High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

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- Regge limit in a conformal theory.
- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- Non-linear evolution equation in the NLO.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies
- NLO BK kernel in $\mathcal{N} = 4$.
- NLO amplitude in $\mathcal{N} = 4$ SYM
- Conclusions
- Outlook: rapidity evolution of TMD's
Conformal four-point amplitude

\[ A(x, y, x', y') = (x - y)^4(x' - y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y') \rangle \]

\[ O = \text{Tr}\{Z^2\} \quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)) \] - chiral primary operator

In a conformal theory the amplitude is a function of two conformal ratios

\[ A = F(R, R') \]

\[ R = \frac{(x - y)^2(x' - y')^2}{(x - x')^2(y - y')^2}, \quad R' = \frac{(x - y)^2(x' - y')^2}{(x - y')^2(x' - y)^2} \]

At large \( N_c \)

\[ A(x, y, x', y') = A(g^2 N_c) \]

\[ g^2 N_c = \lambda \] - 't Hooft coupling
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AdS/CFT gives predictions at large \( \lambda \rightarrow \infty \).
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Our goal is perturbative expansion and resummation of \((\lambda \ln s)^n\) at large energies in the next-to-leading approximation

\[ (\lambda \ln s)^n(c_n^{\text{LO}} + c_n^{\text{NLO}} \lambda) \]
Regge limit in the coordinate space

Regge limit: \( x_+ \to \rho x_+ \), \( x'_+ \to \rho x'_+ \), \( y_- \to \rho y_- \), \( y'_- \to \rho y'_- \) \( \rho, \rho' \to \infty \)

Full 4-dim conformal group: \( A = F(R, r) \)

\[
R = \frac{(x - y)^2(x' - y')^2}{(x - x')^2(y - y')^2} \to \frac{\rho^2 \rho'^2 x_+ x'_+ y_- y'_-}{(x - x')^2(y - y')^2} \to \infty
\]

\[
r = \frac{[(x - y)^2(x' - y')^2 - (x' - y)^2(x - y')^2]^2}{(x - x')^2(y - y')^2(x - y)^2(x' - y')^2}
\to \frac{[(x' - y')^2 x_+ y_- + x'_+ y'_-(x - y)^2 + x_+ y'_-(x' - y)^2 + x'_+ y_-(x - y')^2]^2}{(x - x')^2(y - y')^2 x_+ x'_+ y_- y'_-}
\]
4-dim conformal group versus $SL(2, C)$

Regge limit: $x_+ \to \rho x_+, \; x'_+ \to \rho x'_+, \; y_- \to \rho' y_-, \; y'_- \to \rho' y_- \quad \rho, \rho' \to \infty$

Regge limit symmetry: 2-dim conformal group $SL(2, C)$ formed from $P_1, P_2, M^{12}, D, K_1$ and $K_2$ which leave the plane $(0, 0, z_\perp)$ invariant.
A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2}

L. Cornalba (2007)

\[ f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi\omega} \] - signature factor

\[ \Omega(r, \nu) - \text{solution of the eqn } (\Box_{H_3} + \nu^2 + 1)\Omega(r, \nu) = 0. \]

Explicit form:

\[ \Omega(r, \nu) = \frac{\nu^2}{\pi^3} \int d^2 z \left( \frac{\kappa^2}{(2\kappa \cdot \zeta)^2} \right)^{\frac{1}{2} + i\nu} \left( \frac{\kappa'^2}{(2\kappa' \cdot \zeta)^2} \right)^{\frac{1}{2} - i\nu} \]

\[ \zeta = p_1 + \frac{z^2}{s} p_2 + z_{\perp}, \quad p_1^2 = p_2^2 = 0, \quad 2(p_1, p_2) = s \]

\[ \kappa = \frac{1}{2x_+} (p_1 - \frac{x^2}{s} p_2 + x_{\perp}) - \frac{1}{2y_+} (p_1 - \frac{y^2}{s} p_2 + y_{\perp}), \quad \kappa^2 \kappa'^2 = \frac{1}{R} \]

\[ \kappa' = \frac{1}{2x'_-} (p_1 - \frac{x'^2}{s} p_2 + x'_{\perp}) - \frac{1}{2y'_-} (p_1 - \frac{y'^2}{s} p_2 + y'_{\perp}), \quad 4(\kappa \cdot \kappa')^2 = \frac{r}{R} \]

The dynamics is described by \( \omega(\lambda, \nu) \) and \( F(\lambda, \nu) \).
Pomeron in the conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu \, f_+(\nu(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

Pomeron intercept \( \omega(\nu, \lambda) \) is known in two limits:

1. \( \lambda \to 0 \):

\[ \omega(\nu, \lambda) = \frac{\lambda}{\pi} \chi(\nu) + \lambda^2 \omega_1(\nu) + \ldots \]

\[ \chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right) \] - BFKL intercept,

\( \omega_1(\nu) \) - NLO BFKL intercept \quad Lipatov, Kotikov (2000)

2. \( \lambda \to \infty \):

\[ AdS/CFT \quad \Rightarrow \quad \omega(\nu, \lambda) = 2 - \frac{\nu^2 + 4}{2\sqrt{\lambda}} + \ldots \]

2 = graviton spin , next term - \quad Brower, Polchinski, Strassler, Tan (2006)
Pomeron in the conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu f_+(\omega(\lambda, \nu))F(\lambda, \nu)\Omega(r, \nu)R^{\omega(\lambda, \nu)/2} \]

The function \( F(\nu, \lambda) \) in two limits:

1. \( \lambda \to 0 \):
   \[ F(\nu, \lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + \ldots \]
   \[ F_0(\nu) = \frac{\pi \sinh \pi \nu}{4\nu \cosh^3 \pi \nu} \]
   Cornalba, Costa, Penedones (2007)
   \[ F_1(\nu) = \text{see below} \]

2. \( \lambda \to \infty \):
   AdS/CFT \Rightarrow \omega(\nu, \lambda) = \pi^3 \nu^2 \frac{1 + \nu^2}{\sinh^2 \pi \nu} + \ldots
   L. Cornalba (2007)
\[ A(x, y; x', y') \xrightarrow{s \to \infty} i \frac{1}{2} \int d\nu f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

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We calculate \( F_1(\nu) \) (and confirm \( \omega_1(\nu) \)) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)
Light-cone expansion and DGLAP evolution in the NLO

\[ \mu^2 \text{ - factorization scale (normalization point)} \]

\[ k_\perp^2 > \mu^2 \text{ - coefficient functions} \]

\[ k_\perp^2 < \mu^2 \text{ - matrix elements of light-ray operators (normalized at } \mu^2 \text{)} \]
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OPE in light-ray operators

\[ T\{j_\mu(x)j_\nu(y)\} = \frac{x_\xi}{2\pi^2 x^4} \left[ 1 + \frac{\alpha_s}{\pi} (\ln x^2 \mu^2 + C) \right] \bar{\psi}(x) \gamma_\mu \gamma_\xi \gamma_\nu [x, y] \psi(y) + \mathcal{O}(\frac{1}{x^2}) \]

\[ [x, y] \equiv P e^{ig \int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} - \text{gauge link} \]
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Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\( (x - y)^2 = 0 \)

\[ \mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y]\psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y]\psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y]\psi(y) \]
Expansion of the amplitude in color dipoles in the NLO

The high-energy operator expansion is

\[ T\{\hat{O}(x)\hat{O}(y)\} = \int d^2 z_1 d^2 z_2 \ I^{LO}(z_1, z_2) \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\} \]

\[ + \int d^2 z_1 d^2 z_2 d^2 z_3 \ I^{NLO}(z_1, z_2, z_3) \left[ \frac{1}{N_c} \text{Tr}\{T^n \hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_3} T^n \hat{U}^\eta_{z_3} \hat{U}^{\dagger \eta}_{z_2}\} - \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}\right] \]

In the leading order - conf. invariant impact factor

\[ I^{LO} = \frac{x_+^{-2} y_+^{-2}}{\pi^2 Z_1^2 Z_2^2}, \quad \mathcal{Z}_i \equiv \frac{(x - z_i)^2_\perp}{x_+} - \frac{(y - z_i)^2_\perp}{y_+} \]

CCP, 2007
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$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I_A(k_\perp) \langle B | \text{Tr} \{ U(k_\perp) U^\dag(-k_\perp) \} | B \rangle$$

$$U(x_\perp) = Pe^{ig \int_{-\infty}^{\infty} du \ n^\mu A_\mu(un + x_\perp)}$$

Wilson line
Spectator frame: propagation in the shock-wave background.
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Boosted Field
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Each path is weighted with the gauge factor $P e^{i g \int d x A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
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\[ x \rightarrow z \text{: free propagation} \times \]
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Each path is weighted with the gauge factor $P e^{i g \int d x_{\mu} A_{\mu}}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

\[ x \rightarrow z: \text{free propagation} \times \]
\[ U^{ab}(z_{\perp}) - \text{instantaneous interaction with the } \eta < \eta_2 \text{ shock wave} \times \]
\[ z \rightarrow y: \text{free propagation} \]
\( Y > \eta \) - rapidity factorization scale

Rapidity \( Y > \eta \) - coefficient function ("impact factor")

Rapidity \( Y < \eta \) - matrix elements of (light-like) Wilson lines with rapidity divergence cut by \( \eta \)

\[
U_x^\eta = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du \, p_1^\mu A_\mu^\eta (up_1 + x_\perp) \right]
\]

\[
A_\mu^\eta (x) = \int \frac{d^4k}{(2\pi)^4} \theta (e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu (k)
\]
The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta$ is not invariant

However, if we define a composite operator ($a$ - analog of $\mu^{-2}$ for usual OPE)

\[
[\text{Tr}\{
\hat{U}_\eta^{z_1} \hat{U}_\eta^{z_2}\}]^\text{conf} = \text{Tr}\{ \hat{U}_\eta^{z_1} \hat{U}_\eta^{z_2}\} + \frac{\lambda}{2\pi^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{Tr}\{ T^n \hat{U}_\eta^{z_1} \hat{U}_\eta^{z_3} T^n \hat{U}_\eta^{z_2} \hat{U}_\eta^{z_2}\} - N_c \text{Tr}\{ \hat{U}_\eta^{z_1} \hat{U}_\eta^{z_2}\} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} + O(\lambda^2)
\]

the impact factor becomes conformal in the NLO.
Operator expansion in conformal dipoles

\[
T\{\hat{O}(x)\hat{O}(y)\} = \int d^2 z_1 d^2 z_2 \, I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}^{\text{conf}}
+ \int d^2 z_1 d^2 z_2 d^2 z_3 \, I^{\text{NLO}}(z_1, z_2, z_3) \left[ \frac{1}{N_c} \text{Tr}\{T^n \hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_3} T^n \hat{U}^\eta_{z_3} \hat{U}^{\dagger \eta}_{z_2}\} - \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\} \right]
\]

\[
I^{\text{NLO}} = - I^{\text{LO}} \frac{\lambda}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \ln \frac{z_{12}^2 e^{2\eta a s}}{z_{13}^2 z_{23}^2} z_3^2 - i\pi + 2C \right]
\]

The new NLO impact factor is conformally invariant
\[
\Rightarrow \quad \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}^{\text{conf}} \text{ is Möbius invariant}
\]

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbation theory.
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to $\eta_2$).
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\[
\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes
\]
Leading order: BK equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \Rightarrow \\
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}
\]

\[
U_{z}^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}
\Rightarrow \text{Evolution equation is non-linear}
\]
Non linear evolution equation

\[ \mathcal{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\mathcal{U}(x_\perp)\mathcal{U}^\dagger(y_\perp)\} \]
Non linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x)\hat{U}^\dagger(y)\} \]

BK equation

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z (x - y)^2}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]

Non-linear evolution equation

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LLA for DIS in pQCD \(\Rightarrow\) BFKL

(LLA: \(\alpha_s \ll 1, \alpha_s \eta \sim 1\))
Non-linear evolution equation

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LLA for DIS in pQCD \Rightarrow BFKL

\[ \text{(LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1) \]

LLA for DIS in sQCD \Rightarrow BK eqn

\[ \text{(LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1) \]

(s for semiclassical)
Formally, a light-like Wilson line

$$
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}
$$

is invariant under inversion (with respect to the point with $x^- = 0$).
Conformal invariance of the BK equation

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Indeed,

\[(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow \text{after the inversion } x_\perp \to x_\perp/x_\perp^2 \text{ and } x^+ \to x^+/x_\perp^2\]
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\[ \left[ \infty p_1 + x_\perp, -\infty p_1 + x_\perp \right] \to \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x_\perp^2} \, A_+ \left( \frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2} \right) \right\} = \left[ \infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2} \right] \]
Formally, a light-like Wilson line

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\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}) \right\} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]
\]

\Rightarrow \text{The dipole kernel is invariant under the inversion } V(x_\perp) = U(x_\perp/x_\perp^2)

\[
\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^A} \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\} \right]
\]
Conformal invariance of the BK equation

\[ \hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2) \]

\[ [\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0, \]

\[ [\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \]

\[ z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z, \bar{z}) \]
Conformal invariance of the BK equation

\[ \hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2) \]

\[ [\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0, \]

\[ [\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \]

\[ z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z, \bar{z}) \]

Conformal invariance of the evolution kernel

\[ \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz \; K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}\text{Tr}\{U_x U_y^\dagger\}] \]

\[ \Rightarrow \left[ x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \]
Conformal invariance of the BK equation

\[ \hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2) \]

\[ [\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0, \]

\[ [\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z\partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \]

\[ z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_{\perp}) = U(z, \bar{z}) \]

Conformal invariance of the evolution kernel

\[ \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz \ K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] \text{Tr}\{U_x U_y^\dagger\}] \]

\[ \Rightarrow \left[ x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \]

In the leading order - OK. In the NLO - ?
Non-linear evolution equation in the NLO

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \\
\int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x - y)^2}{(x - z)^2 (z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_z U_y^\dagger\} \right] + \\
\alpha_s^2 \int d^2 z d^2 z' \left( K_4(x, y, z, z') \{U_x, U_{z', z}^\dagger, U_z, U_y^\dagger\} + K_6(x, y, z, z') \{U_x, U_{z', z}^\dagger, U_{z', z}^\dagger, U_z, U_{z'}^\dagger, U_y^\dagger\} \right)
\]

\(K_{NLO}\) is the next-to-leading order correction to the dipole kernel and \(K_4\) and \(K_6\) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
In general

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3) \]
Definition of the NLO kernel

In general

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3) \]

\[ \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3) \]
\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3) \]

\[ \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3) \]

We calculate the “matrix element” of the r.h.s. in the shock-wave background

\[ \langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3) \]
Definition of the NLO kernel

In general

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)
\]

\[
\alpha_s^2 K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)
\]

We calculate the “matrix element” of the r.h.s. in the shock-wave background

\[
\langle \alpha_s^2 K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)
\]

Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

\[
\Rightarrow \left[ \frac{1}{v} \right]_+ \text{ prescription in the integrals over Feynman parameter } v
\]

Typical integral

\[
\int_0^1 dv \left( \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v)} \right) \left[ \frac{1}{v} \right]_+ = \frac{1}{p_\perp^2} \ln \left( \frac{(k-p)_\perp^2}{p_\perp^2} \right)
\]
Gluon part of the NLO BK kernel: diagrams

I. Balitsky (JLAB & ODU)
High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order
JLab April 8, 2011 27 / 41
Diagrams for $1 \to 3$ dipoles transition
Diagrams for $1 \rightarrow 3$ dipoles transition

(XXXI)  
(XXXII)  
(XXXIII)  
(XXXIV)
"Running coupling" diagrams

I. Balitsky  (JLAB & ODU)
High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

JLab  April 8, 2011  30 / 41
1 → 2 dipole transition diagrams

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)
Evolution equation for color dipole in $\mathcal{N} = 4$

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\} = \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_1^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
\times \left[ \text{Tr}\{T^a \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_3} T^a \hat{U}^\eta_{z_3} \hat{U}^\dagger_{z_2}\} - N_c \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\}\right] \\
- \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_2^2}{z_{34}^2} \frac{z_3^2}{z_{13} z_{24}} \left[ 1 + \frac{z_{13}^2 z_{24}^2}{z_{13}^2 z_{24}^2 - z_{34}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
\times \text{Tr}\{[T^a, T^b] \hat{U}^\eta_{z_1} T^a T^b \hat{U}^\dagger_{z_2} + T^b T^a \hat{U}^\eta_{z_1} [T^b', T^a'] \hat{U}^\dagger_{z_2}\}\} (\hat{U}^\eta_{z_3})^{aa'} (\hat{U}^\eta_{z_4} - \hat{U}^\eta_{z_3})^{bb'}
\]

NLO kernel = **Non-conformal term** + **Conformal term.**

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.
Evolution equation for color dipole in $\mathcal{N} = 4$

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} = \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
\times \left[ \text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger \eta} T^a \hat{U}_{z_2}^\eta \hat{U}_{z_3}^{\dagger \eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} \right] - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2 z_{24}^2}{z_{34}^2} \left[ 1 + \frac{z_{12}^2 z_{24}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^a' \hat{U}_{z_2}^{\dagger \eta} + T^b T^a \hat{U}_{z_1}^\eta [T^b', T^a'] \hat{U}_{z_2}^{\dagger \eta}\} (\hat{U}_{z_3}^\eta )^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta )^{bb'}
\]

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant
Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$

$$\frac{d}{d\eta} [\text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\}]^{\text{conf}}$$

$$= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c \pi^2}{4\pi} \right] [\text{Tr}\{T^a \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_3} T^a \hat{U}^\eta_{z_3} \hat{U}^\dagger_{z_2}\} - N_c \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\}]^{\text{conf}}$$

$$- \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 z_{34}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} \right\}$$

$$\times \text{Tr}\{[T^a, T^b] \hat{U}^\eta_{z_1} T^{a'} T^{b'} \hat{U}^\dagger_{z_2} + T^b T^a \hat{U}^\eta_{z_1} [T^{b'}, T^{a'}] \hat{U}^\dagger_{z_2}\} \left[ (\hat{U}^\eta_{z_3})^{a a'} (\hat{U}^\eta_{z_4})^{b b'} - (z_4 \rightarrow z_3) \right]$$

Now Möbius invariant!
To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{U}^n(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^n \hat{U}_y^n\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{U}^n_{\text{conf}}(z_1, z_2) = \hat{U}^n(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \hat{U}^n(z_1, z_3) + \hat{U}^n(z_2, z_3) - \hat{U}^n(z_1, z_2) \right]$$
To find \( A(x, y; x', y') \) we need the linearized (NLO BFKL) equation. With two-gluon accuracy

\[
\hat{U}^n(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^n \hat{U}_y^n\}
\]

Conformal dipole operator in the BFKL approximation

\[
\hat{U}^n_{\text{conf}}(z_1, z_2) = \hat{U}^n(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{z_{12}^2}{z_{13} z_{23}} \ln \frac{a z_{12}^2}{z_{13} z_{23}} [\hat{U}^n(z_1, z_3) + \hat{U}^n(z_2, z_3) - \hat{U}^n(z_1, z_2)]
\]

Define

\[
\hat{U}^a_{\text{conf}}(z_1, z_2)
\]

\[
= \hat{U}^n(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{z_{12}^2}{z_{13} z_{23}} \ln \frac{a e^{2n} z_{12}^2}{z_{13} z_{23}} [\hat{U}^n(z_1, z_3) + \hat{U}^n(z_2, z_3) - \hat{U}^n(z_1, z_2)] + ...
\]

such that \( \frac{d}{d\eta} \hat{U}^a_{\text{conf}}(z_1, z_2) = 0 \).

\( \Rightarrow \) The evolution can be rewritten in terms of \( a \).
NLO BFKL equation in $\mathcal{N} = 4$ SYM

\[ a \frac{d}{da} \hat{U}_{\text{conf}}^a(z_1, z_2) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \frac{z_1^{12}}{z_1^{13} z_2^{13} z_2^{23}} \left[ 1 - \frac{\alpha_s N_c \pi^2}{4\pi 3} \right] \left[ \hat{U}_{\text{conf}}^a(z_1, z_3) + \hat{U}_{\text{conf}}^a(z_2, z_3) - \hat{U}_{\text{conf}}^a(z_1, z_2) \right] \]

\[ + \frac{\alpha_s^2 N_c^2}{8\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_1^{12} z_3^{34}}{z_3^{13} z_4^{13} z_4^{23}} \left\{ 2 \ln \frac{z_1^{12} z_3^{34}}{z_4^{13} z_4^{23}} + \left[ 1 + \frac{z_1^{12} z_3^{34}}{z_4^{13} z_4^{23} - z_4^{23}} \right] \ln \frac{z_1^{12} z_3^{34}}{z_4^{13} z_4^{23}} \right\} \hat{U}_{\text{conf}}^a(z_3, z_4) \]

\[ + \frac{3\alpha_s^2 N_c^2}{2\pi^3} \zeta(3) \hat{U}_{\text{conf}}^a(z_1, z_2) \]

Eigenfunctions are determined by conformal invariance

\[ E_{\nu,n}(z_{10}, z_{20}) = \left[ \frac{z_1^{12}}{z_{10} z_{20}} \right]^{\frac{1}{2} + i\nu + \frac{n}{2}} \left[ \frac{z_1^{12}}{z_{10} z_{20}} \right]^{\frac{1}{2} + i\nu - \frac{n}{2}} \]

The expansion in eigenfunctions

\[ \hat{U}_{\text{conf}}^a(z_1, z_2) = \sum_{n=0}^\infty \int d^2 z_0 \int d\nu E_{\nu,n}(z_{10}, z_{20}) \hat{U}_{z_0,\nu,n}^a \Rightarrow a \frac{d}{da} \hat{U}_{z_0,\nu,n}^a = \omega(n, \nu) \hat{U}_{z_0,\nu,n}^a \]

\[ \omega(n, \nu) \equiv \text{pomeron intercept = eigenvalue of the BFKL equation} \]
Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

\[
\omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[ \chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right],
\]

\[
\delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma)
\]

where \(\gamma = \frac{1}{2} + i\nu\) and

\[
\chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})
\]

\[
\Phi(n, \gamma) = \int_0^1 dt \frac{t^{\gamma - 1 + \frac{n}{2}}}{1 + t} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi'\left(\frac{n + 1}{2}\right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. \\
- \left. \left( \psi(n + 1) - \psi(1) + \ln(1 + t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k + n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k + n)^2} \left[ 1 - (-1)^k \right] \right\}
\]
Pomeron intercept

Pomeron intercept $= \text{the eigenvalue of the BFKL equation}$

$$\omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[ \chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right],$$

$$\delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma)$$

where $\gamma = \frac{1}{2} + i\nu$ and

$$\chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})$$

$$\Phi(n, \gamma) = \int_0^1 \frac{dt}{1 + t} \ t^{\gamma - 1 + \frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi'\left(\frac{n + 1}{2}\right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. - \left. \left( \psi(n + 1) - \psi(1) + \ln(1 + t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k + n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k + n)^2} [1 - (-1)^k] \right\}$$

Coincides with Lipatov & Kotikov

Agrees with $j \to 1$ asymptotics of 3-loop splitting functions

Vogt, Moch, Vermaseren,(2003)
(x − y)^4 T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x − y)^4}{\pi^2} \int d^2 z_1 d^2 z_2 \frac{(x_+ y_+)^{-2}}{Z_1^2 Z_2^2} \left\{ \hat{U}^{\text{conf}} \right\}

- \frac{\lambda}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} - i\pi \right] \left[ \hat{U}^{\text{conf}}(z_1, z_3) + \hat{U}^{\text{conf}}(z_2, z_3) - \hat{U}^{\text{conf}}(z_1, z_2) \right]}

With two-gluon accuracy (\(R \equiv \frac{(x − y)^2 z_{12}^2}{x_+ y_+ + z_1 z_2}\) - conformal ratio \(\equiv u\) from Joao's talk)

(x − y)^4 T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x − y)^4}{\pi^2} \int d^2 z_1 d^2 z_2 \frac{(x_+ y_+)^{-2}}{Z_1^2 Z_2^2} \left\{ 1 

- \frac{\lambda}{2\pi^2} \left[ 4\text{Li}_2(1 - R) - \frac{2\pi^2}{3} + 2 \left( \ln \frac{1}{R} + \frac{1}{R} - 2 \right) \ln \frac{a z_{12}^2}{z_{12}^2} \right] \hat{U}^{\text{conf}}(z_1, z_2) \right\}

The impact factor should not scale with energy \(\Rightarrow a = \frac{x_+ y_+}{(x − y)^2}\) (analog of \(\mu^2 = Q^2\) in DIS)

(x − y)^4 T\{\hat{O}(x)\hat{O}(y)\} = \frac{1}{\pi^2} \int d^2 z_1 d^2 z_2 \frac{R^2}{z_{12}^4} \left\{ 1 

- \frac{\lambda}{2\pi^2} \left[ 4\text{Li}_2(1 - R) - \frac{2\pi^2}{3} + 2 \left( \ln \frac{1}{R} + \frac{1}{R} - 2 \right) \ln \frac{1}{R} \right] \hat{U}^{\text{conf}}(z_1, z_2) \right\} \}
NLO impact factors

The projection onto the conformal eigenfunctions \( \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma (\gamma = \frac{1}{2} + i\nu) : \)

\[
\int dz_1 dz_2 (x - y)^4 T\{ \hat{O}(x) \hat{O}(y) \} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma = \left( \frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2} \right)^\gamma [I_{ALO}^A(\gamma) + I_{ANLO}^A(\gamma)] \hat{U}(z_0, \gamma),
\]

\[
\hat{U}(z_0, \gamma) = \int d^2z_1 d^2z_2 \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma \hat{U}(z_1, z_2)
\]

\[
I_{LO}^A(\gamma) = \frac{\Gamma^2(1 - \gamma)}{\Gamma(2 - 2\gamma)} \Gamma(1 + \gamma) \Gamma(2 - \gamma)
\]

\[
I_{NLO}^A(\gamma) = \frac{\lambda}{8\pi^2} I_{LO}^A \left[-2\psi'(\gamma) - 2\psi'(1 - \gamma) + \frac{2\pi^2}{3} + \frac{\chi(\gamma) - 2}{\gamma(1 - \gamma)} + 2C\chi(\gamma) \right]
\]
The projection onto the conformal eigenfunctions $(\frac{z_{12}^2}{z_{10}^2 z_{20}^2})^\gamma (\gamma = \frac{1}{2} + i\nu)$:

$$\int dz_1 dz_2 (x - y)^4 T \{\hat{O}(x)\hat{O}(y)\} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma = \left(\frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2}\right)^\gamma [I_{LO}^A(\gamma) + I_{NLO}^A(\gamma)] \hat{U}(z_0, \gamma),$$

$$\hat{U}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{U}(z_1, z_2)$$

$$I_{LO}^A(\gamma) = \frac{\Gamma^2(1 - \gamma)}{\Gamma(2 - 2\gamma)} \Gamma(1 + \gamma) \Gamma(2 - \gamma)$$

$$I_{NLO}^A(\gamma) = \frac{\lambda}{8\pi^2} I_{LO}^A \left[ - 2\psi'(\gamma) - 2\psi'(1 - \gamma) + \frac{2\pi^2}{3} + \frac{\chi(\gamma) - 2}{\gamma(1 - \gamma)} + 2C\chi(\gamma) \right]$$

Similarly

$$\int dz_1 dz_2 (x' - y')^4 T \{\hat{O}(x')\hat{O}(y')\} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{1-\gamma} = \left(\frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2}\right)^{1-\gamma} [I_{LO}^B(\gamma) + I_{NLO}^B(\gamma)] \hat{V}(z_0, \gamma),$$

$$\hat{V}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{V}(z_1, z_2)$$

$$I_{LO}^B(\gamma) = \frac{\Gamma^2(1 + \gamma)}{\Gamma(2 + 2\gamma)} \Gamma(1 + \gamma) \Gamma(2 - \gamma)$$

$$I_{NLO}^B(\gamma) = \frac{\lambda}{16\pi^2} I_{LO}^B \left[ - 2\psi'(\gamma) - 2\psi'(1 - \gamma) + \frac{2\pi^2}{3} + \frac{\chi(\gamma) - 2}{\gamma(1 - \gamma)} + 2C\chi(\gamma) \right]$$
The last ingredient is the amplitude of scattering of two conformal dipoles
\( (\gamma \equiv \frac{1}{2} + i \nu) \)

\[
\langle \hat{U}^a(z_0, \gamma) \hat{V}^b(z'_0, \gamma) \rangle = \delta(\nu - \nu') \delta(z_0 - z'_0) (ab)^{\frac{1}{2} \omega(\nu)} [A_{\text{LO}}(\gamma) + A_{\text{NLO}}(\gamma)]
\]

\[
A_{\text{LO}}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma - 1)}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)}, \quad A_{\text{NLO}}(\gamma) = -\frac{\lambda}{4\pi^2 A_{\text{LO}}} \left[ \frac{\chi(\gamma)}{\gamma(1 - \gamma)} + 2C\chi(\gamma) + \frac{\pi^2}{3} \right]
\]

With our choice \( a = \frac{x+y}{(x-y)^2}, \quad b = \frac{x'-y'}{(x'-y')^2} \), \( ab = R \Rightarrow \)

\[
\langle \hat{U}(z_0, \gamma) \hat{V}(z'_0, \gamma) \rangle = \delta(\nu - \nu') \delta(z - z') R^{\frac{1}{2} \omega(\nu)} [A_{\text{LO}}(\gamma) + A_{\text{NLO}}(\gamma)]
\]
Assembling NLO $F(\nu)$

The last ingredient is the amplitude of scattering of two conformal dipoles $(\gamma \equiv \frac{1}{2} + i\nu)$

$$\langle \hat{U}^a(z_0, \gamma)\hat{V}^b(z'_0, \gamma) \rangle = \delta(\nu - \nu')\delta(z_0 - z'_0) (ab)\frac{1}{2}\omega(\nu)[A_{LO}(\gamma) + A_{NLO}(\gamma)]$$

$$A_{LO}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma - 1)}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)}, \quad A_{NLO}(\gamma) = -\frac{\lambda}{4\pi^2}A_{LO}\left[\frac{\chi(\gamma)}{\gamma(1 - \gamma)} + 2C\chi(\gamma) + \frac{\pi^2}{3}\right]$$

With our choice $a = \frac{x+y}{(x-y)^2}$, $b = \frac{x'-y'}{(x'-y')^2}$ $ab = R \Rightarrow$

$$\langle \hat{U}(z_0, \gamma)\hat{V}(z'_0, \gamma) \rangle = \delta(\nu - \nu')\delta(z - z') R\frac{1}{2}\omega(\nu)[A_{LO}(\gamma) + A_{NLO}(\gamma)]$$

Now one can assemble $F(\nu)$ in the next-to-leading order

$$F(\nu) = F_{LO}(\nu) + \lambda F_{NLO}(\nu) + O(\lambda^2) \Rightarrow F_{LO}(\nu) = I_{LO}^A(\nu)A_{LO}(\nu)I_{LO}^B(\nu),$$

$$F_{NLO}(\nu) = I_{NLO}^A(\nu)A_{LO}(\nu)I_{LO}^B + I_{LO}^A(\nu)A_{NLO}(\nu)I_{LO}^B + I_{NLO}^A(\nu)A_{LO}(\nu)I_{NLO}^B(\nu)$$

The result is

$$F(\nu) = \frac{N_c^2}{N_c^2 - 1} \frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu} \left\{1 + \frac{\alpha_s N_c}{\pi} \left[ - 2\psi'\left(\frac{1}{2} + i\nu\right) - 2\psi'\left(\frac{1}{2} - i\nu\right) + \frac{\pi^2}{2} - \frac{8}{1 + 4\nu^2} \right] + O(\alpha_s^2) \right\}$$
Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four $Z^2$ operators is calculated at the NLO order.
Outlook: rapidity evolution of TMD’s

Gluon TMD : \( D(x_B, k_\perp) \sim \int d^2k_\perp e^{ik_\perp \cdot z_\perp} \)
\( \times \int dudv \langle \left[ -\infty, u \right] G_{+i}(z_\perp + up_1)[u, -\infty] \left[ -\infty, u \right]_0 G_{+i}(vp_1)[u, -\infty]_0 e^{i(u-v)x_B \frac{g}{2}} \rangle \)

Compare to \((U_i \equiv U_i^\dagger i\partial_i U)\)
\( \{U_i(z_\perp)U_i(0_\perp)\}^\eta = \int dudv \langle \left[ -\infty, u \right] G_{+i}(z_\perp + up_1)[u, -\infty] \left[ -\infty, u \right]_0 G_{+i}(vp_1)[u, -\infty]_0 e^{i(u-v)x_B \frac{g}{2}} \rangle \)

⇒ same operator with different rapidity cutoff.

Evolution equation (leading order)
\[
\frac{d}{d\eta} (U_i^a(x)U_i^a(y))^\eta = -\frac{\alpha_s}{\pi^2} (\nabla_i^x \int \frac{(x-z, y-z)}{(x-z)^2(y-z)^2} (U_x^\dagger U_y + 1 - U_x^\dagger U_z - U_z^\dagger U_y) \nabla_i^y)^{aa}
\]
\[
-\frac{\alpha_s}{\pi^2} \left[ \int \frac{dz}{(x-z)^2} \left[ f^{abc} (U_{x_i}^\dagger \partial_i U_z)_{bc} U_i^a(y) + N_c U_i^a(x)U_i^a(y) + x \leftrightarrow y \right] \right]
\]

⇒ Rapidity evolution of TMD’s follows from the evolution of color dipoles.