Kaon and pion form factors in two-flavor QCD

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Pion and kaon form factors

- Phenomenological importance
  - Kaon form factor $f_+(0)$ is a theoretical input for the determination of $|V_{us}|$ via $K_{l3}$ decay, reliable estimate from the first principle of QCD is highly desirable.

- Proving ground for precise lattice calculation
  - Form factor is the second simplest quantity (the first is decay constant), necessary step to much more complicated quantities (i.e. $K\to\pi\pi$, …)
  - Precise experimental data available
  - Theoretically well-understood using ChPT
  - Comparison between lattice data and experimental value / ChPT prediction
JLQCD Nf=2 configurations

- Non-perturbatively $O(a)$-improved Wilson fermion + plaquette gauge
- $20^3 \times 48$, $\beta=5.2$, $a^{-1} \sim 0.1$ fm
- 5 quark masses correspond to $\pi/\rho=0.6-0.8$ ($m_\pi=550-1000$ MeV)
- 1,200 configs separated by 10 HMC trajectories
- Jackknife error estimate with bin size 10

- light hadron spectrum, decay constant, quark masses
- B meson decay constant and B parameters
Kaon form factors: method and results
Kaon form factor: definitions

\( K_{l3} \) decay form factors

\[
\langle \pi(p') | V_\mu | K(p) \rangle = f_+(q^2)(p + p')_\mu + f_-(q^2)(p - p')_\mu
\]

- \( f_+(0) \): a theoretical input for the determination of \( |V_{us}| \), a few percent accuracy is needed
- \( f_- \) contribution is proportional to \( m_l^2 \)

Scalar form factor

\[
f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)
\]

Another convention

\[
\xi(q^2) = f_-(q^2) / f_+(q^2)
\]
Previous estimates of $f_+(0)$

ChPT analysis by Leutwyler-Roos (1984)

$$f_+(0) = 1 + f_2 + f_4 = 1 - 0.023 - 0.016(8) = 0.961(8)$$

- standard value in the phenomenological analysis
- leading correction is determined unambiguously
- next-to-leading order correction is estimated from a model of the wave function of the pseudoscalar meson

First quenched result by Becirevic et al. (2005)

$$f_+(0) = 1 + f_2 + f_4^q = 1 - 0.023 + (-0.017 \pm 0.005_{\text{stat}} \pm 0.007_{\text{syst}}) = 0.960(9)$$

- non-perturbatively O(a)-improved Wilson and plaquette gauge, $24^3 \times 56$, $\beta=6.2$
- double ratio method is used

Several groups have been carrying out the unquenched study
Lattice calculation: method

three step calculation

\[
f_+(0) = f_+(q_{\text{max}}^2) \left[ 1 + \xi(q_{\text{max}}^2) \frac{m_K - m_\pi}{m_K + m_\pi} \right] \times \frac{f_+(0)}{f_+(q_{\text{max}}^2)} \left[ 1 + \xi(0) \frac{m_K - m_\pi}{m_K + m_\pi} \right] \times \frac{1}{1 + \xi(0) \frac{m_K - E_\pi}{m_K + E_\pi}}
\]

1. determine \( f_0(q_{\text{max}}^2) \)
2. interpolation to \( q^2=0 \)
3. subtract unnecessary contribution from \( \xi(0) \)

double ratio of correlation functions is used in each step
→ renormalization factors and bulk of statistical errors cancel
double ratio I: determine \( f_0(q_{\text{max}}^2) \)

\[
R_1(t) = \frac{C_{\pi\nu\pi}(t, T / 2; 0, 0)C_{\pi\nu\pi}(t, T / 2; 0, 0)}{C_{\pi\nu\pi}(t, T / 2; 0, 0)C_{\pi\nu\pi}(t, T / 2; 0, 0)} \rightarrow \frac{\langle \pi(0)|V_4|K(0)\rangle\langle K(0)|V_4|\pi(0)\rangle}{\langle \pi(0)|V_4|\pi(0)\rangle\langle K(0)|V_4|K(0)\rangle} = \left[ \frac{m_K + m_\pi}{2\sqrt{m_K m_\pi}} f_0(q_{\text{max}}^2) \right]^2
\]

three-point function

\[
C_{\pi\nu\pi}(t_x, t_y; \bar{p}, \bar{q}) = \sum_{\bar{x}, \bar{y}} \langle \pi(t_y, \bar{y}) V_\mu(t_x, \bar{x}) K(0) \rangle e^{i\bar{q} \cdot \bar{y}} e^{-i \bar{p} \cdot \bar{x}}
\]

- Double ratio used before for semileptonic \( B \) decay by the Fermilab group
- measures SU(3) breaking at \( q_{\text{max}}^2 = (m_K - m_\pi)^2 \)
- larger deviation from 1 for larger mass differences

\[ K_{\text{sea}} = 0.1355 \text{ (lightest)} \]
double ratio II: interpolate to $q^2 = 0$

\[
R_2(t; \bar{p}) = \frac{C_{\pi^+K}(t, T/2; \bar{p}, \bar{p})}{C_{\pi^+K}(t, T/2; \bar{0}, \bar{0})} \frac{\langle \pi(p)|V_4|K(0) \rangle}{\langle \pi(0)|V_4|K(0) \rangle} \rightarrow \frac{m_K + E_\pi}{m_K + m_\pi} f_+(q^2) \left[ 1 + \xi(q^2) \frac{m_K - E_\pi}{m_K + E_\pi} \right] \\
\frac{m_K + m_\pi}{m_K + m_\pi} f_+(q_{\text{max}}^2) \left[ 1 + \xi(q_{\text{max}}^2) \frac{m_K - m_\pi}{m_K + m_\pi} \right]
\]

- 2pt. function in the denominator is to cancel the energy mismatch
- exactly 1 in zero-recoil case
- interpolation with a quadratic function

\[
t = q^2 = (m_K - E_\pi)^2 - |\bar{p}|^2, \quad \left( |\bar{p}| = \frac{2\pi}{20}, \frac{2\pi}{20}\sqrt{2}, \frac{2\pi}{20}\sqrt{3} \right)
\]

$K_{\text{sea}} = 0.1355$ (lightest)
double ratio III: subtract $\xi(0)$ contribution

\[
R_3(t; \bar{p}) = \frac{C_{\pi V,K}(t, T/2; \bar{p}, \bar{p})}{C_{\pi V,K}(t, T/2; \bar{p}, \bar{p})} \frac{\langle \pi(p) | V_k | K(0) \rangle}{\langle \pi(p) | V_4 | K(0) \rangle} \rightarrow \frac{1 - \xi(q^2)}{m_K + E_K - \xi(q^2) \frac{m_K - E_K}{m_\pi + E_\pi}}
\]

- $q^2$ dependence is very small and seems to be independent of the strange quark mass
- Extrapolation to $q^2=0$ is done by assuming linear dependence

$K_{sea} = 0.1355$ (lightest)
chiral extrapolation (quad.)

fit the data with simple quadratic function

\[ f_+(0) = 1 - \left( c_0 + c_1 \left[ (r_0 m_K)^2 + (r_0 m_\pi)^2 \right] \right) \left[ (r_0 m_K)^2 - (r_0 m_\pi)^2 \right]^2 \]

\[ f_+(0) = 0.967(6) \text{ (preliminary)} \]
chiral extrapolation (ChPT)

fit with one-loop ChPT plus a quadratic function

\[
f_+(0) = 1 + \frac{3}{2} H_{\kappa\pi}(0) + \frac{3}{2} H_{\kappa\eta}(0) - \left( c_0 + c_1 \left[ (r_0 m_K)^2 + (r_0 m_\pi)^2 \right] \right) \left[ (r_0 m_K)^2 - (r_0 m_\pi)^2 \right]^2
\]

\[
H_{pQ}(0) = -\frac{1}{128\pi^2 f^2} \left[ m_p^2 + m_Q^2 + \frac{2m_p^2 m_Q^2}{m_p^2 - m_Q^2} \ln \frac{m_Q^2}{m_p^2} \right]
\]

: unambiguously determined

\[f_+(0) = 0.952(6) \text{ (preliminary)}\]
chiral extrapolation (pqChPT)

partially quenched ChPT formula by Becirevic et al. (2005)

\[ f_2^{pq} = -\frac{2m_K^2 + m_\pi^2}{32\pi^2 f^2} - \frac{3m_K^2 m_\pi^2 \ln \frac{m_\pi^2}{m_K^2}}{64\pi^2 f^2 (m_K^2 - m_\pi^2)} + \frac{m_K^2 (4m_K^2 - m_\pi^2) \ln \left(2 - \frac{m_\pi^2}{m_K^2}\right)}{64\pi^2 f^2 (m_K^2 - m_\pi^2)} \]

\[ f_+(0) = f_+^{pq}(0) - f_2^{pq} + f_2 = 0.945(6) \text{ (preliminary)} \]
the chiral logarithm is significant only in the region where $m_{\pi}^2 \ll m_K^2$, while the data region $\frac{1}{2} < m_{\pi}^2/m_K^2 < 2$ is well described by the quadratic form

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leutwyler-Roos (1984)</td>
<td>0.961(8)</td>
</tr>
<tr>
<td>Becirevic et al. (2005)</td>
<td>0.960(9)</td>
</tr>
<tr>
<td>This work (quad.)</td>
<td>0.967(6)</td>
</tr>
<tr>
<td>This work (ChPT + quad.)</td>
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</tr>
</tbody>
</table>
Vector charge radius

- charge radius is a slope of the form factor near $q^2=0$

$$f_+^{K\pi}(q^2) = 1 + \frac{1}{6} \langle r^2 \rangle_V^{K\pi} q^2 + \cdots$$

- one-loop ChPT predicts

$$\langle r^2 \rangle_V^{K\pi} = \langle r^2 \rangle_V^{\pi} - \frac{1}{64\pi^2 f^2} \left[ 3h \left( \frac{m_\pi^2}{m_k^2} \right) + 3h \left( \frac{m_\pi^2}{m_k^2} \right) + \frac{5}{2} \ln \frac{m_k^2}{m_\pi^2} + \frac{5}{2} \ln \frac{m_\eta^2}{m_\pi^2} - 6 \right]$$

$$\langle r^2 \rangle_V^{\pi} = \frac{12L_0}{f^2} - \frac{1}{32\pi^2 f^2} \left[ 2 \ln \frac{m_\pi^2}{\mu^2} + \ln \frac{m_k^2}{\mu^2} + 3 \right]$$

- we fit the data with ChPT + linear

$$\langle r^2 \rangle_V^{K\pi} = 0.26(3) \text{ fm}^2 \text{ (preliminary)}$$

cf) the exp. value

$$\langle r^2 \rangle_V^{K\pi} = 0.331(8) \text{ fm}^2$$
Scalar charge radius

- The same analysis can be applied to the scalar charge radius
  \[ \langle r^2 \rangle^\pi_\pi = 0.37(6) \text{ fm}^2 \text{ (preliminary)} \]

- overshoots the exp. value (PDG2004)
  \[ \langle r^2 \rangle^\pi_\pi = 0.21(3) \text{ fm}^2 \]
Pion form factors and charge radii
Pion form factor

- Pion electromagnetic form factor

\[ \left\langle \pi(p') \right| J_{em}^\mu \left| \pi(p) \right\rangle = G_\pi(q^2)(p + p')^\mu \]

- normalized as \( G_\pi(0) = 1 \) by charge conservation

- Scalar form factor

\[ \left\langle \pi(p') \right| \bar{q} q \left| \pi(p) \right\rangle = G_S(q^2) \]

- These are calculated from double ratio II, which is used to study \( q^2 \) dependence

\[ R_2(t; \bar{p}) = \frac{C_{\pi\gamma\pi}(t, T/2; \bar{p}, \bar{p})}{C_{\pi\gamma\pi}(t, T/2; \bar{0}, \bar{0})} \frac{m_\pi + E_\pi}{2m_\pi} \frac{G_\pi(q^2)}{G_\pi(0)} \]
Vector charge radius

- charge radius is a slope of the form factor near $q^2=0$
  \[ G_\pi(q^2) = 1 + \frac{1}{6} \langle r^2 \rangle_\pi q^2 + \ldots \]

- exp. data are well described by VMD

- two different fit forms
  - free pole
    \[ G_\pi(q^2) = \frac{1}{1 - c_0 (r_0^2 q^2)} + c_1 (r_0^2 q^2)^2 \]
  - measured pole
    \[ G_\pi(q^2) = \frac{1}{1 - (r_0^2 q^2)/(r_0^2 m_V^2)} + d_0 (r_0^2 q^2) \]

Both can describe the data well
Vector charge radius: chiral extrapolation

- One-loop ChPT predicts the chiral logarithm

\[
\langle r^2 \rangle^\pi_V = \frac{12 L^r_9}{f^2} - \left. \frac{1}{(4\pi f)^2} \ln \frac{m^2_\pi}{\Lambda} + \frac{3}{2} \right]
\]

- Fit the data using ChPT + quadratic

\[
\langle r^2 \rangle^\pi_V = 0.396(10) \text{ fm}^2 \text{ (preliminary)}
\]

- Significantly smaller than the experimental value

\[
\langle r^2 \rangle^\pi_V = 0.452(11) \text{ fm}^2 \text{ (PDG2004)}
\]
Scalar charge radius: chiral extrapolation

- for the scalar charge radius, chiral logarithm is rather stronger
  \[
  \langle r^2 \rangle_S^\pi = \frac{12L_0}{f^2} - \frac{15}{2} \left( \frac{1}{4\pi f} \right)^2 \left[ \ln \frac{m_{\pi}^2}{\Lambda} + \frac{3}{2} \right]
  \]

- fit the data using ChPT + quadratic
  \[
  \langle r^2 \rangle_S^\pi = 0.60(15) \text{ fm}^2 \text{ (preliminary)}
  \]

- consistent with the experimental value \( \langle r^2 \rangle_S^\pi = 0.61(4) \text{ fm}^2 \) (PDG2004)

- However, sensitive to the details of the fit function
Summary

- a two-flavor QCD calculation of kaon and pion form factors is presented
- form factors can be calculated with good precision using double ratio method
- systematic errors should be under control in the future study
- to make contact ChPT, much lighter sea quarks will be necessary