\( \eta \rightarrow 3\pi \) and quark masses

Stefan Lanz

Department of Astronomy and Theoretical Physics, Lund University

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Outline

1. Introduction
2. Dalitz plot measurements
3. Theoretical work
4. Our dispersive analysis
5. Comparison of results
Introduction

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2. Dalitz plot measurements

3. Theoretical work

4. Our dispersive analysis

5. Comparison of results
Light quark masses

- not directly accessible to experiment due to confinement
- $m_{u,d,s} \ll \text{scale of QCD} \Rightarrow \text{small contribution to hadronic quantities}$
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  - $(\text{meson mass})^2 = (\text{spontaneous } \chi_{SB}) \times (\text{explicit } \chi_{SB})$
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  - quark condensate $\langle \bar{q}q \rangle$ [Gell-Mann, Oakes & Renner ’68]
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  - quark condensate \( \langle \bar{q}q \rangle \)
  - quark masses \( m_q \)
Gell-Mann–Oakes–Renner relations

\( m^2_{\pi^+} = B_0 (m_u + m_d) \)
\( m^2_{\pi^0} = B_0 (m_u + m_d) \)
\( m^2_{K^+} = B_0 (m_u + m_s) \)
\( m^2_{K^0} = B_0 (m_d + m_s) \)
\( m^2_{\eta} = B_0 \frac{m_u + m_d + 4m_s}{3} \)
Gell-Mann–Oakes–Renner relations

- $m_{\pi^+}^2 = B_0 (m_u + m_d)$
- $m_{\pi^0}^2 = B_0 (m_u + m_d) + \frac{2\epsilon}{\sqrt{3}} B_0 (m_u - m_d) + \ldots$
- $m_{K^+}^2 = B_0 (m_u + m_s)
- $m_{K^0}^2 = B_0 (m_d + m_s)$
- $m_{\eta}^2 = B_0 \frac{m_u + m_d + 4m_s}{3} - \frac{2\epsilon}{\sqrt{3}} B_0 (m_u - m_d) + \ldots$
Gell-Mann–Oakes–Renner relations

\[ m_{\pi^+}^2 = B_0 (m_u + m_d) + \Delta_{em}^{\pi} + \ldots \]

\[ m_{\pi^0}^2 = B_0 (m_u + m_d) + \frac{2\varepsilon}{\sqrt{3}} B_0 (m_u - m_d) + \ldots \]

\[ m_{K^+}^2 = B_0 (m_u + m_s) + \Delta_{em}^K + \ldots \quad \Delta_{em}^{\pi/K} \sim (35 \text{ MeV})^2 \]

\[ m_{K^0}^2 = B_0 (m_d + m_s) \quad \Delta_{em}^{\pi} = \Delta_{em}^K \quad \text{[Dashen '69]} \]

\[ m_{\eta}^2 = B_0 \frac{m_u + m_d + 4m_s}{3} - \frac{2\varepsilon}{\sqrt{3}} B_0 (m_u - m_d) + \ldots \]
Gell-Mann–Oakes–Renner relations

- $m_{\pi^+}^2 = B_0(m_u + m_d) + \Delta_{\pi em} + \ldots$
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- $m_{K^0}^2 = B_0(m_d + m_s)$
- $m_\eta^2 = B_0 \frac{m_u + m_d + 4m_s}{3} - \frac{2\epsilon}{\sqrt{3}}B_0(m_u - m_d) + \ldots$
- $\Rightarrow (m_u - m_d)$ well hidden
Quark masses from the lattice

- More on this from others
- Relations between meson masses and quark masses from QCD
- $m_u - m_d$ needs handle on e.m. effects
  - Input from phenomenology (e.g., Kaon mass difference)
  - Put photons on the lattice
- Recent review from FLAG
What has $\eta \rightarrow 3\pi$ to do with quark masses?

- $\eta \rightarrow 3\pi$ depends on $m_q$ in special way:
  - violates isospin
  - generated by $\mathcal{L}_{IB} = -\frac{m_u - m_d}{2} (\bar{u}u - \bar{d}d)$
  - $\Delta I = 1$ operator
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  - violates isospin
  - generated by $\mathcal{L}_{IB} = -\frac{m_u - m_d}{2}(\bar{u}u - \bar{d}d)$
  - $\Delta I = 1$ operator

- $\Rightarrow$ decay amplitude proportional to $(m_u - m_d)$

- $\Rightarrow$ measure for strength of isospin breaking in QCD
Electromagnetic corrections

- $Q_u \neq Q_d \Rightarrow$ e.m. interactions break isospin
- $\Rightarrow$ can contribute to $\eta \to 3\pi$
Electromagnetic corrections

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- but: contribution very small

- one-loop contributions known and small

- recent claim that $\eta \rightarrow 3\pi^0$ is mainly e.m. based on incomplete 2-loop calculation

[Sutherland '66, Bell & Sutherland '68]

[Baur, Kambor, Wyler '96, Ditsche, Kubis, Meißen '09]

[Nehme, Zein '11]
Electromagnetic corrections

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- $\Rightarrow$ can contribute to $\eta \rightarrow 3\pi$
- but: contribution very small
- one-loop contributions known and small
- recent claim that $\eta \rightarrow 3\pi^0$ is mainly e.m. based on incomplete 2-loop calculation
- $\Rightarrow$ clean access to $(m_u - m_d)$

[Sutherland '66, Bell & Sutherland '68]

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The quark mass ratio $Q$

- $A_{\eta \rightarrow 3\pi} \propto B_0(m_u - m_d)$
The quark mass ratio $Q$

$$A_{\eta \to 3\pi} \propto B_0 (m_u - m_d) = \left\{ \begin{array}{l} \frac{1}{Q^2} \frac{m_K^2 (m_K^2 - m_{\pi}^2)}{m_{\pi}^2} + O(M^3) \\ Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \end{array} \right.$$
The quark mass ratio $Q$

\[ A_{\eta \to 3\pi} \propto B_0 (m_u - m_d) = \left\{ \begin{array}{l}
\frac{1}{Q^2} \frac{m_K^2 (m_K^2 - m_\pi^2)}{m_\pi^2} + O(\mathcal{M}^3) \\
-\frac{1}{R} (m_K^2 - m_\pi^2) + O(\mathcal{M}^2) \end{array} \right. \]

- $Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$
- $R = \frac{m_s - \hat{m}}{m_d - m_u}$
The quark mass ratio $Q$

$A_{\eta \to 3\pi} \propto B_0(m_u - m_d) = \left\{ \begin{aligned} \frac{1}{Q^2} \frac{m_K^2(m_K^2 - m_{\pi}^2)}{m_{\pi}^2} + O(M^3) \\ -rac{1}{R}(m_K^2 - m_{\pi}^2) + O(M^2) \end{aligned} \right.$

$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$

$R = \frac{m_s - \hat{m}}{m_d - m_u}$

define normalised amplitude: $A(s, t, u) = -\frac{1}{Q^2} \frac{m_K^2(m_K^2 - m_{\pi}^2)}{2\sqrt{3}m_{\pi}^2 F_{\pi}^2} M(s, t, u)$

$\Gamma_{\text{exp}} \propto \int |A(s, t, u)| \propto 1/Q^4$
What else is interesting?

- slow convergence of chiral series:

\[
\Gamma_c = 66 \text{ eV} + 94 \text{ eV} + \ldots = 296 \text{ eV}
\]

- current algebra [Cronin ‘67, Osborn & Wallace ‘70]
- one-loop $\chi$PT [Gasser & Leutwyler ‘84]

experiment [PDG ‘12]
What else is interesting?

- **slow convergence** of chiral series:

\[ \Gamma_c = 66 \text{ eV} + 94 \text{ eV} + \ldots = 296 \text{ eV} \]

- **current algebra**
  - [Cronin ’67, Osborn & Wallace ’70]

- **one-loop \( \chi \)PT**
  - [Gasser & Leutwyler ’84]

- enhanced by **large final state rescattering effects**
  - [Roiesnel & Truong ’81]
What else is interesting?

- possible tension among charged and neutral channel experiments
What else is interesting?

- possible tension among charged and neutral channel experiments
- charged and neutral channel amplitudes are related:
  \[ A_n(s, t, u) = A_c(s, t, u) + A_c(t, u, s) + A_c(u, s, t) \]
- ⇒ allows for consistency check among measurements
- more on this later...
What else is interesting?

\[ |A_n(s, t, u)|^2 \propto 1 + 2\alpha Z \]

- Crystal Barrel@LEAR (1998) [Abele et al. '98]
- Crystal Ball@BNL (2001) [Tippens et al. '01]
- SND (2001) [Achasov et al. '01]
- WASA@CELSIUS (2007) [Bashkanov et al. '07]
- WASA@COSY (2008) [Adolph et al. '09]
- Crystal Ball@MAMI-B (2009) [Unverzagt et al. '09]
- Crystal Ball@MAMI-C (2009) [Prakhov et al. '09]
- KLOE (2010) [Ambrosino et al. '10]
- PDG average [PDG '12]

\[ \eta \rightarrow 3\pi \text{ and quark masses} \]
What else is interesting?

\begin{itemize}
  \item $\chi PT \mathcal{O}(\rho^4)$ [GL '85, Bijnens&Gasser '02]
  \item $\chi PT \mathcal{O}(\rho^6)$ [Bijnens&Ghorbani '07]
  \item Kambor et al. [Kambor et al. '96]
  \item Kampf et al. [Kampf et al. '11]
  \item NREFT [Schneider et al. '11]
  \item GAMS-2000 (1984) [Alde et al. '84]
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\end{itemize}

\begin{align*}
  \eta \rightarrow 3\pi \text{ and quark masses}
\end{align*}
Kinematics

- $s = (p_{\pi^+} + p_{\pi^-})^2$
- $t = (p_{\pi^0} + p_{\pi^-})^2$
- $u = (p_{\pi^0} + p_{\pi^+})^2$
- $s + t + u = m_{\eta}^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0$

$\Rightarrow$ only two independent variables
Kinematics

\[ s = (p_{\pi^+} + p_{\pi^-})^2 \]

\[ t = (p_{\pi^0} + p_{\pi^-})^2 \]

\[ u = (p_{\pi^0} + p_{\pi^+})^2 \]

\[ s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0 \]

⇒ only two independent variables ,
e.g., \( s \) & \( t - u \propto \cos \theta_s \)
Adler Zero

- soft pion theorem, i.e., valid in $SU(2)$ chiral limit
- decay amplitude has a zero if
  - $p_{\pi^+} \to 0 \iff s = u = 0, \ t = m_\eta^2$
  - $p_{\pi^-} \to 0 \iff s = t = 0, \ u = m_\eta^2$

[ Adler ’65 ]
Adler Zero

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- for $m_\pi \neq 0$ Adler zeros at
  - $s = u = \frac{4}{3} m_\pi^2, \ t = m_\eta^2 + m_\pi^2 / 3$
  - $s = t = \frac{4}{3} m_\pi^2, \ u = m_\eta^2 + m_\pi^2 / 3$

- protected by $SU(2)$ chiral symmetry $\Rightarrow$ no $O(m_s)$ corrections
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Dalitz plot variables

\[ X = \frac{\sqrt{3}}{2m_\eta Q_c} (u - t) \]

\[ Y = \frac{3}{2m_\eta Q_c} \left( (m_\eta - m_{\pi^0})^2 - s \right) - 1 \]

\[ Q_c = m_\eta - 2m_{\pi^+} - m_{\pi^0} \]

\[ Z = X^2 + Y^2 \]
KLOE measurement of the charged channel

- only modern high-statistics Dalitz plot measurement

\[ \sim 1.3 \times 10^6 \, \eta \rightarrow \pi^+ \pi^- \pi^0 \, \text{events from} \, e^+ e^- \rightarrow \phi \rightarrow \eta \gamma \]

[Figure from Ambrosino et al. '08]

Stefan Lanz (Lund University)
KLOE result for Dalitz plot parameters

- Dalitz plot parametrisation:

\[ |\mathcal{A}_c(s, t, u)|^2 \propto 1 + aY + bY^2 + cX + dX^2 + eXY + fY^3 + gX^3 + hX^2Y + lXY^2 \]
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- result:

\[
\begin{align*}
  a &= -1.090^{+0.009}_{-0.020} \\
  b &= 0.124 \pm 0.012 \\
  d &= 0.057^{+0.009}_{-0.017} \\
  f &= 0.14 \pm 0.02
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\]

### charge conjugation symmetry under

\[X \leftrightarrow -X\]

\[h\] consistent with zero

#### older experiments:

- AGS@BNL
- Princeton-Pennsylvania Accelerator
- Crystal Barrel@LEAR

#### upcoming analyses:

- WASA@COSY
- KLOE
MAMI-C measurement of the neutral channel

- \( \sim 3 \times 10^6 \eta \rightarrow 3\pi^0 \) events from \( \gamma p \rightarrow \eta p \)
- smallest uncertainties on \( \alpha \)
- similar but independent measurement from MAMI-B

\[
|A_n(s, \ell, u)|^2 \propto 1 + 2\alpha Z + 6\beta Y \left( X^2 - \frac{Y^2}{3} \right) + 2\gamma Z^2
\]

[ figure from Prakhov et al. ’09 ]
MAMI-C measurement of the neutral channel

- $\sim 3 \times 10^6 \eta \rightarrow 3\pi^0$ events from $\gamma p \rightarrow \eta p$
- Smallest uncertainties on $\alpha$
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$|A_n(s, t, u)|^2 \propto 1 + 2\alpha Z + 6\beta Y \left( X^2 - \frac{Y^2}{3} \right) + 2\gamma Z^2$
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## What has been done?

<table>
<thead>
<tr>
<th></th>
<th>$Q$</th>
<th>$\alpha$</th>
<th>tension</th>
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<tr>
<td>e.m. contributions in $\chi$PT</td>
<td>✓</td>
<td>✓</td>
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<td>two-loop $\chi$PT</td>
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<td>resummed $\chi$PT</td>
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Theoretical work

NREFT analysis

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non-relativistic EFT

- expansion in small $\pi$ three momenta in $\eta$ rest frame
- explicitly includes two pion rescattering processes
- inputs:
  - $O(p^4) \eta \rightarrow 3\pi$ amplitude from $\chi$PT
  - empirical $\pi\pi$ scattering phases
- results only for shape, but not normalisation

[ Schneider, Kubis & Ditsche '11 ]
## NREFT analysis

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- **$\alpha = -0.025 \pm 0.005 \Rightarrow$** correct sign, marginal agreement with experiment
- **tension between charged and neutral channel experiments:**

\[ \alpha \leq \frac{1}{4} (b + d - \frac{1}{4} a^2) \]  

[ Schneider, Kubis & Ditsche '11 ]

[ Bijnens & Ghorbani '07 ]
## NREFT analysis

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**Theoretical work**

- $\alpha = -0.025 \pm 0.005 \Rightarrow$ correct sign, marginal agreement with experiment
- tension between charged and neutral channel experiments:
  \[
  \alpha = \frac{1}{4}(b + d - \frac{1}{4}a^2) + \Delta
  \]
- $\Delta$ can be calculated in NREFT (no $\eta \rightarrow 3\pi$ input from $\chi$PT needed!)

[ Schneider, Kubis & Ditsche ’11 ]
NREFT analysis

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- $\alpha = -0.025 \pm 0.005 \Rightarrow$ correct sign, marginal agreement with experiment
- tension between charged and neutral channel experiments:
  $$\alpha = \frac{1}{4}(b + d - \frac{1}{4}a^2)$$
- $\Delta$ can be calculated in NREFT (no $\eta \to 3\pi$ input from $\chi$PT needed!)
- from KLOE Dalitz plot parameters: $\alpha = -0.059 \pm 0.007$
- main reason for disagreement: $b_{NREFT} = 0.308 > b_{KLOE} = 0.124$
Dispersive analysis by Kampf et al.

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analytical dispersive analysis relying on two-loop $\chi$PT and KLOE data

- 6 subtraction constants
- two rescattering processes $\Rightarrow$ reproduces two-loop result
Dispersive analysis by Kampf et al.

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analytical dispersive analysis relying on two-loop $\chi$PT and KLOE data

- 6 subtraction constants
- two rescattering processes $\Rightarrow$ reproduces two-loop result
- main result: subtraction constants from fit to KLOE data
  
  (normalisation fixed by imaginary part of two-loop result along $t = u$)
Dispersive analysis by Kampf et al.

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<td>(✓) [ Kampf, Knecht, Novotný &amp; Zdráhal '11 ]</td>
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- Adler zero strongly violated ⇒ incompatible
- with SU(2) chiral symmetry

\[ Q^2 \alpha \]

\[ M(s, 3s_0 - 2s, s) \]

\[ s = u \text{ in } m_\pi^2 \]

\[ \text{one-loop } \chi \text{PT} \quad \text{KKNZ dispersive} \]
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- numerical dispersive

- includes arbitrary number of rescattering processes

[Colangelo, SL, Leutwyler, Pasemar (tbp)]
# Method

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- numerical dispersive

- includes arbitrary number of rescattering processes

- two main steps:
  - derive & solve dispersion relations
  - fix subtraction constants

[Colangelo, SL, Leutwyler, Passemar (tbp)]

[Anisovich & Leutwyler '96]
Our dispersive analysis relies on decomposition

\[ M(s, t, u) = M_0(s) + (s - u)M_1(t) + (s - t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \]

[ Fuchs, Sazdijan & Stern '93, Anisovich & Leutwyler '96 ]
Dispersion relations

- relies on decomposition

\[ \mathcal{M}(s, t, u) = M_0(s) + (s - u)M_1(t) + (s - t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \]

- dispersion relation for each \( M_i(s) \):

\[ M_i(s) = \Omega_i(s) \left\{ P_i(s) + \frac{s^{n_i}}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^{n_i}} \frac{\sin \delta_i(s') \hat{M}_i(s')}{\Omega_i(s') |(s' - s - i\epsilon)|} \right\} \]

- Omnès function: \( \Omega_i(s) = \exp \left\{ \frac{s}{\pi} \int_{4m^2_\pi}^{\infty} ds' \frac{\delta_i(s')}{s'(s' - s - i\epsilon)} \right\} \)
Our dispersive analysis

Dispersion relations

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- input needed for
  - \( \pi \pi \) phase shifts
  - subtraction constants

[ Fuchs, Sazdijan & Stern '93, Anisovich & Leutwyler '96 ]

[ Anisovich & Leutwyler '96 ]

[ Ananthanarayan, Colangelo, Gasser & Leutwyler '01 ]

[ Omnès '58 ]
Our dispersive analysis

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Taylor coefficients

- \( M_l(s) = a_l + b_l s + c_l s^2 + d_l s^3 + \ldots \)

- Taylor coefficients \( \Leftrightarrow \) subtraction constants
Taylor coefficients

- \[ M_l(s) = a_l + b_l s + c_l s^2 + d_l s^3 + \ldots \]

- Taylor coefficients \(\leftrightarrow\) subtraction constants

- \(a_l, b_l, \ldots \in \mathbb{R}\), but \(\alpha_l, \beta_l, \ldots \in \mathbb{C}\)

- Imaginary parts of subtraction constants suppressed

- Splitting into \(M_l(s)\) not unique because of \(s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2\)

\[ \rightarrow \text{gauge freedom} \] to fix some Taylor coefficients arbitrarily
Matching to one-loop $\chi$PT

- Subtraction constants from theory alone:
  - $M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots$
  - $M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots$
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- gauge 4 TC to tree level value

- set 4 TC to one-loop value

⇒ dispersive, one loop
Matching to one-loop $\chi PT$

- Subtraction constants from theory alone:
  - $M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots$
  - $M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots$
  - $M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots$

- Use $\chi PT$ at low energy
- Gauge $\rightarrow$ tree level value
- Set $\chi PT$ to one-loop value

$\Rightarrow$ dispersive, one loop
Fit to data

- use data to further constrain subtraction constants:
  
  - $M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots$
  
  - $M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots$
  
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- Fix 2 TC from fit to data
Fit to data

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- gauge 4 TC to tree level value
- set 4 TC to one-loop value
- fix 2 TC from fit to data
- gauge 1 TC such that \( \delta_2 = 0 \)
Fit to data

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- $\Rightarrow$ dispersive, fit to KLOE
Our dispersive analysis

Results

Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$

![Graph showing the Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$. The graph compares the one-loop chiral perturbation theory (χPT) prediction with the KLOE preliminary data. The red line represents the one-loop χPT, and the black line represents the KLOE data. The y-axis represents $\Gamma(0, Y)$, and the x-axis represents $Y$. The graph shows the deviation of the χPT prediction from the KLOE data.]
Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$

- one-loop $\chi$PT
- KLOE
- dispersive, one loop

$p_{0, Y}$

Stefan Lanz (Lund University)
Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$
Dalitz distribution for $\eta \rightarrow 3\pi^0$

\[ \Gamma(Z) \]

- Preliminary
- One-loop $\chi$PT
- MAMI-C

Stefan Lanz (Lund University)
Dalitz distribution for $\eta \rightarrow 3\pi^0$
Dalitz distribution for $\eta \to 3\pi^0$
Our dispersive analysis

Results

But: electromagnetic corrections

- dispersive amplitude in isospin limit ⇒ needs to be accounted for in fits:
  - Dalitz plot distribution: kinematic effects most important (position of cusps)
  - decay rate: kinematic effects not enough (size of phase space)
But: electromagnetic corrections

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- small effect on $Q$, but branching ratio is off

- roughly estimate e.m. effects on $\Gamma$
  ⇒ e.m. corrections can amend BR

[ Gullström et al. '09, Ditsche et al. '09 ]
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- needs to be improved
Outline

1. Introduction
2. Dalitz plot measurements
3. Theoretical work
4. Our dispersive analysis
5. Comparison of results
Comparison of results

Comparison of $Q$

- Dispersive (Walker) [Anisovich & Leutwyler '96, Walker '98]
- Dispersive (Kambor et al.) [Kambor et al. '96]
- Dispersive (Kampf et al.) [Kampf et al. '11]

- $\chi_{\text{PT}} \mathcal{O}(p^4)$ [Gasser & Leutwyler '85, Bijnens & Ghorbani '07]
- $\chi_{\text{PT}} \mathcal{O}(p^6)$ [Bijnens & Ghorbani '07]

- No Dashen violation [Weinberg '77]
- With Dashen violation [Anant et al. '04, Kastner & Neufeld '08]

$Q$ values:
- 20
- 21
- 22
- 23
- 24
### Comparison of $Q$

<table>
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<tr>
<th>Year</th>
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<tr>
<td>20</td>
<td>dispersive (Walker)</td>
<td>Anisovich &amp; Leutwyler '96, Walker '98</td>
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**Dispersive, one loop**

**Dispersive, fit to KLOE**

**Preliminary**
Comparison of results

\[ \alpha \]

\[ \chi \text{PT } \mathcal{O}(\rho^4) \text{ [ GL '85, Bijnens&Gasser '02 ]} \]
\[ \chi \text{PT } \mathcal{O}(\rho^6) \text{ [ Bijnens&Ghorbani '07 ]} \]
Kambor et al. [ Kambor et al. '96 ]
Kampf et al. [ Kampf et al. '11 ]
NREFT [ Schneider et al. '11 ]

GAMS-2000 (1984) [ Alde et al. '84 ]
Crystal Barrel@LEAR (1998) [ Abele et al. '98 ]
Crystal Ball@BNL (2001) [ Tippens et al. '01 ]
SND (2001) [ Achasov et al. '01 ]
WASA@CELSIUS (2007) [ Bashkanov et al. '07 ]
WASA@COSY (2008) [ Adolph et al. '09 ]
Crystal Ball@MAMI-B (2009) [ Unverzagt et al. '09 ]
Crystal Ball@MAMI-C (2009) [ Prakhov et al. '09 ]
KLOE (2010) [ Ambrosino et al. '10 ]
PDG average [ PDG '10 ]
Comparison of $\alpha$
Conclusion & Outlook

- $\eta \rightarrow 3\pi$ very well suited to gain information on isospin breaking in QCD
- Dispersion relations allow to treat rescattering effects properly
- Dispersive treatment significantly improves one-loop result
- Neutral channel slope parameter can be understood based on charged channel data
- No clear sign of a tension among experiments
- More careful treatment of electromagnetic effects needed