Roy–Steiner equations for $\pi N$ scattering

C. Ditsche\textsuperscript{1}  M. Hoferichter\textsuperscript{1}  B. Kubis\textsuperscript{1}  U.-G. Meißner\textsuperscript{1,2}

\textsuperscript{1}Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn
\textsuperscript{2}Institut für Kernphysik, Institute for Advanced Simulation, and Jülich Center for Hadron Physics, Forschungszentrum Jülich

7\textsuperscript{th} International Workshop on Chiral Dynamics 2012
Jefferson Lab, August 8\textsuperscript{th}

[JHEP 1206 (2012) 043]

(\leftrightarrow \text{see also following talk by M. Hoferichter on [JHEP 1206 (2012) 063]})
1. Motivation: Why Roy–Steiner equations for $\pi N$ scattering?

2. Warm-up: Roy equations for $\pi\pi$ scattering

3. $\pi N$ scattering basics

4. Roy–Steiner equations for $\pi N$ scattering

5. Solving the $t$-channel Muskhelishvili–Omnès problem

6. Summary & Outlook
Motivation: Why $\pi N$ scattering? Why Roy–Steiner equations?

- Renewed interest in $\pi N$ scattering:
  - $\pi N \rightarrow \pi N$ amplitudes e.g. for $\sigma$-term physics
  - $\bar{N}N \rightarrow \pi\pi$ crossed amplitudes e.g. for nucleon form factors

$\Rightarrow$ Need esp. low-energy (pseudophysical) amplitudes which are not very well known

$\Rightarrow$ Roy(–Steiner) equations

$\Rightarrow$ Partial-Wave (Hyperbolic) Dispersion Relations coupled by unitarity and crossing symmetry

$\Rightarrow$ Can study processes at low energies with high precision:
- $\pi\pi$ scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
- $\pi K$ scattering: [Büttiker et al. (2004)]
- $\gamma\gamma \rightarrow \pi\pi$ scattering: [Hoferichter et al. (2011)]

Roy–Steiner equations for $\pi N$ scattering:
- Obtain low-energy (pseudophysical) amplitudes with better precision (update input & give errors)
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)

C. Ditsche (HISKP & BCTP, Uni Bonn)
Motivation: Why $\pi N$ scattering? Why Roy–Steiner equations?

- Renewed interest in $\pi N$ scattering:
  - $\pi N \to \pi N$ amplitudes e.g. for $\sigma$-term physics
  - $\bar{N}N \to \pi \pi$ crossed amplitudes e.g. for nucleon form factors

$\Rightarrow$ Need esp. low-energy (pseudophysical) amplitudes which are not very well known

**Roy(–Steiner) equations** = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **PW(H)DRs** together with unitarity, crossing symmetry, and chiral symmetry
  $\Rightarrow$ Can study processes at low energies with high precision:
  - $\pi \pi$ scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - $\pi K$ scattering: [Büttiker et al. (2004)]
  - $\gamma \gamma \to \pi \pi$ scattering: [Hoferichter et al. (2011)]
**Motivation: Why $\pi N$ scattering? Why Roy–Steiner equations?**

- **Renewed interest in $\pi N$ scattering:**
  - $\pi N \to \pi N$ amplitudes e.g. for $\sigma$-term physics
  - $\bar{N}N \to \pi \pi$ crossed amplitudes e.g. for nucleon form factors

  $\Rightarrow$ Need esp. low-energy (pseudophysical) amplitudes which are not very well known

**Roy(–Steiner) equations**

$\equiv$ Partial-Wave (Hyperbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **PW(H)DRs** together with unitarity, crossing symmetry, and chiral symmetry
  $\Rightarrow$ Can study processes at low energies with **high precision**:

  - $\pi \pi$ scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - $\pi K$ scattering: [Büttiker et al. (2004)]
  - $\gamma \gamma \to \pi \pi$ scattering: [Hoferichter et al. (2011)]

**Roy–Steiner equations for $\pi N$ scattering:**

- Obtain low-energy (pseudophysical) amplitudes with better precision (update input & give errors)
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
Warm-up: Roy equations for $\pi\pi$ scattering (1)

- $\pi\pi \rightarrow \pi\pi$ is fully crossing symmetric in Mandelstam variables $s$, $t$, and $u = 4M_{\pi}^2 - s - t$
- Roy equations respect all available symmetry constraints:
  - Lorentz invariance, unitarity, isospin & crossing symmetry, and (maximal) analyticity
Warm-up: Roy equations for $\pi\pi$ scattering (1)

- $\pi\pi \rightarrow \pi\pi$ is fully crossing symmetric in Mandelstam variables $s$, $t$, and $u = 4M^2_{\pi} - s - t$

- Roy equations respect all available symmetry constraints:
  - Lorentz invariance, unitarity, isospin & crossing symmetry, and (maximal) analyticity

- Start from twice-subtracted fixed-$t$ DRs of the generic form $\leftrightarrow s + t + u = 4M^2_{\pi} = s' + t + u'$

\[
T(s, t) = c(t) + \frac{1}{\pi} \int \frac{ds'}{s'^2} \left\{ \frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right\} \text{Im} T(s', t)
\]

- Determine subtraction functions $c(t)$ via crossing symmetry

- PW expansion ($I \in \{0, 1, 2\}$, $J = \ell$): $T^I(s, t) = 32\pi \sum_{J=0}^{\infty} (2J + 1) P_J \left( \cos \theta(s, t) \right) t^I_J(s)$

- PW decomposition of these DRs yields the Roy equations [Roy (1971)]

\[
t^I_J(s) = k^I_J(s) + \frac{1}{\pi} \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int ds' K^{I'I'}_{JJ'}(s, s') \text{Im} t^{I'}_{J'}(s')
\]

- Kernels: analytically known, contain Cauchy kernel $K^{I'I'}_{JJ'}(s, s') = \frac{\delta^{I'I'} \delta_{JJ'}}{s' - s} + \ldots$
Warm-up: Roy equations for $\pi\pi$ scattering (2)

$$t_I^J(s) = k_I^J(s) + \frac{1}{\pi} \sum_{I'=0}^2 \sum_{J'=0}^\infty \int d\pi^2 \left\{ K_{JJ'}^{II'}(s, s') \right\} \text{Im} t_{J'}^{J'}(s')$$

- **Validity:** $4M_\pi^2 \leq s \leq 60M_\pi^2 \approx (1.08 \text{ GeV})^2$ $\rightarrow$ Mandelstam analyticity $\Rightarrow$ $s \leq 68M_\pi^2 \approx (1.15 \text{ GeV})^2$

- **Subtraction constants** (free parameters) contained in $k_I^J(s)$: $\pi\pi$ scattering lengths
  $\Rightarrow$ Matching to **Chiral Perturbation Theory** [Colangelo et al. (2001)]
Warm-up: Roy equations for $\pi\pi$ scattering (2)

$$t^I_J(s) = k^I_J(s) + \frac{1}{\pi} \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4M^2_{\pi}}^{\infty} ds' K_{JJ'}^{II'}(s, s') \text{Im} t^{I'}_{J'}(s')$$

- **Validity**: $4M^2_{\pi} \leq s \leq 60M^2_{\pi} \approx (1.08 \text{ GeV})^2$ $\Rightarrow$ Mandelstam analyticity $\Rightarrow s \leq 68M^2_{\pi} \approx (1.15 \text{ GeV})^2$

- **Subtraction constants** (free parameters) contained in $k^I_J(s)$: $\pi\pi$ scattering lengths
  $\Rightarrow$ Matching to **Chiral Perturbation Theory** [Colangelo et al. (2001)]

- **Elastic unitarity** leads to coupled integral equations for the **phase shifts** $\delta^I_J(s)$

$$\text{Im} t^I_J(s) = \sigma^{\pi}(s) \left| t^I_J(s) \right|^2 \theta(t - 4M^2_{\pi})$$

$$\Rightarrow \sigma^{\pi}(s) t^I_J(s) = \frac{e^{2i\delta^I_J(s)} - 1}{2i} = \sin \delta^I_J(s) e^{i\delta^I_J(s)}$$

$$\sigma^{\pi}(s) = \sqrt{1 - \frac{4M^2_{\pi}}{s}}$$

C. Ditsche (HISKP & BCTP, Uni Bonn)
Generically: \( \pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p') \)

Kinematics:
\[
\begin{align*}
s &= (p + q)^2, \\
t &= (p - p')^2, \\
u &= (p - q')^2 \\
u &= 2(m^2 + M_{\pi}^2) - s - t, \\
\nu &= \frac{s-u}{4m}
\end{align*}
\]

Isospin structure:
\[
T^{ba} = \delta^{ba} T^+ + i \epsilon^{bac} \tau^c T^-
\]

Lorentz structure \((I \in \{+, -\})\):
\[
T^I = \bar{u}(p') \left\{ A^I + \frac{q' + q}{2} B^I \right\} u(p)
\]

Crossing symmetry relates amplitudes for \(s/u\)-channel \((\pi N \rightarrow \pi N)\) and \(t\)-channel \((\bar{NN} \rightarrow \pi\pi)\), crossing even and odd amplitudes:
\[
A^{\pm}(\nu, t) = \pm A^{\pm}(-\nu, t), \quad B^{\pm}(\nu, t) = \mp B^{\pm}(-\nu, t)
\]
**πN scattering basics: Subthreshold expansion**

- **Subtraction of pseudovector Born terms:** \( X \mapsto \bar{X} \)
- \( D^\pm = A^\pm + \nu B^\pm \)
- **Expand** crossing **even** amplitudes
  
  \[ X^I(\nu^2, t) \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\} \]

  around **subthreshold point** \( \nu = t = 0 \):

  \[ X^I(\nu^2, t) = \sum_{m,n=0}^{\infty} x_{mn}^I (\nu^2)^m t^n \]
- PWs allow for easy incorporation of unitarity constraints \( \leftrightarrow \) helicity formalism [Jacob/Wick (1959)]
**πN scattering basics: Partial Waves**

- **PWs** allow for easy incorporation of *unitarity* constraints  ⇝  helicity formalism [Jacob/Wick (1959)]

- **s-channel** PW projection:
  \[ z_s = \cos \theta_s , \quad W = \sqrt{s} \]
  \[
  A^I_\ell(s) = \left. \frac{1}{-1} \int_{-1}^1 \, dz_s \, P_\ell(z_s) A^I(s, t) \right|_{t=t(s,z_s)}
  \]
  \[
  f^I_{\ell\pm}(W) = \frac{1}{16\pi W} \left\{ (E + m) \left[ A^I_\ell(s) + (W - m)B^I_\ell(s) \right] + (E - m) \left[ - A^I_{\ell\pm 1}(s) + (W + m)B^I_{\ell\pm 1}(s) \right] \right\}
  \]

- **MacDowell symmetry:**
  \[
  f^I_{\ell+}(W) = -f^I_{(\ell+1)-}(-W) \quad \forall \ell \geq 0 \quad \text{[MacDowell (1959)]}
  \]
**πN scattering basics: Partial Waves**

- **PWs** allow for easy incorporation of **unitarity** constraints \(\leftrightarrow\) helicity formalism [Jacob/Wick (1959)]

- **s-channel** PW projection:
  \[ z_s = \cos \theta_s , \quad W = \sqrt{s} \]
  \[
  A^I_{\ell}(s) = \frac{1}{16\pi W} \left\{ (E + m) [A^I_{\ell}(s) + (W - m)B^I_{\ell}(s)] + (E - m) \left[ -A^I_{\ell+1}(s) + (W + m)B^I_{\ell+1}(s) \right] \right\}
  \]

- **MacDowell symmetry**:
  \[ f^I_{\ell+}(W) = -f^I_{(\ell+1)-}(-W) \quad \forall \ell \geq 0 \quad [\text{MacDowell (1959)}] \]

- **t-channel** PW expansion:
  \[ z_t = \cos \theta_t \]
  \[
  A^I(s,t) \bigg|_{s=s(t,z_t)} = -\frac{4\pi}{p_t^2} \sum_J (2J + 1)(p_tq_t)^J \left\{ P_J(z_t) f^I_+(t) - \frac{m}{\sqrt{J(J+1)}} z_t P_J(z_t) f^I_-(t) \right\}
  \]
  \[
  B^I(s,t) \bigg|_{s=s(t,z_t)} = 4\pi \sum_{J>0} \frac{2J+1}{\sqrt{J(J+1)}} (p_tq_t)^{J-1} P_J(z_t) f^I_-(t)
  \]

- **G-parity** \(\Rightarrow\) even \(J\) for \(I=+\) \((I_t=0)\), odd \(J\) for \(I=-\) \((I_t=1)\)
**PWs** allow for easy incorporation of **unitarity** constraints $\leftrightarrow$ helicity formalism [Jacob/Wick (1959)]

- **$s$-channel** PW projection: 
  \[ z_s = \cos \theta_s , \quad W = \sqrt{s} \]
  \[ A^I_\ell(s) = \frac{1}{16\pi W} \left\{ (E + m) [A^I_\ell(s) + (W - m)B^I_\ell(s)] + (E - m) \left[ -A^I_{\ell\pm 1}(s) + (W + m)B^I_{\ell\pm 1}(s) \right] \right\} \]

- **MacDowell symmetry**: 
  \[ f^I_{\ell\pm}(W) = -f^I_{(\ell+1)\pm}(-W) \quad \forall \ell \geq 0 \] [MacDowell (1959)]

- **$t$-channel** PW expansion: 
  \[ z_t = \cos \theta_t \]
  \[ A^I(s,t) \big|_{s=s(t,z_t)} = -\frac{4\pi}{p_t^2} \sum_J (2J + 1)(p_t q_t)^J \left\{ P_J(z_t) f^I_J(t) - \frac{m}{\sqrt{J(J+1)}} z_t P_J(z_t) f^J_- (t) \right\} \]
  \[ B^I(s,t) \big|_{s=s(t,z_t)} = 4\pi \sum_{J>0} \frac{2J+1}{\sqrt{J(J+1)}} (p_t q_t)^{J-1} P_J(z_t) f^J_- (t) \]

- **$G$-parity** $\Rightarrow$ even $J$ for $I = +$ ($I_t = 0$), odd $J$ for $I = -$ ($I_t = 1$)

- **$s$-channel** PW expansion and **$t$-channel** PW projection in analogy
Roy–Steiner equations for $\pi N$ scattering: Hyperbolic DRs

(Unsubtracted) **Hyperbolic DRs:** $\leftrightarrow (s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R} \Rightarrow b = b(s, t, a)$

\[ A^+(s, t) = \frac{1}{\pi} \int_{(m+M_\pi)^2}^{\infty} ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im} A^+(s', t')}{t' - t} \]

\[ B^+(s, t) = N^+(s, t) + \frac{1}{\pi} \int_{(m+M_\pi)^2}^{\infty} ds' \left[ \frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im} B^+(s', t') + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\nu}{\nu'} \frac{\text{Im} B^+(s', t')}{t' - t} \]

\[ N^+(s, t) = g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right] \]

and similarly for $A^-, B^-, N^-$ [Hite/Steiner (1973)]
Roy–Steiner equations for $\pi N$ scattering: Hyperbolic DRs

(Unsubtracted) **Hyperbolic DRs:** \( (s-a)(u-a) = b = (s'-a)(u'-a) \) with \( a, b \in \mathbb{R} \) \( \Rightarrow b = b(s, t, a) \)

\[
A^+(s, t) = \frac{1}{\pi} \int_{(m+M_\pi)^2}^{\infty} ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - s} - \frac{1}{s' - a} \right] \text{Im}A^+(s', t') + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im}A^+(s', t')}{t' - t}
\]

\[
B^+(s, t) = N^+(s, t) + \frac{1}{\pi} \int_{(m+M_\pi)^2}^{\infty} ds' \left[ \frac{1}{s' - s} - \frac{1}{s' - s} \right] \text{Im}B^+(s', t') + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\nu}{\nu'} \frac{\text{Im}B^+(s', t')}{t' - t}
\]

\[
N^+(s, t) = g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right]
\]

and similarly for \( A^-, B^-, N^- \) [Hite/Steiner (1973)]

**Why HDRs?**

- Combine all physical regions \( \leftrightarrow \) important for reliable continuation to the subthreshold region [Stahov (1999)]
- Imaginary parts are only needed in regions where the corresponding PW decompositions converge
- **Range of convergence** can be maximized by tuning the free hyperbola parameter \( a \)
- Especially powerful for the determination of the $\sigma$-term [Koch (1982)]
Roy–Steiner equations for $\pi N$ scattering: Hyperbolic DRs

(Unsubtracted) Hyperbolic DRs: \( (s-a)(u-a) = b = (s'-a)(u'-a) \) with \( a, b \in \mathbb{R} \) \( \implies b = b(s, t, a) \)

\[
A^+(s, t) = \frac{1}{\pi} \int ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int dt' \frac{\text{Im} A^+(s', t')}{t' - t}
\]

\[
B^+(s, t) = N^+(s, t) + \frac{1}{\pi} \int ds' \left[ \frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im} B^+(s', t') + \frac{1}{\pi} \int dt' \frac{\nu}{v'} \frac{\text{Im} B^+(s', t')}{t' - t}
\]

\[
N^+(s, t) = g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right]
\]

and similarly for \( A^-, B^-, N^- \) [Hite/Steiner (1973)]

Why HDRs?

- Combine all physical regions \( \leftrightarrow \) important for reliable continuation to the subthreshold region [Stahov (1999)]
- Imaginary parts are only needed in regions where the corresponding PW decompositions converge
- Range of convergence can be maximized by tuning the free hyperbola parameter \( a \)
- Especially powerful for the determination of the $\sigma$-term [Koch (1982)]

How to derive closed Roy–Steiner system of PWHDRs:

1. Expand $s$-/$t$-channel imaginary parts of HDRs in $s$-/$t$-channel PWs, respectively
2. Project nucleon pole terms and all imaginary parts onto both $s$- and $t$-channel PWs
3. Combine resulting RS equations with the $s$- & $t$-channel (extended) PW unitarity relations
Roy–Steiner equations for $\pi N$ scattering: $s$-channel RS equations

- $s$-channel PW projection of pole terms and $s$-/t-channel-PW-expanded imaginary parts

\[ f^I_{\ell+}(W) = N^I_{\ell+}(W) + \frac{1}{\pi} \int_{4M^2_{\pi}}^{\infty} dt' \sum_J \left\{ G_{\ell J}(W, t') \, \text{Im} f^I_{+}(t') + H_{\ell J}(W, t') \, \text{Im} f^I_{-}(t') \right\} \]

\[ + \frac{1}{\pi} \int_{m+M_{\pi}}^{\infty} dW' \sum_{\ell'=0}^{\infty} \left\{ K^I_{\ell \ell'}(W, W') \, \text{Im} f^I_{\ell'}+(W') + K^I_{\ell \ell'}(W, -W') \, \text{Im} f^I_{(\ell'+1)-}(W') \right\} \]

\[ = -f^I_{(\ell+1)-}(-W) \quad \forall \, \ell \geq 0 \quad \text{[Hite/Steiner (1973)]} \]

- Kernels: analytically known, e.g. \[ K^I_{\ell \ell'}(W, W') = \frac{\delta_{\ell \ell'}}{W' - W} + \ldots \]

- Validity: $\hookrightarrow$ above threshold, assuming Mandelstam analyticity \[ a = -23.19 \, M^2_{\pi} \Rightarrow \]

\[ s \in \left[(m + M_{\pi})^2 = 59.64 \, M^2_{\pi}, \, 97.30 \, M^2_{\pi}\right] \quad \hookrightarrow \quad W \in \left[m + M_{\pi} = 1.08 \, \text{GeV}, \, 1.38 \, \text{GeV}\right] \]
Roy–Steiner equations for $\pi N$ scattering: $t$-channel RS equations

- **$t$-channel** PW projection of pole terms and $s$-/$t$-channel-PW-expanded imaginary parts

$$
(f^J_+)_+^0(t) = \tilde{N}^J_+(t) + \frac{1}{\pi} \int_0^{\infty} dt' \sum_{J'} \left\{ \tilde{K}^1_{JJ'}(t, t') \text{Im} f^I_+(t') + \tilde{K}^2_{JJ'}(t, t') \text{Im} f^I_-(t') \right\}
$$

$$
+ \frac{1}{\pi} \int_{m+M_{\pi}}^{\infty} dW' \sum_{\ell=0}^{\infty} \left\{ \tilde{G}_{J\ell}(t, W') \text{Im} f^I_{\ell+}(W') + \tilde{G}_{J\ell}(t, -W') \text{Im} f^I_{(\ell+1)-}(W') \right\}
$$

$$
(f^J_-)_-^1(t) = \tilde{N}^J_-(t) + \frac{1}{\pi} \int_0^{\infty} dt' \sum_{J'>0} \tilde{K}^3_{JJ'}(t, t') \text{Im} f^I_-(t')
$$

$$
+ \frac{1}{\pi} \int_{m+M_{\pi}}^{\infty} dW' \sum_{\ell=0}^{\infty} \left\{ \tilde{H}_{J\ell}(t, W') \text{Im} f^I_{\ell+}(W') + \tilde{H}_{J\ell}(t, -W') \text{Im} f^I_{(\ell+1)-}(W') \right\}
$$

- Kernels analytically known, e.g. $\tilde{K}^1_{JJ'}(t, t') = \frac{\delta_{JJ'}}{t' - t} + \ldots$, $\tilde{K}^3_{JJ'}(t, t') = \frac{\delta_{JJ'}}{t' - t} + \ldots$

- **Validity**: $\leftrightarrow$ above pseudothreshold, assuming Mandelstam analyticity $a = -2.71 M_{\pi}^2 \Rightarrow$

$$
t \in \left[ 4M_{\pi}^2, 205.45 M_{\pi}^2 \right] \iff \sqrt{t} \in \left[ 2M_{\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV} \right]
$$
\textbf{Roy–Steiner equations for $\pi N$ scattering: Unitarity relations}

- **$s$-channel unitarity relations** ($I_s \in \{1/2, 3/2\}$):

\[
\text{Im} f^I_{\ell \pm} (W) = q_s \left| f^I_{\ell \pm} (W) \right|^2 \theta (W - (m + M_\pi)) \\
+ \frac{1 - \left[ n^I_{\ell \pm} (W) \right]^2}{4q_s} \theta (W - (m + 2M_\pi))
\]
Roy–Steiner equations for $\pi N$ scattering: Unitarity relations

- **$s$-channel unitarity relations** ($I_s \in \{1/2, 3/2\}$):
  \[
  \text{Im} f_{\ell \pm}^{I_s} (W) = q_s \left| f_{\ell \pm}^{I_s} (W) \right|^2 \theta (W - (m + M_\pi)) \\
  + \frac{1 - \left[ n_{\ell \pm}^{I_s} (W) \right]^2}{4q_s} \theta (W - (m + 2M_\pi))
  \]

- **$t$-channel (extended) unitarity relations**: (2-body intermediate states: $\pi \pi$ & $\bar{K} K + \ldots$)
  \[
  \text{Im} f_{\pm}^J (t) = \sigma_t^\pi (t_J^I(t))^* f_{\pm}^I (t) \theta (t - 4M_\pi^2) + c_J 2\sqrt{2} k^J_I \sigma_t^K (g_J^I(t))^* h_{\pm}^J (t) \theta (t - 4M_K^2) + \ldots
  \]

- Only **linear** in $f_{\pm}^J (t)$ ⇒ less restrictive
- **Watson's theorem**: $\arg f_{\pm}^I (t) = \delta_I^K (t)$ [Watson (1954)] for $t < 16M_\pi^2 \lesssim 40M_\pi^2 \approx (0.88 \text{ GeV})^2$
**$s$-channel subproblem:**

- Kernels are diagonal for $I \in \{+, -\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\}$
  - $\Rightarrow$ All PWs are interrelated
- Once the $t$-channel PWs are known
  - $\Rightarrow$ Structure similar to $\pi \pi$ Roy equations

![Diagram showing the relationships between different PWs](image-url)
Roy–Steiner equations for $\pi N$ scattering: Recoupling schemes

**s-channel subproblem:**

- Kernels are diagonal for $I \in \{+, -\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\}$
  - $\Rightarrow$ All PWs are interrelated
- Once the $t$-channel PWs are known
  - $\Rightarrow$ Structure similar to $\pi\pi$ Roy equations

**t-channel subproblem:**

- Only higher PWs couple to lower ones
- Only PWs with even or odd $J$ are coupled
- No contribution from $f_{J'}^+$ to $f_J^-$
  - $\Rightarrow$ Leads to Muskhelishvili–Omnès problem
Roy–Steiner equations for $\pi N$ scattering: $t$-channel subproblem (1)

- Linear combinations $\Gamma^J(t) = m\sqrt{\frac{J}{J+1}} f^J_-(t) - f^J_+(t) \ \forall J \geq 1$

- (unsubtracted) $t$-channel subproblem can be written as

$$f^0_+(t) = \Delta^0_+(t) + \frac{t - 4m^2}{\pi} \int_{4M^2_\pi}^{\infty} dt' \frac{\text{Im} f^0_+(t')}{(t' - 4m^2)(t' - t)} \quad \text{[} f^0_+(4m^2) = 0 \text{] }$$

$$\Gamma^{J \geq 1}(t) = \Delta^J_\Gamma(t) + \frac{t - 4m^2}{\pi} \int_{4M^2_\pi}^{\infty} dt' \frac{\text{Im} \Gamma^J(t')}{(t' - 4m^2)(t' - t)} \quad \text{[} \Gamma^J(4m^2) = 0 \text{] }$$

$$f^{J \geq 1}_-(t) = \Delta^-_J(t) + \frac{1}{\pi} \int_{4M^2_\pi}^{\infty} dt' \frac{\text{Im} f^J_-(t')}{t' - t}$$

with $\text{Im} f^J_\pm(t) = \sigma^\pi_{tJ}(t^J_\pm(t))^* f^J_\pm(t) \theta(t - 4M^2_\pi) + \ldots$

- Inhomogeneities $\Delta(t)$: Born terms, $s$-channel integrals, and higher $t$-channel PWs; e.g.

$$\Delta^-_J(t) = \tilde{N}^-_J(t) + \frac{1}{\pi} \int_{m+M_\pi}^{\infty} dW' \sum_{\ell=0}^{\infty} \left\{ \tilde{H}_{J\ell}(t, W') \text{Im} f^I_\ell(W') + \tilde{H}_{J\ell}(t, -W') \text{Im} f^I_{\ell+1}(W') \right\}$$

$$+ \frac{1}{\pi} \int_{4M^2_\pi}^{\infty} dt' \sum_{J' \geq J+2} \tilde{K}^3_{JJ'}(t, t') \text{Im} f^J_-(t')$$
Roy–Steiner equations for $\pi N$ scattering: $t$-channel subproblem (2)

- In the **low-energy** (pseudophysical) region:
  - Only the lowest $s$-$t$-channel PWs are relevant
  - Can match amplitudes to ChPT [Büttiker/Meißner (2000), Becher/Leutwyler (2001), ...]
  - Neglect inelasticities in both the $\pi\pi$- and the $t$-channel PWs $\Rightarrow \eta^I_J(t) = 1$ & no $\bar{K}K$ + ... $\Rightarrow$ **Watson’s theorem**, single-channel approximation of $t$-channel subproblem
Roy–Steiner equations for $\pi N$ scattering: $t$-channel subproblem (2)

- In the **low-energy** (pseudophysical) region:
  - Only the lowest $s/t$-channel PWs are relevant
  - Can match amplitudes to ChPT [Büttiker/Meißner (2000), Becher/Leutwyler (2001), ...]
  - Neglect inelasticities in both the $\pi\pi$- and the $t$-channel PWs $\leftrightarrow \eta_I(t) = 1$ & no $\bar{K}K + \ldots$
    $\Rightarrow$ **Watson's theorem**, single-channel approximation of $t$-channel subproblem

  - (Single-channel) **Mushkelishvili–Omnès problem** with finite matching point $t_m$
    
    $[\text{Mushkelishvili (1953), Omnès (1958), Büttiker et al. (2004)]}$
    
    $$ \begin{align*}
    f(t) &= \Delta(t) + \frac{1}{\pi} \int_{4M^2_{\pi}}^{t_m} dt' \sin \delta(t') e^{-i\delta(t')} f(t') + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{\text{Im} f(t')}{t' - t} \\
    &\equiv |f(t)| e^{i\delta(t)} \quad \text{for } t \leq t_m < t_{\text{inel}}
    \end{align*} $$

  - Solving for $|f(t)|$ in $4M^2_{\pi} \leq t \leq t_m$ requires: $\delta(t)$ for $4M^2_{\pi} \leq t \leq t_m$ & $\text{Im} f(t)$ for $t \geq t_m$

  - Solution via once-subtracted **Omnès function** with $t_m < \infty$ $\rightarrow \Omega(0) = 1$

    $$ \begin{align*}
    \Omega(t) &= \exp \left\{ \frac{t}{\pi} \int_{4M^2_{\pi}}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\} = \exp \left\{ \frac{t}{\pi} \int_{4M^2_{\pi}}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\} e^{i\delta(t)} \theta(t - 4M^2_{\pi}) \theta(t_m - t)
    \end{align*} $$
In general: Subtractions

- May be necessary to ensure the convergence of DR/MO integrals $\leftrightarrow$ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants $\leftrightarrow$ matching to ChPT
Roy–Steiner equations for $\pi N$ scattering: Subtractions

- In general: Subtractions
  - May be necessary to ensure the convergence of DR/MO integrals $\leftrightarrow$ asymptotic behavior
  - Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
  - Parametrize high-energy information in (a priori unknown) subtraction constants $\leftrightarrow$ matching to ChPT

- Favorable choice for $t$-channel MO problem: subthreshold expansion around $\nu = t = 0$
  - Subtract HDRs for $A^\pm$ and $B^\pm$ at $s = u = m^2 + M^2_{\pi}$ and $t = 0$
  - Done up to full second order; added (partial) third subtraction for $A^\pm$
  - $\Rightarrow$ Obtain sum rules for subthreshold parameters $x_{mn}^I$
  - $\Rightarrow$ General structure of RS/MO problem remains unchanged

- HDRs $\Rightarrow s$-/t-channel RS equations (pole terms & kernels) $\Rightarrow$ t-channel MO problem, e.g. for $P$-waves ($n \geq 1$):

$$
\Gamma^1(t) = \Delta^1_\Gamma(t) \bigg|_{n\text{-sub}} + \frac{t^{n-1}(t - 4m^2)}{\pi} \int dt' \frac{\text{Im} \Gamma^1(t')}{4M^2_{\pi} t'^{n-1}(t' - 4m^2)(t' - t)}
$$

$$
\Gamma^-_1(t) = \Delta^-_1(t) \bigg|_{n\text{-sub}} + \frac{t^n}{\pi} \int dt' \frac{\text{Im} \Gamma^-_1(t')}{4M^2_{\pi} t' t'^{-1}(t' - t)}
$$
Roy–Steiner equations for $\pi N$ scattering: Solution strategy

1. solve RS (MO) equations for $J \leq J_d$ and $t \leq t_m$

2. solve RS equations for $\ell \leq \ell_d$ and $s \leq s_m$

3. $\pi N$ coupling and subthreshold parameters

$\pi \pi$ scattering phases $\delta_f^J(t)$

Higher partial waves
$\text{Im} f^{J>J_d}_J(t \leq t_m)$

Inelasticities for $t \leq t_m$ and $s \leq s_m$

$s$-channel partial waves
$\text{Im} f^{I}(s \geq s_m)$

$t$-channel partial waves
$\text{Im} f^I_\ell(t \geq t_m)$
Here, show results for the $P$-waves, since

- **$S$-wave**: Strong effect from $\bar{K}K$ intermediate states ($f_0(980)$ resonance)
  - $\Rightarrow$ need two-channel MO analysis $\Rightarrow$ following talk
- **$P$-waves**: Single-channel MO approximation well justified in the low-energy region
- **$D$-waves**: Dominated by nucleon pole terms $\leftrightarrow$ in general for all PWs for $t \to 4M^2_\pi$

First step: Check **consistency** with KH80 $t$-channel PWs $\leftrightarrow$ iteration with $s$-channel results t.b.d.
Here, show results for the **P-waves**, since

- **S-wave**: Strong effect from $\bar{K}K$ intermediate states ($f_0(980)$ resonance)
  \[ \Rightarrow \text{need two-channel MO analysis} \Rightarrow \text{following talk} \]
- **P-waves**: Single-channel MO approximation well justified in the low-energy region
- **D-waves**: Dominated by nucleon pole terms \[ \leftrightarrow \text{in general for all PWs for } t \to 4M^2_\pi \]

**First step**: Check **consistency** with KH80 **t-channel** PWs \[ \leftrightarrow \text{iteration with } s\text{-channel results t.b.d.} \]

**Input used**:

- $\pi\pi$ phase shifts $\delta^I_J$ [Caprini/Colangelo/Leutwyler (in preparation)]
- **s-channel**: SAID PWs [Arndt et al. (2008)] for $W \leq 2.5$ GeV, above: Regge model [Huang et al. (2010)]
- KH80 [Höhler (1983)] subthreshold parameters & **coupling** $g^2/(4\pi) = 14.28$
  \[ \leftrightarrow \text{modern value: } g^2/(4\pi) = 13.7 \pm 0.2 \text{ [Baru et al. (2011)]} \]
- **t-channel**: All contributions above $t_m = 0.98$ GeV set to **zero** \[ \Rightarrow \text{solutions fixed} \quad f^I_J(t_m) = 0 \]
$f_+^J$ less well determined in MO framework than $f_-^J$, since

- Effectively one subtraction less $\Rightarrow$ introduced partial third subtraction
- Enhanced sensitivity to subtraction constants $\Leftrightarrow \tilde{N}_+^0 (4M_\pi^2) = \tilde{N}_-^J (4M_\pi^2) = 0$

Estimate systematic uncertainties (1): “fixed-$t$ limit” $|\alpha| \rightarrow \infty \Leftrightarrow$ modulo $t$-channel integrals

Estimate systematic uncertainties (2): Variation of the matching point $t_m \Rightarrow$ similar...

- MO solutions in general consistent with KH80 results
$P$-waves feature in dispersive analyses of the Sachs form factors of the nucleon:

$$\text{Im } G_E^V(t) = \frac{q_i^3}{m\sqrt{t}} (F^V_\pi(t)) \ast f^1_+(t) \theta(t - 4M^2_\pi)$$

$$\text{Im } G_M^V(t) = \frac{q_i^3}{\sqrt{2t}} (F^V_\pi(t)) \ast f^1_- (t) \theta(t - 4M^2_\pi)$$
Summary & Outlook

What has been done:

- Derived a closed system of Roy–Steiner equations (PWHDRS) for $\pi N$ scattering
- Constructed unitarity relations including $\bar{K}K$ intermediate states for the $t$-channel PWs
- Optimized the range of convergence by tuning $a$ for $s$- and $t$-channel each
- Implemented subtractions at several orders
- Solved the $t$-channel (single-channel) MO problem

- $t$-channel RS/MO machinery works $\rightarrow$ modulo the $S$-wave

What needs to be done:

- Two-channel MO analysis for the $S$-wave, effect on scalar form factor $\Rightarrow$ following talk
- Numerical solution of the $s$-channel subproblem using the $t$-channel results as input
- Self-consistent, iterative solution of the full RS system $\Rightarrow$ lowest PWs & low-energy parameters

Possible improvements: Higher subtractions, higher PWs, more inelastic input, ...
Summary & Outlook

What has been done:

- Derived a closed system of Roy–Steiner equations (PWHDRS) for $\pi N$ scattering
- Constructed unitarity relations including $\bar{K}K$ intermediate states for the $t$-channel PWs
- Optimized the range of convergence by tuning $a$ for $s$- and $t$-channel each
- Implemented subtractions at several orders
- Solved the $t$-channel (single-channel) MO problem
  - $t$-channel RS/MO machinery works $\Rightarrow$ modulo the $S$-wave

What needs to be done:

- Two-channel MO analysis for the $S$-wave, effect on scalar form factor $\Rightarrow$ following talk
- Numerical solution of the $s$-channel subproblem using the $t$-channel results as input
  - Self-consistent, iterative solution of the full RS system $\Rightarrow$ lowest PWs & low-energy parameters
- Possible improvements: Higher subtractions, higher PWs, more inelastic input, ...
πN scattering basics

- **Generically:** \( \pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p') \)

- **Kinematics:**
  \[
  s = (p + q)^2, \quad t = (p - p')^2, \quad u = (p - q')^2 \\
  u = 2(m^2 + M_{\pi}^2) - s - t, \quad \nu = \frac{s-u}{4m}
  \]

- **Isospin structure:**
  \[
  T^{ba} = \delta^{ba}T^+ + i\epsilon^{bac}\tau^cT^-
  \]

- **Lorentz structure \((I \in \{+, -\})\):**
  \[
  T^I = \bar{u}(p')\left\{ A^I + \frac{q'+q}{2}B^I \right\} u(p)
  \]

- **Crossing** symmetry relates amplitudes for \( s-/u\)-channel \((\pi N \rightarrow \pi N)\) and \( t\)-channel \((\bar{N}N \rightarrow \pi \pi)\),
  
  crossing even and odd amplitudes:
  \[
  A^\pm(\nu, t) = \pm A^\pm(-\nu, t), \quad B^\pm(\nu, t) = \mp B^\pm(-\nu, t)
  \]

- **Isospin & crossing** ⇒
  \[
  \begin{pmatrix}
  A^I=+ \\
  A^I=-
  \end{pmatrix}
  = \frac{1}{3}
  \begin{pmatrix}
  1 & 2 \\
  1 & -1
  \end{pmatrix}
  \begin{pmatrix}
  A^{I_5=1/2} \\
  A^{I_5=3/2}
  \end{pmatrix}
  = \begin{pmatrix}
  \frac{1}{\sqrt{6}} & 0 \\
  0 & \frac{1}{2}
  \end{pmatrix}
  \begin{pmatrix}
  A^I=0 \\
  A^I=1
  \end{pmatrix}, \quad A \in \{A, B\}
Subtraction of pseudovector Born terms: $X \mapsto \bar{X}$

$D^\pm = A^\pm + \nu B^\pm$

Expand crossing even amplitudes

$X^I(\nu^2, t) \in \left\{ \bar{A}^+ + \frac{a^+}{\nu}, \frac{B^+}{\nu}, \bar{B}^-, \bar{D}^+, D^- \nu \right\}$

around subthreshold point $\nu = t = 0$:

$X^I(\nu^2, t) = \sum_{m,n=0}^{\infty} x^I_{mn}(\nu^2)^m t^n$

Relations between subthreshold parameters $x^I_{mn}$:

$d^+_{mn} = a^+_{mn} + b^+_{m-1,n} \Rightarrow d^+_{0n} = a^+_{0n}$

$d^-_{mn} = a^-_{mn} + b^-_{mn}$

Subthreshold expansion of $A^\pm$ and $B^\pm$:

$A^+(\nu, t) = \frac{g^2}{m} + d^+_{00} + d^+_{01} t + a^+_{10} \nu^2 + \mathcal{O}(\nu^4, \nu^2 t, t^2)$

$A^-(\nu, t) = a^-_{00} \nu + a^-_{01} \nu t + \mathcal{O}(\nu^3, \nu t^2)$

$B^+(\nu, t) = \frac{g^2}{M^2_\pi} \frac{4m\nu}{M^2_\pi} + b^+_{00} \nu + \mathcal{O}(\nu^3, \nu t)$

$B^-(\nu, t) = -\frac{g^2}{2m^2} - \frac{g^2}{M^2_\pi} \left[ 2 + \frac{t}{M^2_\pi} \right] + b^-_{00} + b^-_{01} t + \mathcal{O}(\nu^2, \nu^2 t, t^2)$
Roy–Steiner equations for $\pi N$ scattering: Range of convergence

- Assumption: **Mandelstam analyticity** [Mandelstam (1958,1959)]

$$T(s, t) = \frac{1}{\pi^2} \int \int ds' du' \frac{\rho_{su}(s', u')}{(s' - s)(u' - u)} + \frac{1}{\pi^2} \int \int dt' du' \frac{\rho_{tu}(t', u')}{(t' - t)(u' - u)} + \frac{1}{\pi^2} \int \int ds' dt' \frac{\rho_{st}(s', t')}{(s' - s)(t' - t)}$$

with integration ranges defined by the support of the double spectral regions $\rho$

- Boundaries of $\rho$ are given by the lowest graphs

(I) (II) (III) (IV)

- Convergence of PW exps. of imaginary parts

$\Rightarrow$ **Lehman ellipses** for $z = \cos \theta$ [Lehmann (1958)]

- Convergence of PW projs. of full equations

$\Rightarrow$ for given $a$, hyperbolas must not enter any $\rho$

for all needed values of $b$

$\Rightarrow$ Constraints on $b$ yield ranges in $s$ & $t$
Estimate systematic **uncertainties**: Variation of the matching point $t_m$ \( \rightarrow \) effect of $f_j'(t_m) = 0$

- Convergence pattern & internal consistency
- Consistency with KH80

--> MO solutions in general consistent with KH80 results