Towards a high-precision determination of the meson and glueball spectrum

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Overview

• Anisotropic lattice QCD - advantages and disadvantages.
• Determining non-perturbative action parameters for the $N_f = 2$ anisotropic lattice
• A new toolkit for precision meson spectroscopy; The all-to-all propagator.
• Results:
  – Static-light mesons
  – Light mesons
  – The glueballs
• Conclusions
Anisotropic lattices with dynamical quarks
**Anisotropic lattices**

- Cost of a lattice simulation rises very rapidly as the continuum limit is approached (at least like $a^{-4}$).
- The simulation costs are much higher if light quark dynamics are included.
- To compute energies of states, we need to measure temporal correlation functions. To compute decay widths, we need to compute energies of states!
- ...so make temporal and spatial lattice spacings different
- Choose $a_t \ll a_s$ for optimal cost/benefits.
- Anisotropic lattice also useful for heavy quark physics and finite temperature.
- **Unfortunately, there are extra theory overheads for dynamical fermions:**
Action parameters for dynamical anisotropic lattices

- Lattice discretisations break continuum Lorentz symmetries; anisotropic lattices break even more symmetries.
- The number of distinct dimension-4 operators grows in both the quark and gluon sectors.

- A **simultaneous** tuning of two new action parameters is needed to recover the continuum symmetry.
Action parameters for dynamical anisotropic lattices

- Two physical probes of low-energy physics are needed to give two constraints.
- We use the pion dispersion relation and the “sideways potential”

- Need at least three runs to probe the response of the vacuum to changes in the quark and gluon sectors. Our target is $\xi = a_s/a_t = 6$. 
**Action parameters for dynamical anisotropic lattices**

- The self-consistent action parameters have been determined to 3%. We can do better.
Computing all elements of the lattice quark propagator
How to build an all-to-all propagator (1)

- Spectral representation
- Stochastic representation
- Variance reduction - dilution
- Hybrid method

**Spectral representation:**
Compute the eigenvalues and eigenvectors \( \{v^{(i)}, \lambda_i\} \) of \( Q = \gamma_5 D \), then

\[
Q^{-1} = \sum_{i}^{N} \frac{1}{\lambda_i} v^{(i)} \otimes v^{(i)*}
\]

Unfortunately, we can’t compute all eigenvectors easily, so we must truncate this representation at \( N_{\text{ev}} \ll N \).

This truncation is non-unitary and it is difficult to assign a systematic uncertainty in the error introduced by the truncation.
How to build an all-to-all propagator (2)

Stochastic representation:
Fill a vector, $\eta$ with independent random numbers. We use $\mathbb{Z}_4 = \{1, i, -1, -i\}$.

Since

$$\langle \eta_i \eta_j^* \rangle = \delta_{ij}$$

$Z_N$ noise has the useful property that

$$\eta_i \eta_i^* = 1 \text{ (no sum)}$$

Apply the fermion solver $Q \psi = \eta$ and then

$$\langle \psi_i \eta_j^* \rangle = [Q^{-1}]_{ik} \delta_{kj} = [Q^{-1}]_{ij}$$

This gives an unbiased estimator of every entry in the quark propagator. Unfortunately, the variance of the estimator is large.
Variance reduction (1)

Variance reduction:
- We can reduce the variance of our estimator by gathering statistics, but the noise only falls like $1/\sqrt{m}$ for $m$ noise vectors.
- The exact propagator can be computed using at most $N$ inversions, by putting a single one in all entries of the source vector in turn (i.e., using point propagators from all points!)
- This suggests a trick: break the vector space into $d$ smaller pieces spanned by a sub-set of the basis vectors, e.g.,

$$V_1 = \{e^{(1)} = (1, 0, 0, \ldots), e^{(2)} = (0, 1, 0, \ldots)\}$$

- The “dilution” is defined by the user - e.g., we could choose “time-dilution” where $V_t$ is the $N/N_t$ dimensional space of vectors with support on time-slice $t$. 
Variance reduction (2)

Variance reduction:

- The basis is complete, so $V = V_1 \oplus V_2 \oplus \ldots \oplus V_d$ and $\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} \ldots + \eta^{(d)}$ where $\eta^{(2)} = S_i \eta$ and $S_i$ is the projector into vector space $V_i$.
- Since $S_i^2 = S_i$ and $\sum_i S_i = I$ we can write

$$I = \sum_i S_i = \sum_i S_i^2 = \sum_i S_i \langle \eta \otimes \eta^* \rangle S_i$$

- Another representation of the propagator can be written

$$Q^{-1} = \sum_i Q^{-1} S_i \langle \eta \otimes \eta^* \rangle S_i = \sum_i \langle \psi^{(i)} \otimes \eta^{(i)*} \rangle$$

- The variance is reduced by explicit cancellation of terms whose expectation value is zero. In the “homeopathic limit” when $d = N$, the exact all-to-all propagator is recovered.
A hybrid method:

- Most of the interesting physics is contained in the lowest few eigenvectors.
- Can we use the $N_{ev}$ lowest eigenvectors and correct for the truncation using a stochastic estimator?
- Consider breaking $V$ into the sub-spaces $V_L$, which is spanned by the lowest $N_{ev}$ eigenvectors and $V_H$, spanned by the rest.
- Since $Q$ leaves these vector spaces invariant, we can write the inverse

$$Q^{-1} = \bar{Q}_L + \bar{Q}_H = Q^{-1}P_L + Q^{-1}P_H$$

- $\bar{Q}_L$ is just the truncated eigenvector representation.
- Estimate $\bar{Q}_H$ stochastically with dilute noise. The action of $\bar{Q}_H$ can be computed from

$$\bar{Q}_H = Q^{-1}P_H = Q^{-1}(1 - P_L)$$
A hybrid method (2)

The recipe:
1. Compute $N_{ev}$ eigenvectors and eigenvalues.
2. Generate a noise vector, $\eta$ and dilute it: $\{\eta^{(1)}, \eta^{(2)}, \ldots\}$
3. For each noise vector, compute
   \begin{equation*}
   \psi^{(i)} = Q^{-1}(1 - P_L)\eta^{(i)}
   \end{equation*}
   by first orthogonalising w.r.t the eigenvectors, and then applying the fermion inverse to the resulting vector.
4. Now, $Q^{-1}$ can be estimated from
   \begin{equation*}
   Q^{-1} = \sum_i^{N_{ev}} \frac{1}{\lambda_i} \psi^{(i)} \otimes \psi^{(i)*} + \sum_d^{N_d} \psi^{(d)} \otimes \eta^{(d)*}
   \end{equation*}
This form suggests packing the two sets of vectors together into a “hybrid list”.
Practical implementation (1)

\[ Q^{-1} = \sum_{i}^{N_{ev}} \frac{1}{\lambda_i} v(i) \otimes v(i)^* + \sum_{d}^{N_d} \psi(d) \otimes \eta(d)^* \]

Write two “hybrid lists” of vectors:

\[ w(i) = \left\{ \gamma_5 \frac{v(1)}{\lambda_1}, \cdots, \gamma_5 \frac{v(N)}{\lambda_N}, \gamma_5 \eta^{(1)}, \cdots, \gamma_5 \eta^{(N_d)} \right\} \]

\[ u(i) = \left\{ v(1), \cdots, v(N), \psi(1), \cdots, \psi(N_d) \right\} \]

and then

\[ M^{-1} = \sum_{i}^{N_h} u(i) \otimes w(i)^* \]
Practical implementation (2)

Computing physical correlation functions:
- We don’t usually want to compute the quark propagator directly.
- Computing two-point functions

\[ C_\pi(\Delta t) = \sum_t \langle \sum_y \bar{d}(y, t + \Delta t) \gamma_5 u(y, t + \Delta t) | \sum_x \bar{u}(x, t) \gamma_5 d(x, t) \rangle \]

becomes:

\[ C_\pi(\Delta t) = \sum_t \langle \sum_y w_{[1]}^*(y, t + \Delta t) \gamma_5 u_{[2]}(y, t + \Delta t) \sum_x w_{[2]}^*(x, t) \gamma_5 u_{[1]}(x, t) \rangle \]

- \{u_{[1]}, w_{[1]}\} and \{u_{[2]}, w_{[2]}\} are independent hybrid lists.
Practical implementation (3)

- Flavour singlet correlators can be computed

- Disconnected diagram for eg $\eta'$ correlation function is

$$D(\Delta t) = \sum_t \left\langle \sum_y w_{[1]}^*(y, t + \Delta t) \gamma_5 u_{[1]}(y, t + \Delta t) \sum_x w_{[2]}^*(x, t) \gamma_5 u_{[2]}(x, t) \right\rangle$$

- Hybrid list indices are traced over individually for both the source and the sink.

- Correlations with gluonic operators can be measured.
Practical implementation (4)

- ALL book-keeping can be hidden from end-user, who just writes a function to compute $a^* \gamma_5 b$ on a time-slice.
- The measurement and propagator calculations are decoupled (apart perhaps from consideration of which dilution is best).
- Operator construction is unrestricted - smearings, extended operators, etc. can be decided on at the final measurement phase.
- Large bases of operators can be built and cross-correlated to optimise ground-state (and excited-state) overlaps.
- More dilution can be applied post-hoc, making use of work done so far.
- More eigenvectors can be computed post-hoc, without the need to re-compute inversions.
New results
Static-light mesons

- HQET - lowest order in $1/m_Q$ - heavy quark propagator is a time-like Wilson line.

$G_{1u}$ channel - ground-state is lightest meson
Static-light mesons

P-wave channels $G_1g$ and $H_g$
Static-light mesons

D-wave channel $H_u$

- ground state
- first excited state
Static-light mesons

Energy splittings: is there an inversion in the D-wave?
Light mesons - pseudoscalar

Fitted mass

η' (disconnected)
pion

χ²/NDf.

t_{min}
Light mesons - scalar

\[ \chi^2 / N_{d.f.} \]

- Fitted mass
- 0^{++} isoscalar (disconnected)
- 0^{++} isovector

\[ t_{\text{min}} \]

0 5 10 15 20 25 30 35 40
0 0.04 0.08 0.12 0.16 0.2
0 0.04 0.08 0.12 0.16 0.2
Effective mass, $E_{\text{eff}} a_t$

Consistent here with Yang-Mills theory
Future projects
Future directions (1)

- **Example (1):** the diagrams needed to compute the width of the $\rho$ meson
- **Example (2):** the diagrams needed to compute the width of the glueball
- **Example (3):** the diagrams needed to compute QCD contribution to $B \rightarrow \pi \ell \nu$
Future directions (2)

A diagram of the form

Becomes the evaluation of $\text{Tr} \left( \mathcal{O}_\pi(t') \times \mathcal{O}_\pi(t') \times \mathcal{O}_\rho(t) \right)$. 
Future directions (3)

A diagram of the form

\[ G \]

Becomes the evaluation of

\[ G(t') \times \text{Tr} \left( O_\pi(t) \times O_\pi(t) \right). \]
Future directions (4)

A diagram of the form

\[ \text{EW current} \]

becomes the evaluation of

\[ \text{Tr} \ w^* (t) \Gamma U(t, t') u(t') \otimes C_\pi (t'') \]
Conclusions

- Simulations of $N_f = 2$ QCD on anisotropic lattices are feasible. The technical obstacles can be overcome.
- All-to-all propagator methods greatly enhance the scope of what can be calculated on the lattice.
- High-precision spectroscopy in the static-light and light sectors is easily accessible.
- High-statistics runs are now needed, but are affordable.
- Lighter fermion masses don’t seem to be a problem.