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Towards a high-precision determination of the meson and glueball spectrum

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JLab

Overview

- Anisotropic lattice QCD - advantages and disadavantages.
- Determining non-perturbative action parameters for the $N_f = 2$ anisotropic lattice
- A new toolkit for precision meson spectroscopy; The all-to-all propagator.
- Results:
  - Static-light mesons
  - Light mesons
  - The glueballs
- Conclusions

Anisotropic lattices with dynamical quarks

Anisotropic lattices

- Cost of a lattice simulation rises very rapidly as the continuum limit is approached (at least like a^{-4}).
- The simulation costs are much higher if light quark dynamics are included.
- To compute energies of states, we need to measure temporal correlation functions. To compute decay widths, we need to compute energies of states!
- Want good resolution of the temporal correlation between operators, but also want to control costs. . .
- . . . so make temporal and spatial lattice spacings different
- Choose $a_t \ll a_s$ for optimal cost/benefits.
- Anisotropic lattice also useful for heavy quark physics and finite temperature.
- Unfortunately, there are extra theory overheads for dynamical fermions:

Action parameters for dynamical anisotropic lattices

- Lattice discretisations break continuum Lorentz symmetries; anisotropic lattices break even more symmetries.
- The number of distinct dimension-4 operators grows in both the quark and gluon sectors.

Continuum	$SO(4)$	Tr F^2	$\bar{\psi}\gamma_\mu\partial_\mu\psi$
	$\downarrow$	$\downarrow$	$\downarrow$
Isotropic	$\mathcal{H}$	Tr F^2	$\bar{\psi}\gamma_\mu\partial_\mu\psi$
	$\downarrow$	$\swarrow \quad \searrow$	$\swarrow \quad \searrow$
Anisotropic	$\mathcal{O}_h$	Tr E^2 Tr B^2	$\bar{\psi}\gamma_i\partial_i\psi$ $\bar{\psi}\gamma_0\partial_0\psi$

- A **simultaneous** tuning of two new action parameters is needed to recover the continuum symmetry.

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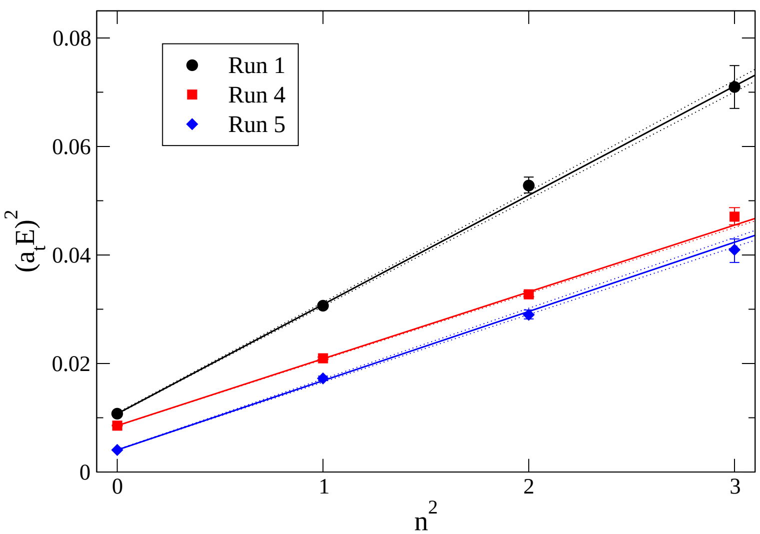
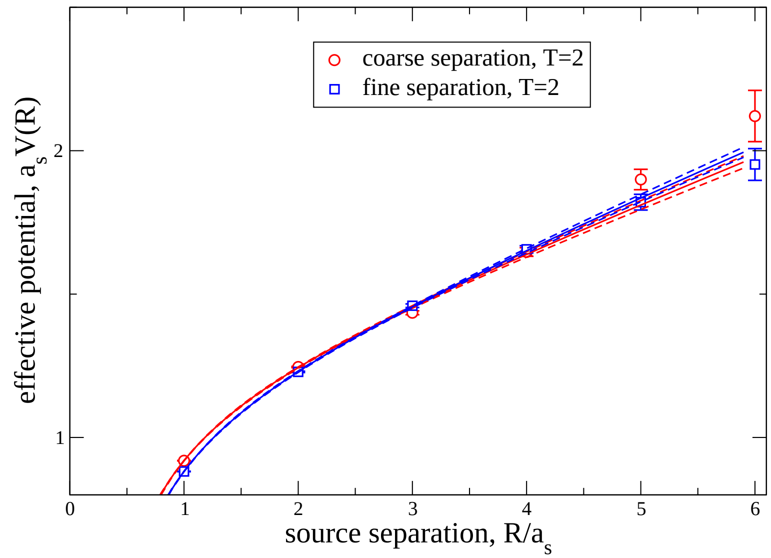
Action parameters for dynamical anisotropic lattices

- Two physical probes of low-energy physics are needed to give two constraints.
- We use the pion dispersion relation and the "sideways potential"

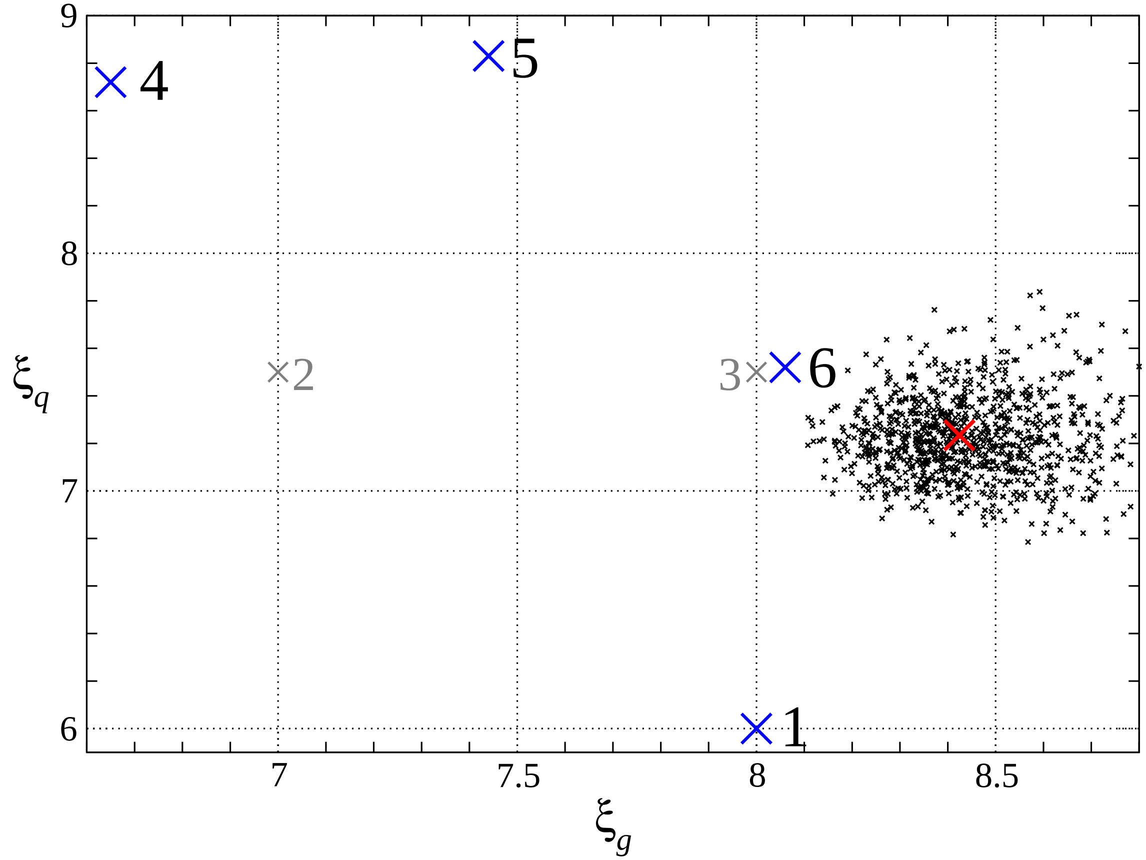




- Need at least three runs to probe the response of the vacuum to changes in the quark and gluon sectors. Our target is $\xi = a_s/a_t = 6$.

Action parameters for dynamical anisotropic lattices

- The self-consistent action parameters have been determined to 3%. We can do better.



Computing all elements of the lattice quark propagator

How to build an all-to-all propagator (1)

- Spectral representation
- Stochastic representation
- Variance reduction - dilution
- Hybrid method

Spectral representation:

Compute the eigenvalues and eigenvectors $\{v^{(i)}, \lambda_i\}$ of $Q = \gamma_5 D$, then

$$Q^{-1} = \sum_i^N \frac{1}{\lambda_i} v^{(i)} \otimes v^{(i)*}$$

Unfortunately, we can't compute all eigenvectors easily, so we must truncate this representation at $N_{\text{ev}} \ll N$.

This truncation is non-unitary and it is difficult to assign a systematic uncertainty in the error introduced by the truncation.

How to build an all-to-all propagator (2)

Stochastic representation:

Fill a vector, η with independent random numbers. We use $Z_4 = \{1, i, -1, -i\}$.

Since

$$\langle \eta_i \eta_j^* \rangle = \delta_{ij}$$

Z_N noise has the useful property that

$$\eta_i \eta_i^* = 1 \text{ (no sum)}$$

Apply the fermion solver $Q\psi = \eta$ and then

$$\langle \psi_i \eta_j^* \rangle = [Q^{-1}]_{ik} \delta_{kj} = [Q^{-1}]_{ij}$$

This gives an unbiased estimator of every entry in the quark propagator. Unfortunately, the variance of the estimator is large.

Variance reduction (1)

Variance reduction:

- We can reduce the variance of our estimator by gathering statistics, but the noise only falls like $1/\sqrt{m}$ for m noise vectors.
- The exact propagator can be computed using at most N inversions, by putting a single one in all entries of the source vector in turn (ie. using point propagators from all points!)
- This suggests a trick: break the vector space into d smaller pieces spanned by a sub-set of the basis vectors, eg.

$$V_1 = \{e^{(1)} = (1, 0, 0, \ldots), e^{(2)} = (0, 1, 0, \ldots)\}$$

- The "dilution" is defined by the user - eg. we could choose "time-dilution" where V_t is the N/N_t dimensional space of vectors with support on time-slice t.

Variance reduction (2)

Variance reduction:

- The basis is complete, so $V = V_1 \oplus V_2 \oplus \ldots V_d$ and $\eta = \eta^{(1)} + \eta_{(2)} + \eta_{(3)} \ldots + \eta_{(d)}$ where $\eta^{(2)} = \mathcal{S}_i \eta$ and S_i is the projector into vector space V_i.
- Since $\mathcal{S}_i^2 = \mathcal{S}_i$ and $\sum_i \mathcal{S}_i = I$ we can write

$$I = \sum_i \mathcal{S}_i = \sum_i \mathcal{S}_i^2 = \sum_i \mathcal{S}_i \langle \eta \otimes \eta^* \rangle \mathcal{S}_i$$

- Another representation of the propagator can be written

$$Q^{-1} = \sum_i Q^{-1} \mathcal{S}_i \langle \eta \otimes \eta^* \rangle \mathcal{S}_i = \sum_i \langle \psi^{(i)} \otimes \eta^{(i)*} \rangle$$

- The variance is reduced by explicit cancellation of terms whose expectation value is zero. In the "homeopathic limit" when $d = N$, the exact all-to-all propagator is recovered.

A hybrid method (1)

A hybrid method:

- Most of the interesting physics is contained in the lowest few eigenvectors.
- Can we use the N_{ev} lowest eigenvectors and correct for the truncation using a stochastic estimator?
- Consider breaking V into the sub-spaces V_L, which is spanned by the lowest N_{ev} eigenvectors and V_H, spanned by the rest.
- Since Q leaves these vector spaces invariant, we can write the inverse

$$Q^{-1} = \bar{Q}_L + \bar{Q}_H = Q^{-1}\mathcal{P}_L + Q^{-1}\mathcal{P}_H$$

- $\bar{Q}_L$ is just the truncated eigenvector representation.
- Estimate $\bar{Q}_H$ stochastically with dilute noise. The action of $\bar{Q}_H$ can be computed from

$$\bar{Q}_H = Q^{-1}\mathcal{P}_H = Q^{-1}(1 - \mathcal{P}_L)$$

A hybrid method (2)

The recipe:

1. Compute N_{ev} eigenvectors and eigenvalues.
2. Generate a noise vector, η and dilute it: $\{\eta^{(1)}, \eta^{(2)}, \ldots\}$
3. For each noise vector, compute

$$\psi^{(i)} = Q^{-1}(1 - \mathcal{P}_L)\eta^{(i)}$$

   by first orthogonalising w.r.t the eigenvectors, and then applying the fermion inverse to the resulting vector.

4. Now, Q^{-1} can be estimated from

$$Q^{-1} = \sum_i^{N_{\text{ev}}} \frac{1}{\lambda_i} v^{(i)} \otimes v^{(i)*} + \sum_d^{N_d} \psi^{(d)} \otimes \eta^{(d)*}$$

This form suggests packing the two sets of vectors together into a "hybrid list".

Practical implementation (1)

$$Q^{-1} = \sum_{i}^{N_{\text{ev}}} \frac{1}{\lambda_i} v^{(i)} \otimes v^{(i)*} + \sum_{d}^{N_d} \psi^{(d)} \otimes \eta^{(d)*}$$

Write two "hybrid lists" of vectors:

$$w^{(i)} = \left\{ \gamma_5 \frac{v^{(1)}}{\lambda_1}, \cdots, \gamma_5 \frac{v^{(N)}}{\lambda_N}, \gamma_5 \eta^{(1)}, \cdots, \gamma_5 \eta^{(N_d)} \right\}$$

$$u^{(i)} = \left\{ v^{(1)}, \cdots, v^{(N)}, \psi^{(1)}, \cdots, \psi^{(N_d)} \right\}$$

and then

$$M^{-1} = \sum_{i}^{N_h} u^{(i)} \otimes w^{(i)*}$$

Practical implementation (2)

Computing physical correlation functions:

- We don't usually want to compute the quark propagator directly.
- Computing two-point functions

$$C_\pi(\Delta t) = \sum_t \langle \sum_y \bar{d}(y, t+\Delta t)\gamma_5 u(y, t+\Delta t) | \sum_x \bar{u}(x,t)\gamma_5 d(x,t)\rangle$$

becomes:

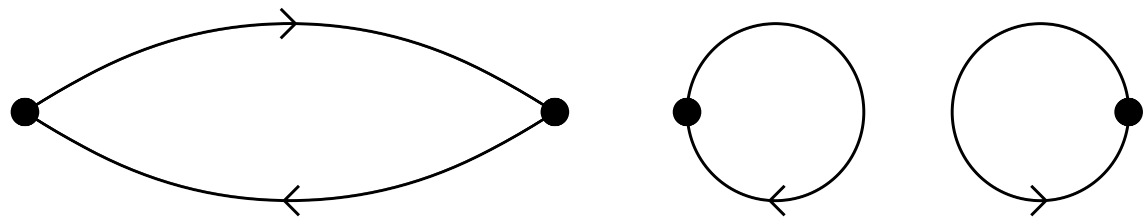
$$C_\pi(\Delta t) = \sum_t$$

$$\langle \sum_y w^*_{[1]}(y, t+\Delta t)\gamma_5 u_{[2]}(y, t+\Delta t) \sum_x w^*_{[2]}(x,t)\gamma_5 u_{[1]}(x,t)\rangle$$

- $\{u_{[1]}, w_{[1]}\}$ and $\{u_{[2]}, w_{[2]}\}$ are independent hybrid lists.

Practical implementation (3)

- Flavour singlet correlators can be computed



- Disconnected diagram for eg η' correlation function is

$$D(\Delta t) = \sum_t$$

$$\langle \sum_y w^*_{[1]}(y, t+\Delta t)\gamma_5 u_{[1]}(y, t+\Delta t) \;\; \sum_x w^*_{[2]}(x,t)\gamma_5 u_{[2]}(x,t)\rangle$$

- Hybrid list indices are traced over individually for both the source and the sink.
- Correlations with gluonic operators can be measured.

Practical implementation (4)

- ALL book-keeping can be hidden from end-user, who just writes a function to compute $a^*\gamma_5 b$ on a time-slice.
- The measurement and propagator calculations are decoupled (apart perhaps from consideration of which dilution is best).
- Operator construction is unrestricted - smearings, extended operators, etc. can be decided on at the final measurement phase.
- Large bases of operators can be built and cross-correlated to optimise ground-state (and excited-state) overlaps.
- More dilution can be applied post-hoc, making use of work done so far.
- More eigenvectors can be computed post-hoc, without the need to re-compute inversions.

New results

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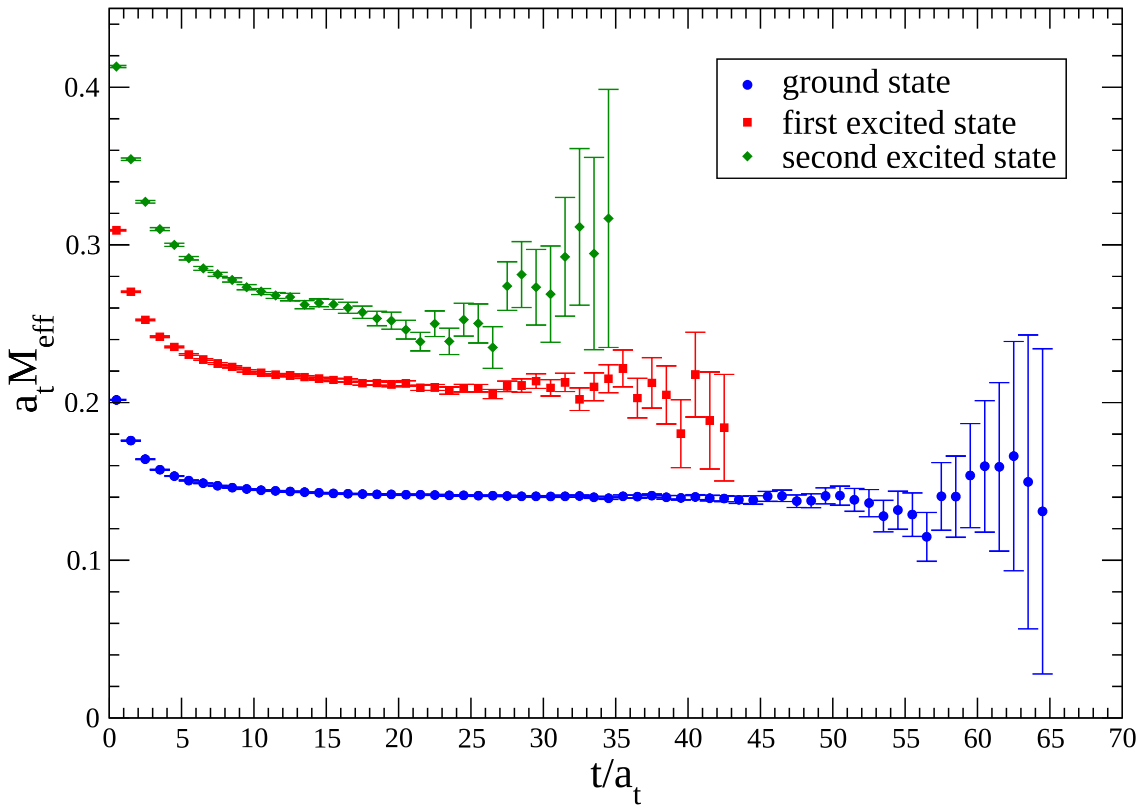
Static-light mesons

- HQET - lowest order in $1/m_Q$ - heavy quark propagator is a time-like Wilson line.



G_{1u} channel - ground-state is lightest meson

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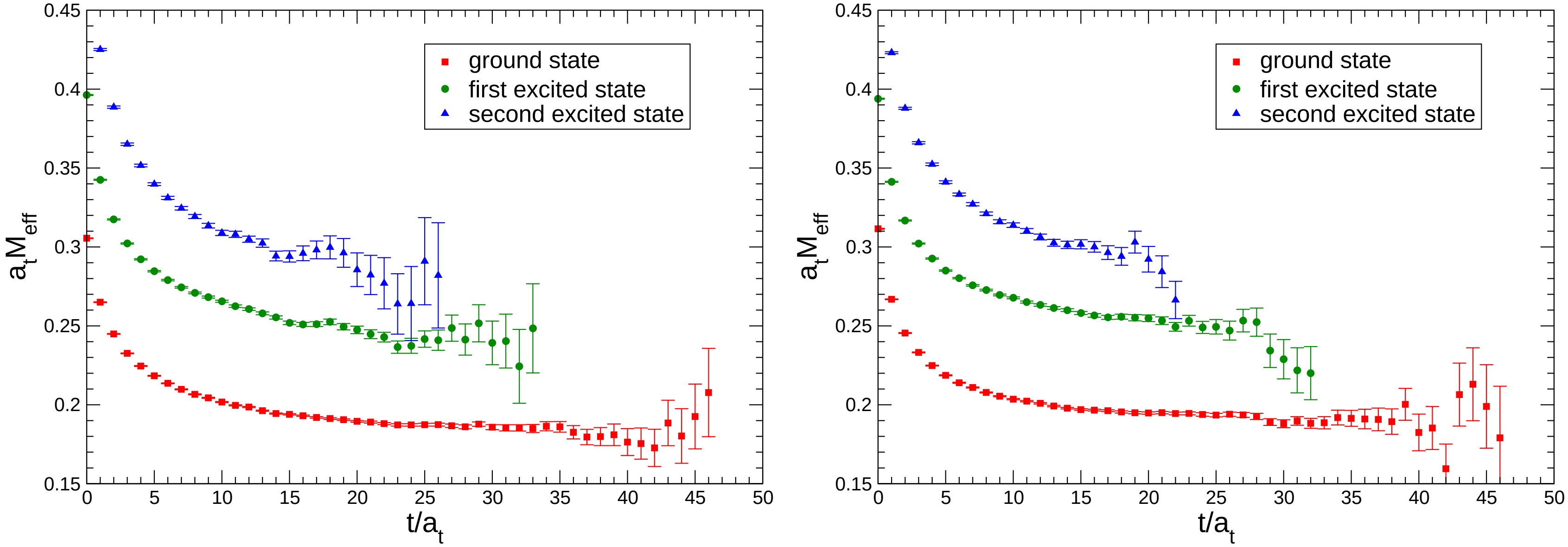
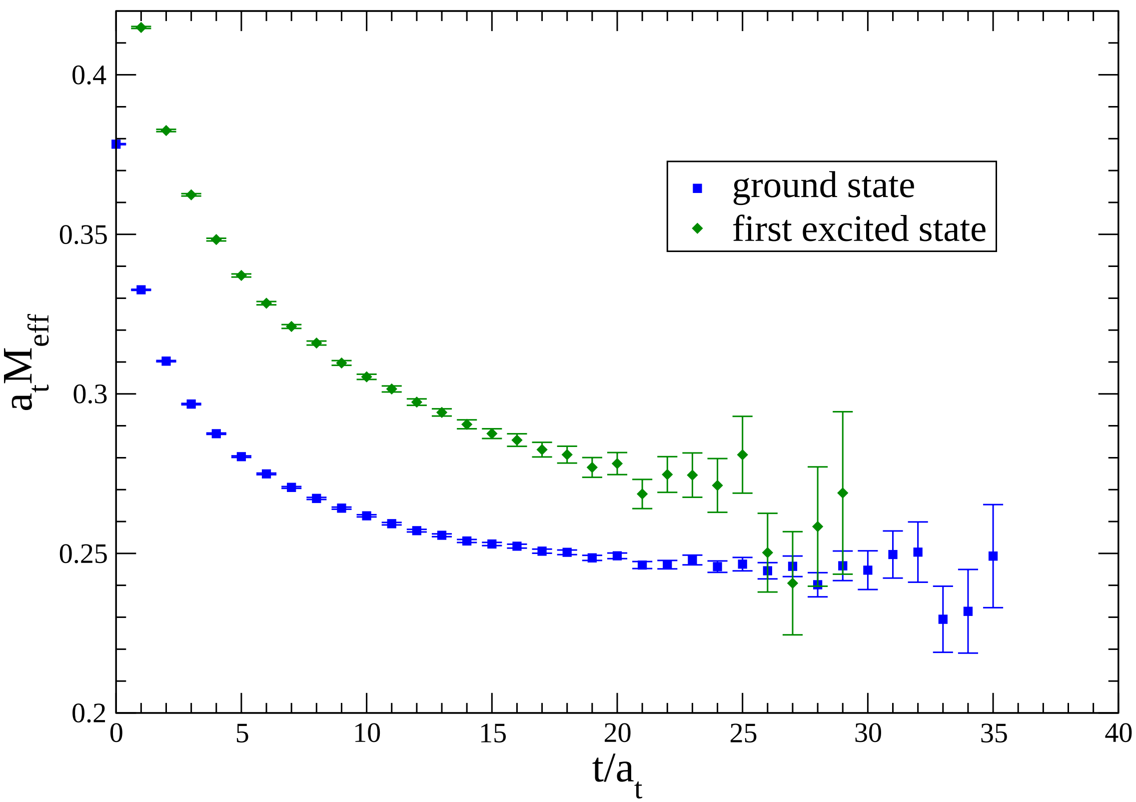
Static-light mesons



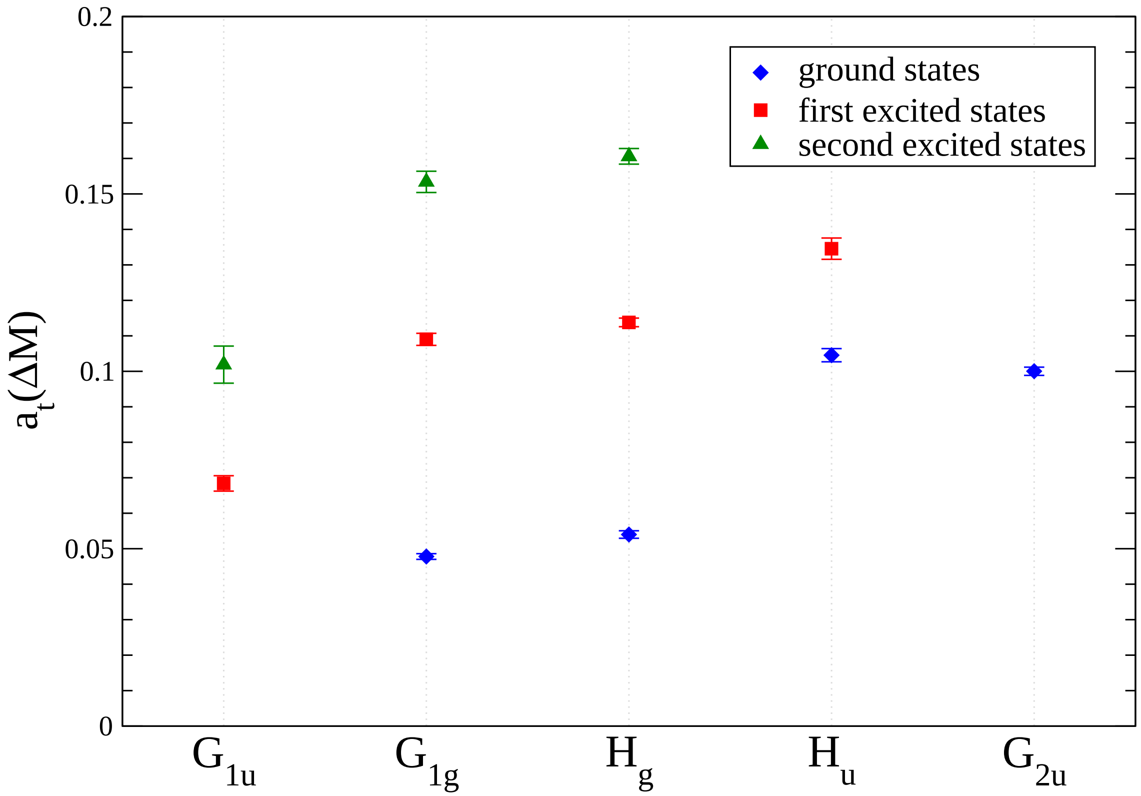
P-wave channels G_{1g} and H_g

Static-light mesons



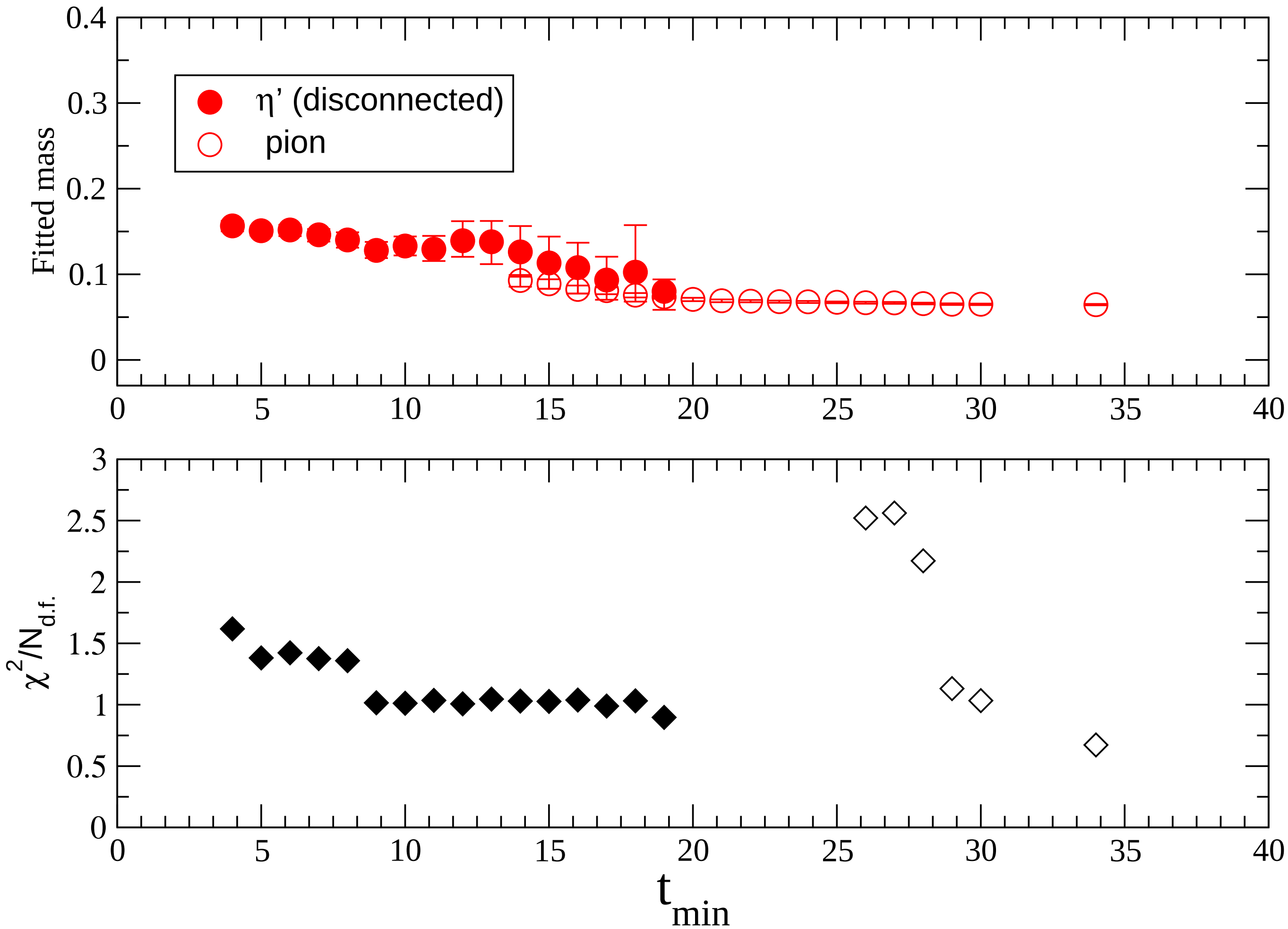
D-wave channel H_u

Static-light mesons

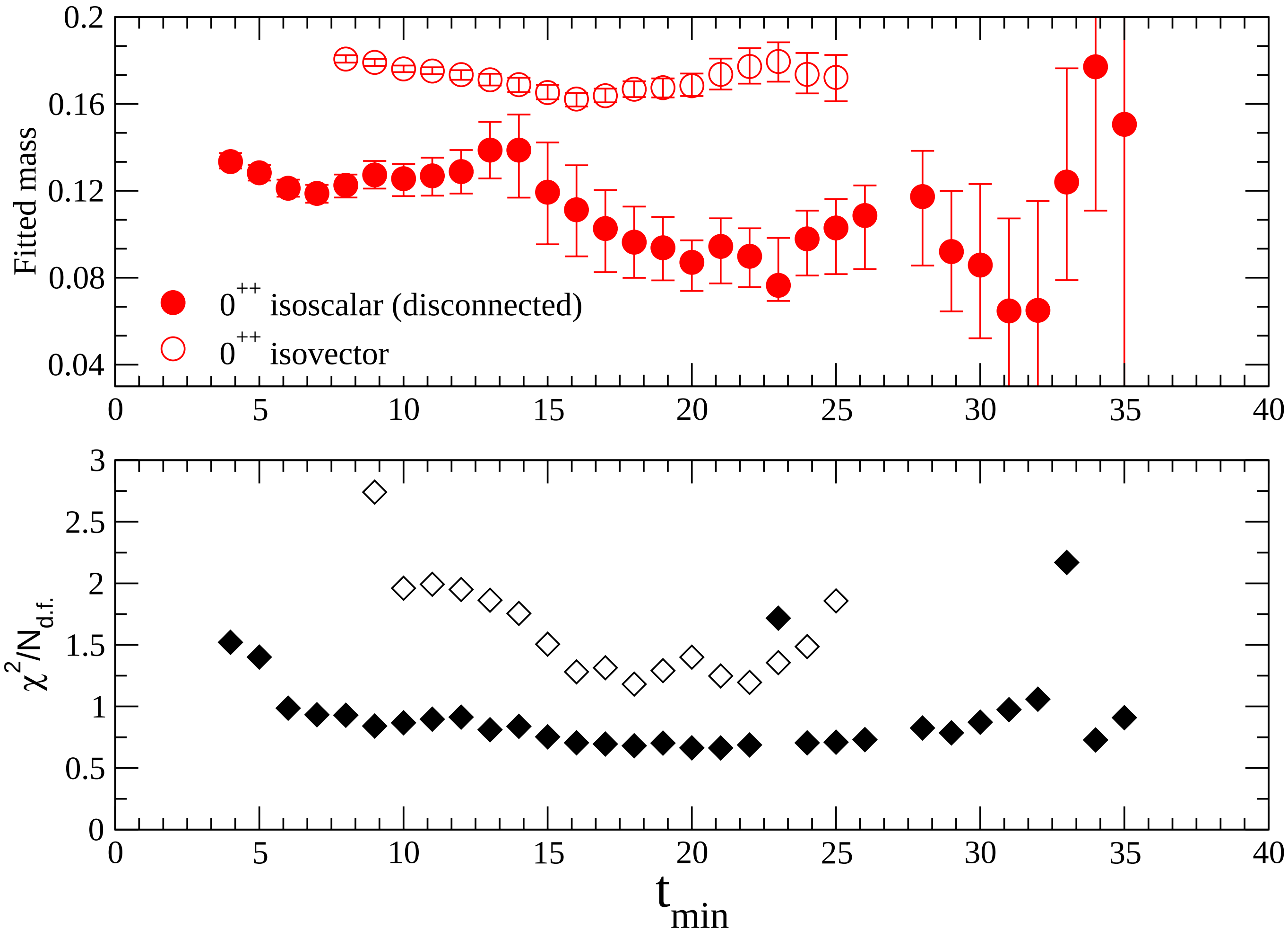


Energy splittings: is there an inversion in the D-wave?

Light mesons - pseudoscalar



Light mesons - scalar



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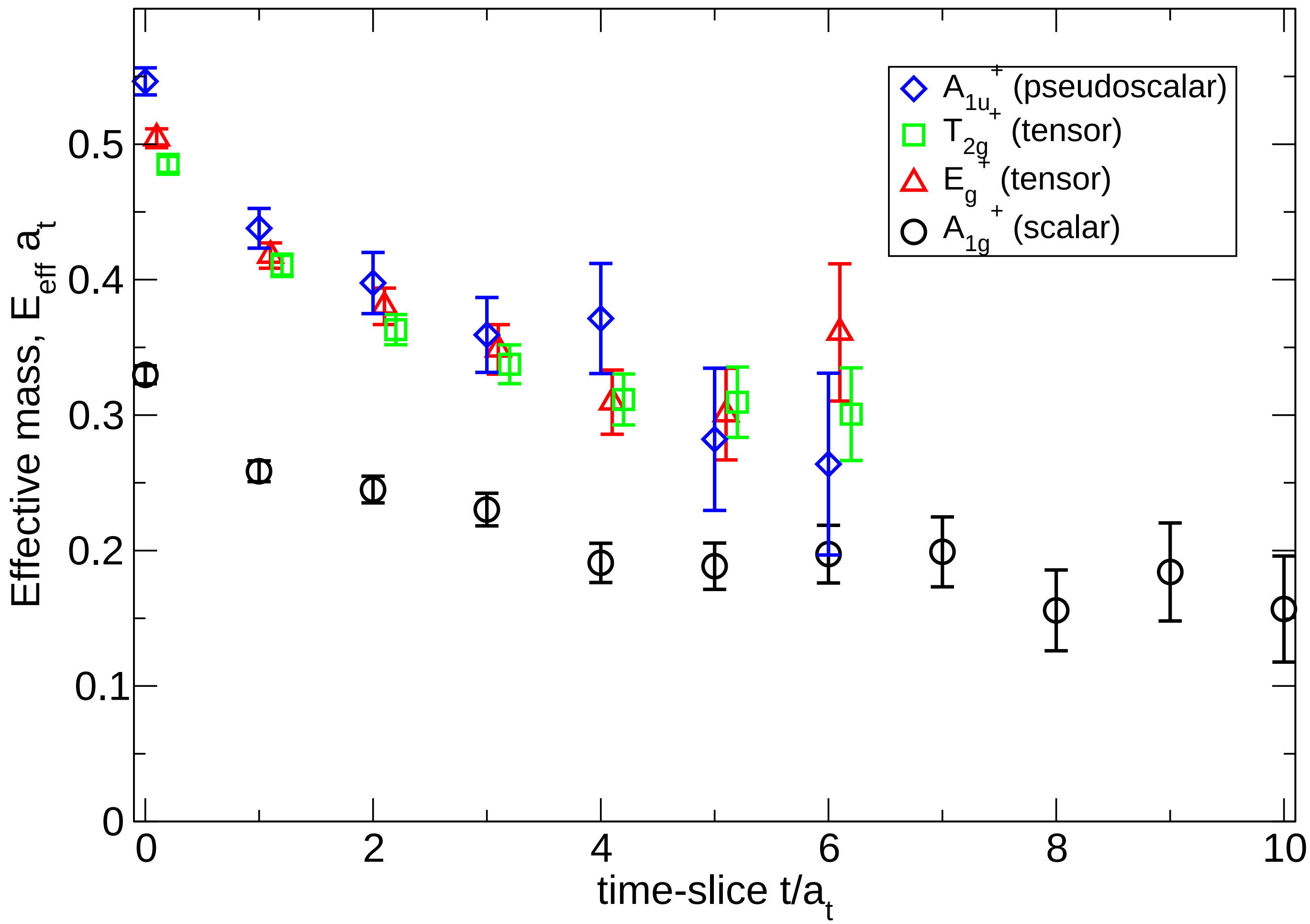
Glueballs



Consistent here with Yang-Mills theory

Future projects

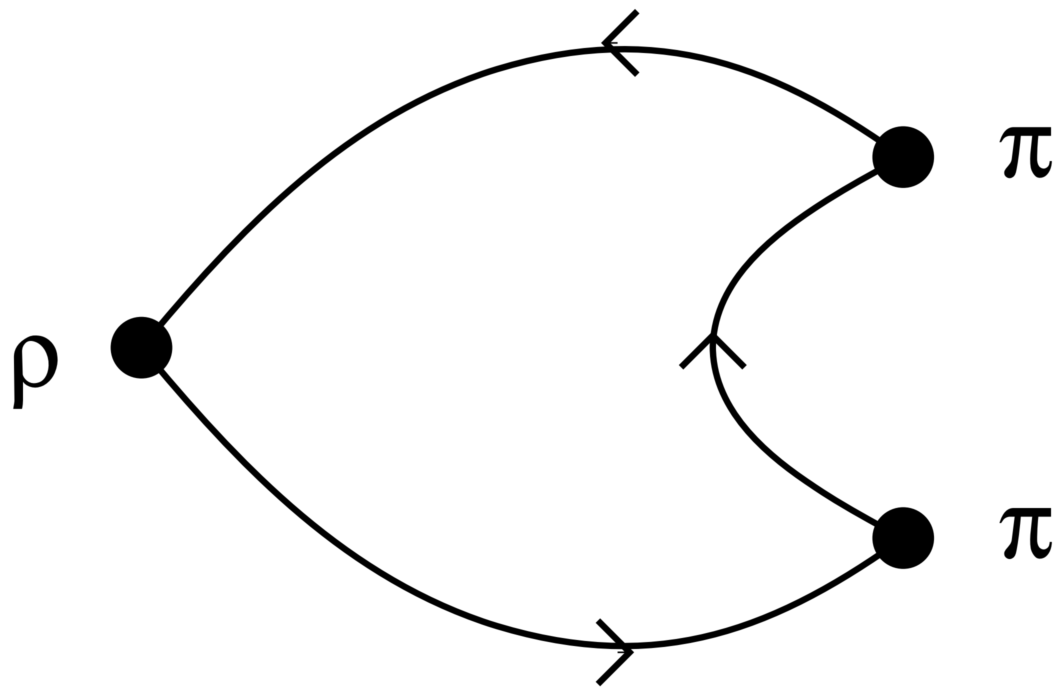
Future directions (1)

Exploit the freedom to build correlation functions easily

- Example (1): the diagrams needed to compute the width of the ρ meson
- Example (2): the diagrams needed to compute the width of the glueball
- Example (3): the diagrams needed to compute QCD contribution to $B \to \pi \ell \nu$

Future directions (2)

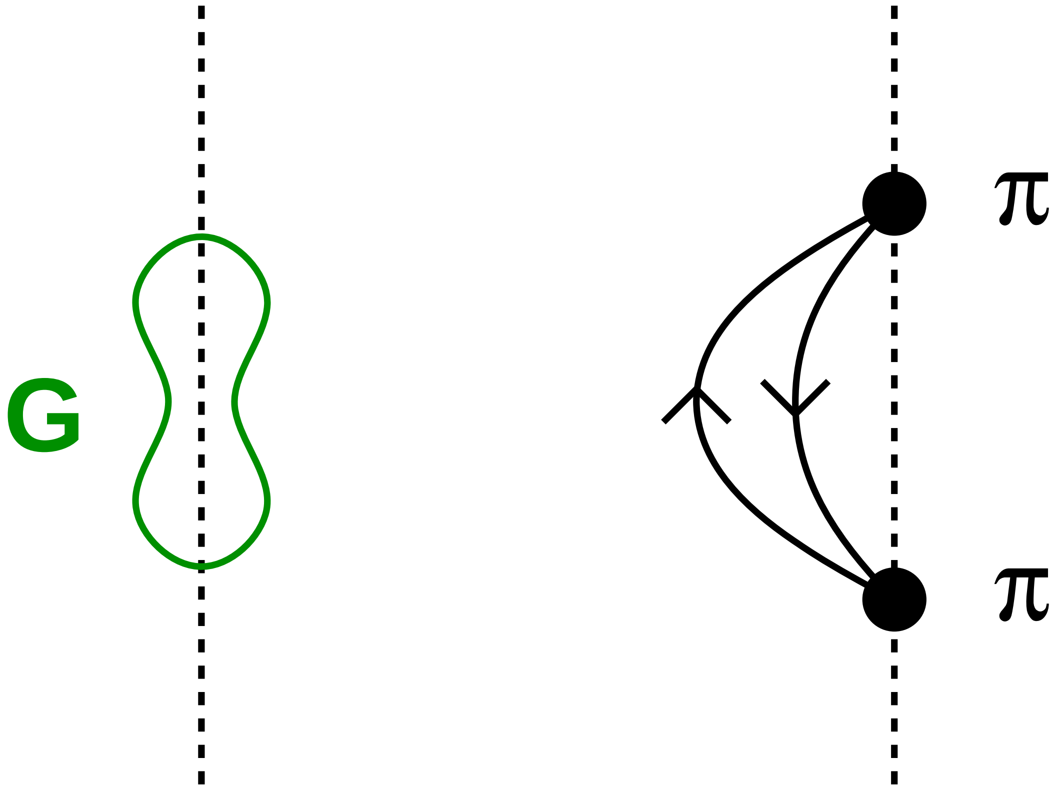
A diagram of the form



Becomes the evaluation of Tr $(\mathcal{O}_\pi(t') \times \mathcal{O}_\pi(t') \times \mathcal{O}_\rho(t))$.

Future directions (3)

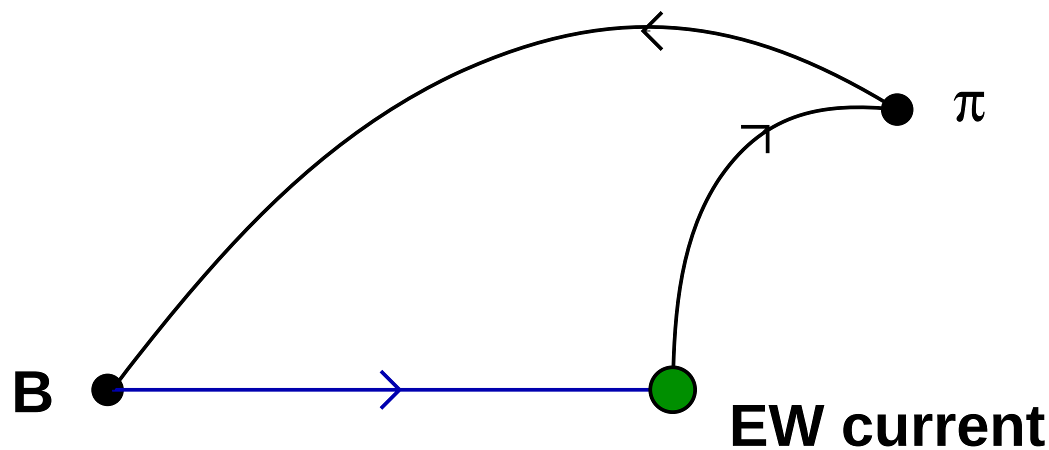
A diagram of the form



Becomes the evaluation of $G(t')\times$ Tr $(\mathcal{O}_\pi(t) \times \mathcal{O}_\pi(t))$.

Future directions (4)

A diagram of the form



becomes the evaluation of $\mathrm{Tr}\ w^*(t)\Gamma U(t,t')u(t') \otimes C_\pi(t'')$

Conclusions

- ◇ Simulations of $N_f = 2$ QCD on anisotropic lattices are feasible. The technical obstacles can be overcome.
- ◇ All-to-all propagator methods greatly enhance the scope of what can be calculated on the lattice.
- ◇ High-precision spectroscopy in the static-light and light sectors is easily accessible.
- ◇ High-statistics runs are now needed, but are affordable.
- ◇ Lighter fermion masses don't seem to be a problem.