Low-energy aspects of amplitude analysis: chiral perturbation theory and dispersion relations

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Techniques of Amplitude Analysis

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Schedule (tentative)

Lecture 1: Introduction to chiral perturbation theory

- chiral symmetry
- construction of effective Lagrangian
- power counting

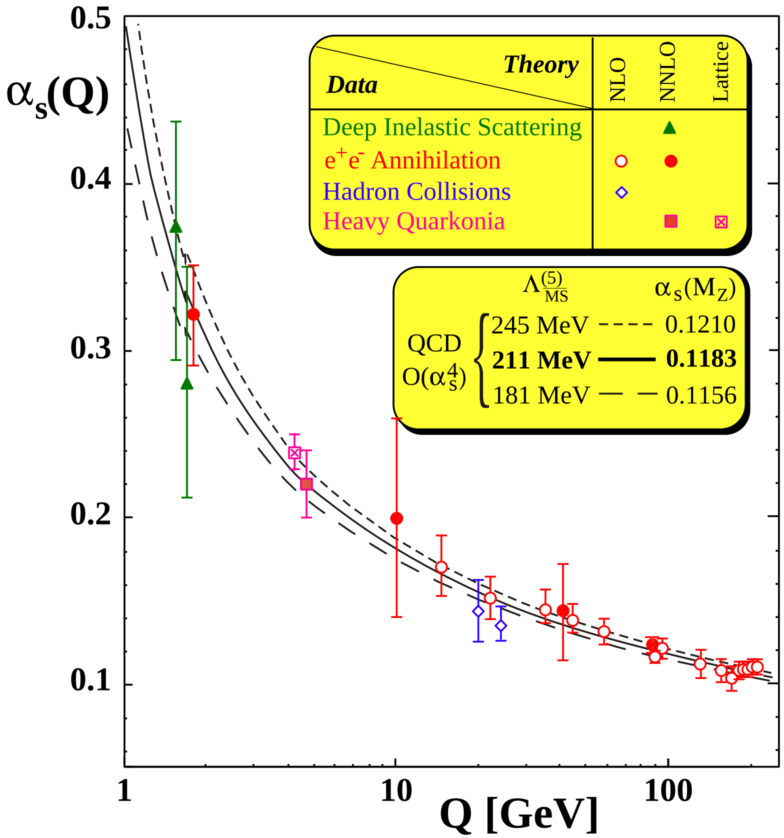
Lecture 2: The pion vector form factor

- dispersion relations, calculation of discontinuities. . .
- Omnès solution
- application(s)

Lecture 3: Dispersion relations for 3-body decays

- quark-mass ratios and $\eta \to 3\pi$
- construction of a solution based on Omnès functions

“Strong” and “weak” QCD



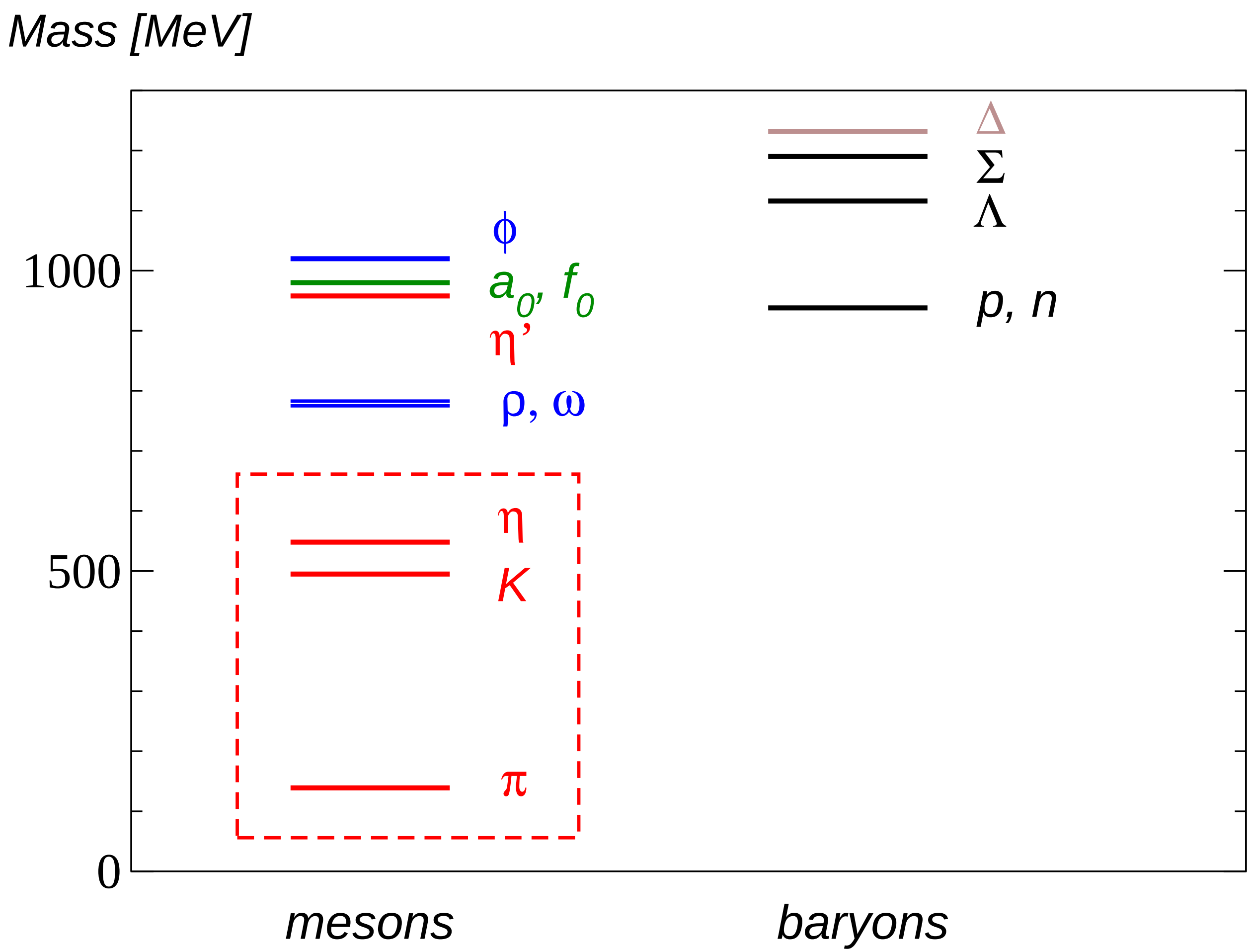
- anti-screening:
  strong coupling becomes weak at high energies!
- asymptotic freedom at high energies ("weak QCD")
- confinement at low energies ("strong QCD"):

  no quarks + gluons, only (colour-neutral) hadrons

  baryons (rgb) + mesons ($r\bar{r}$)

- perturbation theory in α_s at low energies: impossible!

QCD: the spectrum of hadrons



$\longrightarrow$ what does this spectrum have to do with the theory of quarks and gluons?

Introduction to chiral perturbation theory

Slides: QCD running coupling
hadron spectrum

- QCD Lagrangian: $\mathcal{L}_{QCD} = -\frac{1}{2} \mathrm{Tr}\, G^a_{\mu\nu} G^{\mu\nu,a} + \bar{q}\,(i\not{D} - \mathcal{M})\, q$

$$D_\mu = \partial_\mu + i g A^a_\mu \frac{\lambda^a}{2}, \quad G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

$$q^T = (u, d, s \ldots), \quad \mathcal{M} = \mathrm{diag}(m_u, m_d, m_s \ldots)$$

- decompose quark fields according to their chirality:

$$q = \frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q = P_L q + P_R q = q_L + q_R$$

$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ anticommutes with all γ_i, $\gamma_5^2 = \mathbb{1}$

$P_{L/R} = \frac{1 \mp \gamma_5}{2}$ projection operators:

$$P^2_{L/R} = P_{L/R}, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = 1$$

- in the massless limit, chirality = helicity:

solutions of free Dirac equation

$$u(p,\pm) = \sqrt{E+m} \begin{pmatrix} \chi_\pm \\ \frac{\sigma \cdot p}{E+m} \chi_\pm \end{pmatrix} \xrightarrow{E \gg m} \sqrt{E} \begin{pmatrix} \chi_\pm \\ \sigma \cdot \hat{p}\, \chi_\pm \end{pmatrix} \quad \hat{p} = \frac{p}{|p|}$$

$$= \sqrt{E} \begin{pmatrix} \chi_\pm \\ \pm \chi_\pm \end{pmatrix} \equiv u_\pm(p)$$

$\sigma \cdot \hat{p}\, \chi_\pm = \pm \chi_\pm$: spin parallel / antiparallel to momentum

standard convention for Dirac matrices:

$$P_R = \frac{1}{2} \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} \qquad P_L = \frac{1}{2} \begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}$$

- insert $q = q_L + q_R$ in $\mathcal{L}_{QCD}$:

$$\mathcal{L}_{QCD}[q] = i\,\bar{q}_L \not{D} q_L + i\,\bar{q}_R \not{D} q_R - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L$$

ChPT 2) – massless fermion has conserved helicity
– interaction with gluons conserves helicity
– only mass term couples left- to right-handed

- a peak ahead:

$$m_u \approx 2.2\,\text{MeV},\quad m_d \approx 4.7\,\text{MeV},\quad m_s \approx 95\,\text{MeV}$$

compare to $m_p = 938\,\text{MeV} \gg 2m_u + m_d$

<u>so</u>: maybe good idea to approximate

$$\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} - \bar{q}\mathcal{M}q \qquad \text{for the 3 light quarks}$$

by the Lagrangian <u>"in the chiral limit"</u> $\mathcal{L}^0_{QCD}$, and treat the mass term as a perturbation

- $\mathcal{L}^0_{QCD}$ has an additional global symmetry:

$$q_R \mapsto R q_R,\quad q_L \mapsto L q_L,\quad R, L \in U(3)_{R/L}$$

rewrite as $U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$

18 generators; <u>conserved currents</u> according to Noether:

$$V^a_\mu = R^a_\mu + L^a_\mu = \bar{q}\gamma_\mu \frac{\lambda^a}{2} q$$

$$A^a_\mu = R^a_\mu - L^a_\mu = \bar{q}\gamma_\mu\gamma_5 \frac{\lambda^a}{2} q$$

$$V^0_\mu = R^0_\mu + L^0_\mu = \bar{q}\gamma_\mu q \quad \rightarrow \text{quark/baryon number}$$

$$A^0_\mu = R^0_\mu - L^0_\mu = \bar{q}\gamma_\mu\gamma_5 q \quad \rightarrow \text{broken by anomaly}$$

- conserved <u>charges</u> $Q = \int d^3x\, J^0(x)$ commute with H:

$[H, Q] = 0$

state $|\psi_p\rangle$, $H|\psi_p\rangle = E_p|\psi_p\rangle$

$\rightsquigarrow H e^{iQ}|\psi_p\rangle = e^{iQ}H|\psi_p\rangle = E_p(e^{iQ}|\psi_p\rangle)$

$\rightsquigarrow$ another state of same mass $\rightsquigarrow$ degenerate multiplet

- problem with <u>axial</u> generators:

  $e^{iQ_5^a}|\psi_p\rangle$ is a state of <u>opposite parity</u> w.r.t. $|\psi_p\rangle$

e.g. $m_N \approx 938$ MeV, $m_{S_{11}} \approx 1535$ MeV ???

- careful analysis in QFT reveals: have implicitly assumed <u>ground state</u> / vacuum to be invariant under the symmetry $\rightsquigarrow$ needs not be true!

<u>Spontaneous symmetry breaking / Goldstone theorem</u>

- simple example: multi-component field $\underset{\sim}{\phi} = (\phi_1, \dots, \phi_n)$

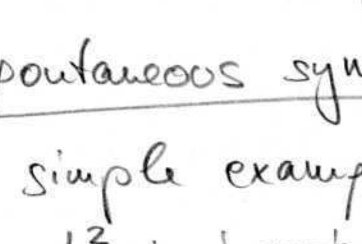
  $\underset{\sim}{\phi}^2$ is invariant under (global) $O(n)$ symmetry:

  $$\mathcal{L} = \frac{1}{2}\partial_\mu \underset{\sim}{\phi}\cdot\partial^\mu \underset{\sim}{\phi} - V(\underset{\sim}{\phi}^2)$$

  suppose minimum of V is <u>not</u> at $\underset{\sim}{\phi} = \underset{\sim}{0}$:

  $$V(\underset{\sim}{\phi}^2) = \frac{g}{4}(\underset{\sim}{\phi}^2 - v^2)^2$$

  min. at $\underset{\sim}{\phi}^2 = v^2$: sphere S^{n-1}, any point with



  $|\underset{\sim}{\phi}| = v$ is point of minimum energy. Choose 1 such point: pot. energy remains 0 in $n-1$ dir.

  $$\underset{\sim}{\phi}_0 = (0, \dots, 0, v), \quad \underset{\sim}{\phi} = (\underset{\sim}{\phi}_\perp, v+f)$$

  $$\rightsquigarrow \underset{\sim}{\phi}^2 = \underset{\sim}{\phi}_\perp^2 + (v+f)^2, \quad V(\underset{\sim}{\phi}^2) = \frac{g}{4}\left(\underset{\sim}{\phi}_\perp^2 + f^2 + 2vf\right)^2$$

  $$= \frac{g}{4}\left(\underset{\sim}{\phi}_\perp^2 + f^2\right)^2 + g v^2 f^2 + g v\left(\underset{\sim}{\phi}_\perp^2 + f^2\right) f$$

  $\rightsquigarrow$ quadratic term in f, mass $m_f^2 = 2gv^2$

  <u>no</u> masses for fields $\underset{\sim}{\phi}_\perp$

- Goldstone phenomenon: symm. <u>broken</u> in ground state, but there are massless modes

ChPT4

- in general: Lagrangian invariant under symmetry group G, ground state invariant under subgroup $H \subset G$

$\leadsto$ $\dim G - \dim H$ massless / Goldstone modes

in the example: $\dim G = \dim O(n) = \frac{1}{2} n(n-1)$
$\dim H = \dim O(n-1) = \frac{1}{2}(n-1)(n-2)$

$\leadsto$ $\dim G - \dim H = n-1 = \dim S^{n-1}$

Consequences for QCD

slide on Mexican hat pot.

- no parity doubling in the hadron spectrum, but (approximate) $SU(3)$ multiplets (and almost perfect isospin):

$$SU(3)_L \times SU(3)_R \xrightarrow{SSB} SU(3)_V$$

Goldstone theorem: expect 8 massless modes ($= \dim SU(3)_A$)
chiral symm. not exact (quark masses finite) $\leadsto$ not massless, but the lightest particles in the spectrum
axial generators broken $\leadsto$ those are pseudoscalars

$$\pi^\pm, \pi^0, K^\pm, \overset{(-)}{K}{}^0, \eta$$

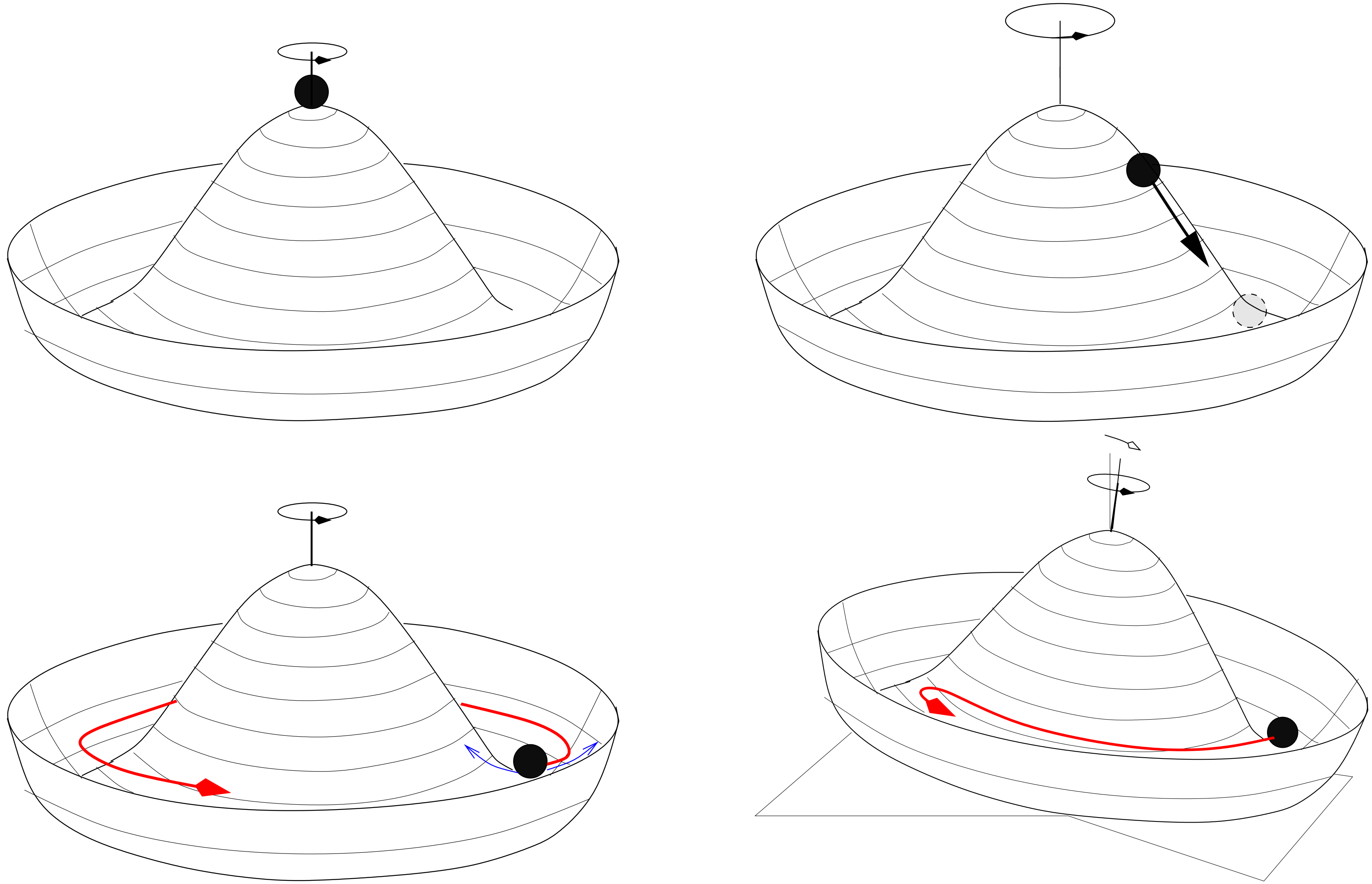
- alternative version: consider only $SU(2)$ chiral limit $m_u = m_d = 0$, keep m_s fixed at its physical value

$$SU(2)_L \times SU(2)_R \xrightarrow{SSB} SU(2)_V \ (= \text{isospin})$$

$\leadsto$ 3 Goldstone bosons = 3 pions

+ : as $m_{u/d} \ll m_s$, this perturbation theory will converge much faster
− : as only the pions have a special role (not K/η), theory will be less predictive

Illustration: spontaneous symmetry breaking



figures courtesy of A. Wirzba

Task: • construct theory for these Goldstone bosons (ChPT 5)
• incorporate all symmetry constraints from QCD
• effective (not fundamental) theory:
  – valid at small energies (below the mass scale of "normal", non-Goldstone states)
  – "mass gap" $M_{GB} \underset{\sim}{\ll} M_{hadr} \approx 1\,\text{GeV}$
  $\rightsquigarrow$ expand in small masses / momenta over the characteristic hadronic scale

Construction of the EFT: 4 steps

1. common field

$$U = \exp\left(\frac{i\phi}{F}\right), \quad \phi_a \lambda_a = \phi = \sqrt{2}\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

F: dimensionful constant, to be determined later

2. specify transformation behavior under chiral group:
$$U \longmapsto L U R^{\dagger} \qquad (*)$$
3. construct Lagrangian $\mathcal{L}[U]$ invariant under $(*)$
4. organise according to number of derivatives
$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \dots$$

• $U \mapsto L U R^{\dagger} \quad \rightsquigarrow \quad U^{\dagger} \mapsto R U^{\dagger} L^{\dagger}$

global trafo: $\partial_\mu U \mapsto \partial_\mu (L U R^{\dagger}) = L(\partial_\mu U) R^{\dagger}$ etc.

$\mathcal{L}^{(0)}$? $\quad \mathrm{Tr}(UU^{\dagger}) \longmapsto \underbrace{\mathrm{Tr}(L U R^{\dagger} R U^{\dagger} L^{\dagger})}_{\text{trace cyclic}} = \mathrm{Tr}(UU^{\dagger})$

but U unitary, $UU^{\dagger} = \mathbb{1}$
$\rightsquigarrow$ irrelevant constant

ChPT 6 · $\mathcal{L}^{(2)}$ contains one single term:

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \equiv \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle$$

invariance as above; normalisation:

$$U = \exp\left(\frac{i\phi}{F}\right) = 1 + \frac{i\phi}{F} - \frac{\phi^2}{2F^2} + \dots$$

$$\rightsquigarrow \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle = \frac{1}{4} \langle \partial_\mu \phi \partial^\mu \phi \rangle \quad \Big| \langle \lambda_a \lambda_b \rangle = 2\delta_{ab}$$

$$= \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a = \frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \pi^+ \partial^\mu \pi^- + \dots$$

· what is F? calculate Noether currents from $\mathcal{L}^{(2)}$

$$\rightsquigarrow \left.\begin{aligned} V^\mu_a &= -i \frac{F^2}{4} \langle \lambda_a [\partial^\mu U, U^\dagger] \rangle \\ A^\mu_a &= i \frac{F^2}{4} \langle \lambda_a \{\partial^\mu U, U^\dagger\} \rangle \end{aligned}\right\} \begin{aligned} U &= 1 + \frac{i\phi}{F} + \dots \\ \partial_\mu U &= \frac{i}{F} \partial_\mu \phi + \dots \end{aligned}$$

$$\rightsquigarrow A^\mu_a = i \frac{F^2}{4} \langle \lambda_a \{ \frac{i}{F} \partial^\mu \phi, 1 \} \rangle + \mathcal{O}(\phi^3)$$

$$= -F \partial^\mu \phi_a + \mathcal{O}(\phi^3)$$

take matrix element $\langle 0 | A^\mu_a | \phi_b(p) \rangle = i p^\mu \delta_{ab} F$

$\rightsquigarrow$ F is pion decay constant, measured in

$\pi^- \longrightarrow \mu^- \nu_\mu \quad \rightsquigarrow F_\pi = 92.2$ MeV

<u>Explicit symmetry breaking by quark masses</u>

$$\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M}^\dagger q_L$$

<u>would</u> be invariant under chiral symmetry <u>if</u>

$$\mathcal{M} \longmapsto L \mathcal{M} R^\dagger$$

so <u>assume</u> this, construct $\mathcal{L}_{eff}[U, \partial_\mu U, \partial^2 U \dots, \mathcal{M}]$ invariant $\rightsquigarrow$ <u>explicit</u> symm. breaking like in QCD

· one term linear in $\mathcal{M}$, no derivatives

$\leadsto \mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U^\dagger + \mathcal{M}^\dagger U) \rangle$

low-energy const. parity

mass term: expand U again,

$$\text{const} - \frac{B}{2} \langle \mathcal{M} \phi^2 \rangle + \mathcal{O}(\phi^4)$$

calculate the trace, read off masses:

$$\left.\begin{aligned} M_{\pi}^2 &= B(m_u + m_d) \\ M_{K^\pm}^2 &= B(m_u + m_s) \\ M_{K^0}^2 &= B(m_d + m_s) \end{aligned}\right\} \quad \text{Gell-Mann–Oakes–Renner: } M_{GB}^2 \propto m_q$$

$$M_\eta^2 = \frac{B}{3}(m_u + m_d + 4m_s)$$

Gell-Mann–Okubo: $4M_K^2 = 3M_\eta^2 + M_\pi^2$

$$\left(0.97 = 0.92 \ [\mathrm{GeV}^2] \right)$$

- quark mass ratios: correct for electromagnetic effects,

Dashen: $(M_{\pi^\pm}^2 - M_{\pi^0}^2)_{em} = (M_{K^\pm}^2 - M_{K^0}^2)_{em} + \mathcal{O}(e^2 m_q)$

$$\leadsto \frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 0.55$$

$$\frac{m_s}{m_d} = \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20.2$$

- what else from $\mathcal{L}^{(2)}$? ππ-scattering

isospin + crossing:

$$\mathcal{M}(\pi^a \pi^b \to \pi^c \pi^d) = \delta_{ab}\delta_{cd} A(s,t,u) + \delta_{ac}\delta_{bd} A(t,u,s) + \delta_{ad}\delta_{bc} A(u,s,t)$$

calculate single invariant amplitude from $\mathcal{L}^{(2)}$:

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}$$

- param.-free
- "Adler zero"

ChPT 8) transform into amplitudes of definite isospin:

$$T^{I=0} = 3A(s,t,u) + A(t,u,s) + A(u,s,t) \qquad S, D \ldots \text{ waves}$$

$$T^{I=1} = A(t,u,s) - A(u,s,t) \qquad P, F \ldots \text{ waves}$$

$$T^{I=2} = A(t,u,s) + A(u,s,t) \qquad S, D \ldots \text{ waves}$$

S-wave scattering lengths:

$$a_0^I = \frac{1}{32\pi} T^I\left(s = 4M_\pi^2, t = u = 0\right)$$

$$\rightsquigarrow \quad a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} \simeq 0.16 \;, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} \simeq -0.045$$

ChPT at higher orders / power counting

- so far: $\mathcal{L}^{(2)}$; higher orders?
- tree level; what about loops?
- $\pi\pi$ scatt. amp. real; unitarity? $\operatorname{Im} t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}}\, |t_\ell^I|^2$

  so $t_\ell^I = \mathcal{O}(p^2) \rightsquigarrow \operatorname{Im} t_\ell^I = \mathcal{O}(p^4) \rightsquigarrow$ generated by loops
- arbitrary loop diagram from $\mathcal{L}_{eff} = \sum_d \mathcal{L}^{(d)}$

  L loops, I propagators / internal lines, V_d vertices of order d

$$\rightsquigarrow \quad A \propto \int (d^4p)^L \frac{1}{(p^2)^I} \prod_d (p^d)^{V_d} \propto p^\nu$$

$$\rightsquigarrow \quad \nu = 4L - 2I + \sum_d d\, V_d$$

now

$$\underbrace{L}_{\#\text{ undetermined momenta}} = \underbrace{I - \sum_d V_d}_{\text{momentum conservation at each vertex}} + \underbrace{1}_{\text{overall mom. cons. factored out}}$$

eliminate I $\rightsquigarrow$ $\boxed{\nu = \sum_d V_d (d-2) + 2L + 2}$

slide: example $\pi\pi$

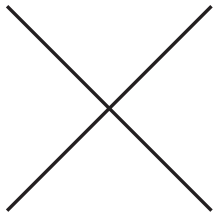
Weinberg's power counting argument

$$\boxed{\nu = \sum_d V_d(d-2) + 2L + 2}$$

- example: $\pi\pi$ scattering

$\boxed{\nu = 2}$

only lowest-order tree graphs: $V_{d>2} = 0$, $L = 0$



Weinberg's power counting argument

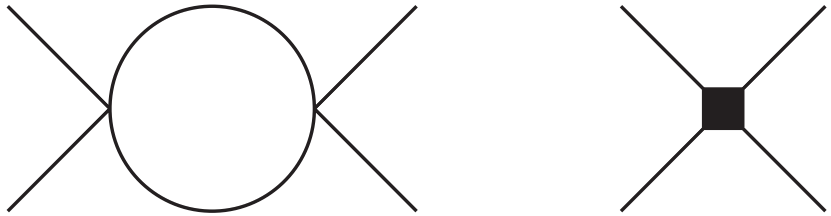
$$\boxed{\nu = \sum_d V_d(d-2) + 2L + 2}$$

- example: $\pi\pi$ scattering

$$\boxed{\nu = 4}$$

one-loop graphs with $\mathcal{L}^{(2)}$: $V_{d>2} = 0$, $L = 1$

<u>or</u> one insertion from $\mathcal{L}^{(4)}$: $V_4 = 1$, $V_{d>4} = 0$, $L = 0$



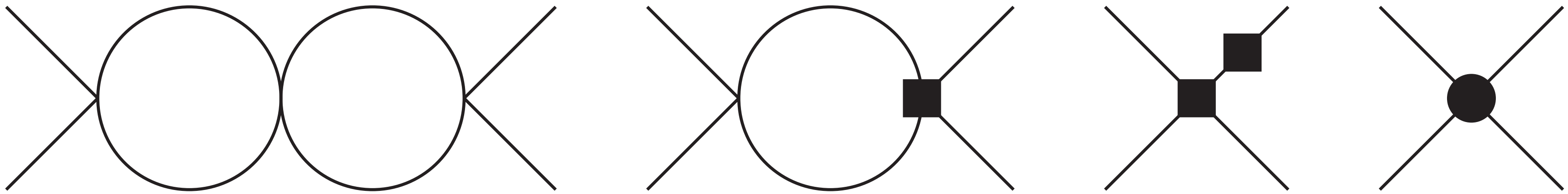
Weinberg's power counting argument

$$\boxed{\nu = \sum_d V_d(d-2) + 2L + 2}$$

- example: $\pi\pi$ scattering

$\boxed{\nu = 6}$

two-loop graphs with $\mathcal{L}^{(2)}$: $V_{d>2} = 0,\ L = 2$
<u>or</u> one-loop with one vertex from $\mathcal{L}^{(4)}$: $V_4 = 1,\ V_{d>4} = 0,\ L = 1$
<u>or</u> two insertions from $\mathcal{L}^{(4)}$: $V_4 = 2,\ V_{d>4} = 0,\ L = 0$
<u>or</u> one insertion from $\mathcal{L}^{(6)}$: $V_4 = 0,\ V_6 = 1,\ V_{d>6} = 0,\ L = 0$



The Lagrangian $\mathcal{L}^{(4)}$ in SU(2)

$$\mathcal{L}^{(4)} = \frac{\ell_1}{4}\langle D_\mu U^\dagger D^\mu U\rangle^2 + \frac{\ell_2}{4}\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle + \frac{\ell_3}{16}\langle \chi^\dagger U + \chi U^\dagger\rangle^2$$
$$+ \frac{\ell_4}{4}\langle D_\mu U D^\mu \chi^\dagger + D_\mu \chi D^\mu U^\dagger\rangle + \ell_5 \langle F_{R,\mu\nu} U^\dagger F_L^{\mu\nu} U\rangle$$
$$+ \frac{i\ell_6}{2}\langle F_R^{\mu\nu} D_\mu U^\dagger D_\nu U + F_L^{\mu\nu} D_\mu U D_\nu U^\dagger\rangle - \frac{\ell_7}{16}\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + \mathcal{L}_{\text{WZW}}$$

Symbols:

- $D_\mu U = \partial_\mu U - i[v_\mu, U] - i\{a_\mu, U\}$ covariant derivative

  $\chi = 2B(s + ip)$, $s = \mathcal{M} + \ldots$ (pseudo)scalar sources

  $F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]$, $F_L^{\mu\nu} = \ldots$ field strength tensors

  $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$ right-/left-handed currents
- Wess–Zumino–Witten term / chiral anomaly $\mathcal{L}_{\text{WZW}}$:
  - ▷ of odd intrinsic parity / odd number of Goldstone bosons
  - ▷ describes processes such as $\pi^0 \to \gamma\gamma$, $\gamma\pi^- \to \pi^0\pi^-$ . . .

The Lagrangian $\mathcal{L}^{(4)}$ in SU(2)

$$\mathcal{L}^{(4)} = \frac{\ell_1}{4}\langle D_\mu U^\dagger D^\mu U\rangle^2 + \frac{\ell_2}{4}\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle + \frac{\ell_3}{16}\langle \chi^\dagger U + \chi U^\dagger\rangle^2$$
$$+ \frac{\ell_4}{4}\langle D_\mu U D^\mu \chi^\dagger + D_\mu \chi D^\mu U^\dagger\rangle + \ell_5 \langle F_{R,\mu\nu} U^\dagger F_L^{\mu\nu} U\rangle$$
$$+ \frac{i\ell_6}{2}\langle F_R^{\mu\nu} D_\mu U^\dagger D_\nu U + F_L^{\mu\nu} D_\mu U D_\nu U^\dagger\rangle - \frac{\ell_7}{16}\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + \mathcal{L}_{\text{WZW}}$$

Physics:

- $\ell_{1,2} = \mathcal{O}(\partial^4)$: needs 4 pions $\Rightarrow$ (e.g.) D-wave $\pi\pi$ scattering
- $\ell_3 = \mathcal{O}(m_q^2)$, $\ell_4 = \mathcal{O}(\partial^2 m_q)$: "symmetry breakers", control m_q-dependence of M_π^2, F_π
- ℓ_5: requires 2 currents: radiative π decay $\pi^+ \to \ell^+ \nu_\ell \gamma$
- ℓ_6: t-dependence / radius of π vector (charge) form factor
- ℓ_7: isospin-breaking correction $\propto (m_u - m_d)^2$ to $M_{\pi^0}^2$

The pion (vector) form factor

FF 1

or: on the limits of chiral perturbation theory

$$\langle \pi^a(p)\pi^b(p') \,|\, \underbrace{\bar{q}\,\tfrac{\tau^3}{2}\gamma_\mu q}_{\text{electromagnetic current}} \,|\, 0 \rangle = i\varepsilon^{a3b}(p'-p)_\mu F_\pi^V(s)$$

$$s = (p+p')^2$$

- ChPT at leading order: $\partial_\mu \longrightarrow D_\mu$

$$\frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger\rangle\,, \quad D_\mu U = \partial_\mu U + i[v_\mu, U]$$

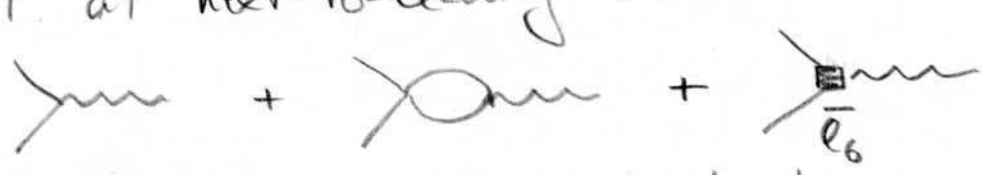
here: vector current $v_\mu \longrightarrow -eA_\mu Q$ ← charge matrix

$$\rightsquigarrow \; ieA_\mu\left(\pi^+\partial^\mu\pi^- - \pi^-\partial^\mu\pi^+ + \ldots\right)$$

← kaons, more pions...

$\rightsquigarrow$ point-like couplings to the meson charges, $\hat{=}$ scalar QED

- ChPT at next-to-leading order:



$\bar{\ell}_6$: $\mathcal{O}(p^4)$ low-energy constant needed to absorb UV divergence of loop diagram

$$F_\pi^V(s) = 1 + \frac{1}{6}\langle r^2\rangle_\pi^V s + \mathcal{O}(s^2)\,, \quad \langle r^2\rangle_\pi^V = \frac{1}{(4\pi F_\pi)^2}(\bar{\ell}_6 - 1)$$

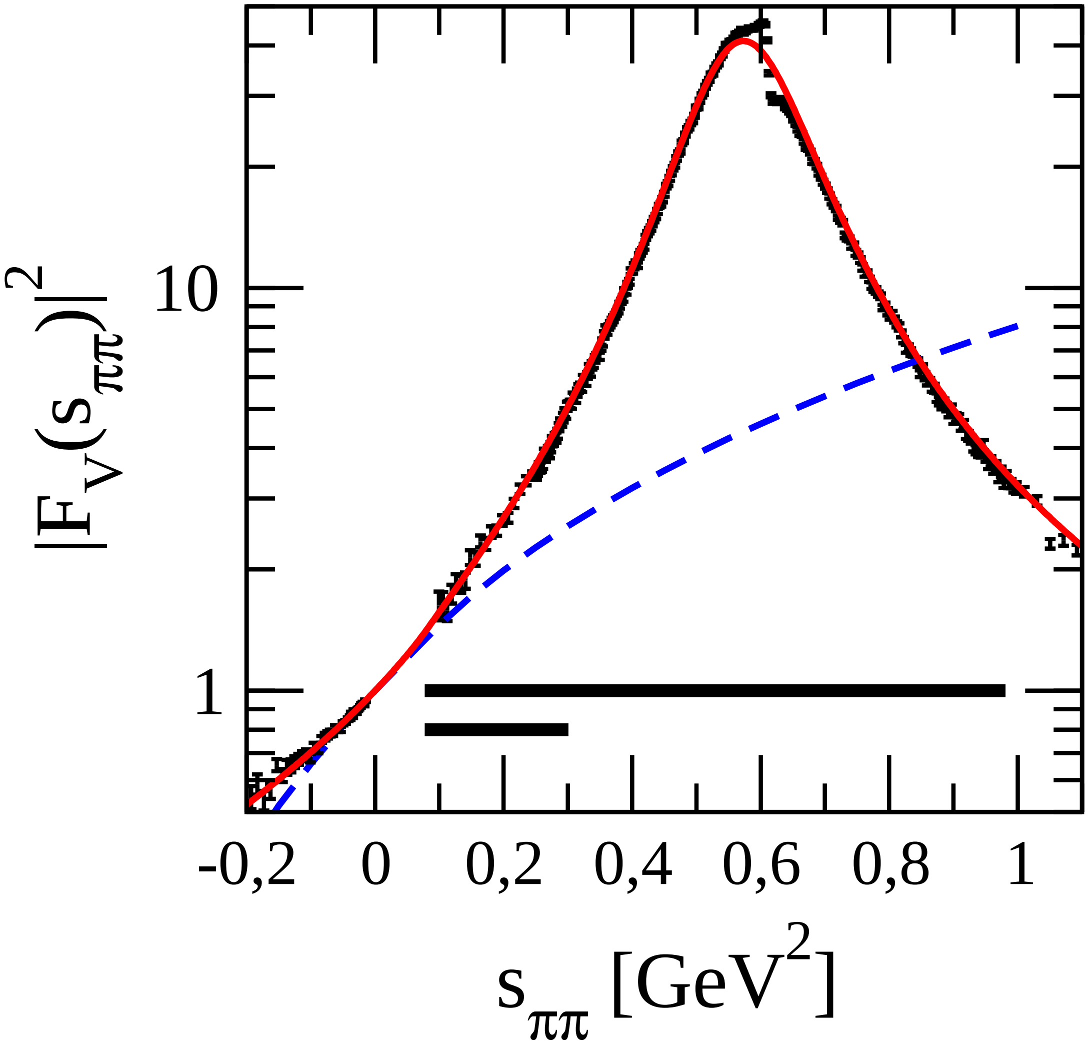
$\rightsquigarrow$ fix finite remainder to pion charge radius

$$\boxed{\text{slide: } F_\pi^V(s) \text{ one-loop vs. phenomenology}}$$

$\rightsquigarrow$ fails very soon: ρ-resonance not reproduced
($\bar{\ell}_6$ parameterises low-energy tail of ρ, linearised)

$\rightsquigarrow$ yet higher orders provide higher-order polynomial terms

The pion vector form factor



ChPT at one loop

data

Omnès representation

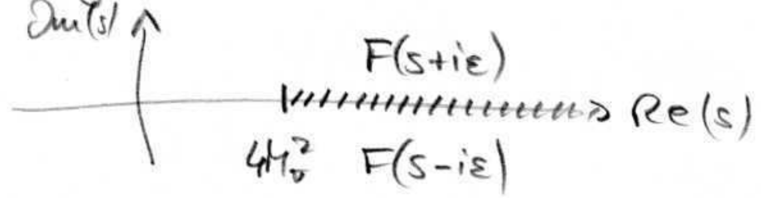
Stollenwerk et al. 2012

FF 2) • study the <u>non-analytic</u> piece of the form factor as contained in  !

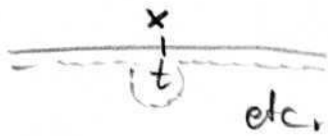
explicit calculation of the loop function yields

$$F(s \pm i\varepsilon) = \text{regular} + \frac{s(4M_\pi^2 - s)}{96\pi^2 F_\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dx}{x - s \mp i\varepsilon} \frac{1}{x} \sqrt{1 - \frac{4M_\pi^2}{x}}$$

$F(s)$ analytic function in a <u>cut</u> plane:



• calculate $\operatorname{disc} F(s) = F(s+i\varepsilon) - F(s-i\varepsilon)$

using $\displaystyle\int \frac{dx}{x - s \mp i\varepsilon} = \fint \frac{dx}{x-s} \pm i\pi$ 

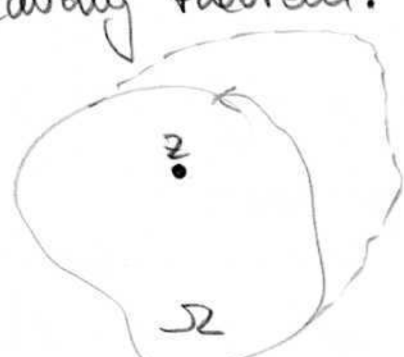
or $\displaystyle\frac{1}{x - s \mp i\varepsilon} = \frac{P}{x-s} \pm i\pi\,\delta(x-s)$

so $\displaystyle\operatorname{disc} F(s) = 2i\pi \frac{s(4M_\pi^2 - s)}{96\pi^2 F_\pi^2} \frac{1}{s} \sqrt{1 - \frac{4M_\pi^2}{s}}$

$= 2i \operatorname{Im} F(s+i\varepsilon)$

• relevance of the discontinuity: <u>dispersion relations</u>

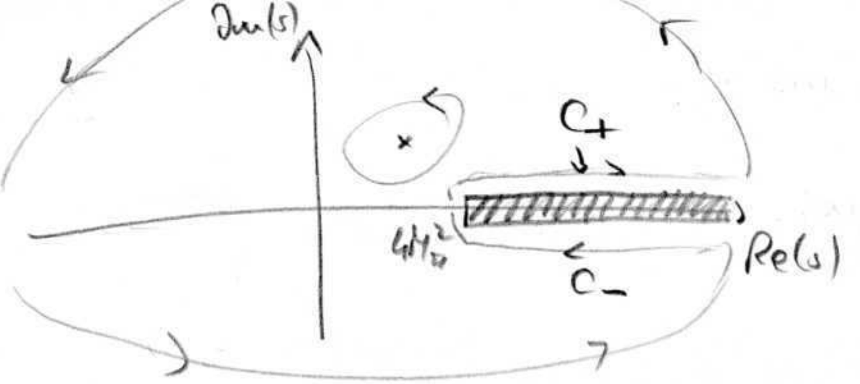
• Cauchy theorem: $\displaystyle f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{dy}{y-z} f(y)$



in particular, as $\oint_{\partial\Omega} dy\, f(y) = 0$ if f is holomorphic, and as $\frac{f(y)}{y-z}$ is holomorphic except at $y=z$,

we can <u>deform</u> contour ad lib. as long as we don't hit a singularity!

(FF3



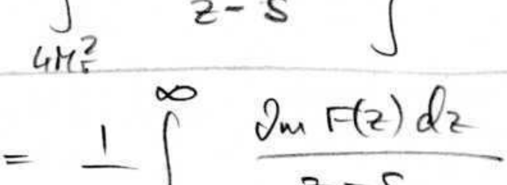
assume that $F(s) \to 0$ sufficiently fast on the large semi-circles

$$\rightsquigarrow \boxed{F(s) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{F(z)\,dz}{z-s}} \longrightarrow \frac{1}{2\pi i}\int_{C_+ + C_-} \frac{F(z)\,dz}{z-s}$$

$$= \frac{1}{2\pi i}\left\{\int_{4M_\pi^2}^{\infty} \frac{F(z+i\varepsilon)\,dz}{z-s} - \int_{4M_\pi^2}^{\infty} \frac{F(z-i\varepsilon)\,dz}{z-s}\right\}$$

$$\boxed{= \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{disc}\,F(z)\,dz}{z-s} = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\,F(z)\,dz}{z-s}}$$

- it turns out there is a much simpler way to calculate the discontinuity / the imaginary part of a diagram than to do the full loop calculation



$$i\mathcal{M} = \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{[\ell^2-M^2+i\varepsilon][(p-\ell)^2-M^2+i\varepsilon]}$$

Cutkosky / cutting rules: propagators can only yield imaginary parts on-shell !

$\rightsquigarrow$ replacement $\frac{1}{p^2-M^2+i\varepsilon} \longrightarrow -2\pi i\,\delta(p^2-M^2)$ yields disc!

(see e.g. Peskin/Schroeder)

$$\mathrm{disc}\left[-\!\bigcirc\!-\right] = i\int \frac{d^4\ell}{(2\pi)^4}\,2\pi\,\delta(\ell^2-M^2)\,2\pi\,\delta((p-\ell)^2-M^2)$$

$d^4\ell = d\ell^0\,\underline{\ell}^2\,d|\underline{\ell}|\,d\Omega_\ell$ CMS: $p=(\sqrt{s},\underline{0})$

$(p_s-\ell)^2-M^2 \xrightarrow{\ell^2=M^2} s-2\sqrt{s}\,\ell^0$

FF4

$$= \frac{i}{4\pi^2}\int \frac{\underline{\ell}^2\, d|\underline{\ell}|\, d\Omega_\ell}{2\ell^0}\,\delta(s-2\sqrt{s}\,\ell^0)$$

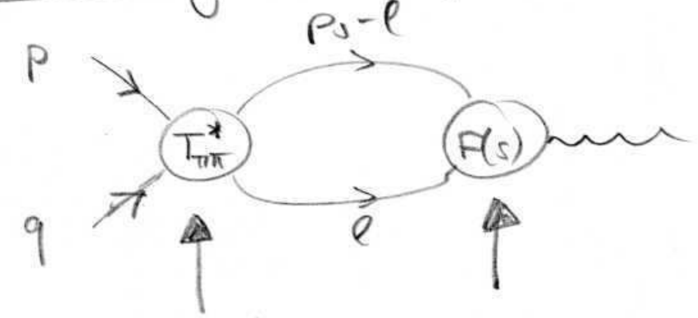
$$|\underline{\ell}|\, d|\underline{\ell}| = \ell^0\, d\ell^0 \quad \text{as } (\ell^0)^2 = \underline{\ell}^2 + M^2$$

$$= \frac{i}{8\pi^2}\int \sqrt{(\ell^0)^2 - M^2}\, d\ell^0\, d\Omega_\ell\, \delta(s-2\sqrt{s}\,\ell^0)$$

$$= \frac{i}{8\pi^2}\,\frac{\sqrt{s/4 - M^2}}{2\sqrt{s}}\int d\Omega_\ell = \frac{i}{32\pi^2}\sqrt{1-\frac{4M^2}{s}}\int d\Omega_\ell$$

$$= \frac{i}{8\pi}\,\sigma \qquad = 2i\, \mathrm{Im}\left[-\bigcirc-\right]$$

Discontinuity of the pion form factor, general case



$$p_s = p+q,\; p_s^2 = s$$

general vertices (beyond ChPT):

$\pi\pi$ scatt. amplitude

pion form factor again!

$$(p-q)_\mu \,\mathrm{disc}\, F(s) = i\int \frac{d^4\ell}{(2\pi)^4}\,\delta(\ell^2 - M_\pi^2)\,\delta\big((p_s-\ell)^2 - M_\pi^2\big)\, T^*_{\pi\pi}(s, z_\ell)$$
$$\times (p_s - 2\ell)_\mu\, F(s)$$

$\cos\theta_\ell$

$$= \ldots = \frac{i}{32\pi^2}\sqrt{1-\frac{4M_\pi^2}{s}}\,F(s)\,\underbrace{\int d\Omega_\ell\, T^*_{\pi\pi}(s, z_\ell)\,(p_s - 2\ell)_\mu}_{\text{how do we calculate this angular integral?}} \qquad = \cos\sphericalangle(\underline{q}, \underline{\ell})$$

ansatz:
$$\int d\Omega_\ell\, T^*_{\pi\pi}(s, z_\ell)\,(p+q-2\ell)_\mu = L_1 (p+q)_\mu + L_2 (p-q)_\mu \qquad (*)$$

kinematics in CMS: $\ell^0 = p^0 = q^0 = \frac{\sqrt{s}}{2}$, $|\underline{\ell}| = |\underline{p}| = |\underline{q}| = \frac{\sqrt{s}}{2}\sigma$

$$z_\ell = \cos\theta_{q\ell} \;\leadsto\; p\ell = \frac{s}{4}(1+\sigma^2 z_\ell)$$
$$q\ell = \frac{s}{4}(1-\sigma^2 z_\ell)$$

– contract (*) with $p_s = p+q$:

$$\int d\Omega_\ell \, T^*_{\pi\pi} \Big[\underbrace{(p+q)^2}_{s} - \underbrace{2\ell(p+q)}_{s} \Big] = 0 \;=\; s \times L_1$$

– contract (*) with $p-q$:

$$\int d\Omega_\ell \, T^*_{\pi\pi}(s,z_\ell)\, 2\ell(q-p) = -\int d\Omega_\ell \, T^*_{\pi\pi}(s,z_\ell)\, \underbrace{s\sigma^2}_{= s-4M_\pi^2} z_\ell$$
$$= (4M_\pi^2 - s)\, L_2$$

<u>so:</u> $$\int d\Omega_\ell \, T^*_{\pi\pi}(s,z_\ell)(p_s - 2\ell)_\mu = 2\pi \int_{-1}^{1} dz \, z\, T^*_{\pi\pi}(s,z)\,(p-q)_\mu \quad (**)$$

• partial-wave expansion of the $\pi\pi$ scatt. amplitude:

$$T_{\pi\pi}(s,z) = \underset{\substack{\uparrow \\ \text{isospin factor} \\ \text{for } \pi^+\pi^-}}{\tfrac{1}{2}}\, 32\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z)\, t_\ell(s)$$

$\leadsto$ $t_\ell(s) = \frac{1}{2}\int_{-1}^{1} dz\, P_\ell(z)\, T_{\pi\pi}(s,z)$

as $\int_{-1}^{1} dz\, P_\ell(z) P_{\ell'}(z) = \frac{2\delta_{\ell\ell'}}{2\ell+1}$; note $P_0(z) = 1$, $P_1(z) = z$

$\leadsto$ we have projected out the P-wave in (**) !

• result: $$\text{disc}\, F(s) = 2i\sigma F(s) \underbrace{t_1^*(s)}_{= \frac{\sin\delta_1(s) e^{-i\delta_1(s)}}{\sigma}} \Theta(s-4M_\pi^2)$$
$$= 2i\, F(s) \sin\delta_1(s)\, e^{-i\delta_1(s)}\, \Theta(s-4M_\pi^2)$$

or $$\boxed{\operatorname{Im} F(s) = F(s) \sin\delta_1(s)\, e^{-i\delta_1(s)}\, \Theta(s-4M_\pi^2)}$$

• Watson's final-state theorem: $F(s) = |F(s)|\, e^{i\delta_F(s)}$

$\leadsto$ require $\boxed{\delta_F(s) = \delta_1(s)}$

the phase of the form factor (below inelastic thresholds) equals the 2-particle scattering phase shift

FF 6) Omnès solution

- a solution to the above can be given analytically
- note : if $\Omega(s)$ is a special solution, $F(s) = \Omega(s)$, then so is $F(s) = P(s)\,\Omega(s)$, P real polynomial
- $$\Omega(s+i\varepsilon) = |\Omega(s)|\, e^{i\delta(s)}$$
  $$\Omega(s-i\varepsilon) = |\Omega(s)|\, e^{-i\delta(s)} = \Omega(s+i\varepsilon)\, e^{-2i\delta(s)}$$
  $$\rightsquigarrow \quad \log \Omega(s-i\varepsilon) = \log \Omega(s+i\varepsilon) - 2i\delta(s)$$
  $$\text{disc} \log \Omega(s) = 2i\delta(s)$$

$\rightsquigarrow$ can write down a dispersion relation for $\log \Omega(s)$!

choose normalisation $\Omega(0) = 1$

$$\rightsquigarrow \quad \log \Omega(s) = \frac{1}{2\pi i} \int_{4M_\pi^2}^{\infty} ds' \underbrace{\text{disc} \log \Omega(s')}_{= 2i\delta(s')} \underbrace{\left(\frac{1}{s'-s} - \frac{1}{s'} \right)}_{= \frac{s}{s'(s'-s)}}$$

$$= \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)} \quad \leftarrow \text{ this is a subtracted DR!}$$

$$\rightsquigarrow \quad \boxed{\Omega(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\delta(s')}{s'(s'-s)} \right\}} \quad \text{Omnès function}$$

- central object for description of FSI between 2 hadrons only! many other applications:
  - scalar ff of the pion $\rightsquigarrow \delta(s) \to \delta_0^0(s)$
  - K_{e3} decays $K^+ \to \pi^0 e^+ \nu_e$ described in terms of 2 form factors; scalar + vector $\rightsquigarrow \delta_0^{1/2}(s), \delta_1^{1/2}(s)$

Exercises: show

(i) $\arg \Omega(s) = \delta(s)$ (simple!)

(ii) assume $\delta(s \to \infty) \to c\pi$; then show
$\Omega(s \to \infty) \sim s^{-c}$.

Side remark: subtractions

- assume form factor satisfies

$$F(s) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{s'-s} \operatorname{Im} F(s')$$

$$= \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{(s'-s+s-s_0)\, ds'}{(s'-s_0)(s'-s)} \operatorname{Im} F(s')$$

$$= \underbrace{\frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{s'-s_0} \operatorname{Im} F(s')}_{①} + \underbrace{\frac{s-s_0}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{(s'-s_0)(s'-s)} \operatorname{Im} F(s')}_{②}$$

- obviously, ① $= F(s_0)$
  $\sim$ "sum rule": $F(s_0) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{s'-s_0} \operatorname{Im} F(s')$
  ②: due to $s'-s_0$ denominator, convergence of dispersive integral improved
- example assumed convergence — may not be granted!
  alternative: derive DR for

$$\underbrace{\frac{F(s) - F(s_0)}{s - s_0}}_{\text{better convergence on "large circle"}} = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{(s'-s_0)(s'-s)} \operatorname{Im} F(s')$$

(assuming $F(s_0) \in \mathbb{R}$)

$$\sim F(s) = F(s_0) + \frac{s-s_0}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{(s'-s_0)(s'-s)} \operatorname{Im} F(s')$$

FF8 can repeat this multiple times; e.g.

$$F(s) = \sum_{j=0}^{n-1} \frac{1}{j!} F^{(j)}(s_0)(s-s_0)^j + \frac{(s-s_0)^n}{\pi} \int_{s_{thr}}^{\infty} \frac{ds' \,\mathrm{Im}\, F(s')}{(s'-s_0)^n (s'-s)}$$

each subtraction (i) improves convergence

(ii) introduces additional parameter

- another application: model-independent analysis of $\eta^{(\prime)} \to \pi^+\pi^-\gamma$; $\pi^+\pi^-$ in a P-wave

$\boxed{\text{slide: spectra with } \Omega(s) \text{ divided out}}$

- physics: chiral anomaly (Wess-Zumino-Witten)
- spectra:

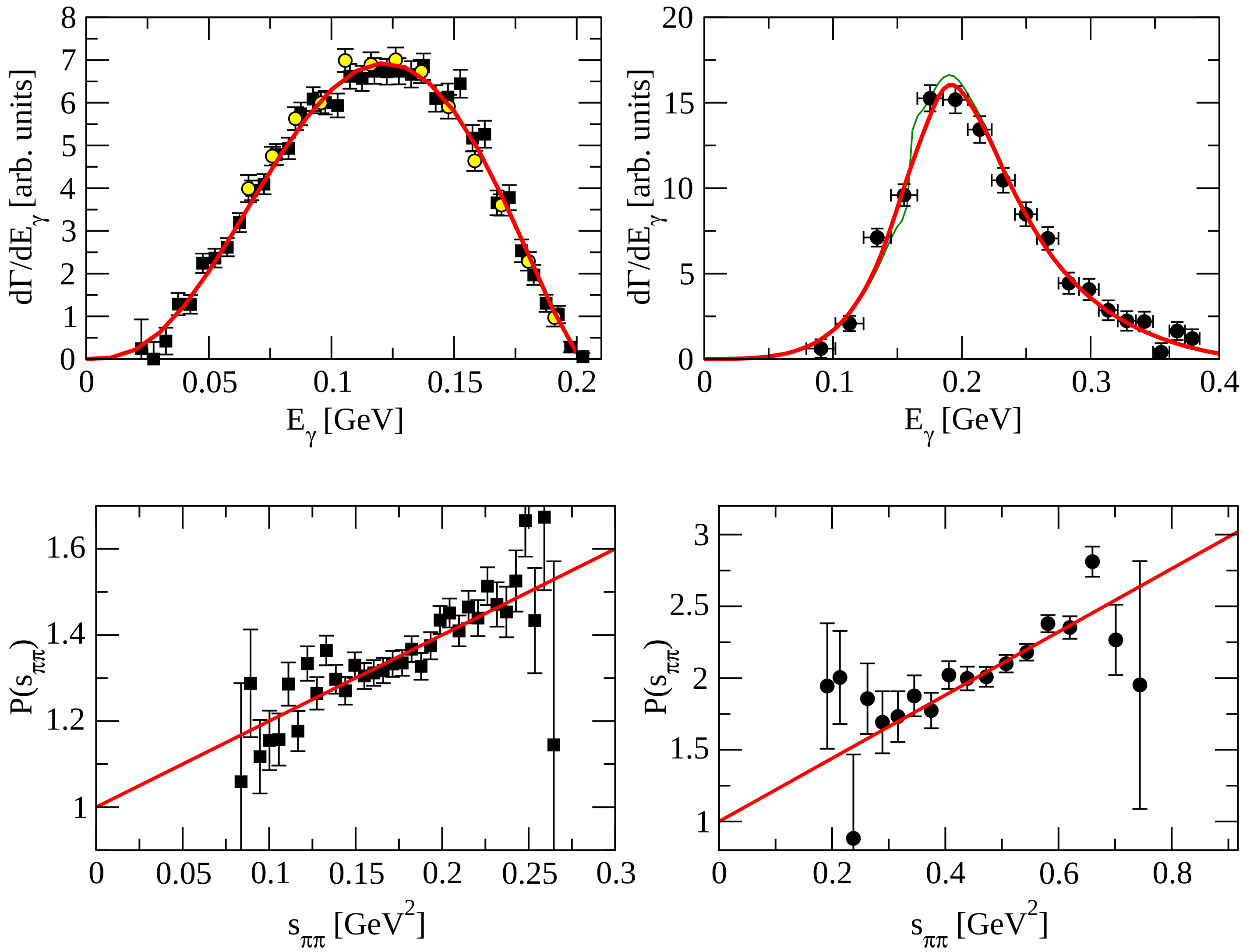
$$\frac{d\Gamma}{ds_{\pi\pi}} = \Big| \underbrace{A \times P(s_{\pi\pi})}_{\text{polynomial}} \times \underbrace{\Omega(s_{\pi\pi})}_{\text{Omnès function}} \Big|^2 \underbrace{\Gamma_0(s_{\pi\pi})}_{\text{phase space}}$$

- universal Ω-factor can be divided out to account for $\pi\pi$ P-wave FSI

$\rightsquigarrow$ data suggest $P(s_{\pi\pi})$ to be linear

try to interpret <u>this</u> in terms of chiral coupling constants

Spectra for $\eta,\ \eta' \to \pi^+\pi^-\gamma$



Stollenwerk et al. 2012

Quark mass ratios & $\eta \to 3\pi$

- masses beyond leading order:

$$M_\pi^2 = B(m_u+m_d)\,\{1+\mathcal{O}(\hat{m},m_s)\}$$
$$M_{K^+}^2 = B(m_u+m_s)\,\{1+\mathcal{O}(\hat{m},m_s)\}$$

- form dimensionless ratios:

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s+\hat{m}}{m_u+m_d}\,\{1+\Delta_M+\mathcal{O}(m_q^2)\}$$

$$\frac{(M_{K^0}^2-M_{K^+}^2)_{\text{strong}}}{M_K^2-M_\pi^2} = \frac{m_d-m_u}{m_s-\hat{m}}\,\{1+\Delta_M+\mathcal{O}(m_q^2)\}$$

$$\Delta_M = \frac{8(M_K^2-M_\pi^2)}{F^2}(2L_8-L_5) + \text{"chiral logarithms"}$$

- form double ratio:

$$Q^2 = \underbrace{\frac{m_s^2-\hat{m}^2}{m_d^2-m_u^2}} = \frac{M_K^2}{M_\pi^2}\,\frac{M_K^2-M_\pi^2}{(M_{K^0}^2-M_{K^+}^2)_{\text{strong}}}\,\{1+\mathcal{O}(m_q^2)\}$$

corrections suppressed by $\underline{\underline{2}}$ orders in quark mass exp.

$$\downarrow$$

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \qquad \left(\text{neglecting } \left(\frac{\hat{m}}{m_s}\right)^2 \sim 1.5\times10^{-3}\right)$$

"Leutwyler's ellipse"

- problem: need to subtract $(M_{K^0}^2-M_{K^+}^2)_{em}$ from physical mass diff. $\rightsquigarrow$ rely on Dashen's theorem

$$(M_{K^+}^2-M_{K^0}^2)_{em} = (M_{\pi^+}^2-M_{\pi^0}^2)_{em} \simeq (M_{\pi^+}^2-M_{\pi^0}^2)_{phys}$$

QMR2

- relying on Dashen's theorem,

$$Q_D = 24.2$$

- but: various calculations suggest corrections to Dashen of $\mathcal{O}(e^2 m_q)$ may be large

$$1 \leq \frac{(M_{K^+}^2 - M_{K^0}^2)_{em}}{(M_{\pi^+}^2 - M_{\pi^0}^2)_{em}} \lesssim 2.5 \quad !$$

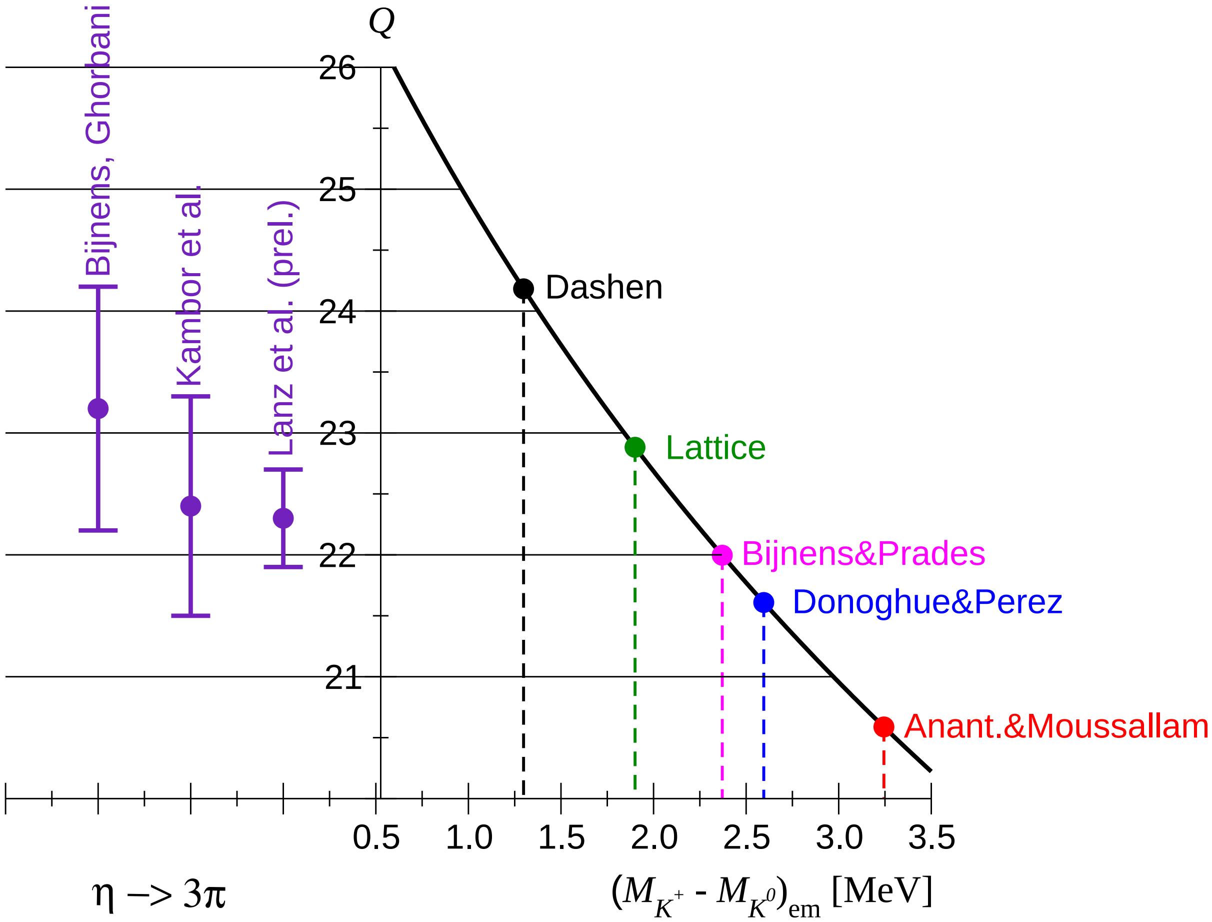
which results in $\quad 20.6 \leq Q \leq 24.2$

$\rightsquigarrow$ not very precise!

Combined result on quark mass ratios (1)

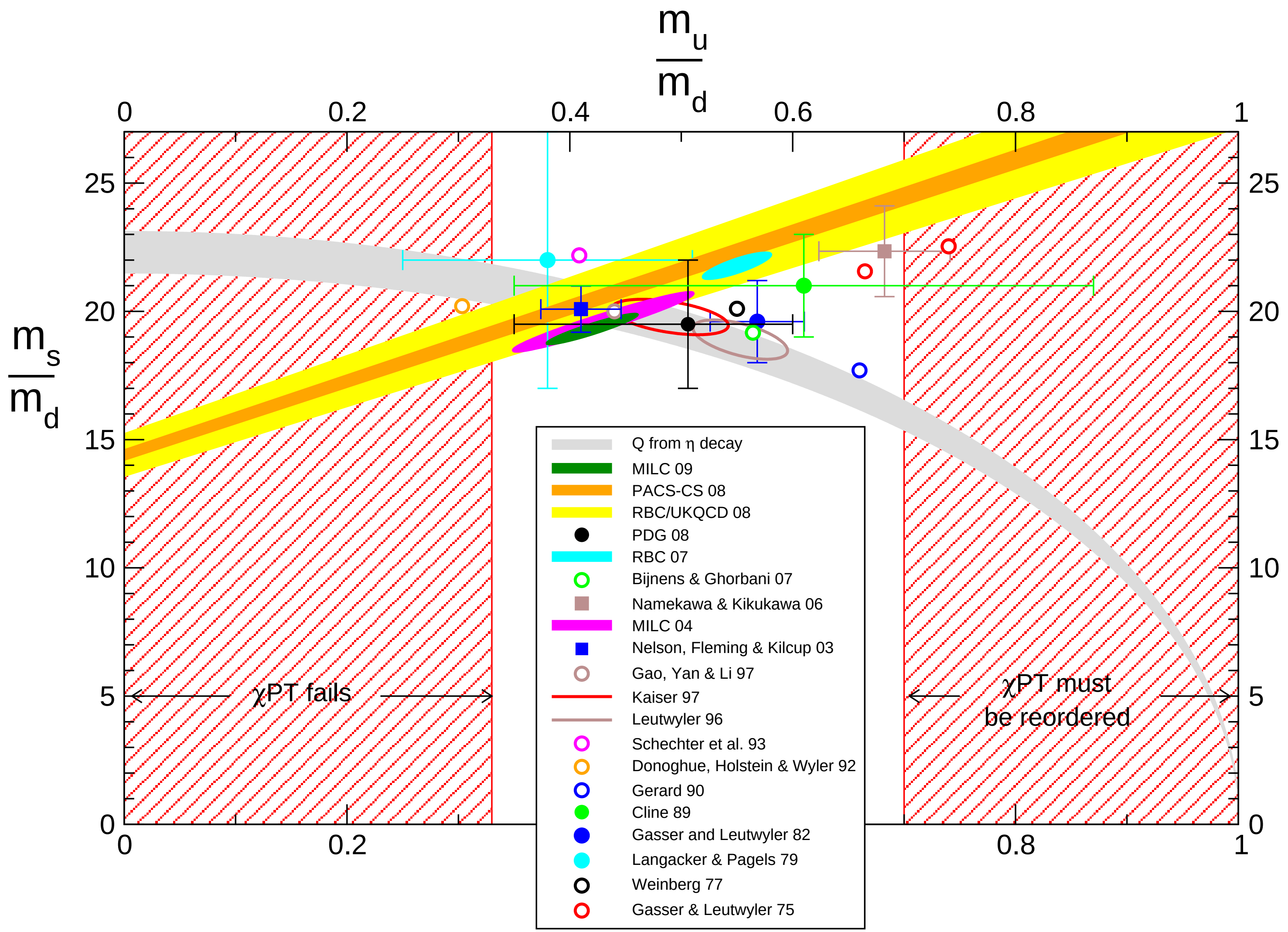
Combined information on Q

$\eta \to 3\pi$ vs. various corrections to Dashen's theorem:



Combined result on quark mass ratios (2)

- additional constraints needed to find position on the ellipse:



Leutwyler 2009

$\eta \to 3\pi$ and dispersion relations

- $\eta \to 3\pi$: η has $I=0$ $\leadsto$ $G=+$, while π have $I=1$ $\leadsto$ $G=-$

  $\leadsto$ violates G-parity / isospin conservation

- ChPT at tree level $\mathcal{O}(p^2)$ (current algebra):

$$A_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) = \underbrace{\frac{B(m_u-m_d)}{3\sqrt{3}F_\pi^2}}_{= -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2-M_\pi^2}{3\sqrt{3}F_\pi^2}} \underbrace{\left\{1+\frac{3(s-s_0)}{M_\eta^2-M_\pi^2}\right\}}_{=M(s,t,u)}$$

$s=(p_{\pi^+}+p_{\pi^-})^2$
$t=(p_{\pi^0}+p_{\pi^-})^2$
$u=(p_{\pi^0}+p_{\pi^+})^2$

$s_0=\frac{1}{3}(M_\eta^2+3M_\pi^2)$: center of Dalitz plot $s=t=u=s_0$

normalisation $M^{CA}(s=t=u=s_0)=1$

$\frac{1}{Q^2}=\frac{m_d^2-m_u^2}{m_s^2-\hat{m}^2}$ double ratio of quark masses

$\leadsto$ wanted !

- $$A_{\eta\to3\pi^0}(s,t,u) = A_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) + A_{\eta\to\pi^+\pi^-\pi^0}(t,u,s) + A_{\eta\to\pi^+\pi^-\pi^0}(u,s,t)$$

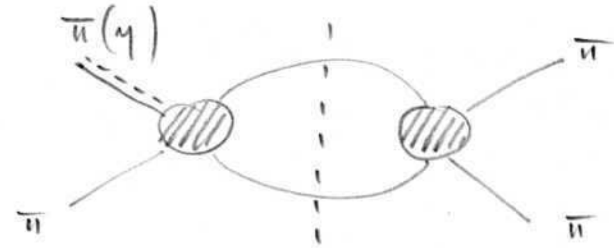
  at leading order: constant, $A^{CA}_{\eta\to3\pi^0}=\frac{B(m_u-m_d)}{\sqrt{3}F_\pi^2}$

- $\eta\to3\pi$ amplitude of very similar structure as $\pi\pi$-scattering: at leading order <u>linear</u> in $s/t/u$

  $\leadsto$ S- and P-waves only

  consequence for <u>imaginary parts</u> :

η DR2



cut-contribution to D and higher waves requires to be $\mathcal{O}(p^4)$ $\rightsquigarrow$ complete diagram $\mathcal{O}(p^8)$ $\hat{=}$ 3-loop order

$\rightsquigarrow$ up-to-and-including 2 loops, $\pi\pi$-scattering / $\eta\to3\pi$ only have discontinuities in S- and P-waves

- "reconstruction theorem": both ($\pi\pi$ / $\eta\to3\pi$) can be decomposed in terms of <u>single-variable functions</u> with a <u>right-hand cut only</u> as

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \tfrac{2}{3}M_2(s) \qquad (*)$$

indices 0,1,2: $\pi\pi$ isospin, $I=0/2$: S-wave
$I=1$: P-wave

($s-u = 4p_t^2\cos\theta_t$ $\rightsquigarrow$ P-wave characteristic)
$\rightsquigarrow$ <u>$t,u(s,\theta_s)$ explicitly here!</u> (from p. 4)

<u>Remarks:</u>

(i) analogy $\pi\pi$ / $\eta\to3\pi$: η decay effected by part of QCD Hamiltonian $\sim \frac{m_u-m_d}{2}\,\bar{q}\tau^3 q$

$\rightsquigarrow$ $\frac{m_u-m_d}{2}\langle\pi^i\pi^j\pi^k|\,\bar{q}\tau^\ell q\,|\eta\rangle \propto M^{ijk,\ell}_{\eta\to3\pi}$

$\bar{q}\tau^\ell q$ transforms under isospin like an additional pion $\rightsquigarrow$ $M^{ijk}_{\eta\to3\pi} \doteq M^{ijk,3}_{\eta\to3\pi} = M(s,t,u)\,\delta^{ij}\delta^{k3} + M(t,s,u)\,\delta^{ik}\delta^{j3} + M(u,t,s)\,\delta^{i3}\delta^{jk}$

in terms of physical pion fields:

$$M^{+-0}_{\eta\to3\pi}(s,t,u) = M^{113}_{\eta\to3\pi}(s,t,u) = M(s,t,u)$$

$$M^{000}_{\eta\to3\pi}(s,t,u) = M^{333}_{\eta\to3\pi}(s,t,u) = M(s,t,u) + M(t,u,s) + M(u,s,t)$$

(ii) isospin-breaking — what about electromagnetism?

- in chiral limit $m_q \to 0$, $\eta\to3\pi$ vanishes
  $\rightsquigarrow$ no contributions of $\mathcal{O}(e^2)$ (Sutherland)
- corrections $\mathcal{O}(e^2 \times m_q)$ are effectively $\mathcal{O}(e^2 m_{u,d})$,
  <u>not</u> $\mathcal{O}(e^2 m_s)$ $\rightsquigarrow$ very small, $\sim 1\%$

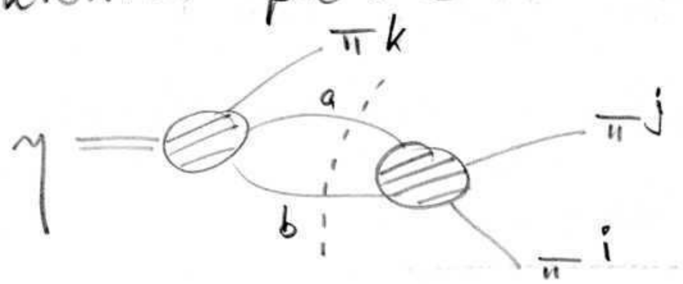
(iii) problem: <u>large</u> final-state interactions in $\eta\to3\pi$

predictions for width $\dfrac{\Gamma^{\text{1-loop}}}{\Gamma^{\text{tree}}} \approx 2.5$!

$\rightsquigarrow$ try to resum these using <u>dispersion relations</u>

The discontinuity equation for $\eta\to3\pi$

- intuitive picture :



- rescattering in "all 3 channels" (s,t,u)
- everything is projected onto the right (S-, P-) partial waves

so
$$\text{disc}\, M^{ijk}_{\eta\to3\pi}(s,t,u) \propto \int d_{LIPS} \Big\{ T^{ab,ij}_{\pi\pi}(s,\theta_s)^* M^{abk}_{\eta\to3\pi}(s,t_s',u_s') + T^{ab,ik}_{\pi\pi}(t,\theta_t)^* M^{ajb}_{\eta\to3\pi}(s_t',t,u_t') + T^{ab,jk}_{\pi\pi}(u,\theta_u)^* M^{iab}_{\eta\to3\pi}(s_u',t_u',u) \Big\}$$

η DR4 • most of the phase-space integration can be done trivially

$$\sim \quad d_{LIPS} \propto \int d\Omega \sqrt{1-\frac{4M_\pi^2}{s}}\,(\ldots)(s)$$
$$+ \sqrt{1-\frac{4M_\pi^2}{t}}\,(\ldots)(t) + \sqrt{1-\frac{4M_\pi^2}{u}}\,(\ldots)(u)$$

• $\pi\pi$ - amplitudes : (i) insert isospin projectors onto $I=0,1,2$

(ii) expand in partial waves:

$$T^{I=0,2} = 32\pi \,\frac{\sin\delta_{0,2}(s)\, e^{i\delta_{0,2}(s)}}{\sqrt{1-4M_\pi^2/s}} + \cancel{D\text{-waves}}^{\text{neglect}}$$

$$T^{I=1} = 96\pi \cos\theta \,\frac{\sin\delta_1(s)\, e^{i\delta_1(s)}}{\sqrt{1-4M_\pi^2/s}} + \cancel{F\text{-waves}}^{\text{neglect}}$$

(iii) insert isospin decomposition for $M_{\eta\to 3\pi}$ according to (*)

note: the relation between t/u and (s-channel) scattering angle θ_s in $\eta\to 3\pi$ is given by

$$t,u = \tfrac{1}{2}\left(3s_0 - s \pm \kappa(s)\cos\theta_s\right)$$
$$\kappa(s) = 4\,|p_{\pi\pi}|\,|p_{\pi\eta}| = \sqrt{1-\frac{4M_\pi^2}{s}}\,\lambda^{1/2}(s, M_\eta^2, M_\pi^2)$$

<u>Result</u>: all the discontinuity equations for $M_{0,1,2}(s)$ are of the form

$$\boxed{\text{disc}\, M_I(s) = 2i\,\Theta(s-4M_\pi^2)\left\{M_I(s)+\hat{M}_I(s)\right\}\sin\delta_I(s)\,e^{-i\delta_I(s)}}$$

this <u>almost</u> looks like the form factor equation

$$\text{disc}F(s) = 2i\,\Theta(s-4M_\pi^2)\,F(s)\,\sin\delta(s)\,e^{-i\delta(s)}$$

<u>except</u> for the inhomogeneities $\hat{M}_I(s)$:

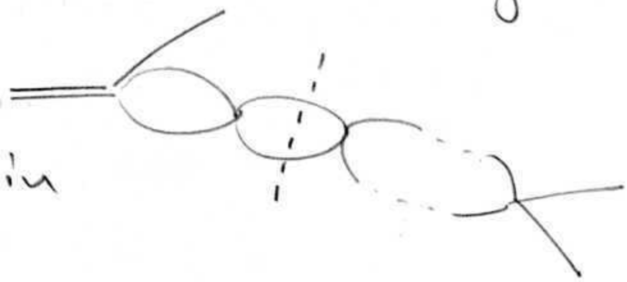
composed of <u>angular averages</u>, e.g. (η DR5

$$\hat{M}_0 = \frac{2}{3}\langle M_0 \rangle + 2(s-s_0)\langle M_1 \rangle + \frac{2}{3}\varkappa(s)\langle z\, M_1(z)\rangle + \frac{20}{9}\langle M_2 \rangle$$

where $\langle z^n M_I \rangle = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I\big(t(s,\theta_s)\big)$

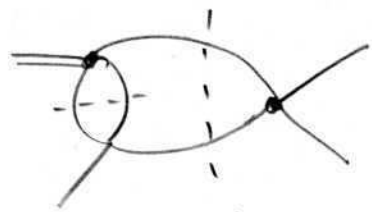
- what is the "physics" behind these inhomogeneities?

homogeneous term:



$\sim$ 2-particle FSI like in form factor

inhomogeneous term:



$\sim$ partial-wave projection of crossed-channel amplitudes

- we <u>know</u> how to solve the homogeneous equation

$$\mathrm{Im}\, M(s) = \Theta(s-4M_\pi^2)\, M(s)\sin\delta(s)\, e^{-i\delta(s)} :$$

$$M(s) = P(s)\,\Omega(s), \quad \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta(s')\,ds'}{s'(s'-s)}\right)$$

$P(s)$: polynomial, $\Omega(s)$: Omnès function

- how do we solve the general case?

ansatz: consider discontinuity of

$\dfrac{M(s)}{\Omega(s)}$; homogeneous: $=0$, solution is a polynomial ηDR6

$$\boxed{\mathrm{disc}\left(\frac{M(s)}{\Omega(s)}\right)} = \frac{M(s+i\varepsilon)}{\Omega(s+i\varepsilon)} - \frac{M(s-i\varepsilon)}{\Omega(s-i\varepsilon)}$$

$$\Omega(s) = |\Omega(s)|\, e^{i\delta(s)}$$

$$= \frac{M(s+i\varepsilon)\, e^{-i\delta(s)} - M(s-i\varepsilon)\, e^{i\delta(s)}}{|\Omega(s)|}$$

$$= \frac{1}{|\Omega(s)|}\left\{\underbrace{\left[M(s+i\varepsilon) - M(s-i\varepsilon)\right]}_{=\,\mathrm{disc}\, M(s)} e^{i\delta(s)} - M(s+i\varepsilon)\underbrace{\left(e^{i\delta(s)} - e^{-i\delta(s)}\right)}_{=\,2i\sin\delta(s)}\right\}$$

$$= 2i\left[M(s) + \hat{M}(s)\right]\sin\delta(s)\, e^{-i\delta(s)}$$

$$\boxed{= 2i\,\frac{\sin\delta(s)\,\hat{M}(s)}{|\Omega(s)|}}$$

that means

$$M(s) = \Omega(s)\left\{P(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\,\frac{\sin\delta(s')\,\hat{M}(s')}{|\Omega(s')|(s'-s)}\right\}$$

- how many subtractions do we need?
  what is the degree of the polynomial $P(s)$?
- → invoke Froissart bound again: $M(s,t,u)$ shouldn't grow faster than $O(s)$ for $s\to\infty$ (neglecting logs)
  so $M_0(s), M_2(s) \sim O(s)$, $M_1(s) \sim O(1)$
- show: assuming $\delta(s\to\infty) \to c\ast\pi$, then
  $\Omega(s) \to \mathrm{const}\ast s^{-c}$

7DR7

- assume $\pi\pi$ - phases to go to multiples of π asymptotically : $\delta_0^0(s\to\infty) = \delta_1^1(s\to\infty) = \pi$
$$\delta_0^2(s\to\infty) = 0$$

so $\Omega_0^0(s), \Omega_1^1(s) \to \mathcal{O}(s^{-1})$, $\Omega_0^2 \to \mathcal{O}(s^0)$

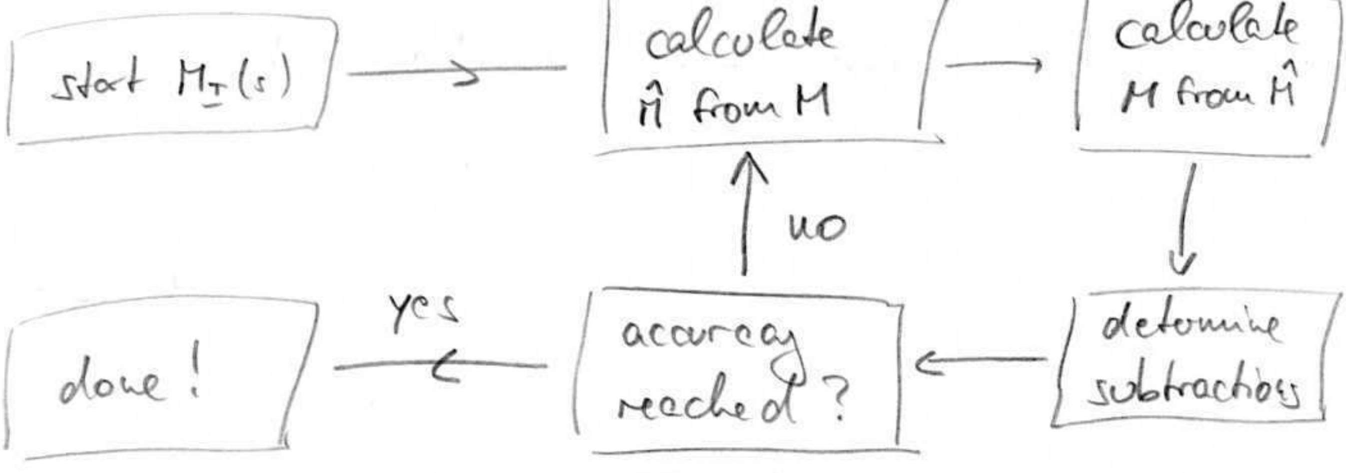
Froissart bound then allows polynomials
$$\left.\begin{aligned} P_0(s) &= \alpha_0 + \beta_0 s + \gamma_0 s^2 \\ P_1(s) &= \alpha_1 + \beta_1 s \\ P_2(s) &= \alpha_2 + \beta_2 s \end{aligned}\right\} \text{reduce \#: can eliminate } \alpha_1, \alpha_2, \beta_2$$

- does this agree with the requirement of having the integral over the inhomogeneity to converge ?

assume $\hat{M}(s)$ to behave like $M(s)$ asymptotically
$$\int \frac{ds'}{s'^n} \frac{\hat{M}(s')}{\Omega(s')(s'-s)} = \begin{cases} I=0: & \int \frac{ds'}{s'^3} \frac{s'}{s'^{-1}(s'-s)} = \int \frac{ds'}{s'^2} \\ I=1: & \int \frac{ds'}{s'^2} \frac{(s')^0}{s'^{-1}(s'-s)} = \int \frac{ds'}{s'^2} \\ I=2: & \int \frac{ds'}{s'^2} \frac{s'}{s'-s} = \int \frac{ds'}{s'^2} \end{cases}$$

$\leadsto$ convergence of dispersion integral works !

- strategy: calculate $M_I(s)$ numerically by a self-consistent procedure :



From unitarity to integral equations: solution

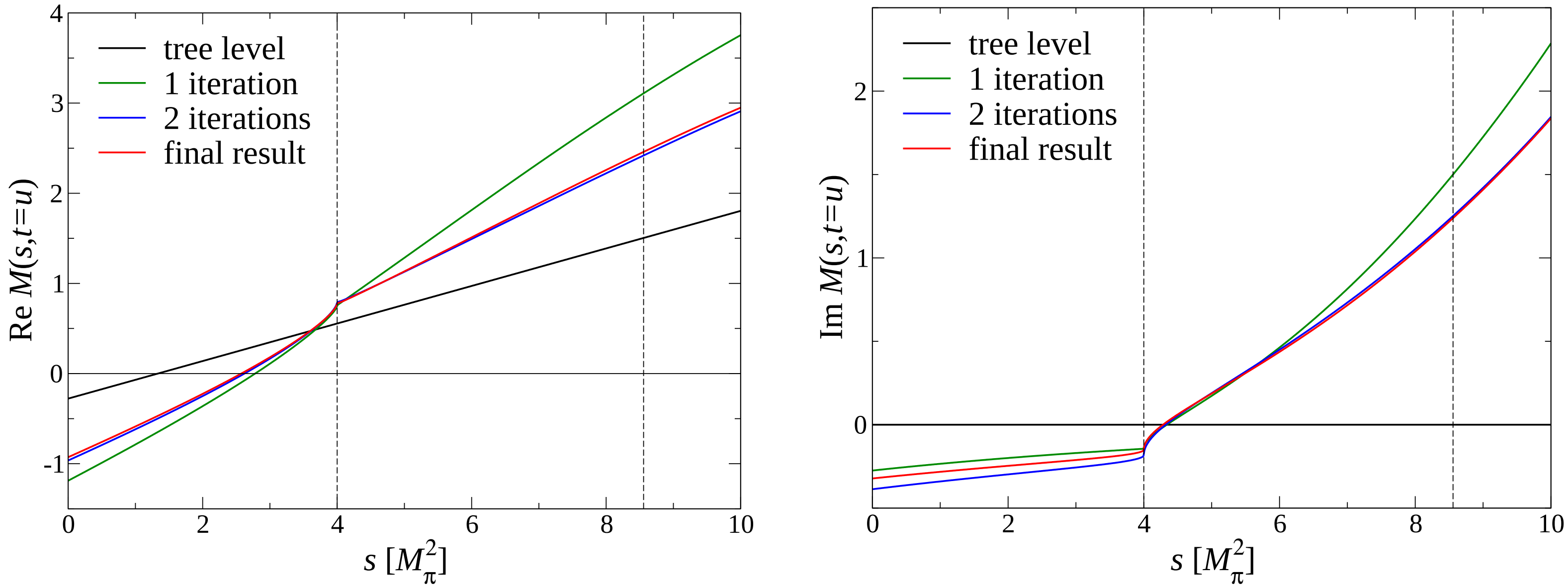
- integral equations including the inhomogeneities $\hat{\mathcal{M}}_I$:

$$\mathcal{M}_0(s) = \Omega_0(s)\left\{\alpha_0+\beta_0\, s+\gamma_0\, s^2+\frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0(s')\hat{\mathcal{M}}_0(s')}{|\Omega_0(s')|(s'-s-i\epsilon)}\right\}$$

  + 2 similar for $\mathcal{M}_{1,2}(s)$; 4 subtraction constants to be fixed

  Khuri, Treiman 1960; Aitchison 1977; Anisovich, Leutwyler 1998

- solve these equations iteratively by a numerical procedure



Schneider, Kubis; compare Colangelo et al. 2010

- fast convergence: close to final result after 2 iterations

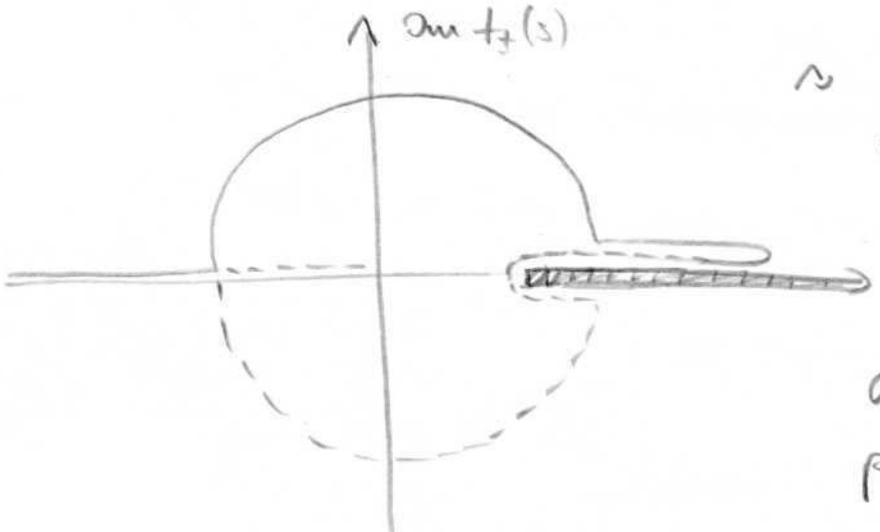
ηDR 8 • detour (ha!): angular integration in $\hat{M}$

$$\langle z^n M\rangle = \frac{1}{2}\int_{-1}^{1} dz\; z^n M(t(s,z)) \quad \Big|\; t = \frac{3s_0 - s + \varkappa z}{2}$$

$$= \frac{1}{\varkappa(s)}\int_{t_-(s)}^{t_+(s)} dt \left(\frac{2t - 3s_0 + s}{\varkappa(s)}\right)^n M(t)$$

$$\varkappa(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}\left[\left((M_\eta + M_\pi)^2 - s\right)\left((M_\eta - M_\pi)^2 - s\right)\right]^{1/2}$$

$$t_\pm(s) = \frac{3s_0 - s \pm \varkappa(s)}{2}$$ complex for $s \in \left((M_\eta - M_\pi)^2, (M_\eta + M_\pi)^2\right)$

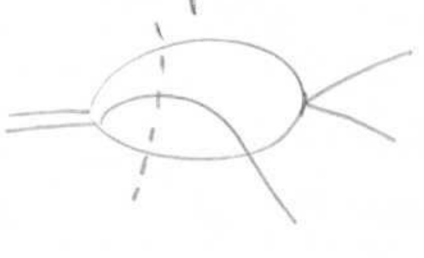


$\rightsquigarrow$ crossed-(t-) channel cut intrudes the angular integration path

• consequence: $\hat{M}$ isn't always real, but complex

$\rightsquigarrow$ M does not have the phase of Ω any more

M is complex even below threshold $s = 4M_\pi^2$:



$\rightsquigarrow$ 3-particle cuts!