Progress toward determining the light baryon SPECTRUM in lattice QCD

- 2005-06 SPECTRUM Collaboration of LHPC
- Goal: Theoretical determination of the spectrum of baryons and other hadrons.
- Means: Lattice QCD
  - Variational method
  - Lots of interpolating field operators
Use quenched QCD to prove methods & look at light baryons

- $16^3 \times 64$ (167 configs) and $24^3 \times 64$ (239 configs) anisotropic lattices

- $m_\pi = 490$ MeV.

- Local operators

- One-link-displaced nonlocal operators

- Calculations performed using Jlab clusters
Outline

• Spin on the lattice
• A lattice view of the physical spectrum
• Variational method
• Stability of eigenvectors
• Pattern of low-lying states at $a_s = 0.1$ fermi
• Evidence for spin $\frac{5}{2}$
• Summary
Energies from correlation functions

\[ C(t) = \sum_x \langle 0 | B(x, t) \overline{B}(0, 0) | 0 \rangle = Z_1^2 e^{-E_1 t} + Z_2^2 e^{-E_2 t} + \ldots \]  

(1)

Quantum numbers of states

• Spin quantum numbers \( J^2 \) and \( J_z \) are not “good”.

• Irreducible representations (irreps) of octahedral group are “good”.

\[ \sum_x \langle 0 | TB_k^{(\Lambda, \lambda)}(x, t) \overline{B}^{(\Lambda', \lambda')}_{k'}(0) | 0 \rangle = C_k^{(\Lambda, \lambda)}(t) \delta_{\Lambda \Lambda'} \delta_{\lambda \lambda'} \]
Double octahedral group, $O^D$ and subduction of $J$.

- $G_{1g}(2), \ H_g(4), \ G_{2g}(2)$ irreps for positive parity.

- $G_{1u}(2), \ H_u(4), \ G_{2u}(2)$ irreps for negative parity.

<table>
<thead>
<tr>
<th>irrep of $O^D$</th>
<th>Spin of $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>1/2 7/2</td>
</tr>
<tr>
<td>$H$</td>
<td>3/2 5/2 7/2</td>
</tr>
<tr>
<td>$G_2$</td>
<td>5/2 7/2</td>
</tr>
</tbody>
</table>

- Spin $\frac{1}{2}$ is in $G_1$ channel, $\frac{3}{2}$ in $H$ channel
- Spin $\frac{5}{2}$ is in both $G_2$ and $H$

Example: As $a_s \to 0$, degenerate states

$$\left( \begin{array}{c} E_n(H) \\ E_m(G_2) \end{array} \right) \longrightarrow E(J = 5/2),$$
Physical positive-parity spectrum subduced

<table>
<thead>
<tr>
<th></th>
<th>G1g</th>
<th>G2g</th>
<th>Hg</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td></td>
<td>N(1990) 7/2+</td>
<td></td>
</tr>
<tr>
<td>1910</td>
<td>Δ(1910) 1/2+</td>
<td></td>
<td>Δ(1920) 3/2+</td>
</tr>
<tr>
<td>1750</td>
<td>Δ(1750) 1/2+</td>
<td>Δ(1905) 5/2+</td>
<td>N(1900) 3/2+</td>
</tr>
<tr>
<td>1710</td>
<td>N(1710) 1/2+</td>
<td></td>
<td>N(1720) 3/2+</td>
</tr>
<tr>
<td>1440</td>
<td></td>
<td>N(1680) 5/2+</td>
<td></td>
</tr>
<tr>
<td>1232</td>
<td></td>
<td></td>
<td>Δ(1600) 3/2+</td>
</tr>
<tr>
<td>939</td>
<td></td>
<td></td>
<td>Δ(1232) 3/2+</td>
</tr>
</tbody>
</table>

- Scattering states starting at $M_N + m_\pi$
  become discrete states on lattice.
Physical negative-parity spectrum subduced

<table>
<thead>
<tr>
<th></th>
<th>G1u</th>
<th>G2u</th>
<th>Hu</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>Δ(1900) 1/2-</td>
<td>Δ(1930) 5/2-</td>
<td>Δ(1940) 3/2-</td>
</tr>
<tr>
<td>1500</td>
<td>N(1650) 1/2-</td>
<td>N(1700) 3/2-</td>
<td>Δ(1700) 3/2-</td>
</tr>
<tr>
<td>1000</td>
<td>Δ(1620) 1/2-</td>
<td>Δ(1675) 5/2-</td>
<td>N(1520) 3/2-</td>
</tr>
<tr>
<td></td>
<td>N(1535) 1/2-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- With $m_\pi = 490$ MeV, $M_N + m_\pi$ s-wave state is close to $N(\frac{1}{2}^-, 1535)$. 
Need for nonlocal operators.

• Three spin $\frac{1}{2}$ local operators can produce $S = \frac{1}{2}$ or $S = \frac{3}{2}$.

• Example: $N_{121} = (u_1d_2 - d_1u_2)u_1$ is $S = \frac{1}{2}$ Nucleon operator.

• No higher spins can be produced.

• No $G_2$ irreps can be produced.

• Nonlocal operators: discretizations of $L = 1, L = 2, \ldots$.

• $L = 1 \oplus S = \frac{3}{2} \rightarrow J = \frac{5}{2} \rightarrow G_2$.
Simplest nonlocal operators

- Displace one quark relative to the other two.

- Form irreps of displacements: \( D_{\lambda_1}^{\Lambda_1} \)

- Use Clebsch-Gordan coefficients for octahedral group to make overall irreps

\[
O^{(\Lambda, \lambda)}_k = \sum_{\lambda_1,\lambda_2} C(\Lambda, \lambda_1, \lambda_2) D_{\lambda_1}^{\Lambda_1} \Psi^{(\Lambda_2, \lambda_2)}_k.
\]
One-link displacement irrep operators $\mathcal{D}^{\Lambda}_{\lambda}$

\[
\begin{pmatrix}
\hat{A}_1 B & \hat{D}_+ B & \hat{D}_- B & \hat{D}_0 B & \hat{E}_0 B & \hat{E}_2 B
\end{pmatrix}^T \equiv \\
\frac{1}{\sqrt{6}}(d_x B + d_y B + d_z B + d_{-x} B + d_{-y} B + d_{-z} B) \\
\frac{i}{2}[(d_x B - d_{-x} B) + i(d_y B - d_{-y} B)] \\
-\frac{i}{2}[(d_x B - d_{-x} B) - i(d_y B - d_{-y} B)] \\
-\frac{i}{\sqrt{2}}(d_z B - d_{-z} B) \\
\frac{1}{\sqrt{12}}[2(d_z B + d_{-z} B) - (d_x B + d_{-x} B) - (d_y B + d_{-y} B)] \\
\frac{1}{2}[(d_x B + d_{-x} B) - (d_y B + d_{-y} B)]
\]

Combinations of displacements that transform according to the $A_1$, $T_1$, and $E$ single-valued irreps of the octahedral group provide lattice discretizations of the spherical harmonics $Y_{\ell m}$,

\[
\hat{A}_1 \sim Y_{00},
\]

\[
\hat{D}_{+,0,-} \sim Y_{11}, \quad Y_{10}, \quad Y_{1-1},
\]

\[
\hat{E}_{0,2} \sim Y_{20}, \quad (Y_{22} + Y_{2-2}).
\]
Charge Conjugation

\[
C^{(Λλ)}_{kk'}(t) = \mathcal{P}_{k'}\delta_{P,0}\left[ \sum_n \theta(t) \langle 0 | B^{(Λλ)}_k | n \rangle \langle n | \bar{B}^{(Λλ)}_{k'} | 0 \rangle e^{-E_nt} \\
- \sum_{\bar{n}} \theta(-t) \langle 0 | \bar{B}^{(Λλ)}_{k'} | \bar{n} \rangle \langle \bar{n} | B^{(Λλ)}_k | 0 \rangle e^{E_{\bar{n}t}} \right],
\]

(3)

Charge conjugation relations \(|\bar{n}\rangle = C|n\rangle e^{i\phi}\) produce a relation between correlation functions

\[
C^{(Λλ)}_{kk'}(t) = \delta_{P,0} \sum_n \left[ \theta(t) \mathcal{P}^{(Λ)}_{k'} \langle 0 | B^{(Λλ)}_k | n \rangle \langle n | \bar{B}^{(Λλ)}_{k'} | 0 \rangle e^{-E_nt} \\
- \eta_t \theta(T-t) \mathcal{P}^{(Λc)}_{k'} \langle 0 | B^{(Λc)}_k | n \rangle^* \langle n | \bar{B}^{(Λc)}_{k'} | 0 \rangle^* e^{-E_{\bar{n}(T-t)}} \right].
\]

(4)

The forward propagating signal of a correlation function is equal to the backward propagating signal of the parity-reversed, complex-conjugated correlation function within the factor \(-\eta_t\), i.e.,

\[
C^{(Λλ)}_{kk'}(t) = -\eta_t C^{(Λcλc)*}_{kk'}(T-t).
\]

(5)
Doubling of statistics

• For $+$ parity operators use $\overline{B}^{(\Lambda)}$.

• For $-$ parity operators use $\overline{B}^{(\Lambda_c)}$.

• Each provides a correlation function in $0 < t < T/2$ for BOTH parities.

• They are independent samples: $\approx 10$ time slices apart.

• One is from $t < T/2$ and the other is from $t > T/2$.

• Number of samples is double the number of gauge configurations.

• $167 \rightarrow 334$ on $16^3$, $239 \rightarrow 478$ on $24^3$. 
Variational method

• Diagonalize matrices of correlation functions $C_{k,k'}(t)$

• Extract spectrum of energies.

1.) Renormalize: $\tilde{C}_{k,k'}^{(\Lambda)}(t_0) = 1$.

$$\tilde{C}_{k,k'}^{(\Lambda)}(t) = N_k C_{k,k'}^{(\Lambda)}(t) N_{k'},$$

2.) Solve generalized eigenvalue eq.

$$\sum_{k'} \tilde{C}_{k,k'}^{(\Lambda)}(t) v_{k'}^{(n)} = \alpha^{(n)}(t, t_0) \sum_{k'} \tilde{C}_{k,k'}^{(\Lambda)}(t_0) v_{k'}^{(n)},$$

3.) Obtain principal eigenvalues

$$\alpha^{(n)}(t, t_0) \approx e^{-E_n(t-t_0)} \left( 1 + O(e^{-|\delta E| t}) \right),$$
Eigenvectors

Eigenvectors diagonalize the renormalized correlation matrix:

\[ v_k^{(n)T}(t, t_0) \tilde{C}_{kk'}^{(\Lambda)}(t) v_{k'}^{(n')}(t, t_0) = \alpha^{(n)}(t, t_0) \delta_{nn'} \]

Diagonal operators:

\[ \mathcal{O}_{n}^{(\Lambda \lambda)} = \sum_k \tilde{v}_k^{(n)}(t, t_0) B_k^{(\Lambda \lambda)} \]

Normalization:

\[ \tilde{v}_k^{(n)}(t, t_0) \equiv Z_n(t) N_k v_k^{(n)}(t, t_0) \]

such that \( \sum_k |\tilde{v}_k^{(n)}(t, t_0)|^2 = 1 \).

Diagonal correlation function:

\[ \tilde{C}_{nn'}^{(\Lambda)}(t) = |Z_n(t)|^2 \alpha^{(n)}(t, t_0) \delta_{n,n'} \]

Weights of basis operators: \( |\tilde{v}_k^{(n)}(t, t_0)|^2 \).
Volume dependence

- Single particle states at zero total momentum:
  \[ \frac{C(L_1)}{C(L_2)} \approx 1. \]

- Two-particle states at zero total momentum:
  \[ \frac{C(L_1)}{C(L_2)} \approx \frac{L_2^3}{L_1^3} \rightarrow 3.37 \]

- Does \( Z_n^2 \) help to identify scattering states?
Stability check: $N G_{1g}$

- $a_t = 6 GeV^{-1}$
Stability check: N $G_{2g}$
Stability check:  N $H_g$

Dimension of matrix 18
Stability check: \( \Delta G_{1g} \)
Stability check: \( \Delta H_g \)
Stability check: N  \( G_{1u} \)
Stability check: N \[ G_{2u} \]

Dimension of matrix
Stability check: N \( H_u \)
Stability check: $\Delta G_{1u}$
Stability check: $\Delta H_u$
Gaussian smearing of quark field:

\( \sigma = 3.0, \ N=20. \)

\[
\hat{G}^{(N)}(x, x') = \sum_y (\delta_{x,y} + \sigma^2 \frac{\nabla^2_{x,y}}{4N}) \hat{G}^{(N-1)}(y, x'),
\]

\[
\hat{G}^{(0)}(x, x') = \delta_{x,x'}.
\]
Volume dependence $N$ $H_u$

![Graph showing the volume dependence $N$ $H_u$ with data points and error bars. The graph plots $Z(16)/Z(24)$ against volume with a logarithmic scale on the x-axis.]
\[ \frac{Z^2(16)/Z^2(24)}{Z^2_N(16)/Z^2_N(24)} \]

![Graph showing volume ratio of spectral weight against state number]

1: N(G_{1g}) gnd; 2: N(G_{1g}) 1st; 3: N(G_{1g}) 2nd;
4: N(G_{2g}) gnd; 5: N(G_{2g}) 1st; 6: N(G_{2g}) 2nd;
7: N(H_g) gnd; 8: N(H_g) 1st; 9: N(G_{1u}) gnd;
10: N(G_{1u}) 1st; 11: N(G_{2u}) gnd; 12: N(H_u) gnd;
13: N(H_u) 1st; 14: N(H_u) 2nd; 15: \Delta(G_{1g}) gnd;
16: \Delta(G_{1g}) 1st; 17: \Delta(G_{2g}) gnd; 18: \Delta(H_g) gnd;
19: \Delta(H_g) 1st; 20: \Delta(H_g) 2nd; 21: \Delta(H_g) 3rd;
22: \Delta(G_{1u}) gnd; 23: \Delta(G_{2u}) gnd; 24: \Delta(H_u) gnd.
Eigenvectors: \( N G_{1g} \)

\[
\begin{align*}
\sqrt{2} \hat{D} \overline{N}_{1,1/2}^{G_{1g}, 1} - \frac{1}{\sqrt{3}} \hat{D}^0 \overline{N}_{3,3/2}^{H_u} + \frac{1}{\sqrt{6}} \hat{D} + \overline{N}_{3,3/2}^{H_u} \\
\sqrt{2/3} \hat{D} + \overline{N}_{1,1/2}^{G_{1u}, 1} - \frac{1}{\sqrt{3}} \hat{D}^0 \overline{N}_{1,1/2}^{G_{1u}, 1} \\
\overline{N}_{1,1/2}^{G_{1g}, 2}
\end{align*}
\]
Eigenvectors: \( N H_u \)

\[
\begin{align*}
\square & \quad \sqrt{3/5} \hat{D}^0 \overline{N_{3/2, 3/2}^{H_u}} - \sqrt{2/5} \hat{D}^+ \overline{N_{3/2, 1/2}^{H_u}} & \text{MA} \\
\triangle & \quad \sqrt{3/5} \hat{D}^0 \overline{N_{3/2, 3/2}^{H_g}} - \sqrt{2/5} \hat{D}^+ \overline{N_{3/2, 1/2}^{H_g}} & \text{S} \\
\Diamond & \quad \sqrt{3/5} \hat{D}^0 \overline{N_{3/2, 3/2}^{H_g}} - \sqrt{2/5} \hat{D}^+ \overline{N_{3/2, 1/2}^{H_g}} & \text{MA} \\
\bigcirc & \quad \hat{D}^+ \overline{N_{1/2, 1/2}^{G_{1g, 1}}} & \text{MA}
\end{align*}
\]
Pattern of lowest energies in each channel

- \( \Delta (1905) \ 5/2^+ \)
- \( \Delta (1750) \ 1/2^+ \)
- \( N(1680) \ 5/2^+ \)
- \( \Delta (1232) \ 3/2^+ \)
- \( N(939) \ 1/2^+ \)
- \( \Delta (1930) \ 5/2^- \)
- \( \Delta (1700) \ 3/2^- \)
- \( N(1675) \ 5/2^- \)
- \( \Delta (1620) \ 1/2^- \)
- \( N(1535) \ 1/2^- \)
- \( N(1520) \ 3/2^- \)

\( \Delta G_{1g} \)
\( \Delta G_{2g} \)
\( N G_{1u} \)
\( N G_{2u} \)
\( Δ H_g \)
\( Δ H_u \)

• Omits, e.g., \( N(\frac{1}{2}, 1440) \).
Summary

- Quasi-local + One-link operators

- Lowest $I = \frac{1}{2}$ and $I = \frac{3}{2}$ energies

- 23 energies found on both lattice volumes

- Similar eigenvectors

- Volume ratios? Scattering states?

- Spin $\frac{5}{2}$ states seen in $G_2$ spectra.

- Partner $H$ states are seen at $a_s = 0.1F$.

- See subduction pattern but want to confirm because scaling may be poor.

- Pattern of lowest energies is similar to the pattern of lowest physical resonance states.