The RHMC Algorithm

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Outline

- Approximation theory
  - Polynomials versus Rationals
- Symplectic Integrators
  - Multiple timescale integrators
  - Instabilities
  - Higher-order integrators
- Multiple pseudofermions
  - Hasenbusch's trick
- RHMC
  - Comparison with R algorithm
    - Testimonials from satisfied customers
  - Domain Wall fermions
  - Wilson-like fermions
- Berlin Wall
- Conclusions

Чебышев's theorem

- Чебышев: There is always a unique rational function of any degree (n,d) which minimises

$$\|r-f\|_\infty = \max_{0\le x\le 1} \left| r(x) - f(x) \right|$$

- The error |r(x) – f(x)| reaches its maximum at exactly n+d+2 points on the unit interval

Чебышев rationals: Example

- A realistic example of a rational approximation is

$$\frac{1}{\sqrt{x}} \approx 0.3904603901 \frac{(x+2.3475661045)(x+0.1048344600)(x+0.0073063814)}{(x+0.4105999719)(x+0.0286165446)(x+0.0012779193)}$$

  - This is accurate to within almost *0.1%* over the range [*0.003,1*]
- Using a partial fraction expansion of such rational functions allows us to use a multishift linear equation solver, thus reducing the cost significantly.
- The partial fraction expansion of the rational function above is

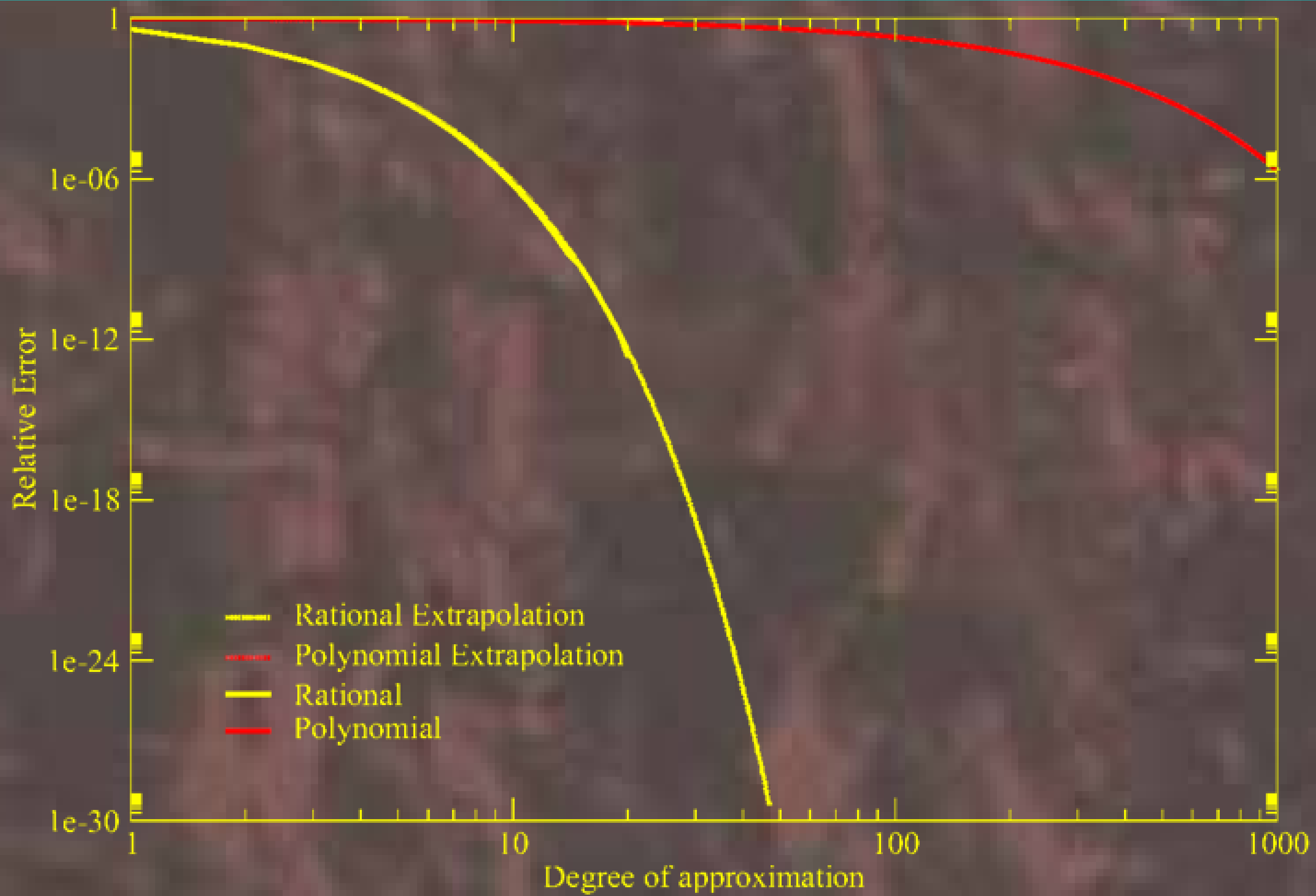
$$\frac{1}{\sqrt{x}} \approx 0.3904603901 + \frac{0.0511093775}{x+0.0012779193} + \frac{0.1408286237}{x+0.0286165446} + \frac{0.5964845033}{x+0.4105999719}$$

  - This appears to be numerically stable.

Polynomials v Rationals: Theory

- Золотарев's formula has L_∞ error $\Delta \le e^{\frac{n}{\ln \varepsilon}}$
- Optimal L_2 approximation with weight $\frac{1}{\sqrt{1-x^2}}$ is
$$\sum_{j=0}^{n} \frac{(-)^j 4}{(2j+1)\pi} T_{2j+1}(x)$$
    - This has L_2 error of $O(1/n)$
- Optimal L_∞ approximation cannot be too much better (or it would lead to a better L_2 approximation)

Polynomials v Rationals: Data



Symplectic Integrators: I

- Baker-Campbell-Hausdorff (BCH) formula
  - If A and B belong to any (non-commutative) algebra then $e^A e^B = e^{A+B+\delta}$, where δ constructed from commutators of A and B (i.e., is in the Free Lie Algebra generated by $\{A,B\}$)
  - More precisely, $\ln\left(e^A e^B\right) = \sum_{n\geq 1} c_n$ where $c_1 = A + B$ and

$$c_{n+1} = \frac{1}{n+1}\left\{ -\tfrac{1}{2}\left[c_n, A-B\right] + \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{B_{2m}}{(2m)!} \sum_{\substack{k_1,\ldots,k_{2m}\geq 1 \\ k_1+\cdots+k_{2m}=n}} \left[c_{k_1},\left[\ldots,\left[c_{k_{2m}}, A+B\right]\ldots\right]\right] \right\}$$

Symplectic Integrators: II

- Explicitly, the first few terms are

$$\ln\left(e^A e^B\right) = \{A+B\} + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\left\{[A,[A,B]] - [B,[A,B]]\right\} - \tfrac{1}{24}\left[B,[A,[A,B]]\right] + \tfrac{1}{720}\left\{\begin{array}{l} -\left[A,\left[A,\left[A,[A,B]\right]\right]\right] - 4\left[B,\left[A,\left[A,[A,B]\right]\right]\right] \\ -6\left[[A,B],\left[A,[A,B]\right]\right] + 4\left[B,\left[B,\left[A,[A,B]\right]\right]\right] \\ -2\left[[A,B],\left[B,[A,B]\right]\right] + \left[B,\left[B,\left[B,[A,B]\right]\right]\right] \end{array}\right\} + \cdots$$

- In order to construct reversible integrators we use symmetric symplectic integrators
- The following identity follows directly from the BCH formula

$$\ln\left(e^{A/2} e^B e^{A/2}\right) = \{A+B\} + \tfrac{1}{24}\left\{[A,[A,B]] - 2[B,[A,B]]\right\} + \tfrac{1}{5760}\left\{\begin{array}{l} 7\left[A,\left[A,\left[A,[A,B]\right]\right]\right] + 28\left[B,\left[A,\left[A,[A,B]\right]\right]\right] \\ +12\left[[A,B],\left[A,[A,B]\right]\right] + 32\left[B,\left[B,\left[A,[A,B]\right]\right]\right] \\ -16\left[[A,B],\left[B,[A,B]\right]\right] + 8\left[B,\left[B,\left[B,[A,B]\right]\right]\right] \end{array}\right\} + \cdots$$

Symplectic Integrators: III

- We are interested in finding the classical trajectory in phase space of a system described by the Hamiltonian $H(q,p) = T(p) + S(q) = \tfrac{1}{2}p^2 + S(q)$
  - The basic idea of such a <u>symplectic integrator</u> is to write the time evolution operator as

$$\exp\left(\tau\frac{d}{dt}\right) \equiv \exp\left(\tau\left\{\frac{dp}{dt}\frac{\partial}{\partial p} + \frac{dq}{dt}\frac{\partial}{\partial q}\right\}\right)$$

$$= \exp\left(\tau\left\{-\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right\}\right) \equiv e^{\tau\hat{H}}$$

$$= \exp\left(\tau\left\{-S'(q)\frac{\partial}{\partial p} + T'(p)\frac{\partial}{\partial q}\right\}\right)$$

Symplectic Integrators: IV

- Define $Q \equiv T'(p)\frac{\partial}{\partial q}$ and $P \equiv -S'(q)\frac{\partial}{\partial p}$ so that $\hat{H} = P + Q$
- Since the kinetic energy T is a function only of p and the potential energy S is a function only of q, it follows that the action of $e^{\tau P}$ and $e^{\tau Q}$ may be evaluated trivially

$$e^{\tau Q} : f(q,p) \mapsto f\left(q + \tau T'(p), p\right)$$
$$e^{\tau P} : f(q,p) \mapsto f\left(q, p - \tau S'(q)\right)$$

Symplectic Integrators: V

- From the BCH formula we find that the PQP symmetric symplectic integrator is given by

$$
\begin{aligned}
U_0(\delta\tau)^{\tau/\delta\tau} &= \left(e^{\frac{1}{2}\delta\tau P} e^{\delta\tau Q} e^{\frac{1}{2}\delta\tau P}\right)^{\tau/\delta\tau} \\
&= \left(\exp\left[(P+Q)\delta\tau - \tfrac{1}{24}\left([P,[P,Q]] + 2[Q,[P,Q]]\right)\delta\tau^3 + O\left(\delta\tau^5\right)\right]\right)^{\tau/\delta\tau} \\
&= \exp\left[\tau\left((P+Q) - \tfrac{1}{24}\left([P,[P,Q]] + 2[Q,[P,Q]]\right)\delta\tau^2 + O\left(\delta\tau^4\right)\right)\right] \\
&= e^{\tau\hat{H}'} = e^{\tau(P+Q)} + O\left(\delta\tau^2\right)
\end{aligned}
$$

- In addition to conserving energy to $O(\delta\tau^2)$ such symmetric symplectic integrators are manifestly area preserving and reversible

Symplectic Integrators: VI

- For each symplectic integrator there exists a nearby Hamiltonian H' which is *exactly conserved*
  - This is obtained by replacing commutators with Poisson brackets in the BCH formula
  - For the PQP integrator we have

$$
\begin{aligned}
H' &= P + Q - \tfrac{1}{24}\left(\{P,\{P,Q\}\} + 2\{Q,\{P,Q\}\}\right)\delta\tau^2 + O\left(\delta\tau^4\right) \\
&= H + \tfrac{1}{24}\left\{ 2p^2 S'' - S'^2 \right\}\delta\tau^2 \\
&\quad + \tfrac{1}{720}\left\{ -p^4 S^{(4)} + 6p^2\left(S'S''' + 2S''^2\right) - 3S'^2 S'' \right\}\delta\tau^4 + O\left(\delta\tau^6\right)
\end{aligned}
$$

    - Note that H' cannot be written as the sum of a p-dependent kinetic term and a q-dependent potential term
    - As H' is conserved, δH is of $O(\delta\tau^2)$ for arbitrary length trajectories

Symplectic Integrators: VII

- Multiple timescales
  - Split the Hamiltonian into pieces $H(q,p) = T(p) + S_1(q) + S_2(q)$
  - Define $Q \equiv T'(p)\frac{\partial}{\partial q}$ and $P_i \equiv -S_i'(q)\frac{\partial}{\partial p}$ so that $\hat{H} = P_1 + P_2 + Q$
  - Introduce a symmetric symplectic integrator of the form

$$U_{\mathrm{SW}}(\delta\tau)^{\tau/\delta\tau} = \left(e^{\frac{1}{2}\delta\tau P_1} \left[e^{\frac{1}{2n}\delta\tau Q} e^{\frac{1}{n}\delta\tau P_2} e^{\frac{1}{2n}\delta\tau Q} \right]^{n_2} e^{\frac{1}{2}\delta\tau P_1} \right)^{\tau/\delta\tau}$$

    - If $\frac{\|P_1\|}{2} \approx \frac{\|P_2\|}{n}$ then the instability in the integrator is tickled equally by each sub-step
    - This helps if the most expensive force computation does not correspond to the largest force

Integrator Instability: Theory

- Consider a leapfrog integrator for free field theory
  - The evolution is given by

$$U(\tau) = \begin{pmatrix} \cos\left[\kappa(\delta\tau)\tau\right] & \dfrac{\sin\left[\kappa(\delta\tau)\tau\right]}{\rho(\delta\tau)} \\ -\rho(\delta\tau)\sin\left[\kappa(\delta\tau)\tau\right] & \cos\left[\kappa(\delta\tau)\tau\right] \end{pmatrix}$$

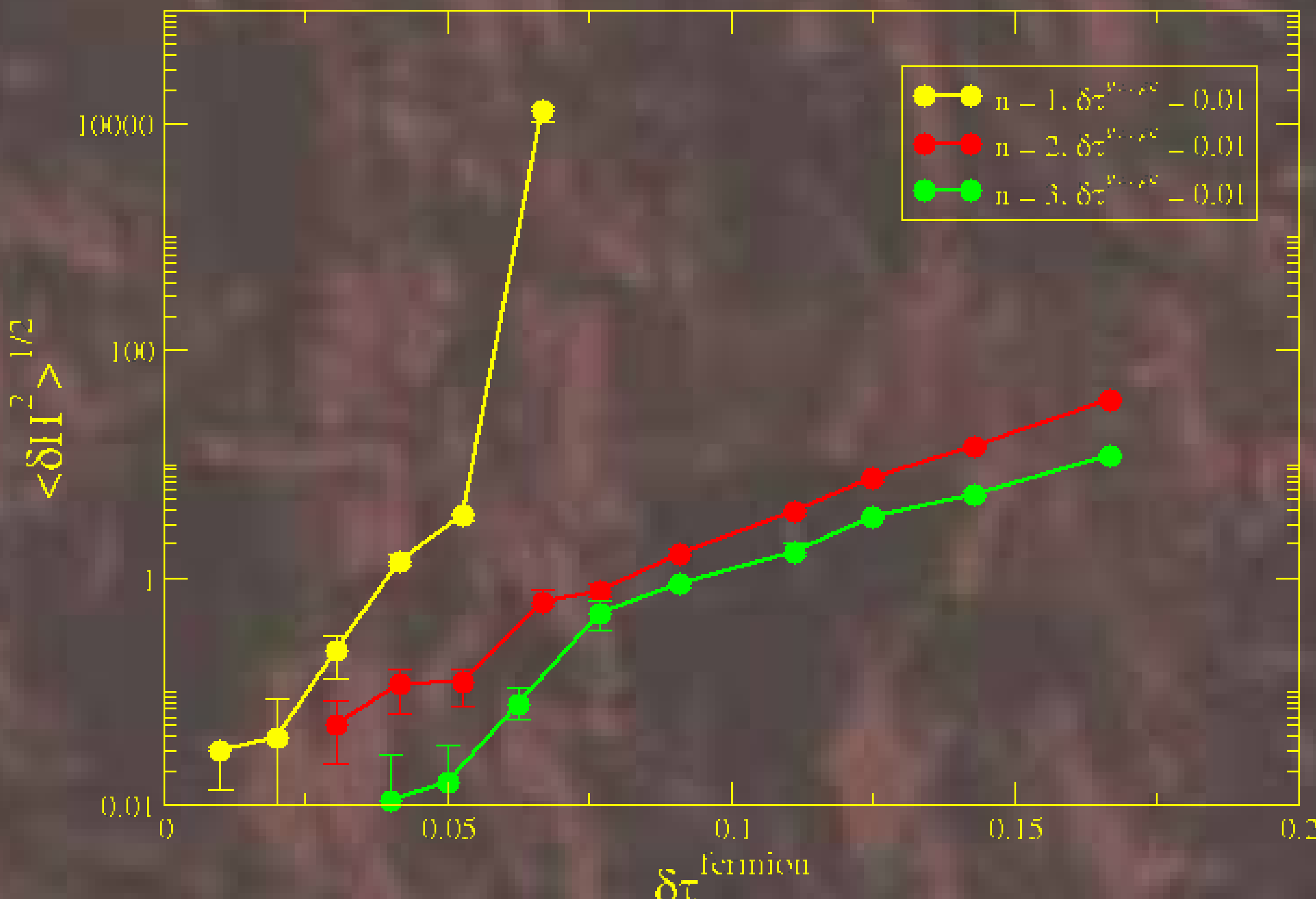
  - where

$$\kappa(\delta\tau) = \frac{\cos^{-1}\left(1 - \tfrac{1}{2}\delta\tau^2\right)}{\delta\tau}, \rho(\delta\tau) = \sqrt{1 - \tfrac{1}{4}\delta\tau^2}$$

  - This grows/oscillates with exponents

$$\nu = \pm\frac{1}{\delta\tau}\operatorname{Re}\ln\left[\tfrac{1}{2}\delta\tau^2 - 1 \pm \sqrt{\tfrac{1}{4}\delta\tau^2 - 1}\right]$$

Integrator Instability: Data



Higher-Order Integrators: I

- Campostrini and Rossi introduced an integrator with arbitrarily high-order errors
  - But the longest constituent step is longer than the overall step size, so integrator instabilities are worse
- Omelyan introduced an integrator to minimise the $\delta\tau$ error for a given number of sub-steps
  - I. P. Omelyan, I. M. Mryglod and R. Folk, Comput. Phys. Commun. 151 (2003) 272
  - Tetsuya Takaishi and Philippe de Forcrand, "Testing and tuning new symplectic integrators for Hybrid Monte Carlo algorithm in lattice QCD," hep-lat/0505020

Higher-Order Integrators: II

- These techniques help if the force is extensive (i.e., a bulk effect)
  - Because the step size needs to be adjusted so that $V\,\delta\tau^{\,n}$ is constant for a fixed HMC acceptance rate
- They do not help if the force is due to one (or a small number) of light modes
  - Here the HMC acceptance goes to zero because the symplectic integrator becomes unstable for a single mode
    - Hasenbusch's trick reduces the maximum force if it is due to noise coming from the pseudofermion fields
    - But not if the intrinsic fermionic force contribution is large.
- The volume dependence of the spectral density of the Wilson Dirac operator needs to be investigated

Non-linearity of CG solver

- Suppose we want to solve $A^2x=b$ for Hermitian A by CG
  - It is better to solve $Ax=y$, $Ay=b$ successively
  - Condition number $\kappa(A^2) = \kappa(A)^2$
  - Cost is thus $2\kappa(A) < \kappa(A^2)$ in general
- Suppose we want to solve $Ax=b$
  - Why don't we solve $A^{1/2}x=y$, $A^{1/2}y=b$ successively?
- The square root of A is uniquely defined if $A>0$
  - This is the case for fermion kernels
- All this generalises trivially to n^{th} roots
  - No tuning needed to split condition number evenly
- How do we apply the square root of a matrix?

Rational matrix approximation

- Functions on matrices
  - Defined for a Hermitian matrix by diagonalisation
  - $H = U D U^{-1}$
  - $f(H) = f(U D U^{-1}) = U f(D) U^{-1}$
- Rational functions do not require diagonalisation
  - $\alpha H^m + \beta H^n = U(\alpha D^m + \beta D^n) U^{-1}$
  - $H^{-1} = U D^{-1} U^{-1}$

No Free Lunch Theorem

- We must apply the rational approximation with each CG iteration
  - $M^{1/n} \approx r(M)$
  - The condition number for each term in the partial fraction expansion is approximately $\kappa(M)$
  - So the cost of applying $M^{1/n}$ is proportional to $\kappa(M)$
  - Even though the condition number $\kappa(M^{1/n})=\kappa(M)^{1/n}$
  - And even though $\kappa(r(M))=\kappa(M)^{1/n}$
- So we don’t win this way…

Pseudofermions

- We want to evaluate a functional integral including the fermionic determinant det M
- We write this as a bosonic functional integral over a pseudofermion field with kernel M^{-1}

$$\det M \propto \int d\phi^* \, d\phi \, e^{-\phi^* M^{-1} \phi}$$

Multipseudofermions

- We are introducing extra noise into the system by using a single pseudofermion field to sample this functional integral
  - This noise manifests itself as fluctuations in the force exerted by the pseudofermions on the gauge fields
  - This increases the maximum fermion force
  - This triggers the integrator instability
  - This requires decreasing the integration step size
- A better estimate is det $M = [\det M^{1/n}]^n$

$$\det M^{\frac{1}{n}} \propto \int d\phi^* \, d\phi \, e^{-\phi^* M^{-\frac{1}{n}} \phi}$$

Hasenbusch’s method

- Clever idea due to Hasenbusch
  - Start with the Wilson fermion action $M=1 - \kappa H$
  - Introduce the quantity $M'=1 - \kappa' H$
  - Use the identity $M = M'(M'^{-1}M)$
  - Write the fermion determinant as det M $=$ det M' det $(M'^{-1}M)$
  - Introduce separate pseudofermions for each determinant
  - Adjust κ' to minimise the cost
- Easily generalises
  - More than two pseudofermions
  - Wilson-clover action

Violation of NFL Theorem

- So let's try using our n^{th} root trick to implement multipseudofermions
  - Condition number $\kappa(r(M))=\kappa(M)^{1/n}$
  - So maximum force is reduced by a factor of $n\kappa(M)^{(1/n)-1}$
  - This is a good approximation if the condition number is dominated by a few isolated tiny eigenvalues
  - This is so in the case of interest
- Cost reduced by a factor of $n\kappa(M)^{(1/n)-1}$
  - Optimal value $n_{\text{opt}} \approx \ln \kappa(M)$
  - So optimal cost reduction is $(e \ln\kappa) / \kappa$

Rational Hybrid Monte Carlo: I

- RHMC algorithm for fermionic kernel $\left(\mathcal{M}^\dagger\mathcal{M}\right)^{\frac{1}{2n}}$
  - Generate pseudofermion from Gaussian heatbath

$$P(\xi) \propto e^{-\frac{1}{2}\xi^\dagger\xi} \qquad \chi = \left(\mathcal{M}^\dagger\mathcal{M}\right)^{\frac{1}{4n}}\xi$$

$$P(\chi) \propto \int_{-\infty}^{\infty} d\xi\, e^{-\frac{1}{2}\xi^\dagger\xi}\,\delta\left(\chi - \left(\mathcal{M}^\dagger\mathcal{M}\right)^{\frac{1}{4n}}\xi\right) \propto e^{-\frac{1}{2}\chi^\dagger\left(\mathcal{M}^\dagger\mathcal{M}\right)^{-\frac{1}{2n}}\chi}$$

  - Use accurate rational approximation $r(x) \approx \sqrt[4n]{x}$
  - Use less accurate approximation for MD, $\tilde{r}(x) \approx \sqrt[2n]{x}$
    - $\tilde{r}(x) \neq r(x)^2$, so there are no double poles
  - Use accurate approximation for Metropolis acceptance step

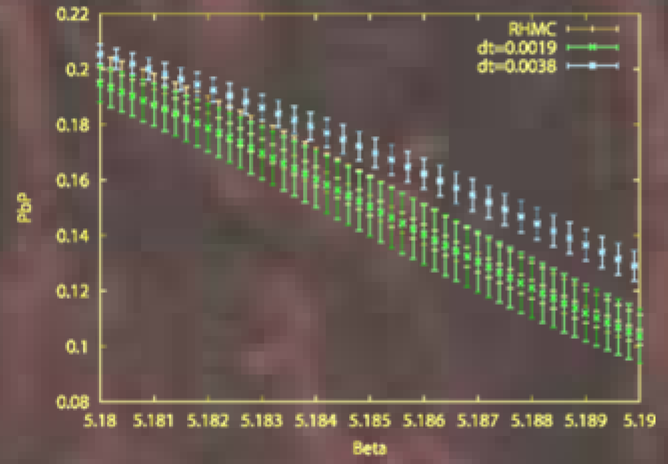
Rational Hybrid Monte Carlo: II

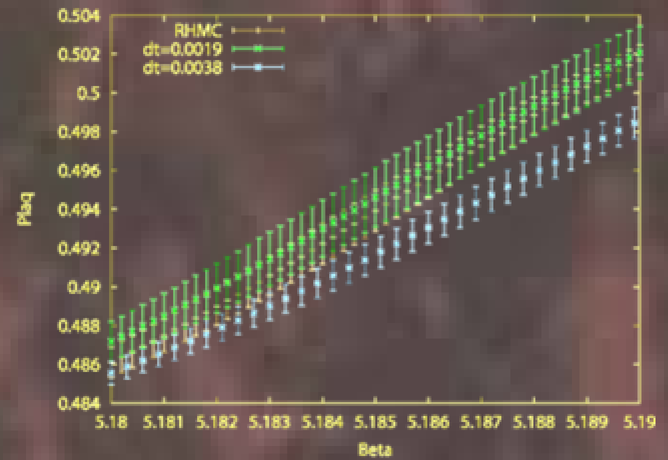
- Apply rational approximations using their partial fraction expansions
- Denominators are all just shifts of the original fermion kernel
  - All poles of optimal rational approximations are real and positive for cases of interest (Miracle #1)
  - Only simple poles appear (by construction!)
- Use multishift solver to invert all the partial fractions using a single Krylov space
  - Cost is dominated by Krylov space construction, at least for *O(20)* shifts
- Result is numerically stable, even in 32-bit precision
  - All partial fractions have positive coefficients (Miracle #2)
- MD force term is of the usual form for each partial fraction
- Applicable to any kernel

Comparison with *R* algorithm: I

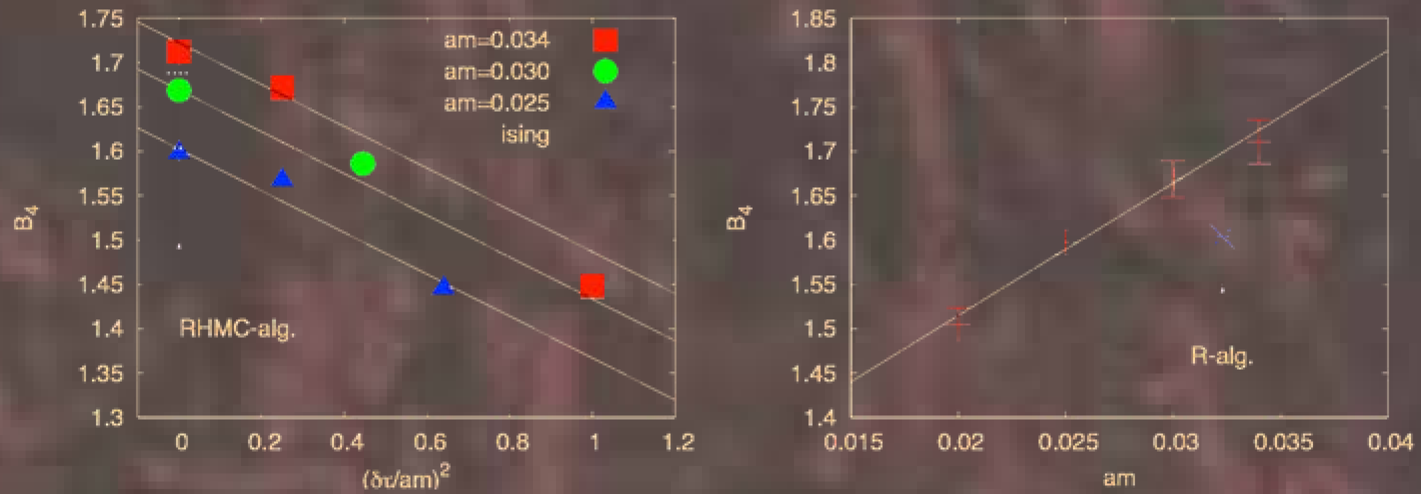
Binder cumulant of chiral condensate, B_4, and RHMC acceptance rate A from a finite temperature study (2+1 flavour naïve staggered fermions, Wilson gauge action, $V = 8^3\times 4$, $m_{ud} = 0.0076$, $m_s = 0.25$, $\tau = 1.0$)

Algorithm	δt	A	B_4
R	0.0019		1.56(5)
R	0.0038		1.73(4)
RHMC	0.055	84%	1.57(2)



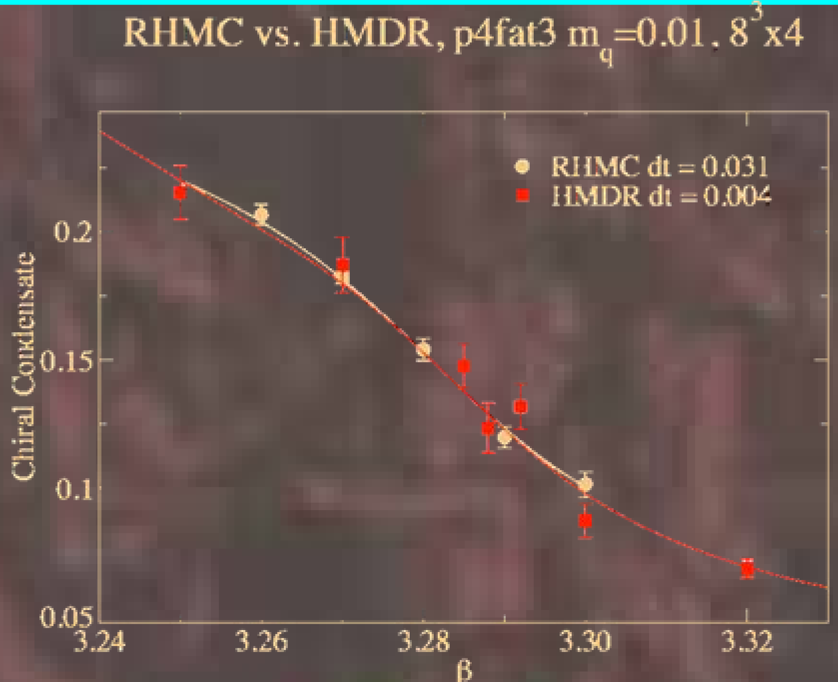


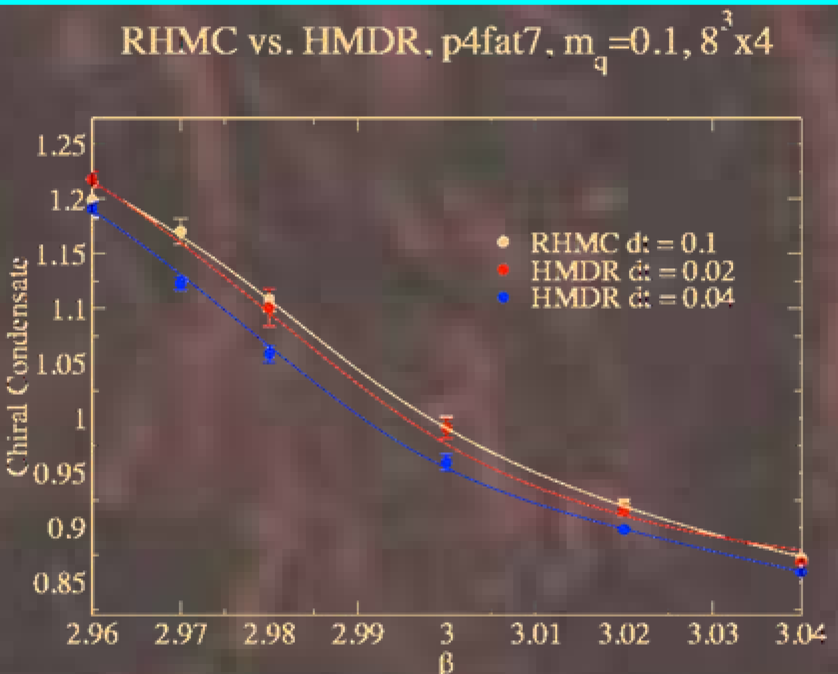
Comparison with *R* algorithm: II



- Naïve Staggered Fermions, $N_f = 3$, $V = 8^3 \times 4$
  - Binder cumulant increases as step-size is reduced
  - Step-size extrapolation is vital for R algorithm
  - 25% reduction in critical quark mass at $\delta\tau = \frac{1}{2} m_\ell$
  - 20% change in renormalized quark mass
- "An exact algorithm is mandatory" (de Forcrand-Philipsen)

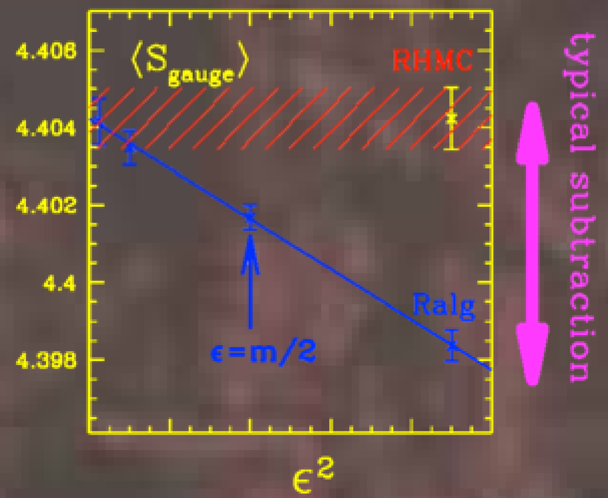
Comparison with *R* algorithm: III

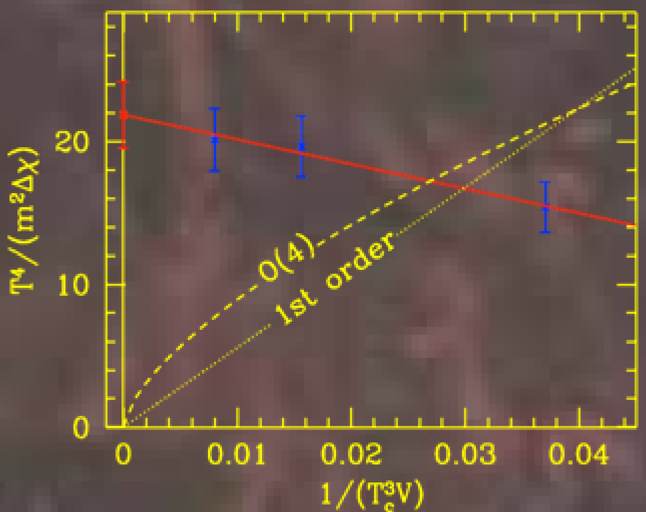




- RBC-Bielefeld
  - P4 staggered fermions
  - RHMC allows an O(10) increase in step-size
  - Speedup greater as $m_\ell \to 0$

Comparison with *R* algorithm: IV



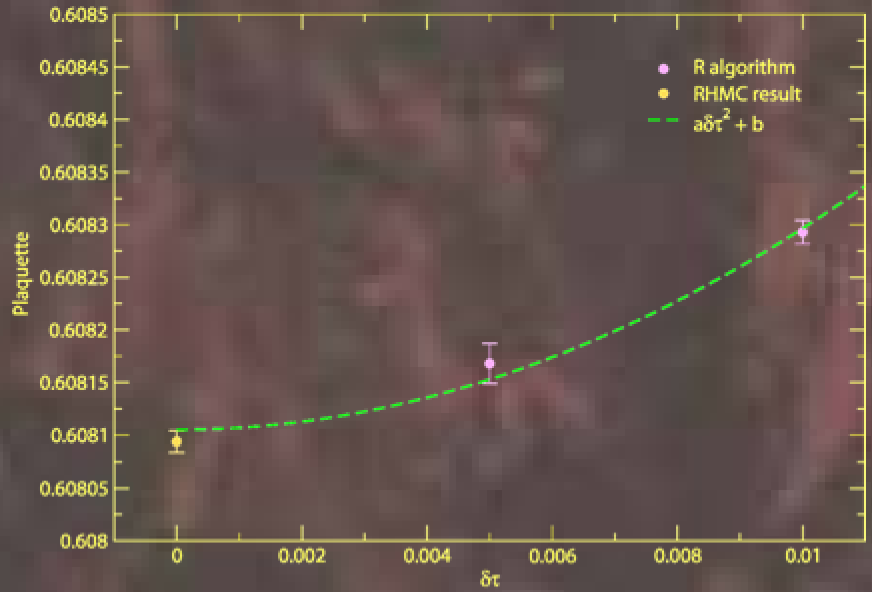


- Wuppertal—Budapest
  - Stout smeared staggered, V = 16^4, m_π = 320 MeV
    - Subtraction required for equation of state and order of the transition
    - At $\delta\tau \simeq \frac{2}{3} m_\ell$ finite step-size error ~ magnitude of subtraction
    - RHMC is order of magnitude faster than the R algorithm
    - Order of transition in continuum limit at physical quark masses for the first time

Comparison with *R* algorithm: V

The different masses at which domain wall results were gathered, together with the step-sizes δt, acceptance rates A, and plaquettes P ($V = 16^3 \times 32 \times 8$, DBW2 gauge action, $\beta = 0.72$)

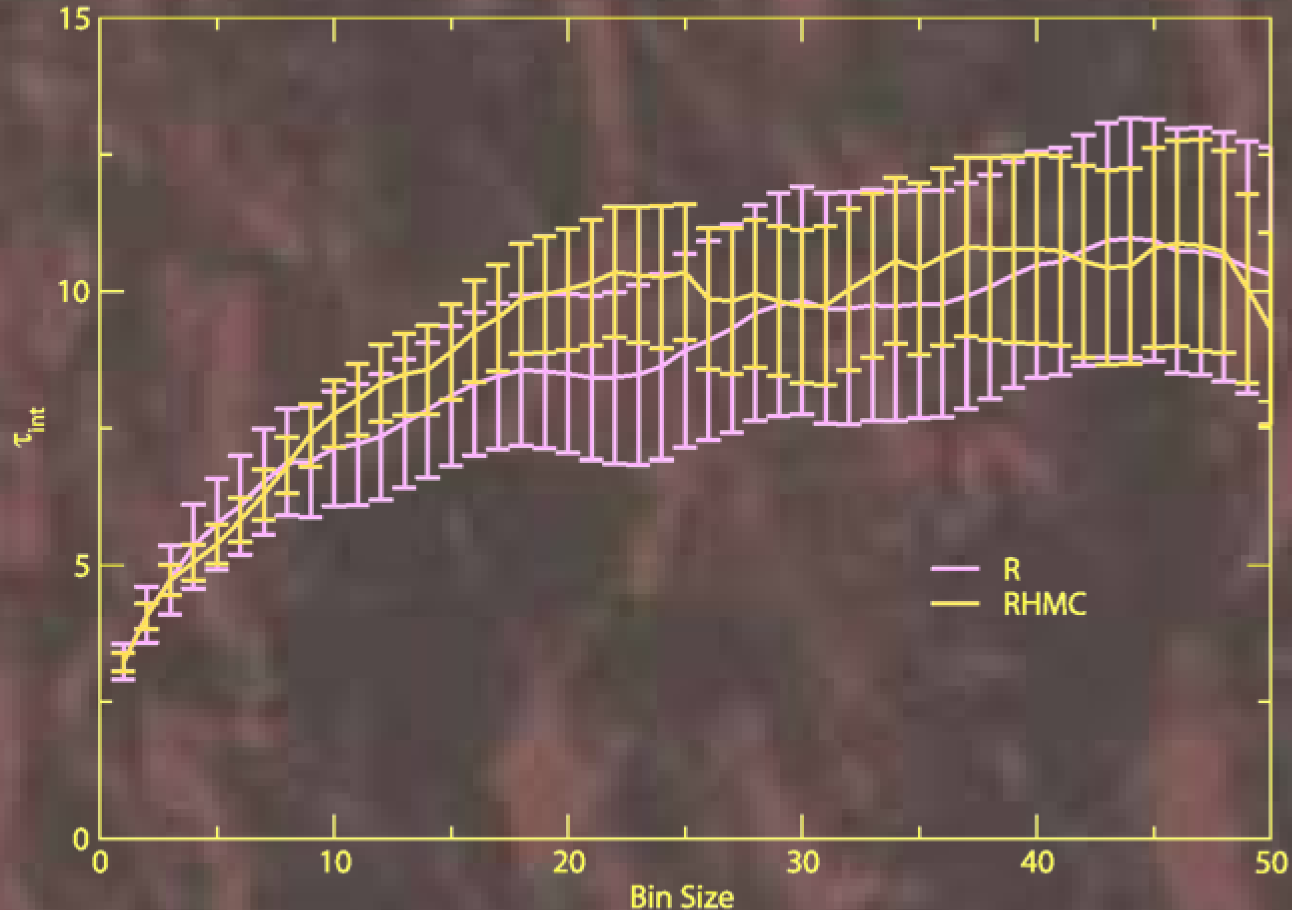
Algorithm	m_{ud}	m_s	δt	A	P
R	0.04	0.04	0.01		0.60812(2)
R	0.02	0.04	0.01		0.60829(1)
R	0.02	0.04	0.005		0.60817
RHMC	0.04	0.04	0.02	65.5%	0.60779(1)
RHMC	0.02	0.04	0.0185	69.3%	0.60809(1)



The step-size variation of the plaquette with $m_{ud} = 0.02$

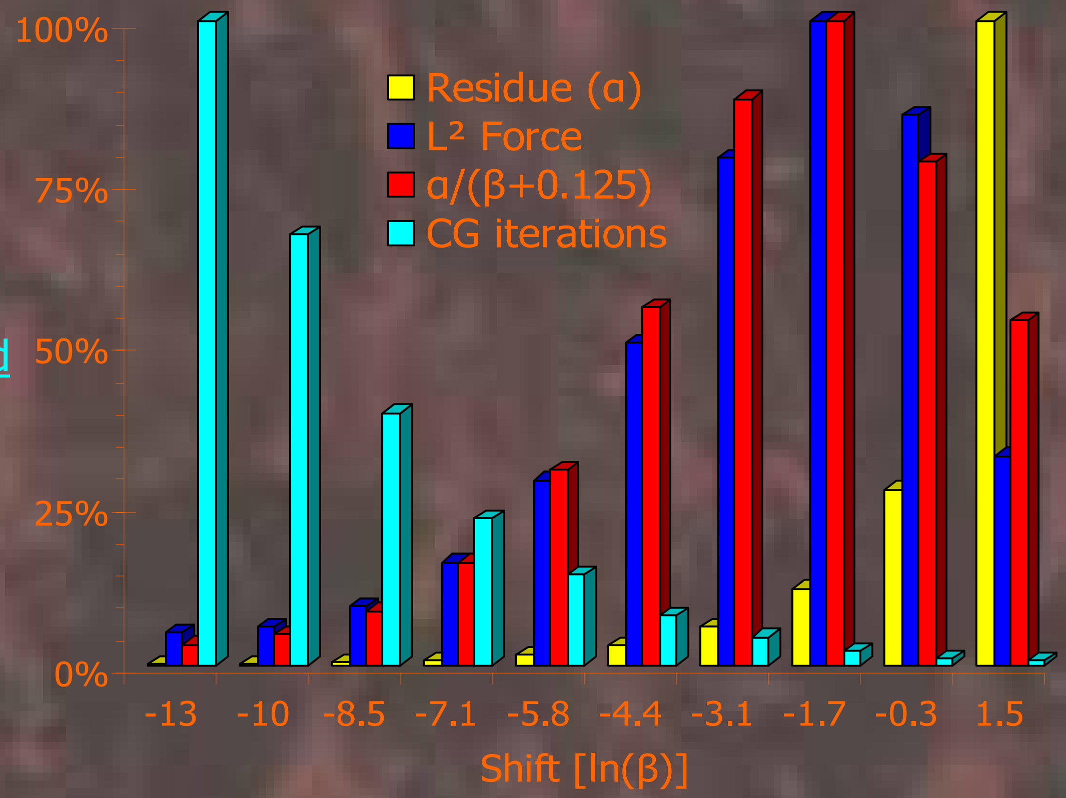
Comparison with *R* algorithm: VI

The integrated autocorrelation time of the 13th time-slice of the pion propagator from the domain wall test, with $m_{ud} = 0.04$



Multipseudofermions with multiple timescales

- Semiempirical observation: The largest force from a single pseudofermion does not come from the smallest shift
  - For example, look at the numerators in the partial fraction expansion we exhibited earlier
- Use a coarser timescale for expensive smaller shifts
- Invert small shifts less accurately
  - Cannot use chronological inverter with multishift solver anyhow



$$\frac{1}{\sqrt{x}} \approx 0.3904603901 + \frac{0.0511093775}{x + 0.0012779193} + \frac{0.1408286237}{x + 0.0286165446} + \frac{0.5964845033}{x + 0.4105999719}$$

Performance Comparison

$V = 24^3 \times 32, \beta = 5.6, N_f = 2,$ Wilson fermions

Cost in units of $A_{plaq} \times N_{MV} \times 10^4$

κ	**RHMC**	**Urbach *et al.***	**Orth *et al.***
0.15750	9.6	9.0	19.1
0.15800	29.9	17.4	128
0.15825	52.5	56.5	–

DWF (RBRC—UKQCD)

- 2+1 Flavour determinant

$$\left(\frac{\det \mathcal{M}_\ell^\dagger \mathcal{M}_\ell}{\det \mathcal{M}_{PV}^\dagger \mathcal{M}_{PV}}\right)\left(\frac{\det \mathcal{M}_s^\dagger \mathcal{M}_s}{\det \mathcal{M}_{PV}^\dagger \mathcal{M}_{PV}}\right)^{\frac{1}{2}} = \left(\frac{\det \mathcal{M}_\ell^\dagger \mathcal{M}_\ell}{\det \mathcal{M}_s^\dagger \mathcal{M}_s}\right)\left(\frac{\det \mathcal{M}_s^\dagger \mathcal{M}_s}{\det \mathcal{M}_{PV}^\dagger \mathcal{M}_{PV}}\right)^{\frac{3}{2}}$$

- Mass (Hasenbusch) preconditioning using *s* quark
- Use multiple timescale integrator
  - Gauge, triple strange, light
- CG count reduced by factor of 10
- CPU time reduced by factor of 6
- Light quarks cost about 10% of total
- Cost has weak mass dependence

2+1 ASQTAD Staggered Fermions

- Mass preconditioning using *s* quark

$$\det \mathcal{M}_\ell^{\frac{1}{2}} \mathcal{M}_s^{\frac{1}{4}} = \left(\frac{\det \mathcal{M}_\ell}{\det \mathcal{M}_s} \right)^{\frac{1}{2}} \det \mathcal{M}_s^{\frac{3}{4}}$$

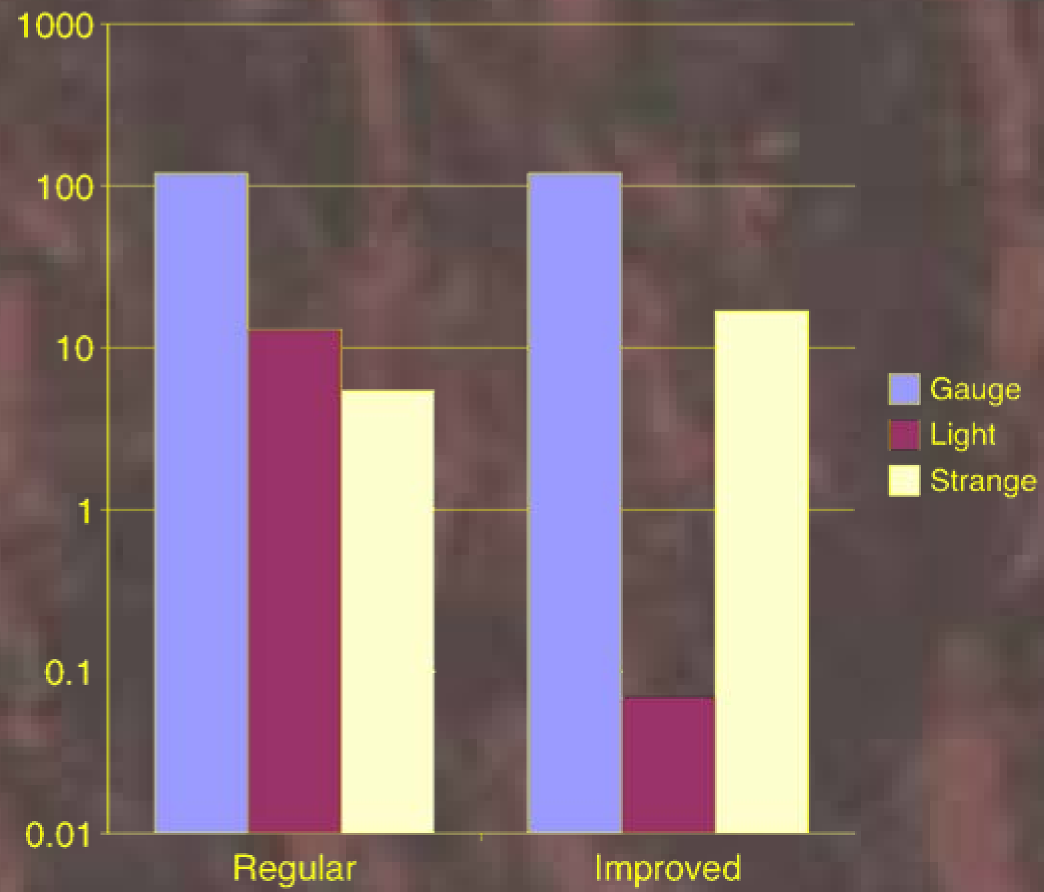
- Mass is just a shift for staggered fermions

$$S_F = \phi_\ell^\dagger \left(\frac{\mathcal{M}_s}{\mathcal{M}_\ell} \right)^{\frac{1}{2}} \phi_\ell + \phi_s^\dagger \mathcal{M}_s^{-\frac{3}{4}} \phi_s = \phi_\ell^\dagger \left(\frac{\mathcal{M}_\ell + \delta m^2}{\mathcal{M}_\ell} \right)^{\frac{1}{2}} \phi_\ell + \phi_s^\dagger \mathcal{M}_s^{-\frac{3}{4}} \phi_s$$

$$= \phi_\ell^\dagger r_1 \left(\mathcal{M}_\ell \right) \phi_\ell + \phi_s^\dagger r_2 \left(\mathcal{M}_s \right) \phi_s$$

2+1 ASQTAD Staggered Fermions

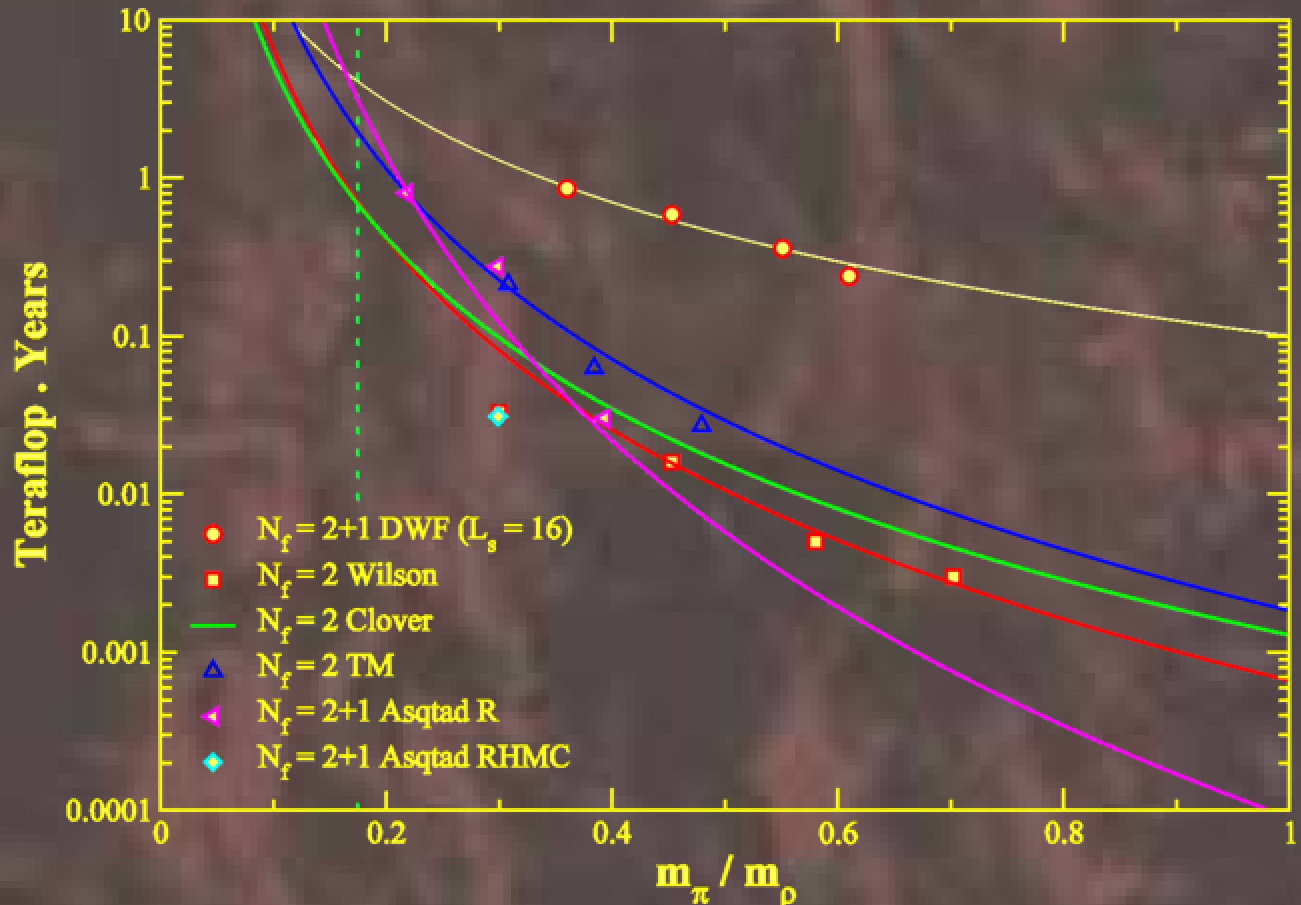
- Use multiple timescales
  - Gauge, triple strange, light
- Dominant cost from triple strange
  - Operator derivative cost » CG cost
  - Further mass preconditioning detrimental
- Use *n* th root trick on triple strange
  - Optimum solution uses mass preconditioning and *n* th root trick
  - Test at current run parameters
    - $V = 24^3 \times 64,\ \beta = 6.76,$ $m_\ell = 0.005,\ m_s = 0.05$
  - Speed up factor 8 over R algorithm **and** exact



Berlin Wall

- Comparison of cost of fermion algorithms
  - N_f = 2+1 DWF RHMC (RBC-UKQCD)
  - N_f = 2 mass preconditioned Wilson (Urbach *et al.*)
  - N_f = 2 mass preconditioned Clover (QCDSF)
  - N_f = 2+1 mass preconditioned Clover + RHMC (Wuppertal-Jülich)
  - N_f = 2 mass preconditioned Twisted Mass (ETM)
  - N_f = 2+1 ASQTAD R (MILC)
  - N_f = 2+1 ASQTAD RHMC (Clark-Kennedy)
- All data scaled to $V = 24^3 \times 40$, $a = 0.08$
- Cost for generating 10^3 independent configurations
  - Independent plaquette measurements

Berlin Wall



Teraflop . Years
10
1
0.1
0.01
0.001
0.0001
0
0.2
0.4
0.6
0.8
1
m_π / m_ρ
$N_f = 2+1$ DWF ($L_s = 16$)
$N_f = 2$ Wilson
$N_f = 2$ Clover
$N_f = 2$ TM
$N_f = 2+1$ Asqtad R
$N_f = 2+1$ Asqtad RHMC

Conclusions (RHMC)

- Advantages of RHMC
  - Exact
    - No step-size errors; no step-size extrapolations
  - Significantly cheaper than the *R* algorithm
  - Allows easy implementation of Hasenbusch (multipseudofermion) acceleration
    - Combination of both can be helpful
  - Further improvements possible
    - Such as multiple timescales for different terms in the partial fraction expansion
- Disadvantages of RHMC
  - ???