INVESTIGATIONS INTO WIRE SAG
IN THE CLAS DRIFT CHAMBER

PART I: A RUBBERBAND ANALOGY FOR WIRE SAG

by

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1. **Purpose**

The purpose of this investigation is to determine the sagging characteristics of the field and sense wires when the end-plates of the drift chamber deflects. These deflections may be caused by gravity, vibrations or non-uniformity in the construction of the end-plates. Such deflections may cause the field and sense wires to tense up and yield/break, or become loose and sag excessively.

Analytical expressions will be derived that show that the stiffness constant $\frac{EA}{l}$ of the wires, plays an important role in determining the wire sag and wire stresses. To provide a better feel of how wire sags as a function of end-plate motion, an analogy drawn from a rubberband will be used.

2. **Introduction**

The field and sense wires in the drift chambers of the CLAS are strung at some predetermined tension to maintain a desirable level of sag. Some questions on sag magnitudes have been discussed at length by Mestayer [1]. In another note [2], a question was posed as to the necessity of springs for proper wire performance under various conditions. This present note in its various parts, is an attempt to answer the question posed in note [2] by investigating the sagging characteristics of wires (in the absence of springs) as a function of end-plate motion.

Two types of wire materials are used - Tungsten (W) for sense wires and Aluminum (Al) for field wires. Due to differences in material and physical properties between sense and field wires, the tensions imposed on the two types of wires may be very different depending on the criterion used in tensioning these wires.

To further complicate the matter, the tension imposed on a given wire, for a given sag is also a strong function of the length of the wire [3]. Since the field and sense wires in the CLAS drift chamber vary in length from 0.3 m to over 3.0 m, the tensions between wires of the same type (sense wires for instance) may vary greatly over the drift chamber, depending on the criterion chosen for wire tensions.

3. **Wire Sag Calculations**

The drift chamber wires are strung perpendicular to the mid-plane of the chamber. At this mid-plane, the field wires form hexagonal patterns with the sense wire at the center of each hexagonal cell. It is important for the field wires that form the corners of the hexagonal pattern to be equidistant from its neighbors as well as lie on the circumference of a circle, with the sense wire at its center. Such a regular hexagonal pattern will provide a symmetrical drift-time characteristic over the hexagonal cell. A recent note [4] examined the distortion characteristics on the hexagonal cells based on layout considerations of the superlayers.
With wire sag, however, the regular hexagonal pattern at the mid-plane may not be adhered to since the length of the wires that form each hexagonal cell is slightly different from each other. This may call for tensioning the wires differently to maintain a constant sag for all the wires within each cell, thereby maintaining consistency in sag for the entire drift chamber as well. While this may seem like a desirable criterion for wire tensioning there are other considerations such as differential sag, wire yield or runaway sags as well as error in tensioning, that should be considered into the criterion for wire tensioning as well. This subject will be topic of discussion in a subsequent part Part II of this note. This note will concentrate on laying the analytical foundations on wire sag so that such issues as tensioning criteria, differential wire sags, permissible range of operation of wires, stresses, tension, actual range limits, as well as errors in tensioning can be investigated in a systematic way.

3.1 Aluminum Field Wires

Based on a tension of 100g in a 140µ Al wire [2] over a span of 3.0 m, the sag is calculated to be 0.48 mm as shown in Appendix A. This closely matches with the recommendations in [1] of 0.5 mm. The stress in the 3.0 m Al field wire is found to be an acceptable 9200 psi with a wire extension of 2.7 mm.

If one were to use the constant sag criterion as an illustration for this note, a 300 mm (0.3 m) Al field wire with the same sag would be tensioned at 1.0 g with a wire extension of 2.8µ and a tensile stress of 92 psi, a very low value that may result in instability due to electrostatic forces. These forces however will not be considered in this note.

3.2 Tungsten Sense Wires

Appendix B provides the calculations for a 20µ W sense wire subjected to a tension of 15 g over a 3.0 m span. The sag is calculated to be 0.46 mm, within the recommended sag of 0.5 mm in [1]. Furthermore, the resulting tensile stress of 68 kpsi is acceptable for this material. Wire extension has been calculated to be 4.1 mm.

Based on a constant sag criterion, as an illustration, a short 0.3 m W wire will have to be tensioned at only 0.15 g to achieve the same sag as that for a 3 m W wire. The resulting tensile stress is 680 psi with a wire extension of 4.1µ. Again, such low values of tension for the short wires may result in instability due to electrostatic forces.

3.3 Inadequacies in Sag Formula

The formula for determining wire sags in the prior two sections relates the wire sag to the length of the wire, its linear density as well as the tension imposed on the wire. This formula can be found in [3] and is given by

\[ f = \frac{wL^2}{8T} \] (1)
where: \( f \) is the wire sag
\( L \) is the length of the wire \( \approx \) wire span
\( \omega \) is the wire density (mass/unit length)
\( T \) is the wire tension

Note that while the above equation provides a good feel as to the sag, it is inadequate in predicting the effect of end-plate motion on the sag in these same wires. As an example, suppose the end-plates move apart so that the wire span \( L \) is increased. Conceptually, one also realizes that the wire tension \( T \) has also increased in the process, by an unknown amount. Hence, referring to equation (1) there is insufficient information in determining the resulting new sag. This problem is the subject of discussion in the section that follows. To assist in visualizing the characteristics of sag, a rubberband analogy will be used to magnify the sag characteristics that otherwise are not readily apparent in the Al and W wires for the drift chamber.

4. A Rubberband Analogy for Wire Sag

The wire sags calculated in Appendices A and B, and summarized in Section 3, are for wires at their designed nominal positions. Appendix D provides a summary of the material and physical properties of the Al and W wires used in this note. For each of the wires investigated in Section 3, if the end-plates were to move apart or towards each other, the sags calculated will no longer hold true. How these sags change, turns out to have major implications on the wire tensions as well as the need for springs.

As analogy to the drift chamber wires, consider a rubberband, of natural unstretched length, 0.10 m. Suppose this rubberband is stretched to a length nominal condition, analogous to the condition of the drift chamber wires before end-plate motion.

Suppose the two ends of the rubberband move in such a way that the distance (span) between these two ends vary between 0.08 m and 0.15 m. Immediately one can observe that the rubberband falls into two quite distinct regimes:

(i) Undeformed regime - when the span, \( a \), is between the lower span limit of 0.08 m and approximately 0.10 m.

(ii) Elastic regime - when the span, \( a \), is between approximately 0.10 m and 0.15 m.

A discussion of each of these regimes are carried out in the following sections.

4.1 Undeformed Regime

In this regime of operation, the rubberband may be modelled by an unstretchable wire. The tension and the corresponding extension in length (due to the tension), are very small, so that for all practical purposes, an undeformable wire model would be sufficiently accurate in determining the sag in the rubberband.
Reference [3] provides an accurate relationship between the length, \( L \) of an undeformable wire over a supporting span, \( a \), with a sag, \( f \), and this is given by:

\[
L = A \left\{ \frac{1}{2} (1 + 16n^2)^{\frac{1}{2}} + \frac{1}{8n} \ln[4n + (1 + 16n^2)^{\frac{1}{2}}] \right\}
\]  

(2)

where

\[ n = \frac{f}{a} \]

In this regime, the wire length \( L \) (based on the rubberband example) is equal to 0.1 m. Suppose the span, \( a \), is given to be 0.098 m, then the sag, \( f \), may be determined from solving the resulting nonlinear algebraic equation (2). This may be carried out for various spans, \( a \), between 0.08 m and 0.098 m for this rubberband example. The solution of this nonlinear algebraic equation is carried out numerically using a Newton-Ralphson scheme [5-7].

Figure 1 shows the results of what happens to the sag as the span is reduced in this undeformed regime. Notice the high rate-of-increase in the sag for a small reduction in the span. Figure 2 is the same figure but with the gravitational sag plotted on a logarithmic scale. The data for this example is given in Appendix D with a 3.0 m nominal span aluminum wire tensioned at 66.7% of maximum allowable stress, or 159.0 grams.
Figure 1: Sag as a function of span for a given wire operating in the undeformed region and elastic region.
Figure 2: Gravitational sag as a function of span plotted on a logarithm scale, with data from Figure 1.
4.2 Elastic Regime

In this regime, the rubberband may be stretched to a considerable extent, i.e. it is possible for the span to be larger than the original unstretched length of the rubberband—the latter observation can never be permitted in the undeformed regime. The equations given in most of the literature [1,3] on the relationship between tension and sag follows the following approximate equation:

\[ f = \frac{wL^2}{8T} \]  

(1)

However equation (1) cannot be applied towards determining the wire sag with a given change in the wire span. This is because when the wire span is increased, the wire tension is also increased so that equation (1) above is insufficient to determine the new sag. In other words, a change in the span would not only change the sag, but also, the tension which in turn changes the sag.

Note that for a given wire of length \( L_o \) under tension \( T_o \), its total weight \( W \) remains unchanged should the same wire is subsequently stretched to length \( L \) under tension \( T \). This is due to the law of conservation of mass.

Therefore

\[ W = w_o L_o = wL \]  

(3a)

so that the linear density of the wire changes, and is inversely proportional to its length. In general,

\[ w = \frac{W}{L} \]  

(3b)

Substituting into equation (1):

\[ f = \frac{WL}{8T} \]  

(3c)

A clearer picture can be seen by taking a partial derivative of equation (3c) so that:

\[ df = \frac{W}{8T} \partial L - \left( \frac{WL}{8T^2} \right) \partial T \]  

(3d)

which shows that a change in the sag is caused by a change in the span as well as a change in the tension. However, the change in tension may be approximately related to a change in the span, via Hooke's Law:

\[ dT = EA \frac{dL}{L} \]  

(4)

where

\( E \) is the Young's Modulus of the wire material.
A is the cross-sectional area of the wire

From (4) into (3d)

\[ df = \frac{W}{8T} \left( 1 - \frac{EA}{T} \right) dL \]  

(5)

In effect, equation (5) gives the change in sag for an infinitesimal change in the span, with the wire being elastic. Note that in this equation, the contribution of the elasticity of the wire plays a very important role in sag change.

Equation (5) is still not directly usable in determining the relationship between the wire span \( L \) and the sag \( f \). In Appendix C, a derivation of this relationship is given and the resulting equation is:

\[ \ln L - \frac{WL}{8EAf} = c \]  

(6)

where \( c = \text{constant} \)

To apply equation (6), determine the constant \( c \) based on the original condition of the wire, i.e. length \( L_o \) with sag \( f_o \). Therefore

\[ c = \ln L_o - \frac{WL_o}{8EAf_o} \]  

(7)

where the wire is stretched or relaxed from length \( L_o \) to \( L_1 \) the parameters \( E, A \) and \( W \) remain unchanged so that the new sag \( f_1 \) can be determined from:

\[ \ln L_1 - \frac{WL_1}{8EAf_1} = \ln L_o - \frac{WL_o}{8EAf_o} \]  

(8)

from which, the new sag \( f_1 \) is given by:

\[ f_1 = \frac{WL_1}{8EA} \cdot \frac{1}{\ln \frac{L_1}{L_o} + \frac{WL_o}{8EAf_o}} \]  

(9)

Figure 1 also shows the sag characteristics as the span is increased in the elastic regime. Note that the sag changes very little over a large change in the span in this regime of operation.

The following section summarizes the observations on an elastic wire and its sag characteristics over the two regimes.
4.3 Observations on the Rubberband Analogy

There are several observations that can be noted from this rubberband analogy:

4.3.1 Observation # 1

This observation comes from the plot in Figure 1 which shows that a small change in wire span would result in a large change in the sag. There, it is inadvisable to permit the wire to operate in the undeformed regime. Conversely, it is true that one would like to operate the wire totally in the elastic regime. Note that in this regime, large excursions in the span will only result in very small changes in the sag.

4.3.2 Observation # 2:

From Observation # 1, the sag in the elastic regime decreases very slowly with increase in span. This means that the increase in the span, and the extension in the wire is very nearly the same, i.e. the sag effect is negligible.

Therefore, the Al and W wires considered for the drift chamber, the tensile stress increases very rapidly, very nearly proportional to the $EA$ product of the wire properties.

4.3.3 Observation # 3

Based on Observation # 1, it therefore follows that the greatest risk of wire breakage comes from the shortest wires in the drift chamber. This is because for a given maximum strain of a wire breakage, the total permissible extension is least for the shortest wire. If the total relative end-plate motion is larger than the extensible range of a given wire, that will be under great risk of breakage or being outstretched beyond yield or set to operate in the inextensible region of operation (resulting in a large sag).
4.3.4 Observation # 4

The small range of operation of short wires is due to the high stiffness constants of the wires, given by $\frac{E}{3}$. To increase that range of extension, one way would be to reduce the stiffness constant of the wire (much like in a rubberband) or inserting low stiffness springs in series with the sense and the field wires. The low stiffness constant springs permit large excursions in relative end-plate motion with minimal change in spring force, and hence minimal wire tension change.

The elastic region may be thought of as the preload region, much as in the preloading of bolts in fastening together of two parts. When the preload is zero (equivalent to zero or near zero tension of the wire the condition at natural length) there is a great risk of part separation (equivalent to large wire sags in the undeformed regime). When the preload is sufficiently large, then risk of part separation is reduced but with an increased danger of bolt failure (equivalent to the increased risk of wire breakage or yielding depending on wire material).

5. Conclusion

This note has shown that the sagging characteristics of the drift chamber wires may be separated into regimes - an elastic and an undeformed region. In the elastic regime, the sag of the wire changes very slowly with changes in the relative positions of the end-plates, i.e., the sag change as a function of span change, is small. On the other hand, in the undeformed region, a small change in span of the wire may result in a very large change in the wire sag. For this reason, the operation of the wire has to lie between the yield limit of the wire material and the point of onset of runaway sag.

The analytical equation that governs the change in sag as a function of the change of span provides that relationship to investigate issues such as criteria for tensioning of wires, differential sags as well as the effect of errors in wire tensioning. Such issues will be discussed in subsequent parts of this note.
REFERENCES
Appendix A: Aluminum Field Wire

Wire Diameter: 140\(\mu\)g
Tension, \(T\): 100 g
Density of Al: 2774 kg/m\(^3\)
Therefore:
Cross-sectional area of wire = \(1.5394 \times 10^{-8}\) m\(^2\)
Linear wire density, \(w\) = 0.0427 g/m

(i) 3 m length wire

sag, \(f = \frac{wL^2}{8T}\)

= 0.48 mm

Therefore:
Wire extension = \(\frac{T_1L}{AE}\) = 2.8 mm
Wire stress = 9220 psi

(ii) 300 mm length wire

Assuming sag is constant throughout chamber, so that

\[ f = 0.48 \text{ mm} \]

Then tension in 300 mm wire

\[ T_2 = \frac{wL^2}{8f} = 1 \text{ g} \]

Therefore
Wire extension = 2.8\(\mu\)
and wire stress = 92.2 psi
Appendix B: Tungsten Sense Wire

Wire diameter = 20μ
Wire tension = 15 g
Density of W = 19.4 \times 10^3 \text{ kg/m}^3

Therefore

Cross-sectional area of wire = 3.1416 \times 10^{-10} \text{ m}^2
Linear wire density, w = 6.095 \times 10^{-6} \text{ kg/m}

(i) For a 3 m wire length

sag, f = \frac{wL^2}{8T}
\approx 0.457 \text{ mm} \ (\text{Similar to field wire in Appendix A.})

Wire extension = 3.39 mm
Wire stress = 67769 psi (O.K.)

(ii) For a 300 mm wire length Using the same sag of 0.457 mm, the tension in a 300 mm W wire

\[ T = \frac{wL^2}{8f} \]
\[ = 0.15 \text{ g (too low)} \]

Wire extension = 3.39μ
Wire stress = 677.7 psi
Appendix C

Relationship between wire span and wire sag for the same wire under tension.

From Eq. (5) of section 4.2:

$$df = \frac{W}{8T} \left(1 - \frac{EA}{T} \right) dL$$  \hspace{1cm} (C.1)

However, from Eq. (2)

$$T = \frac{wL^2}{8f} = \frac{WL}{8f}$$  \hspace{1cm} (C.2)

Substituting for $T$ into (C.1), we have:

$$\frac{df}{dL} = \left(\frac{f}{L}\right) \left[1 - \frac{8EA}{W} \left(\frac{f}{L}\right)\right]$$  \hspace{1cm} (C.3)

Note that: $\frac{8EA}{W}$ is a constant for a given wire operating in the elastic region. Denote this constant by $k$, so that

$$\frac{df}{dL} = \left(\frac{f}{L}\right) \left[1 - k \left(\frac{f}{L}\right)\right]$$  \hspace{1cm} (C.4)

To solve the first-order differential equation given by (C.4) perform the following transformation [8]:

$$v = \frac{f}{L}$$  \hspace{1cm} (C.5a)

so that

$$f = vL$$  \hspace{1cm} (C.5b)

Differentiating (C.5b) w.r.t. $L$

$$\frac{df}{dL} = v + L \frac{dv}{dL}$$  \hspace{1cm} (C.5c)
Substituting (C.5c) and (c.5a) into (c.4):

\[ v + L \frac{dv}{dL} = v[1 - kv] \]  \hspace{1cm} (C.8)

Rearrange (C.6)

\[ \frac{dL}{L} = \frac{dv}{-kv^2} \]  \hspace{1cm} (C.7)

Integrating (C.7)

\[ \ln L = \frac{1}{-kv} + C_1 \]  \hspace{1cm} (C.8)

is

\[ \ln L - \frac{L}{kf} = \text{constant} \]  \hspace{1cm} (C.9)

or

\[ \ln L - \frac{WL}{8EAf} = \text{constant} \]  \hspace{1cm} (C.10)
**Appendix D**

Tabulated below is a summary of the material and physical properties of the Al and W wires used in this note.

<table>
<thead>
<tr>
<th>Property</th>
<th>Al Field Wire</th>
<th>W Sense Wire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Diameter</td>
<td>$140\mu$ (5.5E-3 in)</td>
<td>$20\mu$ (7.87E-4 in)</td>
</tr>
<tr>
<td>Cross-sectional Area of Wire</td>
<td>1.539E-8 m$^2$ (2.386E-5 in$^2$)</td>
<td>3.142E-10 m$^2$ (4.87E-7 in$^2$)</td>
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<tr>
<td>Linear Wire Density</td>
<td>4.27E-5 kg/m (2.39E-6 lb/in)</td>
<td>6.10E-6 kg/m (3.41E-7 lb/in)</td>
</tr>
<tr>
<td>Material Density</td>
<td>2774 kg/m$^3$ (0.10 lb/in$^3$)</td>
<td>19400 kg/m$^3$ (0.70 lb/in$^3$)</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>7.045E9 kg/m$^2$ (1.0E7 psi)</td>
<td>4.227E10 kg/m$^2$ (6.0E7 psi)</td>
</tr>
<tr>
<td>Yield Strength</td>
<td>1.55E7 kg/m$^2$ (22.0E3 psi)</td>
<td>10.57E7 kg/m$^2$ (150.0E3psi)</td>
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