta=RP/PD$; an alternative procedure is to recognize that $\tan\theta=(1/\rho)(\partial x/\partial\alpha)$, where we consider α to be a variable, and evaluate the derivative at the actual value of α. This procedure quickly yields from Eq. (24)

$$\theta=\frac{-x_0}{\rho}\sin\alpha+\theta_0\cos\alpha+\frac{\Delta\rho}{\rho}\sin\alpha, \tag{25}$$

since, to first order $\theta=\tan\theta$.

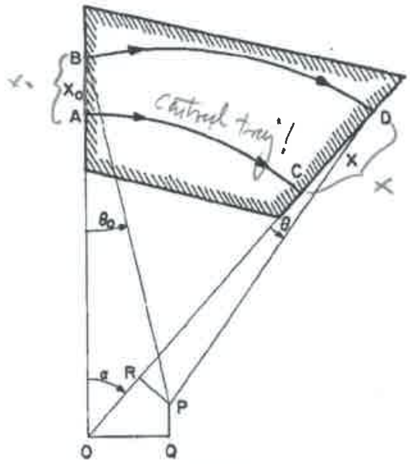


FIG. 3. Construction for calculating trajectories in a uniform field magnet with no vertical focusing. The arc AC is the central trajectory. $x_0=AB$ and $x=CD$. Effects of fringe fields are neglected.

Matrix Formulation

The transformations (24) and (25) are not expressible in the two-dimensional matrix notation used for quadrupole lens. This may be remedied by adding a third component $\Delta p/p$ to the vector space. Remembering that static magnetic fields do not change the magnitudes of particle momenta, and using Eqs. (18), (24), and (25), we see that the appropriate three-dimensional matrix transformation for a uniform field magnet with no vertical focusing is

$$\begin{bmatrix} x \\ \theta \\ \dfrac{\Delta p}{p} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \rho\sin\alpha & \rho(1-\cos\alpha) \\ \dfrac{-\sin\alpha}{\rho} & \cos\alpha & \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \dfrac{\Delta p}{p} \end{bmatrix}. \tag{26}$$

The values of the matrix elements in the third row (0,0,1) will be the same for all transformations involving only static magnetic fields.

In order to calculate trajectories in systems containing both bending magnets and quadrupoles the quadrupole matrices given by Eq. (3) must be extended to the three-dimensional case. However, there are no first-order terms in $\Delta p/p$. In general, any element which does not deflect the central trajectory produces no first-order dispersion. In view of the above, the three-dimensional matrices for quadrupoles $M_q^{(3)}$ may be written

$$M_q^{(3)}=\left[\begin{array}{c|c} M_q^{(2)} & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0 & 0\end{matrix} & 1 \end{array}\right].$$

$M_q^{(2)}$ is the two-dimensional quadrupole matrix, identical with either M_c or M_D of Eqs. (6), depending on whether the element focuses or defocuses. In thin lens approxi-

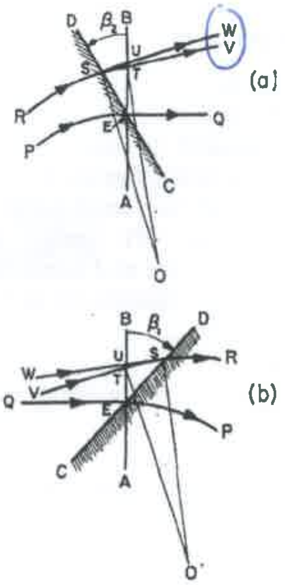


FIG. 4. Construction for calculating the effects of pole edge rotation on the radial trajectories. (a) exit edge rotation, (b) entrance edge rotation.

mation

$$M_q^{(3)}=\begin{bmatrix}1 & 0 & 0\\ -1/f & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad (27)$$

where f is the focal length defined previously.

Effect of Vertical Focusing on Radial Trajectories

The effect of a rotation of the pole edges on the radial trajectories is easily calculated. Figure 4(a) illustrates the effect of rotating the exit pole edge through angle β_2. PQ is the central trajectory and AB is the pole edge when $\beta_2=0$. If AB is the pole edge, the trajectory of a particle with initial coordinates $(x_0,\theta_0,\Delta p/p)$ is RV, which is an arc of radius $\rho+\Delta p$ with its center at 0 between R and T and with a straight line from T to V. The final coordinates of the trajectory are ($x=ET$, $\theta=\sphericalangle OTA$, $\Delta p/p$) as given by Eqs. (24) and (25). If the pole edge is rotated through angle β_2 about the point E on the central trajectory to position CD then the trajectory for the particle $(x_0,\theta_0,\Delta p/p)$ is RW, which consists of the arc RS with center at 0 and the straight line SW. If we call $(x^1,\theta^1,\Delta p/p)$ the final coordinates of this trajectory we see that $x^1=x+\Delta x$, where $\Delta x=TU$, and $\theta^1=\sphericalangle OTA+\sphericalangle TOS=\theta+\sphericalangle TOS$. From the figure we see that SU and ST are of the same order of magnitude as $x(=ET)$, provided β_2 is not much larger than 45°. Thus $\sphericalangle TOS$ is small (of order x/ρ), and $TU(=\Delta x)$ is zero to first order. The perpendicular distance from S to the line AB is $SU\cos\theta=SU$ to first order. Thus

$$SU=ET\tan\beta_2=x\tan\beta_2,$$

and

$$\sphericalangle TOS=SU/(\rho+\Delta\rho)=(x/\rho)\tan\beta_2$$

to first order. Hence the effect of an exit pole edge rotation is expressed by

$$\begin{aligned}x^1&=x,\\ \theta^1&=\theta+(x/\rho)\tan\beta_2.\end{aligned} \quad (28)$$

Substitution of Eqs. (24) and (25) in Eqs. (28) gives the desired transformations. The x^1 transformation is identical with Eq. (24), and

$$\theta^1=-\frac{x_0}{\rho}(\sin\alpha-\cos\alpha\tan\beta_2)+\theta_0(\cos\alpha+\sin\alpha\tan\beta_2)+\frac{\Delta p}{p}[\sin\alpha+(1-\cos\alpha)\tan\beta_2].$$

The effect of rotating the entrance pole edge through angle β_1 may be seen from Fig. 4(b), which is a mirror image of Fig. 4(a) about the line AB. The central trajectory is now QP, and the desired particle trajectory is WR. This particle, whose initial coordinates are ($x_0=UE$, θ_0, $\Delta p/p$) will have the same final coordinates $(x,\theta,\Delta p/p)$ as a particle whose orbit would be VR if the entrance pole edge were not rotated. Thus the desired transformation can be obtained from Eqs. (24) and (25) by replacing x_0 and θ_0 by the values x_0^1 and θ_0^1 appropriate to the trajectory VR. By a method identical to that used for exit pole edge rotation we obtain $x_0^1=x_0$, $\theta_0^1=\theta_0+(x_0/\rho)\tan\beta_1$. Substitution of these relations in Eqs. (24) and (25) yields

$$\begin{aligned}x&=x_0(\cos\alpha+\sin\alpha\tan\beta_1)+\theta_0\rho\sin\alpha+\Delta\rho(1-\cos\alpha)\\ \theta&=-\frac{\theta_0}{\rho}(\sin\alpha-\cos\alpha\tan\beta_1)+\theta_0\cos\alpha+\frac{\Delta p}{p}\sin\alpha\end{aligned} \quad (29)$$

for the case of entrance pole edge rotation and no exit pole rotation. The transformation for the case of both pole edges rotated is obtained by combining Eqs. (28) and (29). The appropriate matrix is

$$M(\alpha,\beta_1\beta_2)=\begin{bmatrix}\dfrac{\cos(\alpha-\beta_1)}{\cos\beta_1} & \rho\sin\alpha & \rho(1-\cos\alpha)\\ -\dfrac{(1-\tan\beta_1\tan\beta_2)\sin(\alpha-\beta_1-\beta_2)}{\rho\cos(\beta_1+\beta_2)} & \dfrac{\cos(\alpha-\beta_2)}{\cos\beta_2} & \sin\alpha+(1-\cos\alpha)\tan\beta_2\\ 0 & 0 & 1\end{bmatrix}. \quad (30)$$

Vertical Motion

A uniform field magnet does not deflect trajectories in the vertical direction if the trajectory is perpendicular to the pole edges. If, however, the pole edges are rotated there are magnetic field components in the x direction of the radial plane near the pole edges (that is, in the fringe field region). The only first-order effect of these field components is to cause an angular deflection of the trajectories in the vertical direction. The deflections are

$$\Delta\phi_0 = -(y_0/\rho)\tan\beta_1 \tag{31}$$

at the entrance pole edge, and

$$\Delta\phi = (y/\rho)\tan\beta_2 \tag{32}$$

at the exit edge. In addition to the usual approximation $|y/\rho|\ll 1$ and $|\phi|\ll 1$, Eqs. (31) and (32) depend on the additional approximation $|\phi\tan\beta|\ll 1$. That is, β must not be near $\pm 90°$.

The vertical direction transformation matrix is obtained by combining the effects of the deflections near the pole edges given by Eqs. (31) and (32) with the rectilinear transformation through the uniform field region of the magnet. To first order the transformation distance is $\rho\alpha$ for all trajectories. Thus

$$M_v = \begin{bmatrix} 1 & 0 \\ -\dfrac{\tan\beta_2}{\rho} & 1 \end{bmatrix} \begin{pmatrix} 1 & \alpha\rho \\ 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{\tan\beta_1}{\rho} & 1 \end{bmatrix},$$

or

$$M_v = \begin{bmatrix} 1-\alpha\tan\beta_1 & \alpha\rho \\ -\dfrac{1}{\rho}(\tan\beta_1+\tan\beta_2) & 1-\alpha\tan\beta_2 \end{bmatrix}. \tag{33}$$

Because the central trajectory is not deflected vertically, there is no vertical dispersion to first order. Thus M_v may be extended to include the $\Delta p/p$ dimension by

$$M_v^{(3)} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{M_v^{(2)}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

IV. BENDING MAGNETS WITH NONUNIFORM FIELDS

We will consider in this section bending magnets in which the magnetic field is given by

$$B' = B(\rho'/\rho)^{-n}. \tag{34}$$

The motion of a particle about its equilibrium orbit in this field is given by the well-known betatron equations[5]

$$x = A_x\sin(1-n)^{1/2}\alpha + B_x\cos(1-n)^{1/2}\alpha$$
$$y = A_y\sin n^{1/2}\alpha + B_y\cos n^{1/2}\alpha,$$

[5] D. W. Kerst and R. Serber, Phys. Rev. 60, 53 (1941).

where x and y are the radial and vertical displacements, respectively, and α is the deflection angle. The constants A_x, B_x, A_y, and B_y are determined by the values of x and y at the point where the particle enters the magnetic field, and by the angles θ_0 and ϕ_0 given by

$$\theta_0 \cong \frac{1}{\rho}\left(\frac{\partial x}{\partial\alpha}\right)_{\alpha=0}$$

$$\phi_0 \cong \frac{1}{\rho}\left(\frac{\partial y}{\partial\alpha}\right)_{\alpha=0}.$$

We easily obtain, from the above, the matrices which transform the particle coordinates from the entrance to the exit of the magnet.

$$M_x = \begin{bmatrix} \cos(1-n)^{1/2}\alpha & \dfrac{\rho}{(1-n)^{1/2}}\sin(1-n)^{1/2}\alpha \\ -\dfrac{(1-n)^{1/2}}{\rho}\sin(1-n)^{1/2}\alpha & \cos(1-n)^{1/2}\alpha \end{bmatrix}$$

$$M_y = \begin{bmatrix} \cos n^{1/2}\alpha & \dfrac{\rho}{n^{1/2}}\sin n^{1/2}\alpha \\ -\dfrac{n^{1/2}}{\rho}\sin n^{1/2}\alpha & \cos n^{1/2}\alpha \end{bmatrix}. \tag{35}$$

Momentum Variation

To obtain the $\Delta p/p$ components for the radial trajectory, we need note that the betatron equations give the displacements with respect to the equilibrium orbit. Using $p=eB\rho$ and Eq. (34), we see that the equilibrium orbit for a particle of momentum $p+\Delta p$ is displaced from the central orbit ($\Delta p=0$) by

$$\Delta\rho \cong \rho/(1-n)(\Delta p/p).$$

Thus the transformation through the magnet for a noncentral-momentum particle is given by

$$\begin{bmatrix} x-\dfrac{\rho}{1-n}\left(\dfrac{\Delta p}{p}\right) \\ \theta \end{bmatrix} = (M_x^{(2)}) \begin{bmatrix} x_0-\dfrac{\rho}{1-n}\dfrac{\Delta p}{p} \\ \theta_0 \end{bmatrix}, \tag{36}$$

where $M_x^{(2)}$ is given by the first of Eqs. (35). Equation (36) yields the three-dimensional transformation

$$M_x^{(3)} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{M_x^{(2)}} & \dfrac{\rho}{1-n}[1-\cos(1-n)^{1/2}\alpha] \\ & & \dfrac{1}{(1-n)^{1/2}}\sin(1-n)^{1/2}\alpha \\ \hline 0 & 0 & 1 \end{array}\right]. \tag{37}$$

We can also obtain the matrix transformations for a

2. *Matrix Elements for a Pure Sextupole Field*

For a pure sextupole, the matrix elements are derived from those of the general case by letting

$$\beta h^3 = k_s^2 = (B_0/a^2)(1/B\rho)$$

and then taking the limit $h \to 0$. The results are:

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= t \\
T_{111} &= -\tfrac{1}{2}k_s^2 t^2 \\
T_{112} &= -\tfrac{1}{3}k_s^2 t^3 \\
T_{122} &= -\tfrac{1}{12}k_s^2 t^4 \\
T_{133} &= \tfrac{1}{2}k_s^2 t^2 \\
T_{134} &= \tfrac{1}{3}k_s^2 t^3 \\
T_{144} &= \tfrac{1}{12}k_s^2 t^4 \\
R_{21} &= 0 \\
R_{22} &= 1 \\
T_{211} &= -k_s^2 t \\
T_{212} &= -k_s^2 t^2 \\
T_{222} &= -\tfrac{1}{3}k_s^2 t^3 \\
T_{233} &= k_s^2 t \\
T_{234} &= k_s^2 t^2 \\
T_{244} &= \tfrac{1}{3}k_s^2 t^3 \\
R_{33} &= 1 \\
R_{34} &= t \\
T_{313} &= k_s^2 t^2 \\
T_{314} &= \tfrac{1}{3}k_s^2 t^3 \\
T_{323} &= \tfrac{1}{3}k_s^2 t^3 \\
T_{324} &= \tfrac{1}{6}k_s^2 t^4 \\
R_{43} &= 0 \\
R_{44} &= 1 \\
T_{413} &= 2k_s^2 t \\
T_{414} &= k_s^2 t^2 \\
T_{423} &= k_s^2 t^2 \\
T_{424} &= \tfrac{2}{3}k_s^2 t^3
\end{aligned}
\qquad (56)
$$

All nonlisted matrix elements are identically zero.

3. *First- and Second-Order Matrix Elements for a Curved, Inclined Magnetic Field Boundary*

Matrix elements for the fringing fields of bending magnets have been derived using an impulse approximation.[7,8] These computations, combined with a correction term[9] to the R_{43} elements (to correct for the finite extent of actual fringing fields), have produced results which are in substantial agreement with precise ray-tracing calculations and with experimental measurements made on actual magnets.

We introduce four new variables (illustrated in Fig. 11); the angle of inclination β_1 of the entrance face of a bending magnet, the radius of curvature R_1 of the entrance face, the angle of inclination β_2 of the exit face, and the radius of curvature R_2 of the exit face. The sign convention of β_1 and β_2 is considered positive for positive focusing in the transverse (y) direction. The sign convention for R_1 and R_2 is positive if the field boundary is convex outward; (a positive R represents a negative sextupole component of strength $k_s^2 L = -(h/2R)\sec^3\beta$). The sign conventions adopted here are in agreement with Penner,[4] and Brown, Belbeoch, and Bounin.[7]

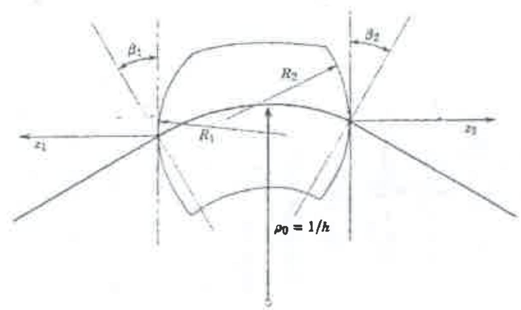


FIG. 11. Field boundaries for bending magnets. Definition of the quantities β_1, β_2, R_1, and R_2 used in the matrix elements for field boundaries of bending magnets. The quantities have a positive sign convention as illustrated in the figure.

Ta je samo za robno polje!

The results of these calculations yield the following matrix elements for the fringing fields of the entrance face of a bending magnet:

$$
\begin{aligned}
R_{11} &= 1\\
R_{12} &= 0\\
T_{111} &= -(h/2)\tan^2\beta_1\\
T_{133} &= (h/2)\sec^2\beta_1\\
R_{21} &= -(1/f_x) = h\tan\beta_1\\
R_{22} &= 1\\
T_{211} &= (h/2R_1)\sec^3\beta_1 - nh^2\tan\beta_1\\
T_{212} &= h\tan^2\beta_1\\
T_{216} &= -h\tan\beta_1\\
T_{233} &= h^2(n + \tfrac{1}{2} + \tan^2\beta_1)\tan\beta_1 - (h/2R_1)\sec^3\beta_1\\
T_{234} &= -h\tan^2\beta_1\\
R_{33} &= 1\\
R_{34} &= 0\\
T_{313} &= h\tan^2\beta_1\\
R_{43} &= -(1/f_y) = -h\tan(\beta_1 - \psi_1)\\
R_{44} &= 1\\
T_{413} &= -(h/R_1)\sec^3\beta_1 + 2h^2n\tan\beta_1\\
T_{414} &= -h\tan^2\beta_1\\
T_{423} &= -h\sec^2\beta_1\\
T_{436} &= h\tan\beta_1 - h\psi_1\sec^2(\beta_1 - \psi_1)
\end{aligned}
\qquad (57)
$$

All nonlisted matrix elements are equal to zero. The quantity ψ_1 is the correction to the transverse focal length when the finite extent of the fringing field is included.[9]

$$\psi_1 = Khg\sec\beta_1(1 + \sin^2\beta_1) + \text{higher order terms in } (hg)$$

where g = the distance between the poles of the magnet at the central orbit (i.e., the magnet gap) and

$$K = \int_{-\infty}^{+\infty} \frac{B_y(z)[B_0 - B_y(z)]}{gB_0^2}\,dz$$

$B_y(z)$ is the magnitude of the fringing field on the magnetic mid-plane at a position z. z is the perpendicular distance measured from the entrance face of the magnet to the point in question. B_0 is the asymptotic value of $B_y(z)$ well inside the magnet entrance. Typical values of K for actual magnets may range from 0.3 to 1.0 depending upon the detailed shape of the magnet profile and the location of the energizing coils.

The matrix elements for the fringing fields of the exit face of a bending magnet are:

$$
\begin{aligned}
R_{11} &= 1\\
R_{12} &= 0\\
T_{111} &= (h/2)\tan^2\beta_2\\
T_{133} &= -(h/2)\sec^2\beta_2\\
R_{21} &= -1/f_x = h\tan\beta_2\\
R_{22} &= 1\\
T_{211} &= (h/2R_2)\sec^3\beta_2 - h^2(n + \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2\\
T_{212} &= -h\tan^2\beta_2\\
T_{216} &= -h\tan\beta_2\\
T_{233} &= h^2(n - \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2 - (h/2R_2\sec^3\beta_2\\
T_{234} &= h\tan^2\beta_2\\
R_{33} &= 1\\
R_{34} &= 0\\
T_{313} &= -h\tan^2\beta_2\\
R_{43} &= -1/f_y = -h\tan(\beta_2 - \psi_2)\\
R_{44} &= 1\\
T_{413} &= -(h/R_2)\sec^3\beta_2 + h^2(2n + \sec^2\beta_2)\tan\beta_2\\
T_{414} &= h\tan^2\beta_2\\
T_{423} &= h\sec^2\beta_2\\
T_{436} &= h\tan\beta_2 - h\psi_2\sec^2(\beta_2 - \psi_2)
\end{aligned}
\qquad (58)
$$

All nonlisted matrix elements are zero.

$$\psi_2 = Khg\sec\beta_2(1 + \sin^2\beta_2) + \text{higher order terms in } (hg)$$

and K is evaluated for the exit fringing field.

4. *Matrix Elements for a Drift Distance*

For a drift distance of length L, the matrix elements are simply as follows:

$$R_{11} = R_{22} = R_{33} = R_{44} = R_{55} = R_{66} = 1$$

$$R_{12} = R_{34} = L$$

All remaining first- and second-order matrix elements are zero.

IV. Some Useful First-Order Optical Results Derived from the General Theory of Section II [10,11]

We have shown in Section II, Eq. (47), that beam transport optics may be reduced to a process of matrix multiplication. To first order, this is represented by the matrix equation

$$x_i(t) = \sum_{j=1}^{6} R_{ij} x_j(0) \tag{59}$$

where

$$x_1 = x,\ x_2 = \theta,\ x_3 = y,\ x_4 = \varphi,\ x_5 = l, \text{ and } x_6 = \delta$$

We have also proved that the determinant $|R| = 1$ results from the basic equation of motion and is a manifestation of Liouville's theorem of conservation of phase space volume.

The six simultaneous linear equations represented by Eq. (59) may be expanded in matrix form as follows:

$$\begin{bmatrix} x(t) \\ \theta(t) \\ y(t) \\ \varphi(t) \\ l(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \varphi_0 \\ l_0 \\ \delta_0 \end{bmatrix} \tag{60}$$

where the transformation is from an initial position $\tau = 0$ to a final position $\tau = t$.

The zero elements $R_{13} = R_{14} = R_{23} = R_{24} = R_{31} = R_{32} = R_{41} = R_{42} = R_{36} = R_{46} = 0$ in the R matrix are a direct consequence of midplane symmetry. If midplane symmetry is destroyed, these elements will in general become nonzero. The zero elements in column five occur because the variables x, θ, y, φ, and δ are independent of the path length difference l. The zeros in row six result from the fact that we have restricted the problem to static magnetic fields, i.e., the scalar momentum is a constant of the motion.

We have already attached a physical significance to the nonzero matrix elements in the first four rows in terms of their identification with characteristic first-order trajectories. We now wish to relate the elements appearing in column six with those in row five and calculate both sets

in terms of simple integrals of the characteristic first-order elements $c_x(t) = R_{11}$ and $s_x(t) = R_{12}$. In order to do this, we make use of the Green's integral, Eq. (43) of Section II, and of the expression for the differential path length in curvilinear coordinates

$$dT = [(dx)^2 + (dy)^2 + (1 + hx)^2(dt)^2]^{1/2} \tag{61}$$

used in the derivation of the equation of motion.

1. *First-Order Dispersion*

The spacial dispersion $d_x(t)$ of a system at position t is derived using the Green's function integral, Eq. (43), and the driving term $f = h(\tau)$ for the dispersion (see Table I). The result is

$$d_x(t) = R_{16} = s_x(t) \int_0^t c_x(\tau) h(\tau)\, d\tau - c_x(t) \int_0^t s_x(\tau) h(\tau)\, d\tau \tag{62}$$

where τ is the variable of integration. Note that $h(\tau)\, d\tau = d\alpha$ is the differential angle of bend of the central trajectory at any point in the system. Thus first-order dispersion is generated only in regions where the central trajectory is deflected (i.e., in dipole elements.) The angular dispersion is obtained by direct differentiation of $d_x(t)$ with respect to t;

$$d_x'(t) = R_{26} = s_x'(t) \int_0^t c_x(\tau) h(\tau)\, d\tau - c_x'(t) \int_0^t s_x(\tau) h(\tau)\, d\tau \tag{63}$$

where

$$c_x'(t) = R_{21} \quad \text{and} \quad s_x'(t) = R_{22}.$$

2. *First-Order Path Length*

The first-order path length difference is obtained by expanding Eq. (61) and retaining only the first-order term, i.e.,

$$l - l_0 = (T - t) = \int_0^t x(\tau) h(\tau)\, d\tau + \text{higher order terms}$$

from which

$$\begin{aligned} l &= x_0 \int_0^t c_x(\tau) h(\tau)\, d\tau + \theta_0 \int_0^t s_x(\tau) h(\tau)\, d\tau + l_0 + \delta \int_0^t d_x(\tau) h(\tau)\, d\tau \\ &= R_{51} x_0 + R_{52} \theta_0 + l_0 + R_{56} \delta \end{aligned} \tag{64}$$

7. Focal Length

It can be readily demonstrated from simple lens theory[4] that the physical interpretations of R_{21} and R_{43} are:

$$c_x'(t) = R_{21} = -1/f_x \quad \text{and} \quad c_y'(t) = R_{43} = -1/f_y \tag{76}$$

where f_x and f_y are the system focal lengths in the x and y planes, respectively, between $\tau = 0$ and $\tau = t$.

8. Evaluation of the First-Order Matrix for Ideal Magnets

From the results of Section III, we conclude that for an ideal magnet the matrix elements of R are simple trigonometric or hyperbolic functions. The general result for an element of length L is

$$R = \begin{bmatrix} \cos k_xL & \frac{1}{k_x}\sin k_xL & 0 & 0 & 0 & \frac{h}{k_x^2}[1-\cos k_xL] \\ -k_x\sin k_xL & \cos k_xL & 0 & 0 & 0 & \left(\frac{h}{k_x}\right)\sin k_xL \\ 0 & 0 & \cos k_yL & \frac{1}{k_y}\sin k_yL & 0 & 0 \\ 0 & 0 & -k_y\sin k_yL & \cos k_yL & 0 & 0 \\ \frac{h}{k_x}\sin k_xL & \frac{h}{k_x^2}[1-\cos k_xL] & 0 & 0 & 1 & \frac{h^2}{k_x^3}[k_xL-\sin k_xL] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{77}$$

where for a dipole (bending) magnet, we have defined

$$k_x^2 = (1-n)h^2 \quad \text{and} \quad k_y^2 = nh^2$$

For a pure quadrupole, the R matrix is evaluated by letting

$$k_x^2 = k_q^2 \quad \text{and} \quad k_y^2 = -k_q^2$$

and taking the limiting case $h \to 0$, where

$$k_q^2 = -nh^2 = (B_0/a)(1/B\rho)$$

Taking these limits, the R matrix for a quadrupole is:

$$R = \begin{bmatrix} \cos k_qL & \frac{1}{k_q}\sin k_qL & 0 & 0 & 0 & 0 \\ -k_q\sin k_qL & \cos k_qL & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh k_qL & \frac{1}{k_q}\sinh k_qL & 0 & 0 \\ 0 & 0 & k_q\sinh k_qL & \cosh k_qL & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{78}$$

Note that the trigonometric and hyperbolic functions will interchange if the sign of B_0 is reversed.

9. The R Matrix Transformed to the Principal Planes

The positions Z of the principal planes of a magnetic element (measured from its ends) may be derived from the following matrix equation:

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ R_{21} & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{43} & 1 & 0 & 0 \\ X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -Z_{2x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{2y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} R \begin{bmatrix} 1 & -Z_{1x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{1y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{79}$$

Solving this equation, we have

$$Z_{1x} = (R_{22}-1)/R_{21} \qquad Z_{2x} = (R_{11}-1)/R_{21}$$
$$Z_{1y} = (R_{44}-1)/R_{43} \qquad Z_{2y} = (R_{33}-1)/R_{43} \tag{80}$$