May 18, 1995

Modified g_2^{WW} formula for E143 Kinematics

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Using an operator product expansion technique, Wandzura and Wilczek [1], derived an approximate sum rule expression for the twist-two component of the $g_2(x, Q^2)$ structure function:

$$g_2^{WW}(x, Q^2) = -g_1(x, Q^2) + \int_0^1 \frac{g_1(y, Q^2)}{y} dy$$

A close examination of an earlier paper by Wandzura [2] indicates that the original $g_2^{WW}(x, Q^2)$ derivation used the assumption that $\gamma^2 = \frac{4M^2x^2}{Q^2} = 0$. We know, from the E143 analysis, that this kinematic term cannot be neglected at E143 kinematics. The purpose of this note is to rederive the $g_2^{WW}(x, Q^2)$ expression, with the γ^2 terms intact. We start with eqs. 46-48 from Wandzura [2]:

$$\nu_0 = 0$$

$$\nu_n = -\frac{4}{n+2} \int_0^1 \frac{dx}{x^2} \left[\frac{2x}{1 + \sqrt{1 + 4M^2x^2/Q^2}} \right]^{n+1}$$

$$\times \left(1 + (n+1)\sqrt{1 + \frac{4M^2x^2}{Q^2}} \right) \left(g_1(x, Q^2) + g_2(x, Q^2) \right), \qquad n = 1, 3, ...,$$

$$\nu_n = -\frac{8}{(n+1)(n+3)(n+4)} \int_0^1 \frac{dx}{x^2} \left[\frac{2x}{1 + \sqrt{1 + 4M^2x^2/Q^2}} \right]^{n+2} \left[(n^2 + 4n + 6) \right.$$

$$\left. + 3(n+2)\sqrt{1 + \frac{4M^2x^2}{Q^2}} + (n+1)(n+3)\frac{4M^2x^2}{Q^2} \right] g_2(x, Q^2), \qquad n = 2, 4, ...,$$

It is convenient to convert these expresions to the more-standard Jaffe [3] notation. This can be done with the relations

$$\frac{a_n}{4} = -\frac{\nu_{n+1}}{4} + \frac{(n+1)\nu_n}{8}, \qquad n = 0, 2, 4, ...$$

$$\frac{1}{4}\frac{n}{n+1}(d_n - a_n) = -\frac{(n+1)\nu_n}{8}, \qquad n = 2, 4, ...$$

to yield:

$$\int_0^1 \frac{dx}{x^2}\left[\frac{2x}{1+\sqrt{1+4M^2x^2/Q^2}}\right]^{n+2}\left[\left(1+\frac{n+2}{n+3}\left(\sqrt{1+\frac{4M^2x^2}{Q^2}}-1\right)\right)g_1(x,Q^2)+\right.$$
$$\left.\left(\frac{(n+1)(n+2)}{(n+3)(n+4)}\left(\sqrt{1+\frac{4M^2x^2}{Q^2}}-1\right)-\frac{(n+1)}{(n+4)}\frac{4M^2x^2}{Q^2}\right)g_2(x,Q^2)\right]=\frac{a_n}{4},\qquad n=0,2,4,...$$

$$\int_0^1 \frac{dx}{x^2}\left[\frac{2x}{1+\sqrt{1+4M^2x^2/Q^2}}\right]^{n+2}\left[1+\frac{3(n+2)}{(n+3)(n+4)}\left(\sqrt{1+\frac{4M^2x^2}{Q^2}}-1\right)+\right.$$
$$\left.\frac{(n+1)}{(n+4)}\frac{4M^2x^2}{Q^2}\right]g_2(x,Q^2)=\frac{1}{4}\frac{n}{n+1}(d_n-a_n),\qquad n=2,4,...,$$

Make a note that the top equation could conceivably have d_n contributions on the right hand side because of the $g_2(x,Q^2)$ appearing in the integrand. However, since the next step will be to set $d_n=0$, this doesn't really matter here. Also, these equations are "screaming" for simplification, so we will introduce a new function, $a(x,Q^2)=\sqrt{1+4M^2x^2/Q^2}-1$. Now setting $d_n=0$ and combining the equations to remove a_n, we get:

$$\int_0^1 x^n dx\left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2}\left[\frac{n}{n+1}\left(1+\frac{n+2}{n+3}a(x,Q^2)\right)g_1(x,Q^2)\right.$$
$$\left.+\left(1+a(x,Q^2)+\frac{1}{n+4}a^2(x,Q^2)\right)g_2(x,Q^2)\right]=0,\qquad n=2,4,..., \tag{1}$$

For $a(x,Q^2)=0$ we get back the original sum rule

$$\int_0^1 x^n dx\left[\frac{n}{n+1}g_1(x,Q^2)+g_2(x,Q^2)\right]=0,\qquad n=2,4,...,$$

For an infinite number of sum rules, it must be that $g_2(x,Q^2)$ can be completely determined as a function of $g_1(x,Q^2)$. It is the purpose of this note to determine this relationship. In the limit

$n \to \infty$, only high x contributions are important:

$$\int_0^1 x^n dx \left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} (1+a(x,Q^2)) \left(g_1(x,Q^2)+g_2(x,Q^2)\right) = 0 \qquad \text{for } n \to \infty$$

$$\Rightarrow g_2(x,Q^2) \approx -g_1(x,Q^2) \qquad \text{at high } x$$

We neglect the $a^2(x,Q^2)$ term in Eq. (1) and define the to-be-determined functions, $h(x,Q^2)$ and $f(x,Q^2)$ by $g_2(x,Q^2) = -g_1(x,Q^2) + h(x,Q^2)\int_x^1 f(y,Q^2)dy$, which satisfies the high x constraint mentioned above.

$$\int_0^1 x^n dx \left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} \left[\left(-\frac{1}{n+1} - \frac{2n+3}{(n+1)(n+3)}a(x,Q^2)\right) g_1(x,Q^2) \right.$$
$$\left. + (1+a(x,Q^2))\, h(x,Q^2) \int_x^1 f(y,Q^2)dy\right] = 0, \qquad n = 2,4,..., \tag{2}$$

Exchanging the order of integration on the portion of the integral with the new functions and using a variable substitution, $u = x/(1+a(x,Q^2)/2)$ yields

$$\int_0^1 x^n dx \left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} (1+a(x,Q^2))\, h(x,Q^2) \int_x^1 f(y,Q^2)dy$$
$$= \int_0^1 f(y,Q^2)dy \int_0^y dx h(x,Q^2) x^n \left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} (1+a(x,Q^2))$$
$$= \int_0^1 f(y,Q^2)dy \int_0^{y/(1+a(y,Q^2)/2)} u^n du \left(\frac{(1+a(x,Q^2))^2 h(x,Q^2)}{1+a(x,Q^2)/2}\right)$$
$$= \int_0^1 f(y,Q^2)dy \int_0^{y/(1+a(y,Q^2)/2)} u^n du = \int_0^1 \frac{dy f(y,Q^2)}{n+1} \left(\frac{y}{1+a(y,Q^2)/2}\right)^{n+1}$$

where $h(x,Q^2)$ is defined to absorb all the offensive x-dependent terms:

$$h(x,Q^2) = \frac{1+a(x,Q^2)/2}{(1+a(x,Q^2))^2} \tag{3}$$

Now we substitute this result back into Eq. (2):

$$\int_0^1 \frac{x^n dx}{n+1}\left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2}\left[\left(-1-\frac{2n+3}{n+3}a(x,Q^2)\right)g_1(x,Q^2)\right.$$
$$\left.+xf(x,Q^2)(1+a(x,Q^2)/2)\right]=0, \qquad n=2,4,..., \tag{4}$$

We judiciously choose $f(x,Q^2)$ such that the first order terms cancel and the second order terms come out nice and integrable. This turns out to be:

$$xf(x,Q^2)(1+a(x,Q^2)/2)=g1(x,Q^2)(1+2a(x,Q^2)+w(x,Q^2), \tag{5}$$

where $w(x,Q^2)$ is the new to-be-determined function. Eq. (4) now becomes:

$$\int_0^1 \frac{x^n dx}{n+1}\left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2}\left[\frac{3a(x,Q^2)g_1(x,Q^2)}{n+3}+w(x,Q^2)\right]=0, \qquad n=2,4,..., \tag{6}$$

Now we try to solve for $w(x,Q^2)$ by letting $w(x,Q^2)=r(x,Q^2)\int_x^1 s(y,Q^2)dy$ and consider just the part of the integral containing the $w(x,Q^2)$ term. We make the same variable substitution used earlier, $u=x/(1+a(x,Q^2)/2)$.

$$\int_0^1 \frac{x^n dx}{n+1}\left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} r(x,Q^2)\int_x^1 s(y,Q^2)dy$$
$$=\int_0^1 \frac{s(y,Q^2)dy}{n+1}\int_0^y \frac{x^n dx}{n+1}\left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} r(x,Q^2)$$
$$=\int_0^1 \frac{s(y,Q^2)dy}{n+1}\int_0^{y/(1+a(x,Q^2)/2)} \frac{du(1+a(x,Q^2)}{u^2(1+a(x,Q^2)/2)}u^{n+2}r(x,Q^2)$$
$$=\int_0^1 \frac{s(y,Q^2)dy}{n+1}\int_0^{y/(1+a(x,Q^2)/2)} u^{n+2}du$$
$$=\int_0^1 \frac{s(y,Q^2)dy}{(n+1)(n+3)}\left(\frac{y}{1+a(y,Q^2)/2}\right)^{n+3}$$

where $r(x, Q^2)$ has been defined:

$$r(x,Q^2) = \frac{1+a(x,Q^2)/2}{1+a(x,Q^2)}u^2 = \frac{x^2}{(1+a(x,Q^2))(1+a(x,Q^2)/2)}$$

Finally we substitute back into Eq. (6) :

$$\int_0^1 \frac{x^n dx}{(n+1)(n+3)} \left[\frac{1}{1+a(x,Q^2)/2}\right]^{n+2} \left[3a(x,Q^2)g_1(x,Q^2) + \frac{s(x,Q^2)x^3}{1+a(x,Q^2)/2}\right] = 0, \qquad n = 2, 4, ..., \tag{7}$$

which can only hold true for all n if

$$3g_1(x,Q^2) = \frac{s(x,Q^2)x^3}{1+a(x,Q^2)/2}.$$

All the unknown functions have been solved for, yielding the final answer for g_2^{WW}:

$$g_2(x,Q^2) = -g_1(x,Q^2) + \frac{1+a(x,Q^2)/2}{(1+a(x,Q^2))^2} \int_x^1 dy \left[\frac{g_1(y,Q^2)(1+2a(y,Q^2))}{y(1+a(y,Q^2)/2)}\right.$$

$$\left. + \frac{y}{(1+a(y,Q^2))(1+a(y,Q^2))} \int_y^1 \frac{du}{u^3} 3g_1(u,Q^2)a(u,Q^2)(1+a(u,Q^2)/2)\right]$$

1 . S. Wandzura and F. Wilczek, Phys. Lett. **72B**, 195 (1977).

2 . S. Wandzura, Nucl. Phys. **B122**, 412 (1977).

3 . R. Jaffe, Comments Nucl. Part. Phys. **19**, 239 (1990).