The Lambda parameter and strange quark mass in two-flavor QCD

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talk based on [arXiv:1205.5380]

in collaboration with
F.Knechtli, B.Leder, M.Marinkovic, S.Schaefer, R.Sommer, F.Virotta

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Introductory remarks

**Toolbox**: lattice regularisation of QCD, including $N_f = 2$ dynamical quarks

1. **Compute fundamental parameters of QCD**, i.e., renormalization group invariant (RGI) quantities

   \[ \Lambda_{\text{QCD}}^{(N_f)} \; ; \; M_i, \; i = u, d, s, \ldots \]

   non-perturbatively

   \[ \sim \Lambda_{\overline{\text{MS}}}^2, m_s(2\text{GeV}) \]

2. **Good control over systematic errors**, like

   autocorrelations; finite volume, discretisation, \ldots, chiral extrapol.

   *(Coordinated Lattice Simulations, CLS, effort)*

3. **Scale setting** through physical quantities as $f_\pi, f_K, m_\Omega, \ldots$

   to convert dim.less numbers $(a f_\pi, a f_K, \ldots)$

   \[ \sim a, L, \ldots \text{in physical units} \]

4. **Compute further quantities of interest**

   decay constants, masses, form factors, \ldots, ALPHAs B-physics program
GENERAL SETUP
Non-perturbative Renormalization

Ingredient: Schrödinger functional as intermediate renormalization scheme

- massless, finite volume renorm. scheme in the continuum
- IR regulator on the lattice (Dirichlet b.c. in time) \( \Rightarrow m = 0 \) on the lattice
- NP definition of a running coupling \( \Rightarrow \bar{g}^2(\mu) \), w/ box size \( L = 1/\mu \)
- \( \mathcal{N}_f = 2 \): QCD running coupling [ALPHA’04] and mass [ALPHA’05] known through finite size scaling technique

NP running coupling:

NP running mass:

\( \Lambda \)-parameter, low energy scale \( \mu \equiv 1/L_1 \) \( \Rightarrow \) RGI quark masses \( M_i \)
Dynamical fermion simulations
e.g.: O7 ensemble, $64^3 \times 128$, $m_\pi \sim 270$ MeV

Lattice framework:
- plaquette gauge action
- mass-degenerate doublet of non-perturbatively improved Wilson fermions

Our criteria:
- FV effects small by construction
\[ Lm_\pi \geq 4.0 \]
- data for chiral extrapolation uses
\[ m_\pi \lesssim 500 \text{ MeV} \]
- three lattice spacings
\[ < 0.08 \text{ fm} \]

Goal: controlled extrapol. to physical point ★

\begin{center}
\begin{tikzpicture}
\begin{axis}[
width=\textwidth,height=\textwidth,
xlabel={$a/\text{fm}$},
ylabel={$m_\pi/\text{MeV}$},
]
\addplot[only marks, mark=*, mark size=2pt] coordinates {
(0,400)
(0.02,500)
(0.04,600)
(0.06,700)
(0.08,800)
};
\end{axis}
\end{tikzpicture}
\end{center}
TWO STRATEGIES FOR CHIRAL EXTRAPOLATIONS
Setting the scale

Standard procedure, still room for improvements though

- calibrate lattice spacing $a$ through dimensionful reference quantity $Q$:
  
  $$a^{-1} [\text{MeV}] = \frac{Q|_{\text{exp}} [\text{MeV}]}{[aQ]_{\text{latt}}}, \quad Q \in \{ f_\pi, f_K, m_N, m_\pi, \ldots \}$$

choose well behaved quantity $Q$ according to

- experimentally available input
- reasonable signal-to-noise ratio
- well-controlled and understood chiral behaviour
- mild cut-off effects
- ...

Our choice: kaon decay constant $f_K$

- milder chiral extrapolation compared to $f_\pi$
- better control over systematic errors (2 strategies)
Two chiral extrapolations

\[ M_s, \quad M_{\text{phys}}, \quad M_{\text{light}} \]

- Strategy 1: \[ M_s = \text{const} \]
- Strategy 2: \[ R_K = \text{const} \]
Strategy 1

fix ratio:

\[
\frac{m_K^2(\kappa_1, h(\kappa_1))}{f_K^2(\kappa_1, h(\kappa_1))} = R_K = \frac{m_{K,\text{phys}}^2}{f_{K,\text{phys}}^2}
\]

- trajectory with \( m_K \approx m_{K,\text{phys}} \) and thus
  \[
  \bar{m}_s + \bar{m}_\text{light} \approx \text{const} + O(m^2)
  \]
- \( \kappa_3 \equiv h(\kappa_1) \) determined by interpolation
- PQ-SU(3) ChPT: \([\text{Sharpe'97}]\)
  \[
  f_K(\kappa_1, h(\kappa_1)) \rightarrow f_{K,\text{phys}}
  \]
- systematic expansion in
  \[
  m_\pi^2, m_K^2 \leq m_{K,\text{phys}}^2
  \]
Strategy 2

fix strange quark’s PCAC mass:

\[ am_{34}(\kappa_1, s(\kappa_1, \mu)) \overset{!}{=} \mu \]

- \( \mu \) fixed for 2 values (independent of \( \kappa_1 \))
- \( \kappa_3 \equiv s(\kappa_1, \mu) \) determined by interpolation
- SU(2) ChPT: [Roessl:1999; AlltonEtAl:2008]

\[
af_K(\kappa_1, s(\kappa_1, \mu)) \rightarrow p(\mu) \\
[am_K]^2(\kappa_1, s(\kappa_1, \mu)) \rightarrow q(\mu)
\]

- solve numerically for \( \mu_s \):

\[
\left. \frac{q(\mu)}{p(\mu)^2} \right|_{\mu=\mu_s} \equiv \frac{m_{K,phys}^2}{f_{K,phys}^2}
\]
Results for chiral extrapolations

- Global fit
- Expansion variable:
  \[ y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, \kappa_3)} \leq 0.1 \]
- ChPT to 1-loop, i.e., LEC or combinations thereof multiply \((y_1 - y_\pi)\)
- \(O(y^2)\) small

\[ f_{K,\text{phys}} = 155 \text{ MeV} \ [\text{FLAG'11}] \] (isospin symmetric limit & QED effects removed)

**Strategy 1:**
- NLO partially quenched SU(3) ChPT
- \(\kappa_3 = h(\kappa_1)\)
- \(a = [af_K]/f_{K,\text{phys}}\)

**Strategy 2:**
- NLO SU(2) ChPT
- \(\kappa_3 = s(\kappa_1, \mu_s)\)
- \(a = p(\mu_s)/f_{K,\text{phys}}\)
Results for chiral extrapolations
Systematics at finest lattice spacing

- cut at $y_1 = 0.1$
  \[ y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1)} \]

- global fit:
  $\alpha_4, \alpha_f$ independent of $\beta$

check systematic from chiral extrapolation:
  - chiral vs. linear fit (see above)
  - $|\text{Strategy 1 - Strategy 2}|$

LEC’s:
  \[ \alpha_4 = 0.57(12), \quad \alpha_f = 1.13(8) \]

taking all systematics into account:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a f_K$</th>
<th>$a[\text{fm}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>0.0593(7)(6)</td>
<td>0.0755(9)(7)</td>
</tr>
<tr>
<td>5.3</td>
<td>0.0517(6)(6)</td>
<td>0.0658(7)(7)</td>
</tr>
<tr>
<td>5.5</td>
<td>0.0382(4)(3)</td>
<td>0.0486(4)(5)</td>
</tr>
</tbody>
</table>
RESULTS
\[ \Lambda, M_s \]
The $\Lambda$ parameter of $N_f = 2$ QCD

Master formula: [ALPHA’05]

$$\frac{\Lambda_{\text{MS}}^{(2)}}{f_K} = \frac{1}{[f_K L_1]_{\text{cont}}} \times \left[ L_1 \Lambda_{\text{SF}}^{(2)} \right]_{\text{cont}} \times \frac{\Lambda_{\text{MS}}^{(2)}}{\Lambda_{\text{SF}}^{(2)}}$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484, \quad \mu_1 = 1/L_1$

Exact relation between schemes:

- $\Lambda_{\text{MS}}^{(2)}/\Lambda_{\text{SF}}^{(2)} = 2.382035(3)$

Non-perturbative running coupling:

- $L_1 \Lambda_{\text{SF}}^{(2)} = 0.264(15)$

Missing piece:

$$[f_K L_1]_{\text{cont}}$$
The $\Lambda$ parameter of $N_f = 2$ QCD

Master formula: [ALPHA’05]

$$\frac{\Lambda^{(2)}_{\text{MS}}}{f_K} = \frac{1}{[f_K L_1]_{\text{cont}}} \times [L_1 \Lambda^{(2)}_{\text{SF}}]_{\text{cont}} \times \frac{\Lambda^{(2)}_{\text{MS}}}{\Lambda^{(2)}_{\text{SF}}}$$

using result from strategy 1

$$L_1 f_K = 0.315(8)(2)$$

$$\downarrow$$

$$\Lambda^{(2)}_{\text{SF}} / f_K = 0.84(6)$$

$$\downarrow$$

$$\Lambda^{(2)}_{\text{MS}} = 310(20) \text{ MeV}$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484$, $\mu_1 = 1/L_1$

Exact relation between schemes:

- $\Lambda^{(2)}_{\text{MS}} / \Lambda^{(2)}_{\text{SF}} = 2.382035(3)$

Non-perturbative running coupling:

- $L_1 \Lambda^{(2)}_{\text{SF}} = 0.264(15)$

Missing piece:

$$[f_K L_1]_{\text{cont}}$$

![Graph showing the missing piece and comparison between strategies 1 and 2](image-url)
The strange quark mass in $N_f = 2$ QCD

RGI strange quark mass:

$$M_s = Z_M m_s = \frac{M}{m(\mu_1)} \times \bar{m}_s(\mu_1)$$

$$= \frac{M}{m(\mu_1)} \times \frac{Z_A}{Z_P(\mu_1)} \times m_s$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484$, $\mu_1 = 1/L_1$

Non-perturbative running mass: [ALPHA'05]

- $M/m(\mu_1) = 1.308(16)$

Strategy 1:

- only combination $m_s + \hat{m}$ directly accessible
- remove average light quark mass
- $\hat{m} \rightarrow$ additional systematic uncertainty
- combination of LEC $\alpha_4$ & $\alpha_6$ from constrained global fit

Strategy 2: (conceptually preferred)

- no additional LEC's involved

$$m_s/f_K = 0.678(12)(5)$$

$$M_s = 138(3)(1)\text{ MeV}, m_{\overline{MS}}(\mu = 2\text{ GeV}) = 102(3)(1)\text{ MeV}$$
The strange quark mass in $N_f = 2$ QCD

RGI strange quark mass:

$$M_s = Z_M m_s = \frac{M}{\bar{m}(\mu_1)} \times \bar{m}_s(\mu_1)$$

$$= \frac{M}{\bar{m}(\mu_1)} \times \frac{Z_A}{Z_P(\mu_1)} \times m_s$$

Renorm. scale set through SF coupling:

\begin{itemize}
  \item $\bar{g}^2(L_1) = 4.484, \quad \mu_1 = 1/L_1$
  \item Non-perturbative running mass: $[\text{ALPHA'05}]$
  \item $M/\bar{m}(\mu_1) = 1.308(16)$
\end{itemize}

Strategy 1:

\begin{itemize}
  \item only combination $\bar{m}_s + \hat{m}$ directly accessible
  \item remove average light quark mass $\hat{m}$
  \item $\sim$ additional systematic uncertainty
  \item combination of LEC $\alpha_4$ & $\alpha_6$ from constrained global fit
\end{itemize}

Strategy 2: (conceptually preferred)

\begin{itemize}
  \item no additional LEC's involved
  \item $\bar{m}_s / f_K = 0.678(12)(5)$
\end{itemize}

$$M_s = 138(3)(1) \text{ MeV}, \quad \bar{m}_s^{\text{MS}}(\mu = 2 \text{ GeV}) = 102(3)(1) \text{ MeV}$$
Summary

- **complete analysis of CLS based** $N_f = 2$ **ensembles**;
  
  $a = (0.05 - 0.08)$ fm

- Conservative error estimates through autocorrelation analysis

- **Scale setting with** $f_K$
  
  - Two strategies for chiral extrapolation in agreement
  
  - simple linear extrapolation also agrees within errors
  
  - small cut-off effects

  $\leadsto$ control over systematic effects in scale setting

- **Main results**

  $\Lambda^{(2)}_{\text{MS}} = 310(20)$ MeV ,

  $M_s = 138(3)(1)$ MeV

  $\bar{m}_s^{\text{MS}} (\mu = 2 \text{ GeV}) = 102(3)(1)$ MeV

  controlled reduction of statistical & systematic error achieved (since 2005)

  consistent with scale setting through $r_0$ (not covered)
Outlook

- unclear situation

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$\Lambda_{\overline{\text{MS}}}$</th>
<th>experiment</th>
<th>theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>238(19) MeV</td>
<td>$m_K, K \rightarrow \mu\nu_\mu, K \rightarrow \pi\mu\nu_\mu$</td>
<td>lattice gauge theory [ALPHA'93]</td>
</tr>
<tr>
<td>2</td>
<td>310(20) MeV</td>
<td>$m_K, K \rightarrow \mu\nu_\mu, K \rightarrow \pi\mu\nu_\mu$</td>
<td>this work</td>
</tr>
<tr>
<td>5</td>
<td>212(12) MeV</td>
<td>world average</td>
<td>perturb. theory [Bethke'11]</td>
</tr>
</tbody>
</table>

$\rightsquigarrow$ remove remaining systematic uncertainty due to quenching $\rightsquigarrow N_f = 3, 4$

? TODO ?

- other low energy constants
- $\hat{m}/m_s$
- ...

Thank you for your attention!

... and many thanks to my colleagues

F.Knechtli, B.Leder, M.Marinkovic, S.Schaefer, R.Sommer, F.Virotta
BACKUP SLIDES
Autocorrelations in HMC generated data

... grow towards the continuum limit

- Autocorrelation depends on observable
  \[ \bar{F} = \frac{1}{N} \sum_{i=1}^{N} F_i \]

- Autocorrelation function
  \[ \Gamma_F(t) = \langle (F_t - \bar{F})(F_0 - \bar{F}) \rangle \]

- Integrated autocorrelation time
  \[ \tau_{\text{int}} = \frac{1}{2} \sum_{t=1}^{\infty} \rho_F(t) \cdot \rho_F(t) = \frac{\Gamma_F(t)}{\Gamma_F(0)} \]

- \( \tau_{\exp}(\beta) = 200 \frac{c_\tau}{R_{\text{act}}} e^{7[\beta-5.3]} \)

- \( \tau_{\exp} \) estimated from \( \beta = 5.3 \) and quenched scaling [Schaefer, Sommer, Virotta’10]

- \( R_{\text{act}} \) – fraction of active links (DD-HMC: \( \sim 30\% \); MP-HMC: 100%)

- \( c_{\tau=2} = 1; c_{\tau=0.5} = 2 \)

Exponential tail accounts for \( \sim 50\% \) of the total uncertainty!
Masses and decay constants
A handle on excited states \((128 \times 64^3 \text{ lattice, } a = 0.045\text{fm}, m_\pi = 268\text{MeV})\)

Taking \(\tau_{\text{exp}}\) into account:

- **1\textsuperscript{st} step:** double exponential fit to \(\{c_1, c_2, m\}\) with \(m' = m + 2m_\pi\)

\[
F(x_0) = c_1 \left( e^{-mx_0} + e^{-m(T-x_0)} \right) + c_2 \left( e^{-m'x_0} + e^{-m'(T-x_0)} \right)
\]

+ criterion: \(\text{stat.error}(aM_{\text{eff}}) \leq 4 \times 1\textsuperscript{st} \text{ excited state contribution} \Rightarrow \ x_0^{\min}\)

- **2\textsuperscript{nd} step:** single exponential fit \((c_2 \equiv 0)\)

\(c_2 \approx 0\) well justified theoretically; see talk by M.Golterman
Results for chiral extrapolations

Strategy 1

![Graph with data points and fitted lines]

- **strat.1**: ▲ □
- **strat.2**: ▲ ● ■
- Cut at \( y_1 = 0.1 \)

\[
y_1 \equiv \frac{m_{\pi}^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, h(\kappa_1))}
\]

- Constrained global fit: \( \alpha_4 \) independent of \( a \)

Partially quenched ChPT:

\[
f_K(\kappa_1, h(\kappa_1)) = f_{K,\text{phys}} \left[ 1 + \bar{L}_K(y_1, y_K) + (\alpha_4 - \frac{1}{4}) (y_1 - y_\pi) + O(y^2) \right],
\]

\[
\bar{L}_K(y_1, y_K) = L_K(y_1, y_K) - L_K(y_\pi, y_K),
\]

\[
L_K(y_1, y_K) = -\frac{1}{2} y_1 \log(y_1) - \frac{1}{8} y_1 \log(2y_K/y_1 - 1).
\]

\( y_3 \equiv y_1(\pi \to K) \) does not appear, since \( y_3 = 2y_K - y_1 + O(y^2) \)

Combination of chiral logs overall much smaller with less curvature compared to \( L_\pi \)
Results for chiral extrapolations

Strategy 2

\[ y_1 \equiv \frac{m^2_\pi(\kappa_1)}{8\pi^2 f^2_K(\kappa_1, s(\kappa_1, \mu))} \]

cut at \( y_1 = 0.1 \)

strat.1: \( \Delta, \bullet, \square \)  \hspace{1cm} strat.2: \( \Delta, \bullet, \blacksquare \)

SU(2) ChPT:

\[ a f_K(\kappa_1, s(\kappa_1, \mu)) = p(\mu) \left[ 1 - \frac{3}{8} [y_1 \log(y_1) - y_\pi \log(y_\pi)] + \alpha_f(\mu) (y_1 - y_\pi) + O(y_1^2) \right] , \]
\[ a^2 m^2_K(\kappa_1, s(\kappa_1, \mu)) = q(\mu) \left[ 1 + \alpha_m(\mu) (y_1 - y_\pi) + O(y_1^2) \right] \]

for three different fixed \( \mu \)

\[ \frac{q(\mu_s)}{p(\mu_s)^2} = \frac{m^2_{K,\text{phys}}}{f^2_{K,\text{phys}}} \]

\[ \Rightarrow \quad a = \frac{p(\mu_s)}{f_{K,\text{phys}}} \]
The strange quark mass in $N_f = 2$ QCD

RGI strange quark mass:

$$M_s = Z_M m_s = \frac{M}{\bar{m}(\mu_1)} \times \bar{m}_s(\mu_1)$$

$$= \frac{M}{\bar{m}(\mu_1)} \times \frac{Z_A}{Z_P(\mu_1)} \times m_s$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484, \quad \mu_1 = 1/L_1$

Non-perturbative running mass: [ALPHA’05]

- $M/\bar{m}(\mu_1) = 1.308(16)$

Strategy 1:

$$\frac{2m_{13}(\kappa_1, h(\kappa_1))}{Z_P f_K(\kappa_1, h(\kappa_1))} = \frac{\bar{m}_s + \hat{m}}{f_{K,\text{phys}}} \left[ 1 + \bar{L}_m(y_1, y_K) + (\alpha_{4,6} - \frac{1}{4}) (y_1 - y_\pi) + \mathcal{O}(y_1^2) \right]$$,

$$\bar{L}_m(y_1, y_K) = L_m(y_1, y_K) - L_m(y_\pi, y_K), \quad \alpha_{4,6} = 3\alpha_4 - 4\alpha_6,$$

$$L_m(y_1, y_K) = -(y_K - \frac{3}{8}y_1) \log(2y_K/y_1 - 1) - y_K \log(y_1)$$

Strategy 2: (conceptually preferred)

$$\frac{M_s}{f_{K,\text{phys}}} = \frac{M}{\bar{m}(L_1)} \times \frac{1}{Z_P(L_1)} \times \left[ 1 - \bar{b}_P \mu_s \right] \frac{\mu_s}{f_{\text{bare}}_{K,\text{phys}}}$$

Strategy 2: (conceptually preferred)
The strange quark mass in $N_f = 2$ QCD

**Strategy 1:**
- use $y_1 = m_\pi^2(\kappa_1)/8\pi^2 f_K^2(\kappa_1) < 0.1$
- remove avg. light quark contribution $\hat{m}$ by correction factor $\rho$: $M_S = (M_S + \hat{M})(1 - \rho)$
- LO\(\chi PT\): $\rho = \hat{M}/(M_S + \hat{M}) \approx \frac{m_\pi^2}{2m_K^2}$
- LEC $\alpha_{4,6}$ from constrained global fit

**Strategy 2:**
- neglect tiny $O(a)$ impr. term, $(\bar{b}_A - \bar{b}_P)am_{\text{sea}}$

\[ \frac{m_S}{f_K} = 0.678(12)(5) \]

\[ M_S = 138(3)(1) \text{ MeV} \]

\[ \overline{m}_S^{MS}(\mu = 2\text{GeV}) = 102(3)(1) \text{ MeV} \]

\[ \frac{M_S}{f_{K,\text{phys}}} = \frac{M}{\overline{m}(L_1)} \times \frac{1}{Z_P(L_1)} \times [1 - \bar{b}_P\mu_s] \frac{\mu_s}{f_{bare}^{K,\text{phys}}} \]

(strategy conceptually preferred)
Generic strategy

... to connect low- & high-energy regime NP’ly

one more important ingredient:

How to connect hadronic observables from low-energies to the widely used MS-scheme?

![Diagram with scales and symbols]

\[ 1/L \ll \mu, E, |p| \ll 1/a \]

\[ \mu_{\text{pert}} \gg \Lambda \]
Generic strategy

... to connect low- & high-energy regime NP’ly

one more important ingredient:

How to connect hadronic observables from low-energies to the widely used MS-scheme?

intermediate renorm. scheme

≡ Schrödinger functional

(finite volume, continuum scheme)

- RG scale evolution solved NP’ly
- thus continuum limit needs to be well controlled (small cutoff effects)
- low-energy scale fixed by imposing

$$\bar{g}^2(L_{\text{max}}) \equiv u_{\text{max}}$$

100 MeV
10 GeV
1 GeV
100 MeV

$$\mu \rightarrow \infty$$

intermediate (SF)

PT (MS)

$$m_Z$$

$$m_\tau$$

$$m_D$$

$$m_p$$

$$m_\pi$$

$$1/L \ll \mu, E, |p| \ll 1/a$$

$$\mu_{\text{pert}} \gg \Lambda$$
Scale dependence of QCD parameters

Running coupling and mass,

**Renormalization group (RG) equations**

1. **coupling**

\[ \mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \rightarrow 0 \sim -\bar{g}^3 (b_0 + b_1 \bar{g}^2 + \ldots) \]

2. **mass**

\[ \frac{\mu}{\bar{m}} \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \quad \bar{g} \rightarrow 0 \sim -\bar{g}^2 (d_0 + d_1 \bar{g}^2 + \ldots) \]

in a massless scheme, \( b_0, b_1, d_0 \) universal

Solution leads to exact equations in mass-independent scheme
Scale dependence of QCD parameters

Running coupling and mass, Renormalization Group Invariants (RGI)

Renormalization group (RG) equations

1. coupling

\[ \mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \xrightarrow{\bar{g} \to 0} -\bar{g}^3 (b_0 + b_1 \bar{g}^2 + \ldots) \]

\[ \Lambda \equiv \mu \left[ b_0 \bar{g}^2 \right]^{-b_1/(2b_0^2)} \exp \left\{ -\int_0^{\bar{g}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\} \]

2. mass

\[ \frac{\mu}{\bar{m}} \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \xrightarrow{\bar{g} \to 0} -\bar{g}^2 (d_0 + d_1 \bar{g}^2 + \ldots) \]

\[ M \equiv \bar{m} \left[ 2b_0 \bar{g}^2 \right]^{-d_0/(2b_0)} \exp \left\{ -\int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\} \]

in a massless scheme, \( b_0, b_1, d_0 \) universal

Solution leads to exact equations in mass-independent scheme