

Dalitz Plot Analysis of Heavy Quark Mesons Decays (1).

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Acknowledgements

□ In these slides I have taken material and ideas from several people. This is a list, in sparse order. Apologies if I forget somebody.

Marco Pappagallo, Tim Gershon, Thomas Latham, Brian Meadows, Bill Dunwoodie, Olivier Schneider, Stefan Spanier, David Asner, ...

□ I tried to include links to many references.

Introduction

- The aim of these lectures is to give information on how to perform in practice a Dalitz analysis which worth while to be published.
- Dalitz analyses are not simple. There are many unknown aspects. Some of them will remain unresolved.
- Even the fitting model is a working approximation. Rather often it is impossible to obtain a statistically good description of the data.
- One should be very careful on how to handle the introduction of new resonances.
- Sometime they can appear as adding extra terms to a polynomial fitting.
- It is possible that something new appears in the data. However before claiming for a new effect, one should try to explain it in terms of what is already known.

A Dalitz analysis program.

- If you like I can provide a Dalitz analysis program. The idea is that “doing things in practice” helps much more than many words.
- Unfortunately this program is written in FORTRAN, but it should not be difficult to use.
- It is set-up for $D^0 \rightarrow K_S^0 \pi^+ \pi^-$ but it can be modified easily.
- It performs Monte Carlo simulations and fits Monte Carlo data, but it can be easily modified to run on data.
- It only needs MINUIT data cards for the free parameters.
- At the end, you can look at the output either using PAW++ or ROOT by converting:
h2root hbookfile rootfile
- It also needs the CERN library, I can supply it if needed.
- The program is stored at:
<http://www.ba.infn.it/~palano/dalitz.tar.gz>
- Notice that files such as *mc_dummy.car*, *mc_data.car* are dummy files and will be replaced during the execution.
- To run the program:

g77 -O2 -fno-automatic -g daly_k0spipi.f -L ~/cern/2005/lib -lgrafX11 -lmathlib -lpawlib -lkernlib -lgraffib -lgeant321 -lphtools -lpacklib_noshift -v

./a.out

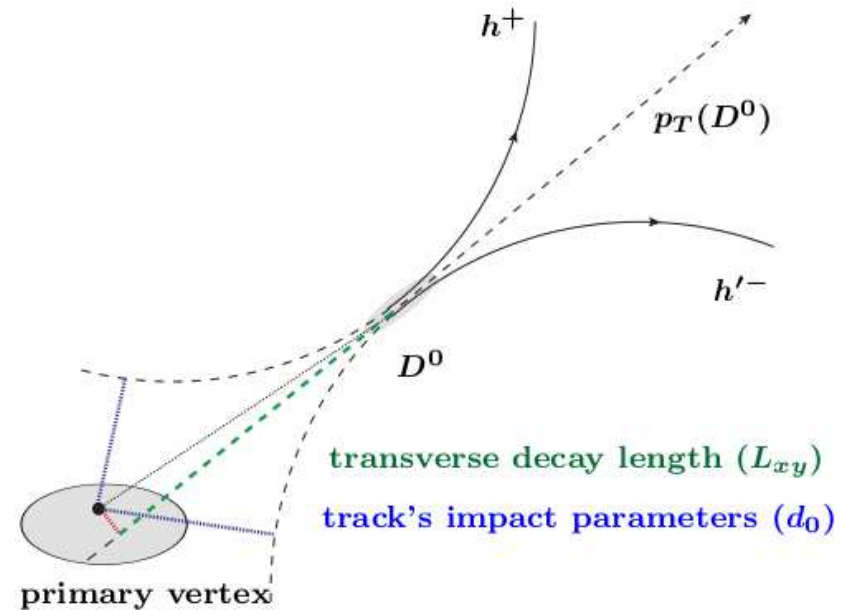
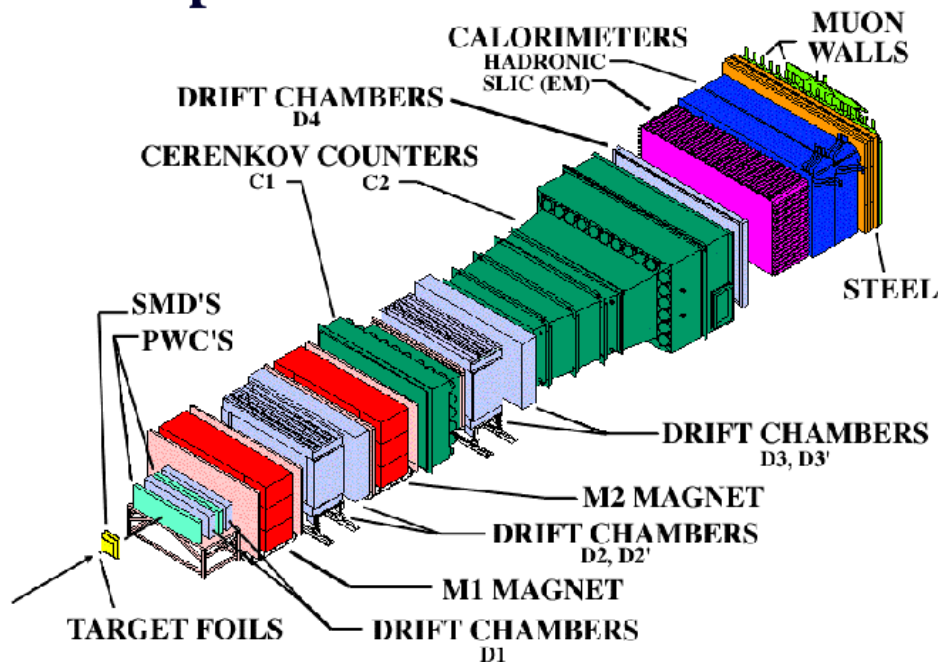
Main Recent Experiments on Heavy Flavours

- The Data which will be presented are coming from several experiments.
- A short summary is given.

Main Experiments

- Fermilab Experiment E791: Study of production and decay of charmed particles produced using a 500 GeV π^- beam on Pt and C targets at the Fermilab Tagged Particle Spectrometer. Data-Taking from June 1988 to January 1992.

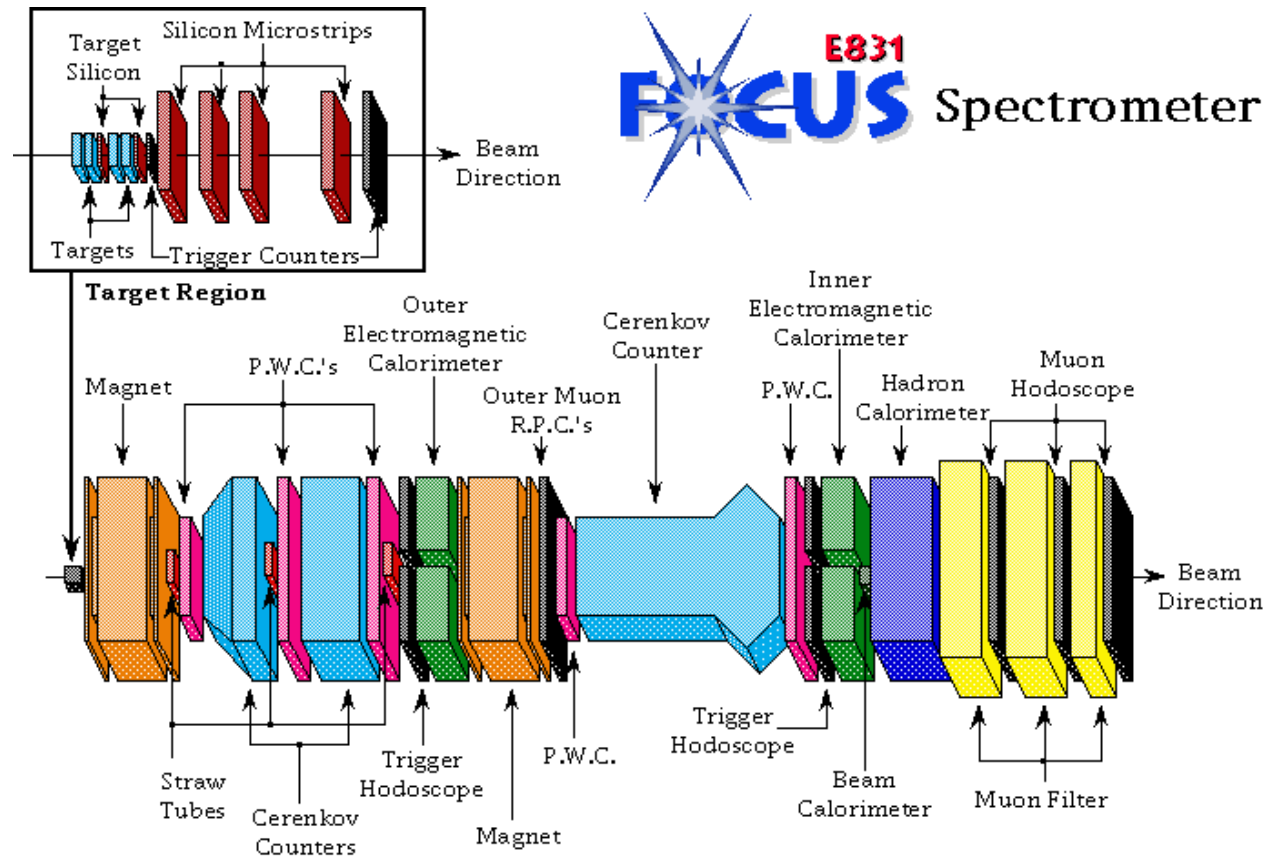
E-791 Spectrometer



Good particle identification and displaced vertex for studying charm decays to charged tracks.

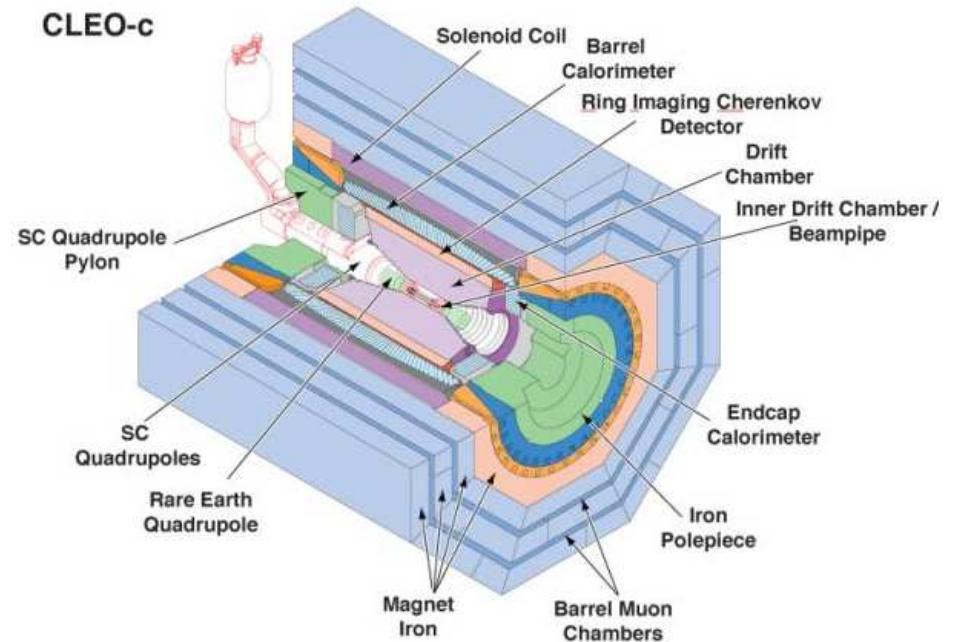
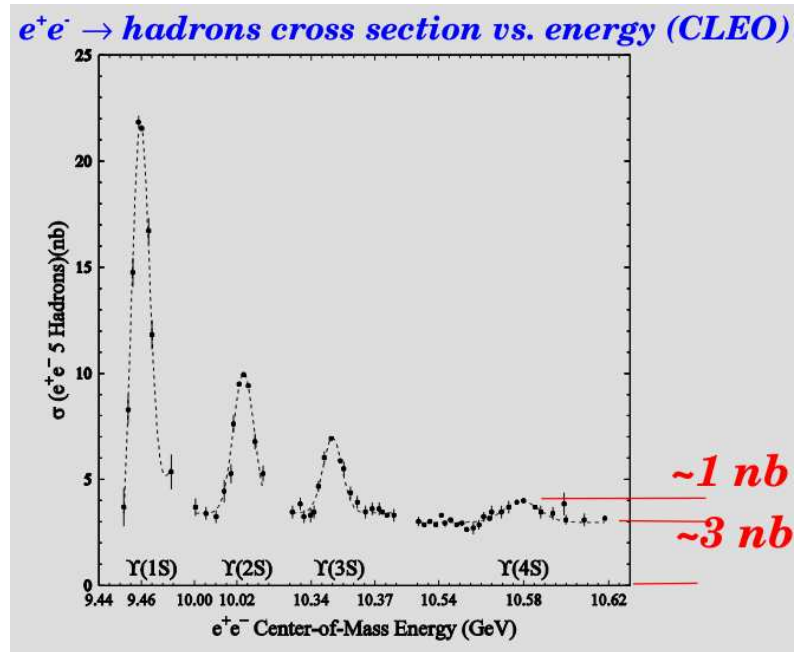
(<http://ppd.fnal.gov/experiments/e791/welcome.html>)

- FOCUS (E831), is a heavy-flavor photoproduction experiment located in the Wide Band Area of Fermilab. It was an upgraded version of its predecessor, E687. The experiment accumulated data during the 1996-1997 fixed target run and has fully reconstructed more than one million charm particles.



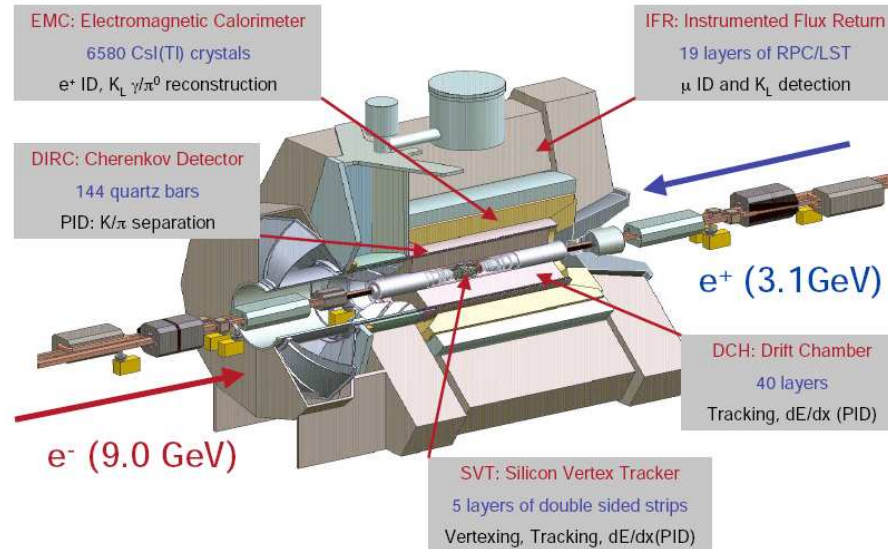
(<http://www-focus.fnal.gov/>)

- CLEO. e^+e^- annihilations. After about twenty successful years of B physics, CLEO and CESR (Cornell) have entered a new phase by moving into the charm region. Data samples at the $\Upsilon(5S)$, $\Upsilon(1-3S)$, 4170 GeV, 3770 GeV, and at the $\psi(2S)$, complemented by nearby continuum samples, have allowed us to perform many sensitive studies of weak and strong physics.

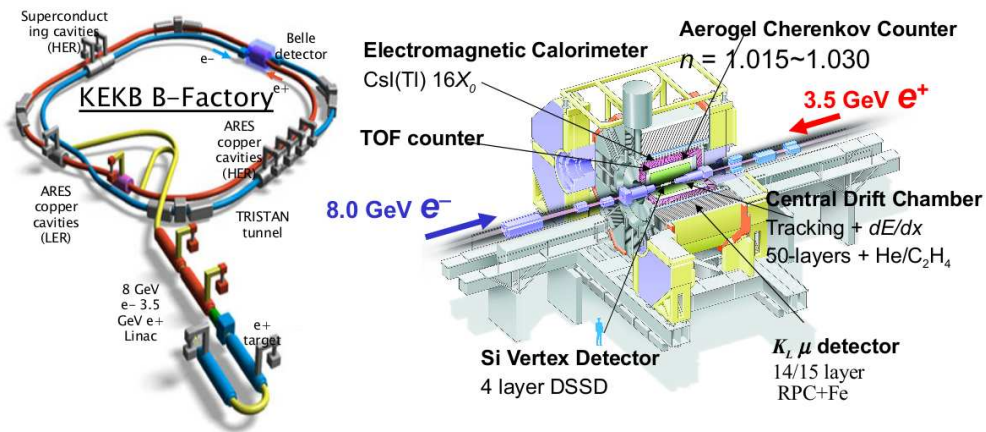


(<http://www.lepp.cornell.edu/Research/EPP/CLEO/>)

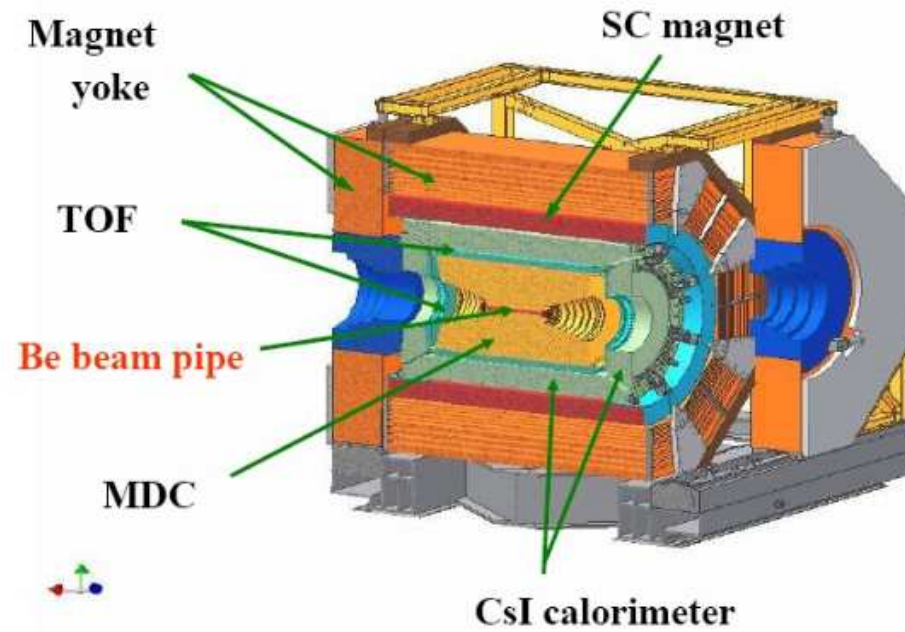
- e^+e^- annihilations. BaBar(SLAC) and Belle(KEK) physics results span a broad range of topics, including B, charm, and τ physics; CP violation; precision CKM measurements; charmonium and bottomonium states; hadron production; and searches for physics beyond the standard model.



Data-taking from 1999 to 2008 (BaBar <http://www.slac.stanford.edu/BFROOT/>), 1999-2010 (Belle (<http://belle.kek.jp/>)).



- Beijing Spectrometer (BES) is a general-purpose detector located in the interaction region at the BEPC storage ring, where the electron and positron beams collide. Study of e^+e^- interactions at the charmonium threshold. (<http://bes.ihep.ac.cn/>)



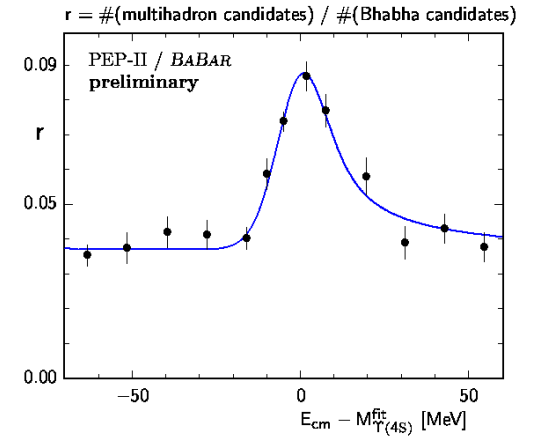
- e^+e^- experiments have the advantage of exploring a broader final states program which includes the detection of K_L^0 , γ 's, π^0 , and η .
- Hadronic experiments have the advantage of higher statistics but sometime tight trigger conditions and therefore more difficult efficiency evaluation.

Dalitz Analyses on Heavy Flavours decays

- Most of the Dalitz Analyses have been performed on charmed and B mesons decays.
- New analyses are recently coming from charmonium decays.
- How to extract clean data samples useful for Dalitz analyses?

B-factories: charm physics in the continuum.

□ At $e^+e^- \rightarrow \Upsilon(4S)$ B-factories, the $\Upsilon(4S)$ resonance sits on a consistent continuum background.



□ Cross sections for different processes, at the $\Upsilon(4S)$:

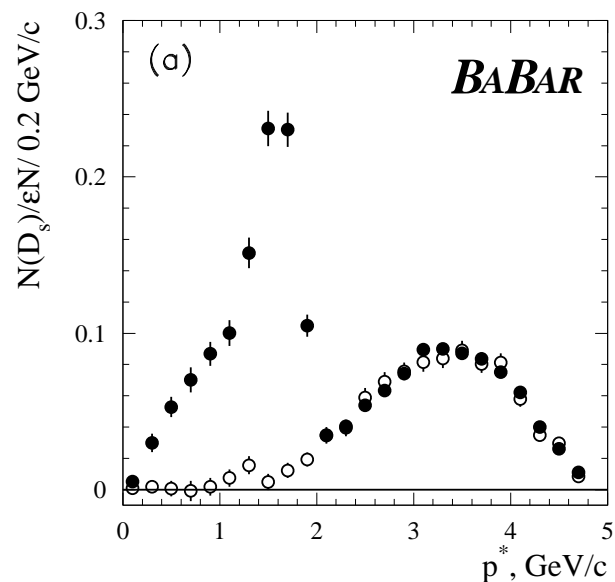
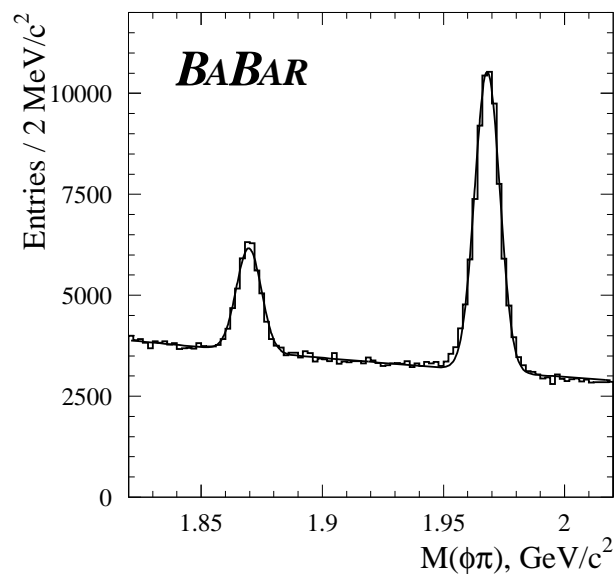
□ Inclusive Charm Physics is performed on events selected from continuum $e^+e^- \rightarrow \bar{c}c$

$e^+e^- \rightarrow$	σ (nb)
$b\bar{b}$	1.05
$c\bar{c}$	1.30
$s\bar{s}$	0.35
$u\bar{u}$	1.39
$d\bar{d}$	0.35
$\tau^+\tau^-$	0.94
$\mu^+\mu^-$	1.16
e^+e^-	≈ 40

Study of D_s^+ in BaBar.

- Example from BaBar: mass distribution and p^* momentum spectrum for $D_s^+ \rightarrow \phi\pi^+$.

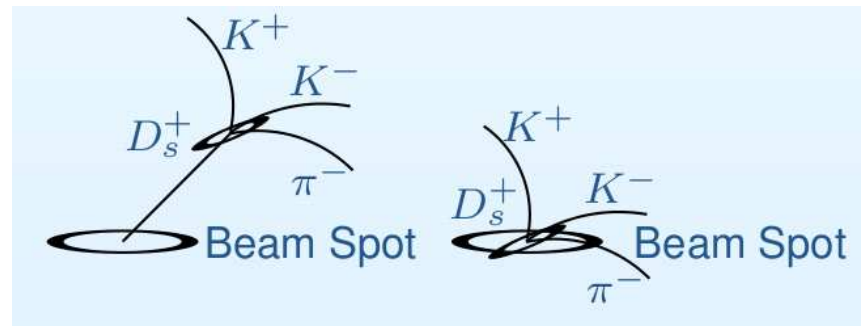
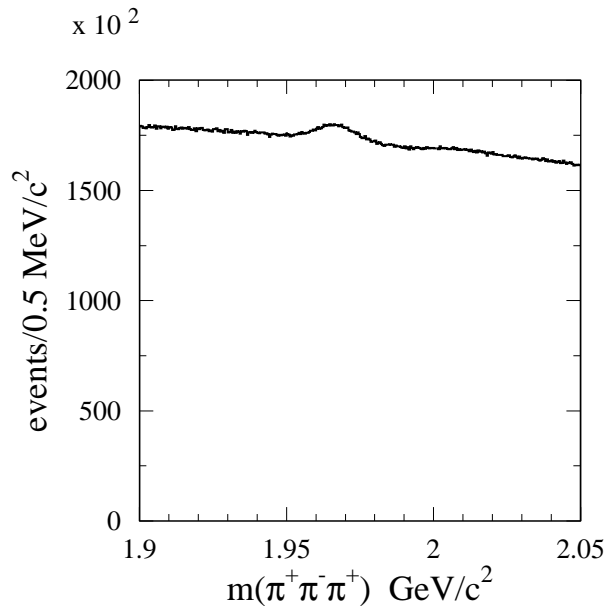
Filled/open points: normalized on/off peak data.



- By using inclusive continuum events combinatorial background is strongly reduced.
- Kinematical selection: the center of mass momentum (p^*) > 2.5 GeV/c.
- To perform Dalitz analyses a high signal to background ratio is needed.

Example of Charmed mesons reconstruction.

- BaBar has performed a Dalitz plot analysis of $D_s^+ \rightarrow \pi^+ \pi^- \pi^+$ (arXiv:0808.0971) and $D_s^+ \rightarrow K^+ K^- \pi^+$ (arXiv:1011.4190).
- Initial $\pi^+ \pi^- \pi^+$ mass spectrum.



- The three tracks are fitted to a common vertex, and the χ^2 fit probability (labeled P_1) must be greater than 0.1 %.
- To help discriminate signal from background, an additional fit which uses the constraint that the three tracks originate from the e^+e^- luminous region (beam spot) is performed. The χ^2 probability of this fit is labeled as P_2 , and it is expected to be large for background and small for D_s^+ signal events, since in general the latter will have a measurable flight distance.

Charmed mesons reconstruction: tagging

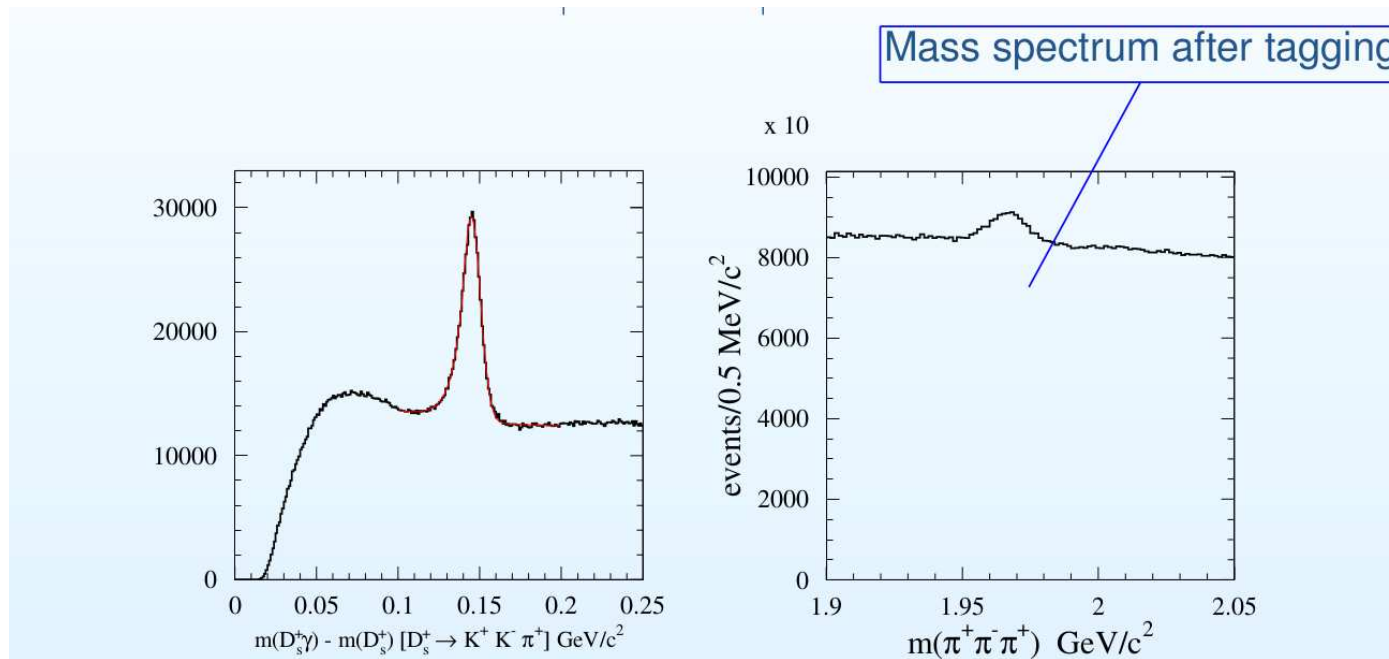
- The combinatorial background is reduced by requiring the D_s^+ to originate from the decay

$$D_s^*(2112)^+ \rightarrow D_s^+ \gamma$$

using the mass difference

$$\Delta m(D_s^+ \gamma) = m(\pi^+ \pi^- \pi^+ \gamma) - m(\pi^+ \pi^- \pi^+)$$

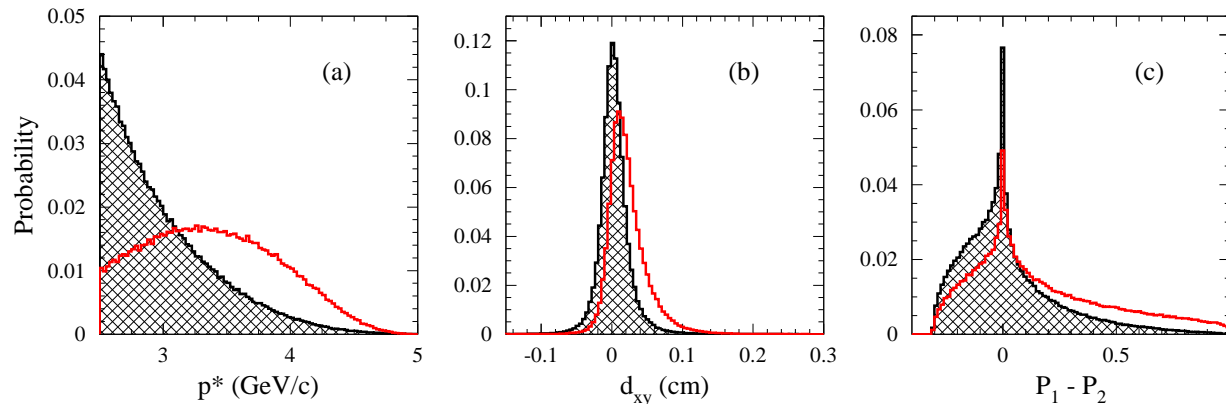
- $\Delta m(D_s^+ \gamma)$ for the $D_s^+ \rightarrow K^+ K^- \pi^+$ decay.



(The use of the Δm instead of the invariant mass improves resolution, particularly in the threshold region.)

Charmed mesons reconstruction.

- Each D_s^+ candidate is characterized by three variables: the center of mass momentum p^* , the difference in probability $P_1 - P_2$, and the signed decay distance d_{xy} between the D_s^+ decay vertex and the beam spot projected in the plane normal to the beam collision axis.
- The distributions for these variables for background are inferred from the $D_s^+ \rightarrow \pi^+ \pi^- \pi^+$ invariant mass sidebands.
- Since these variables are (to a good approximation) independent of the decay mode, the distributions for the three-pion invariant mass signal, are inferred from the $D_s^+ \rightarrow K^+ K^- \pi^+$ decay.
- The distributions of these variables for the $D_s^+ \rightarrow K^+ K^- \pi^+$ decay for signal and background.



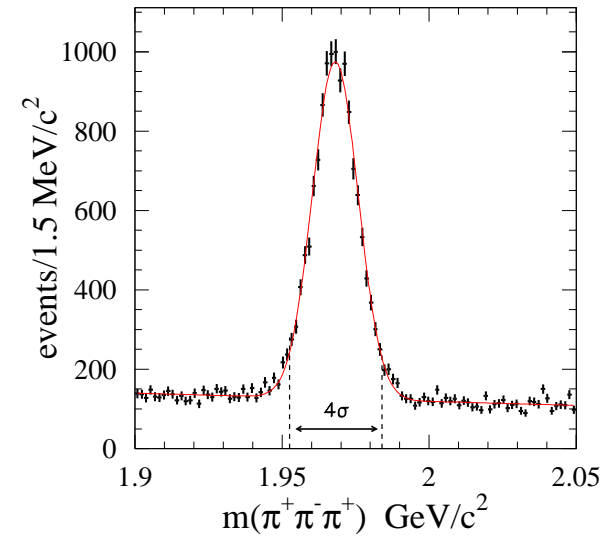
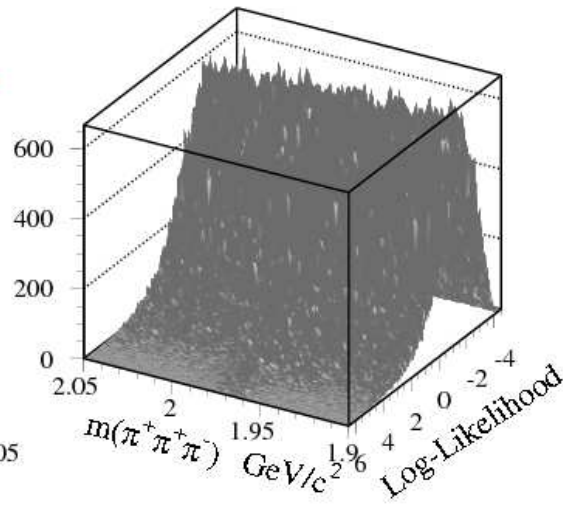
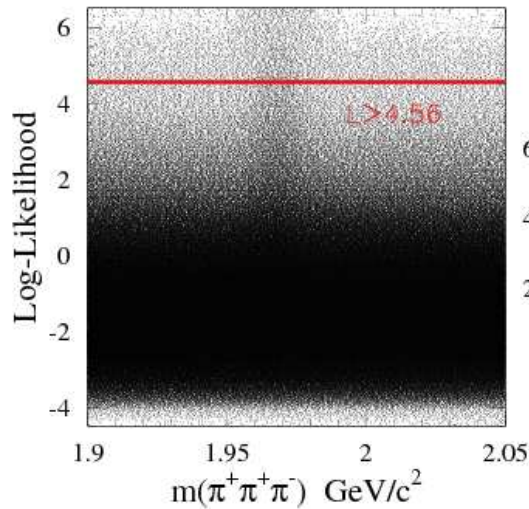
- These normalized distributions are then combined in a likelihood ratio test.

$$\mathcal{L} = \prod_i \frac{(PDF_{signal})_i}{(PDF_{background})_i}$$

where PDF_{signal} and $PDF_{background}$ are the Probability Density Function of the $i - th$ variable for the signal and the sideband regions respectively.

Charmed mesons reconstruction.

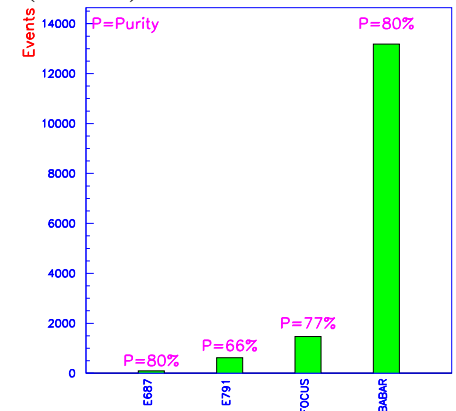
- The cut on the likelihood ratio has been chosen in order to obtain the largest statistics with background small enough to perform a Dalitz plot analysis.
- The scatter plot of the likelihood vs. mass for the selection of the $D_s^+ \rightarrow \pi^+ \pi^- \pi^+$ decay.



- Signal Purity: 80 %.

$$Purity = \frac{Signal}{(Signal + Background)}$$

- $D_s^+ \rightarrow \pi^+ \pi^- \pi^+$ yields and Purities from different experiments.



D^0 reconstruction.

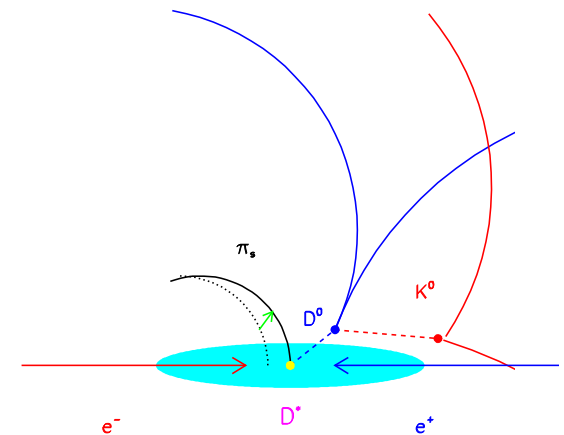
- The D^* tagging works much better for D^0 .
- We want to study the following final state: $D^0 \rightarrow K_s^0 h^+ h^- + c.c.$
- Tagged with D^* . For example:

$$\begin{aligned} D^{*+} &\rightarrow D^0 \pi^+ \\ &\rightarrow \bar{K}^0 \pi^+ \pi^- \end{aligned}$$

$$\begin{aligned} D^{*-} &\rightarrow \bar{D}^0 \pi^- \\ &\rightarrow K^0 \pi^+ \pi^- \end{aligned}$$

- The charge of the slow pion gives the flavor of the K^0 and of the D^0 .
- Refitting of the slow pion. The momentum of the slow pion is usually below 500 MeV/c: badly measured.
- Cartoon of one event:

- Using the event vertex in the fit of the slow pion momentum improves the resolution.



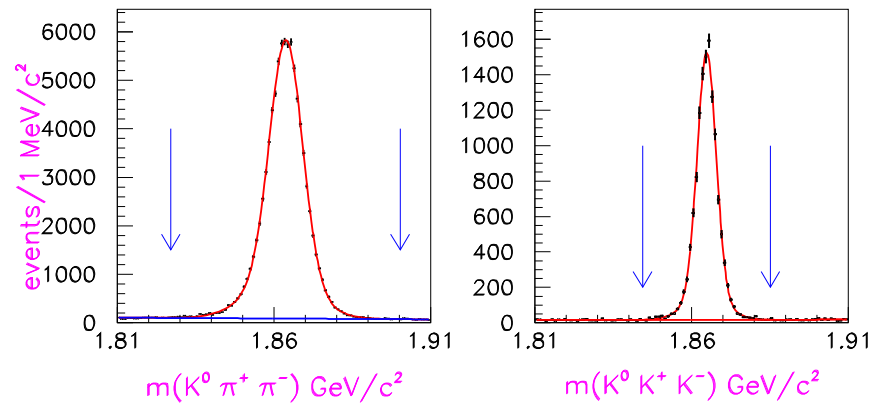
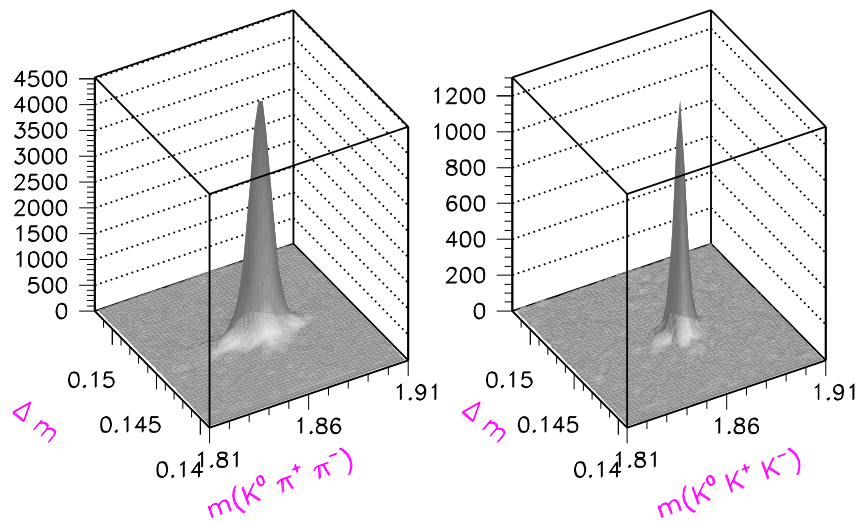
Δm selection:

□ Example from BaBar. (arXiv:hep-ex/0507026)

□ Plot of Δm:

$$\Delta m = m(K_s^0 h^+ h^- \pi^+) - m(K_s^0 h^+ h^-)$$

for: $\bar{K}^0 \pi^+ \pi^-$ and $\bar{K}^0 K^+ K^-$



□ Selecting events within 2σ in Δm ($\sigma = 300 \text{ KeV}/c^2$) we obtain clean tagged $D^0 \rightarrow K_s^0 h^+ h^-$ decays.

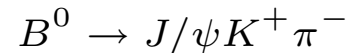
B mesons reconstruction from $\Upsilon(4S)$ decays.

- The selection of B mesons in $\Upsilon(4S) \rightarrow B\bar{B}$ decays makes use of two variables: m_{ES} and ΔE .

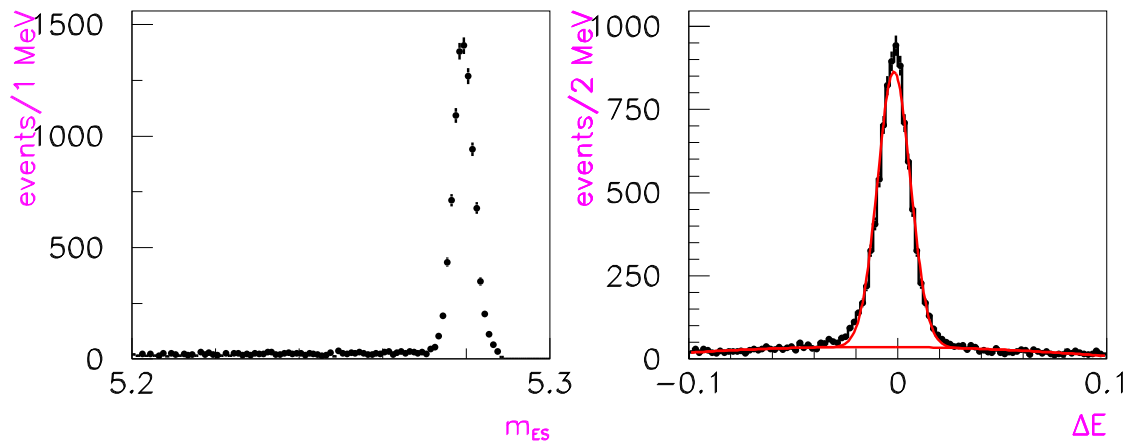
$$\Delta E \equiv E_B^* - \sqrt{s}/2$$

$$m_{ES} \equiv \sqrt{((s/2 + \vec{p}_i \cdot \vec{p}_B)/E_i)^2 - \vec{p}_B^2}$$

- (E_i, \vec{p}_i) is the initial state e^+e^- four-momentum vector in the lab. and \sqrt{s} is the c.m. energy.
- E_B^* is the B meson energy in the c.m., \vec{p}_B is its lab. momentum.
- Example from: (arXiv:0811.0564)



- m_{ES} after $\pm 2\sigma$ selection on ΔE .
- ΔE after $\pm 2\sigma$ selection on m_{ES} .



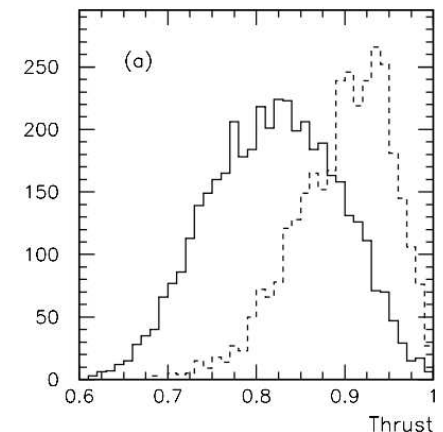
Selection of three-body Charmless B decays

- Charmless B decays have small branching fractions and, in e^+e^- machines are highly affected by $q\bar{q}$ continuum background.
- In many cases the Dalitz analysis is difficult because of the presence of a severe background component.
- Need to make use of multivariate analyses which make use of combinations of shape variables which are able to discriminate signal from continuum events.
- Backgrounds arise primarily from random combinations in continuum events.
- To enhance discrimination between signal and continuum, one method is given by a neural network (NN) to combine several discriminating variables:

- the angle with respect to the beam axis of the B momentum;
- B thrust axis in the $\Upsilon(4S)$ frame. The thrust axis of an event, \hat{T} , is defined to be the direction which maximizes the sum of the longitudinal momenta of the particles. Thrust, T , is related to this direction by

$$T = \frac{\sum_i \hat{T} \cdot \mathbf{p}_i}{\sum_i |\mathbf{p}_i|}$$

The allowed range of T is $(0.5, 1)$, where $T \approx 1$ corresponds to a highly directional event, and $T = 0.5$ corresponds to an isotropic event.

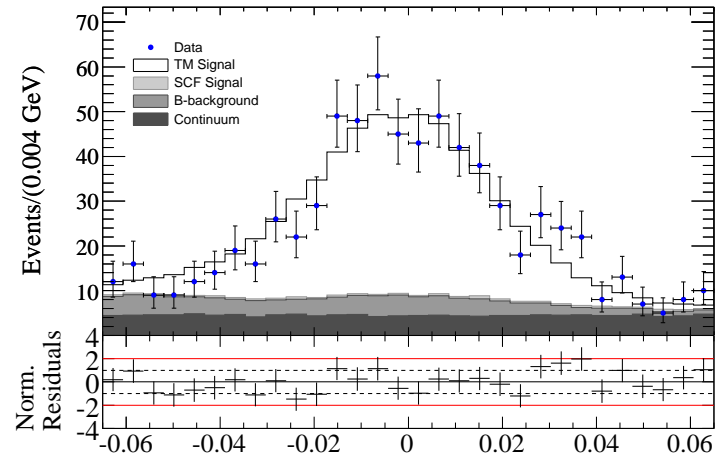
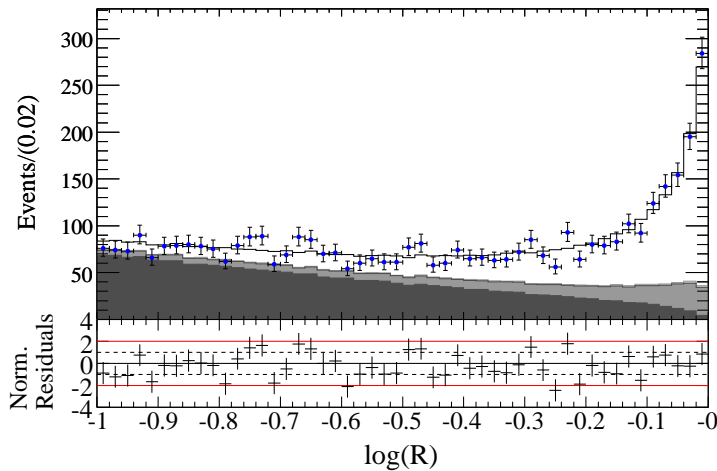


- the zeroth and second order monomials $L_{0,2}$ of the energy flow about the B thrust axis. The monomials are defined by $L_n = \sum_i p_i \times |\cos \theta_i|^n$, where θ_i is the angle with respect to the B thrust axis of track or neutral cluster i and p_i is the magnitude of its momentum. The sum excludes the B candidate and all quantities are calculated in the $\Upsilon(4S)$ frame.

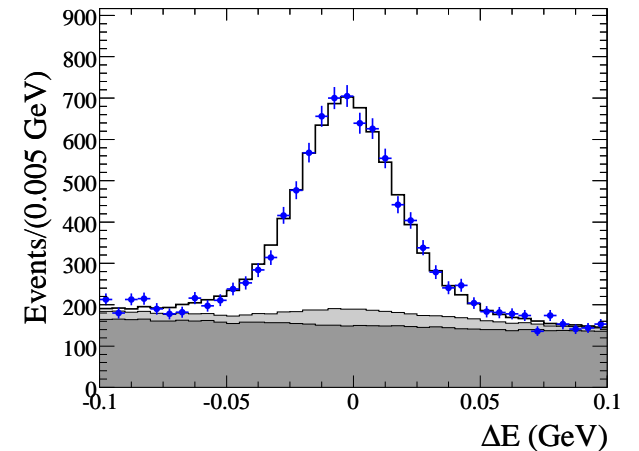
□ The NN is trained using off-resonance data as well as simulated signal events, all of which passed the selection criteria.

Selection of Charmless B decays

- Example from $B^0 \rightarrow K_S^0 \pi^+ \pi^-$ (arXiv:0905.3615).
- The final sample of signal candidates is selected with a requirement on the NN output that retains 90% of the signal and rejects 71% of the background.
- Resulting ΔE signal.

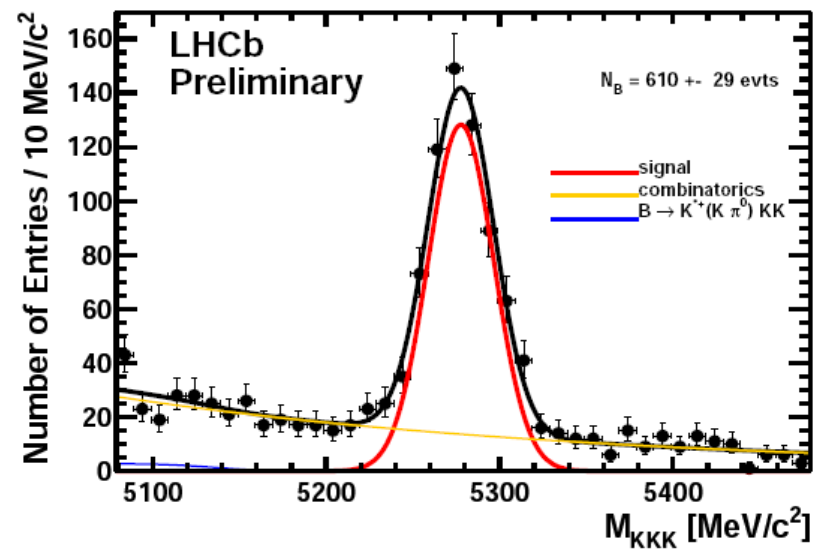
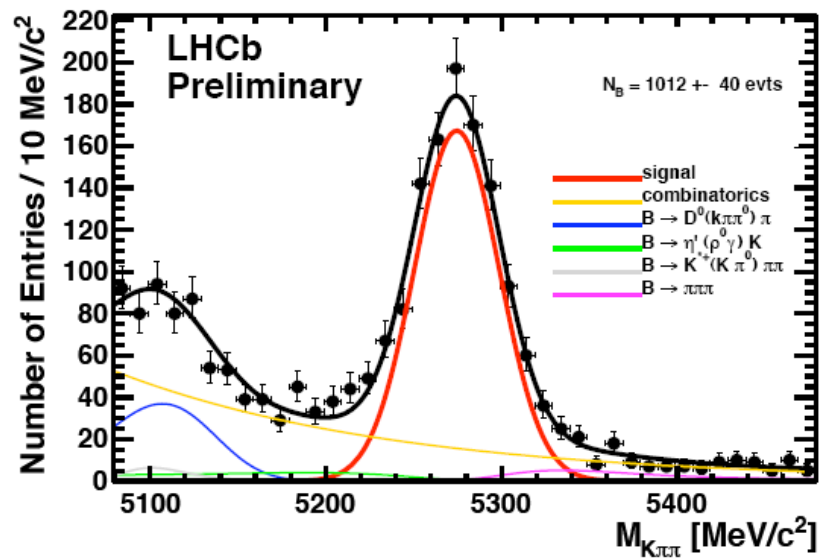


- Example from $B^+ \rightarrow K^+ K^- K^+$ (BaBar).



Selection of Charmless B decays

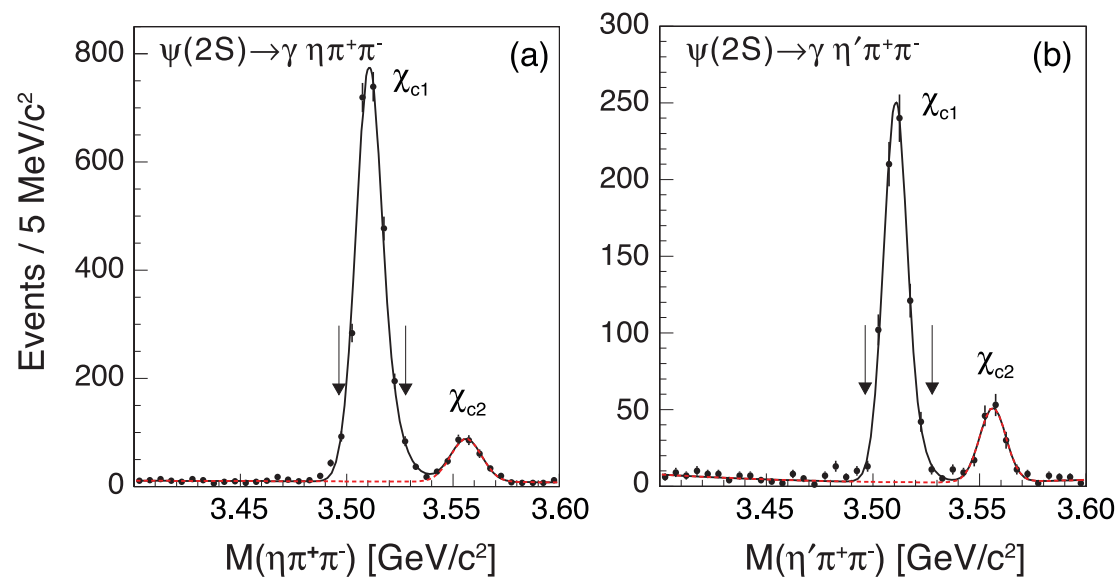
□ In the case of high energy hadronic colliders the power of the decay flight allows to obtain high statistics clean samples. Small data samples from LHCb



Dalitz Analysis of χ_{c1} decays.

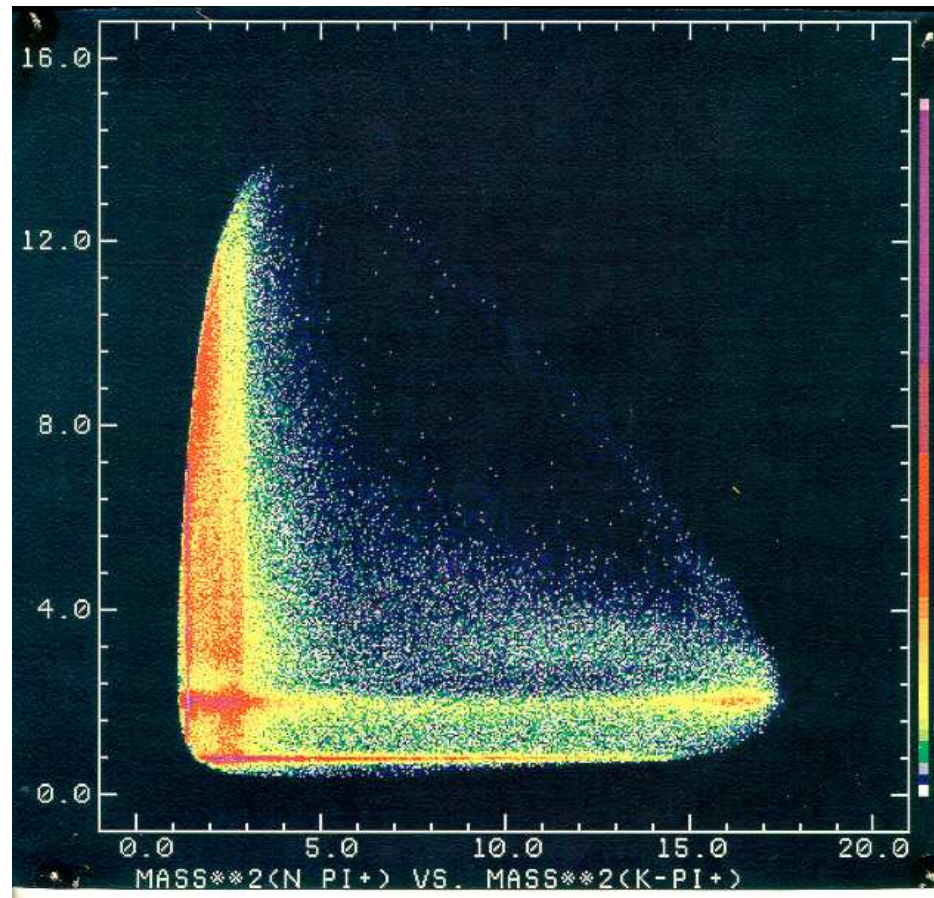
□ CLEO has performed a Dalitz analysis of $\chi_{c1} \rightarrow \eta\pi^+\pi^-$ and $\chi_{c1} \rightarrow \eta'\pi^+\pi^-$ (arXiv:1109.5843), obtained from the decays:

$$\psi(2S) \rightarrow \gamma\chi_{c1}$$



Dalitz analysis

- Dalitz plot analysis has been extensively used in many hadron induced interactions.
- Example from $K^- p \rightarrow K_S^0 \pi^- p$ at 11 GeV/c. (LASS experiment at SLAC, from B. Dunwoodie).

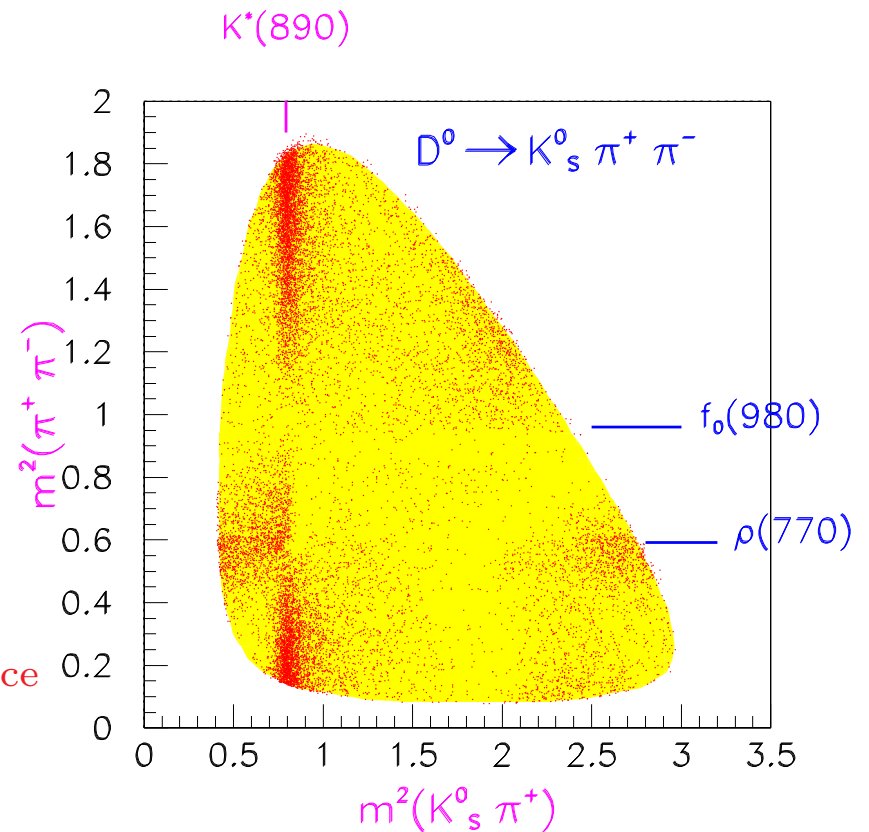


- Presence of complex structures.

Other examples(2).

□ Example from charm decays: $D^0 \rightarrow K_S^0 \pi^+ \pi^-$ (BaBar), very complex structure.

- Scalar resonances appear as uniform bands.
- Spin-1 resonances appear with a two-lobes structure,
- Spin-2 resonances with three-lobes structures.
- Resonances may appear along three different directions.
- Pattern complicated by the presence of interference effects.
- Resonances may also appear as depletions.
- Understanding this Dalitz plot has taken several years of work of many people.

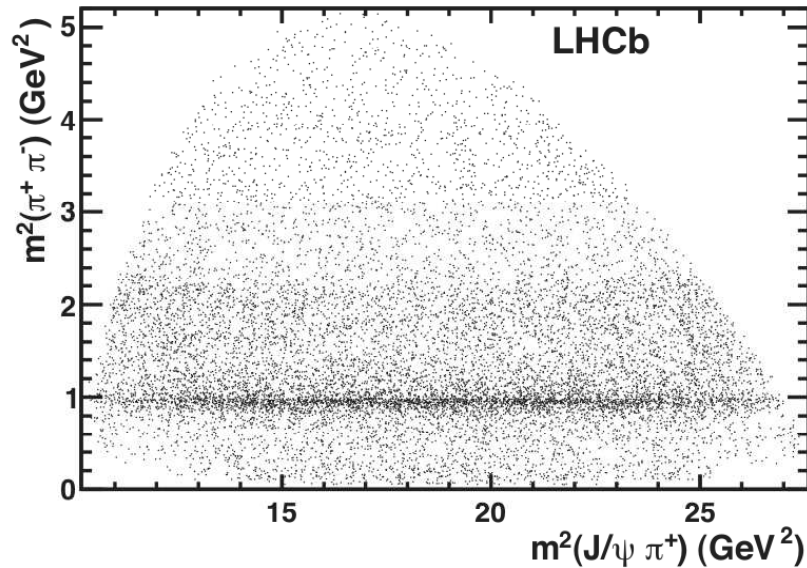


Other examples(3).

- Example from B_s decay: $B_s \rightarrow J/\psi \pi^+ \pi^-$ (LHCb, arXiv:1204.5643). In this case we have a very simple physics.
- We observe only scalar resonances appearing as uniform bands centered at the $f_0(980)$ and $f_0(1400)$ mass values.

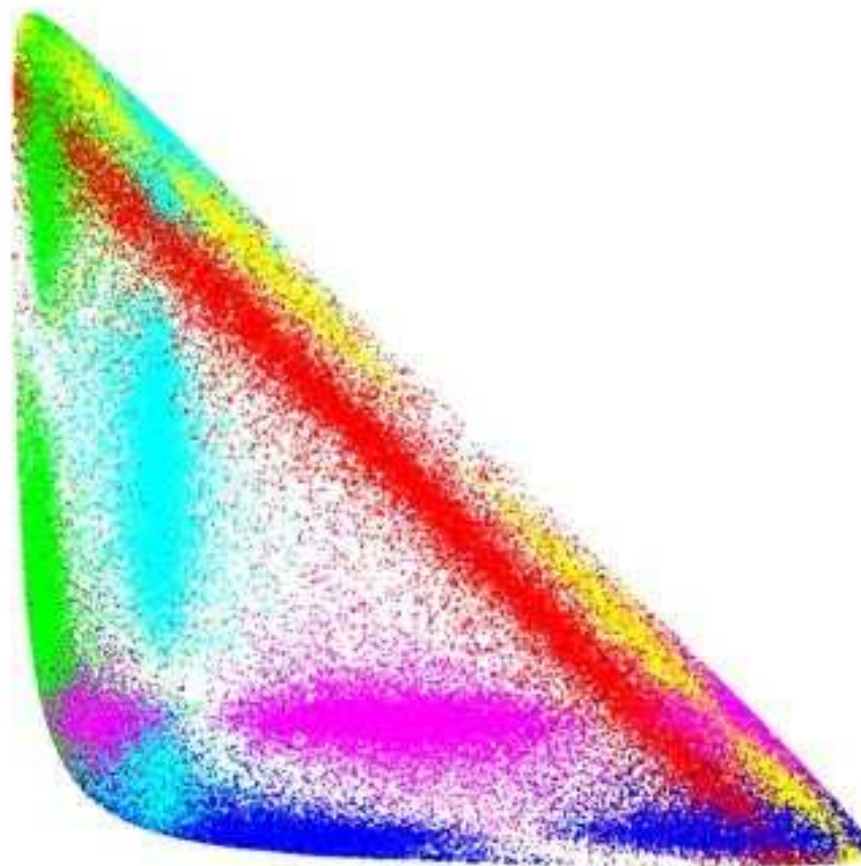
$f_0(1400) \rightarrow$

$f_0(980) \rightarrow$



Other examples(4).

□ An artistic view of a Dalitz Plot from Tom Latham



Dalitz plot Analysis

- Dalitz plot analysis is a powerful technique that involves studying the amplitude of a decay into a three-body final state.
- Compared to two-body decays, the three-body decay has intrinsic degrees of freedom that allow to obtain the amplitudes and relative phases of interfering amplitudes.
- Types of measurement that can take advantage of using the Dalitz plot analysis technique include:
 - Searches for new states.
 - Measurements of their properties masses, widths, quantum numbers.
 - CP violation searches and measurements of related parameters.
 - Partial Wave Analysis.

How to perform a Dalitz analysis

- Obtain a clean data sample. High signal to background ratio is an essential ingredient. Residual background could be wrongly assigned to Non-Resonant background or S-wave contributions. Remove peaking backgrounds as much as possible by studying all the possible final states which could contribute. Remove only those which do not affect the efficiency distribution on the Dalitz plot.
- Obtain an accurate efficiency distribution on the Dalitz plot using high statistics Monte Carlo samples. Uniform efficiencies help but it is not a fundamental ingredient. Important not to have strong efficiency depletions.
- Define the amplitudes in the fit. Several amplitudes are uncertain. Sometime it is necessary to include different parameterizations. Sometime the shape of amplitudes can be obtained directly from the data.

Steps in a Dalitz analysis

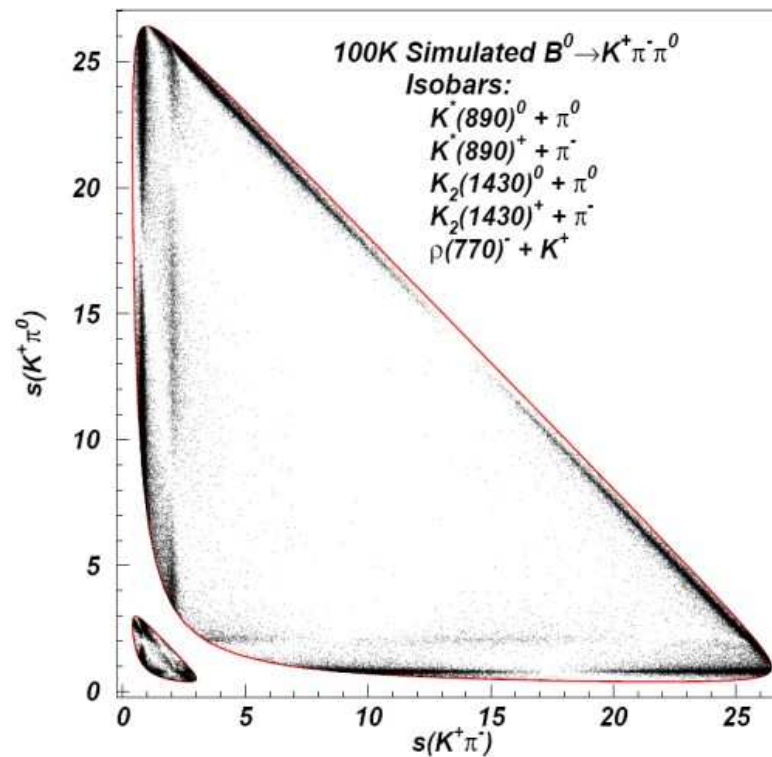
- Read the data, 4-vectors to be converted in the resonance rest-frame.
- Integrate the amplitudes on efficiency biased Monte Carlo data.
- Fit the data by unbinned maximum likelihood including the background model.
- Obtain fractions and relative errors from the fit covariance matrix.
- Project the fit on the data using several methods: Dalitz projections, Legendre polynomials, 2-D χ^2 , likelihood distributions, etc.
- Rather often, several solutions are obtained. Report the best solutions.

Dalitz plot Analysis

- Tree-body decay has internal degrees of freedom. After requiring the energy and momentum conservation to the system of three final state particles, we are left with 5 degrees of freedom.
- In case the initial and final state particles all have spin zero, after taking into account arbitrary rotations, two degrees of freedom remain.
- The amplitude of the decay can thus be represented as a function of two parameters; the scatter plot of this pair of parameters is called Dalitz plot.
- There is a freedom in the choice of a pair of parameters describing the amplitude of a three-body decay.
- Originally, the Dalitz plot was expressed as a function of kinetic energies of two final state particles.
- Recent analyses often use squares of invariant masses of pairs of final state particles. This choice has certain advantages: it allows for easy determination of masses of intermediate states, and the structure of the amplitude becomes apparent.
- Key information: **The phase space term is flat in these variables.**

Dalitz plot Analysis

- Comparison between simulated Dalitz plots from B and D decays.
- Huge difference in phase space. Some more problems in understanding the phase space term in B decays.
- We know with some precision the light resonances spectrum up to about 2.5 GeV. What about higher masses?
- How to model the high mass regions of the B dalitz plot?



Dalitz plot Analysis

□ Assume a particle with the mass M decays into three particles denoted as a , b and c . The differential decay probability is:

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{32M^3} |\overline{\mathcal{A}}|^2 dm_{ab}^2 dm_{bc}^2$$

□ Thus, any non-uniformity observed in the distribution of the variables m_{ab}^2 and m_{bc}^2 is due to dynamic structure of the amplitude \mathcal{A} .

□ The invariant masses of pairs of final state particles are related by the linear dependence:

$$m_{ab}^2 + m_{bc}^2 + m_{ac}^2 = M^2 + m_a^2 + m_b^2 + m_c^2$$

□ The range of invariant masses m_{bc}^2 depends on m_{ab}^2 :

$$\begin{aligned} (m_{bc}^2)_{\max} &= (E_b^* + E_c^*)^2 - (p_b^* - p_c^*)^2, \\ (m_{bc}^2)_{\min} &= (E_b^* + E_c^*)^2 - (p_b^* + p_c^*)^2, \end{aligned} \tag{1}$$

where

$$E_b^* = \frac{m_{ab}^2 - m_a^2 + m_b^2}{2m_{ab}}, \quad E_c^* = \frac{M^2 - m_{ab}^2 - m_c^2}{2m_{ab}} \tag{2}$$

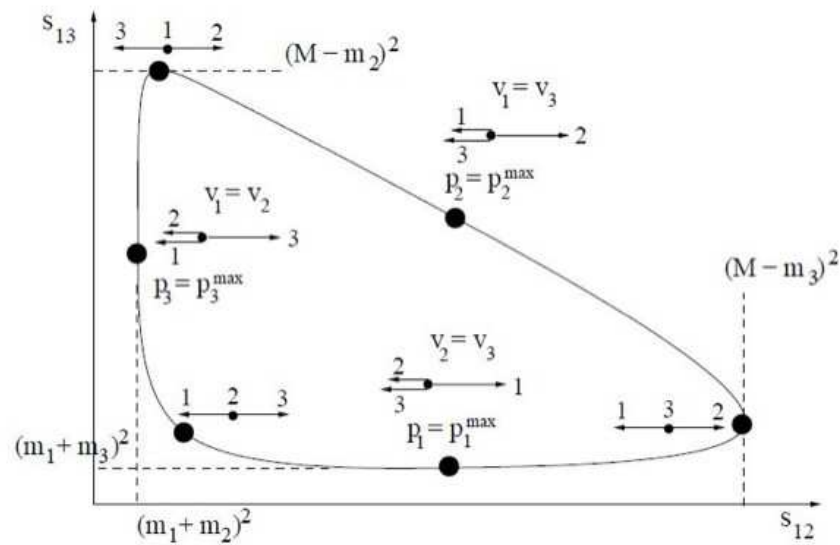
are the energies of particles b and c in the ab rest frame and

$$p_b^* = \sqrt{E_b^{*2} - m_b^2}, \quad p_c^* = \sqrt{E_c^{*2} - m_c^2} \tag{3}$$

are the corresponding momenta.

Dalitz plot Analysis

- The region of kinematically allowed phase space described by these constraints is here.
- Kinematic boundaries of the three-body decay phase space.
- In this example, $B^0 \rightarrow D^0 K^+ \pi^-$ phase space is shown. $a = \pi^-$, $b = D^0$, $c = K^+$.



Dalitz plot Analysis

□ Non-leptonic B and D decays proceed predominantly through resonant two-body decays and this is confirmed by the experiment.

□ For three-body decays of a spin-0 particle (D or B meson) to all pseudo-scalar final state particles abc , the baseline model adopted to describe the decay amplitude $\mathcal{A}(m_{ab}^2, m_{bc}^2)$ consists of a coherent sum of two-body amplitudes (subscript r) and a “non-resonant” (subscript NR) contribution:

$$\mathcal{A}(\mathbf{m}) = \sum_r a_r e^{i\phi_r} \mathcal{A}_r(\mathbf{m}) + a_{\text{NR}} e^{i\phi_{\text{NR}}}. \quad (4)$$

□ The parameters a_r (a_{NR}) and ϕ_r (ϕ_{NR}) are the magnitude and phase of the amplitude for component r (NR).

□ The function \mathcal{A}_r is a Lorentz-invariant expression that describes the dynamic properties of the decay into the multibody final state as a function of position in the Dalitz plane $\mathbf{m} \equiv (m_{ab}^2, m_{bc}^2)$

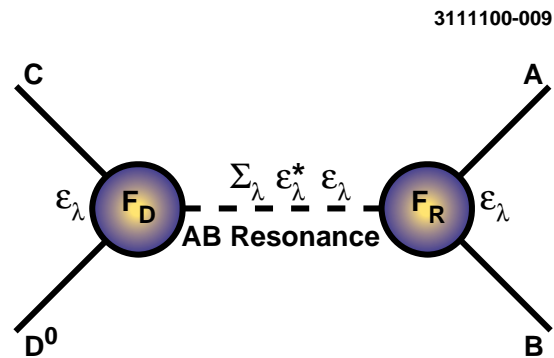
□ See: (arXiv:hep-ex/0011065), (arXiv:hep-ex/0410014).

Isobar formalism

□ In the isobar formalism, the function \mathcal{A}_r describes the decay through a single intermediate resonance r and takes the form

$$\mathcal{A}_r = F_D \times F_r \times T_r \times W_r, \quad (5)$$

where F_D (F_r) is the Blatt-Weisskopf centrifugal barrier factor for the D or B meson (resonance) decay vertex with radius R , T_r is the resonance propagator, and W_r describes the angular distribution in the decay.

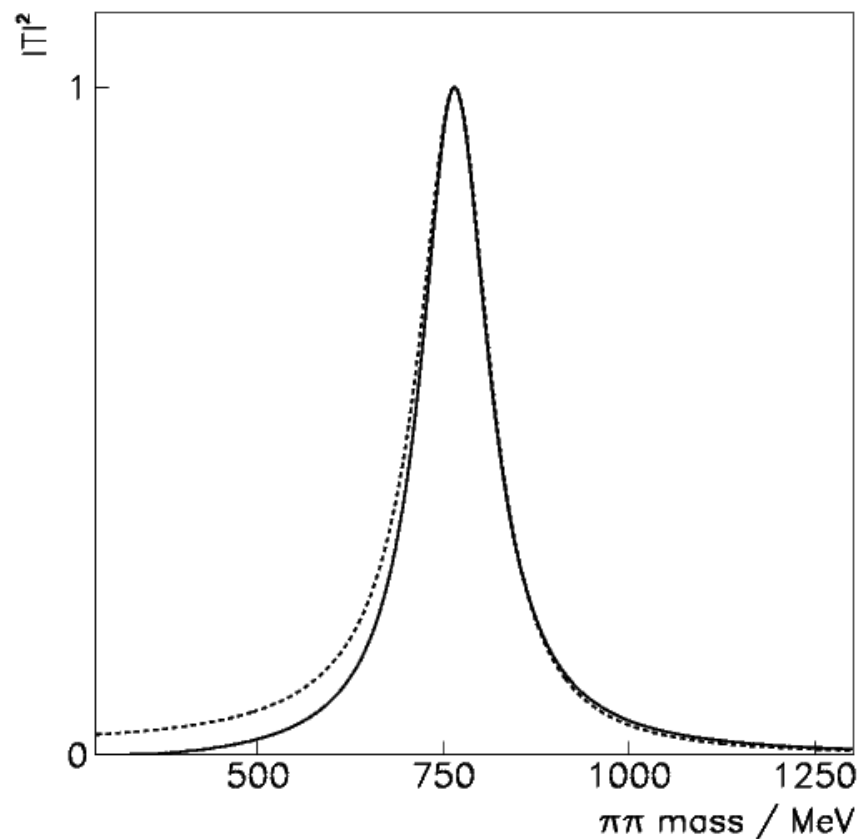


□ The resonance propagator T_r is usually described using a relativistic Breit-Wigner (BW) parameterization with mass-dependent width

$$T_r = \frac{1}{m_r^2 - m_{ab}^2 - im_r \Gamma_{ab}}$$

Breit-Wigner Lineshape.

- Comparison between resonances lineshapes without (dashed) and with a mass-dependent Γ (solid line).



Blatt-Weisskopf penetration factors

□ The expressions for the Blatt-Weisskopf penetration factors depend on spin J of the intermediate resonance

$$F^{(0)} = 1$$

$$F^{(1)} = \frac{\sqrt{1+z_0}}{\sqrt{1+z}}$$

$$F^{(2)} = \frac{\sqrt{z_0^2+3z_0+9}}{\sqrt{z^2+3z+9}}$$

$$F^{(3)} = \frac{\sqrt{z_0^3+6z_0^2+45z_0+225}}{\sqrt{z^3+6z^2+45z+225}}$$

Here $z = r^2 p_R^2$ where r is the hadron scale, and $z_0 = r^2 p_{R0}^2$ where p_{R0} is the R resonance daughter's momentum calculated for the pole mass of the R resonance.

□ The mass-dependent width is given by

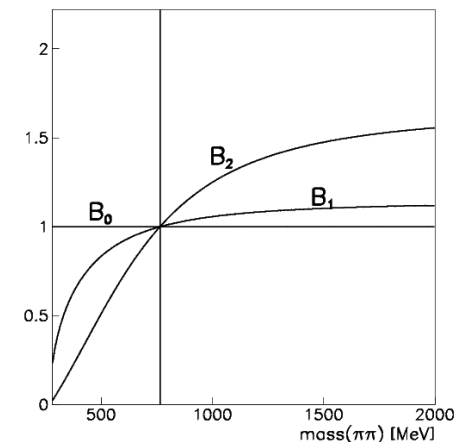
$$\Gamma_{ab} = \Gamma_r \left(\frac{p_{ab}}{p_r} \right)^{2J+1} \left(\frac{m_r}{m_{ab}} \right) F_r^2$$

where Γ_r is the width of the resonance.

□ The value of R is unknown and ranges from 0 to 5.

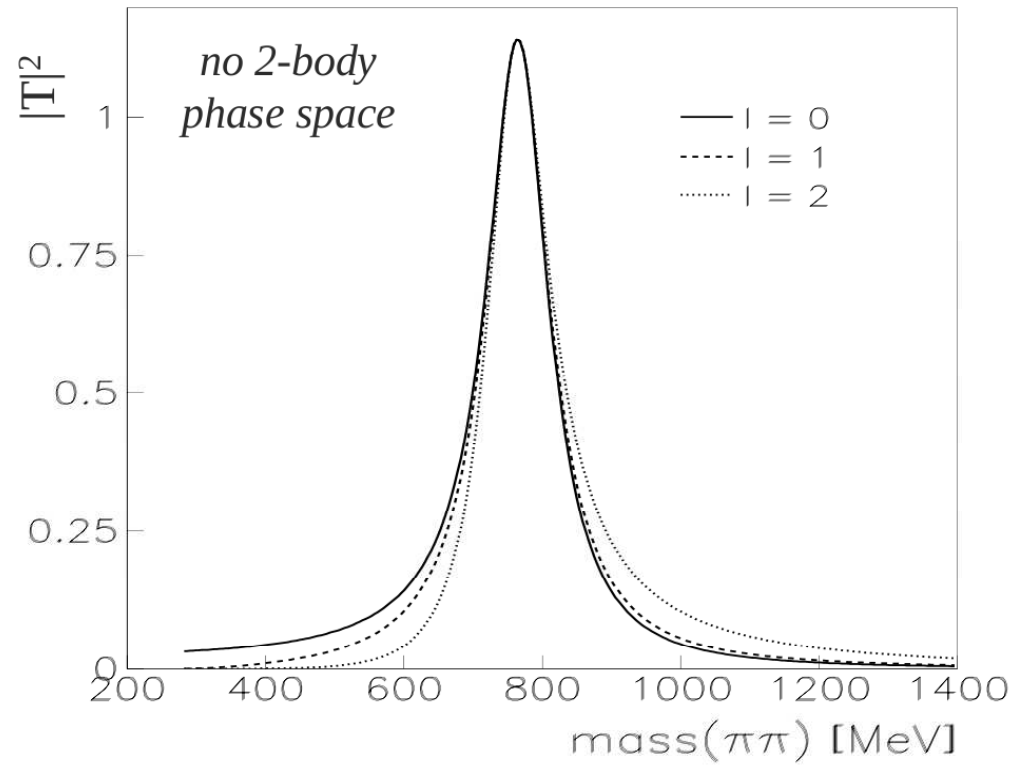
It is usually included in the systematic errors evaluation.

□ Blatt-Weisskopf factors.



Breit-Wigner with Blatt-Weisskopf penetration factors

- Breit-Wigner lineshape with different angular momentum factors.

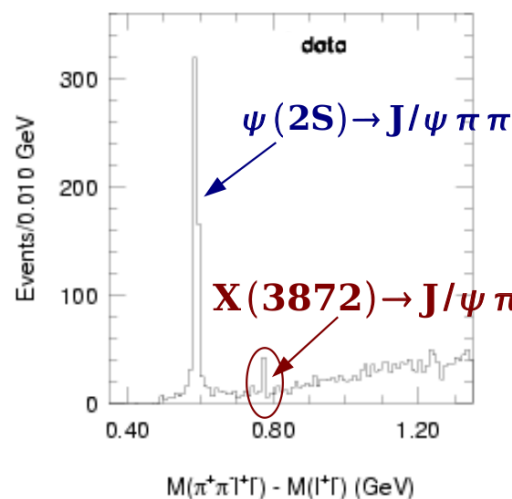


Discovery of X(3872).

□ 2003: Belle discovers X(3872) in B decays (S.-K. Choi *et al.*, Phys. Rev. Lett. **91**, 262001 (2003)).

X(3872) first observation

$B \rightarrow K \pi^+ \pi^- J/\psi$ using 140 fb^{-1}



$N = 34 \pm 7$
significance 10σ

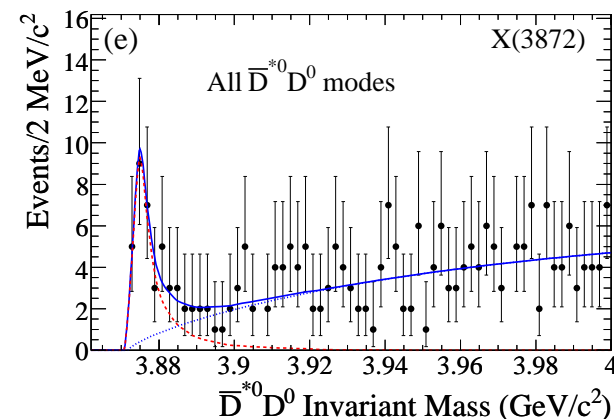
$(3872.0 \pm 0.6 \pm 0.5) \text{ MeV}/c^2$
 $\sim m_{D^0} + m_{D^{*0}}$
 $[m_{D^0} + m_{D^{*0}} = (3871.81 \pm 0.36) \text{ MeV}/c^2]$
 $\Gamma < 2.3 \text{ MeV} \text{ (90\% C.L.)}$

X is narrow and doesn't decay to $D\bar{D}$

□ Confirmed by CDF, D0, BaBar.

□ BaBar: Observation of $X(3872) \rightarrow D^* \bar{D}$

(B. Aubert *et al.*, Phys. Rev. D **77**, 011102(R) (2008))

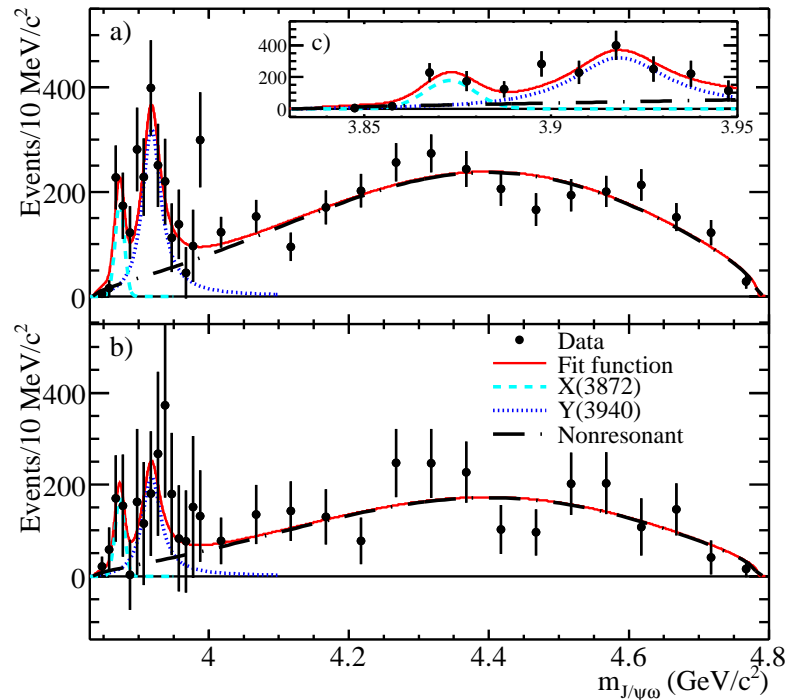


An interesting physics result.

□ BaBar study of:(arXiv:1005.5190)

$$B^{0,+} \rightarrow K^{0,+} J/\psi \omega$$

□ The $J/\psi \omega$ mass spectrum shows a signal of $X(3872)$ whose spin-parity has still to be defined.



□ The $X(3872) \rightarrow J/\psi \omega$ is close to threshold.

An interesting physics result.

□ MC simulation of a $X(3872) \rightarrow J/\psi\omega$ with a Breit-Wigner with S – wave and P – wave Blatt-Weisskopf factors.

□ Plot of the expected $\pi^+\pi^-\pi^0$ mass spectrum compared with the data.

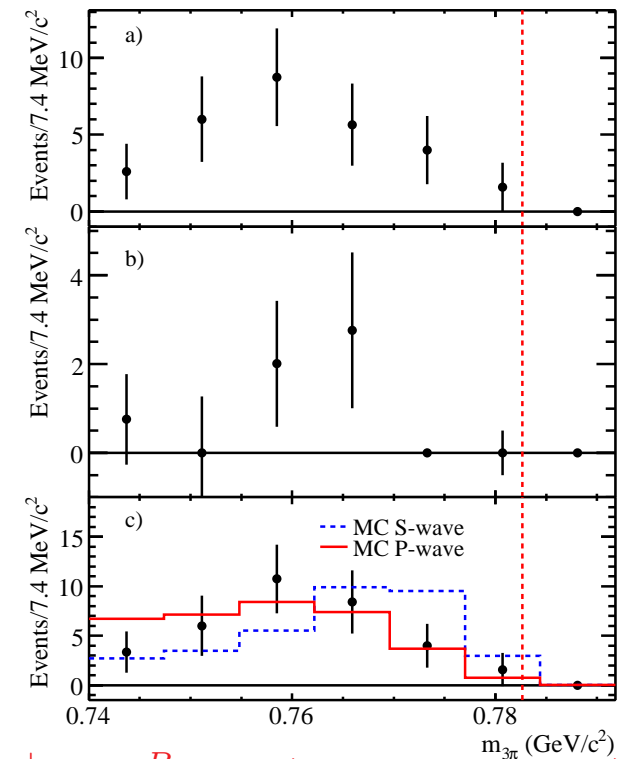
□ One unit of orbital angular momentum creates a centrifugal barrier factor $q^2/(1 + R^2q^2)$ in the description of the $J/\psi\omega$ final state, where $R = 3 \text{ GeV}^{-1}$ is the P -wave Blatt-Weisskopf barrier factor radius.

□ Values in the range $0 < R < 5 \text{ GeV}^{-1}$ yield no difference.

□ For the dashed histogram the χ^2 -distribution probability is $P = 7.1\%$, while for the solid histogram $P = 61.9\%$.

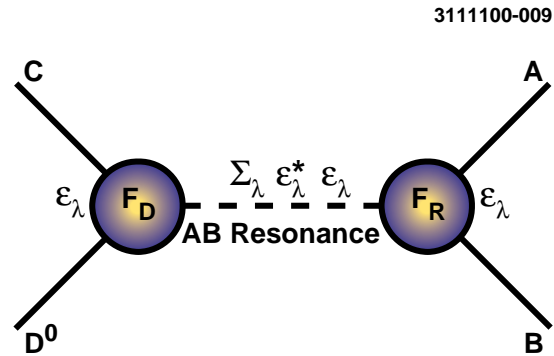
□ It follows that the observed distribution favors the P -wave description both quantitatively and qualitatively.

□ This analysis therefore (together with the CDF analysis ($J^P = 1^+$ or $J^P = 2^-$) (hep-ex/0612053)) favor $J^P = 2^-$ for the $X(3872)$ meson.



Angular dependence

- The angular dependence W_r is described using either Zemach tensors or the helicity formalism.



- In the helicity formalism, the expressions for scalar, vector and tensor states are:

$$W_0 = 1$$

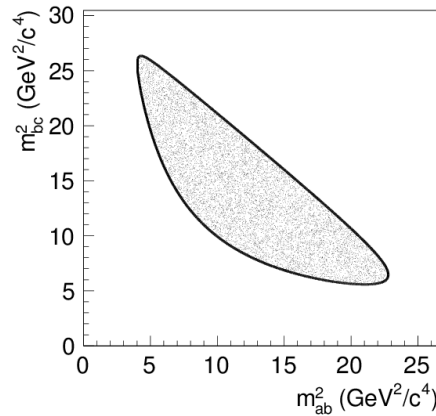
$$W_1 = m_{ac}^2 - m_{bc}^2 + \frac{(M^2 - m_c^2)(m_b^2 - m_a^2)}{m_r^2}$$

$$W_2 = \left[m_{bc}^2 - m_{ac}^2 + \frac{(M^2 - m_c^2)(m_b^2 - m_a^2)}{m_r^2} \right]^2 - \frac{1}{3} \left[m_{ab}^2 - 2M^2 - 2m_c^2 + \frac{(M^2 - m_c^2)^2}{m_r^2} \right] \times$$

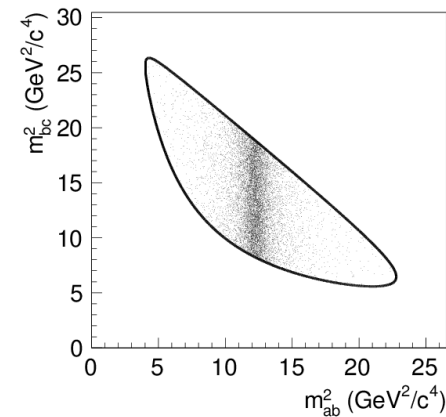
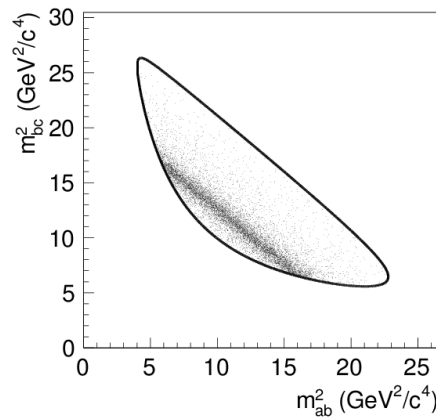
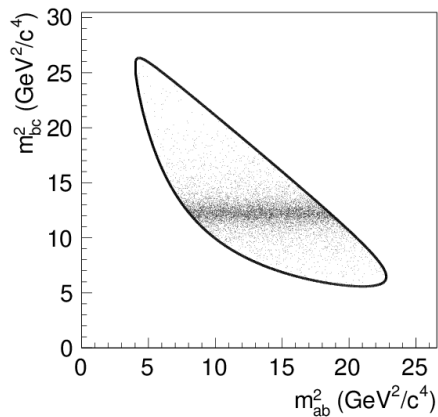
$$\left[m_{ab}^2 - 2m_a^2 - 2m_b^2 + \frac{(m_a^2 - m_b^2)^2}{m_r^2} \right].$$

Isobar formalism

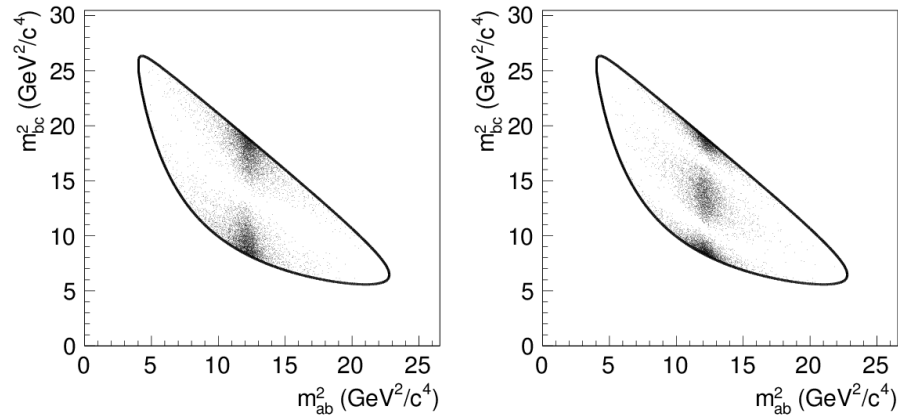
- The figure shows how various intermediate two-body states appear in the Dalitz plot.
- Unlike the uniform distribution that the phase-space decay has, scalar resonances appear as bands in the Dalitz plot plane.



Scalar in bc channel, Scalar in ac channel, Scalar in ab channel

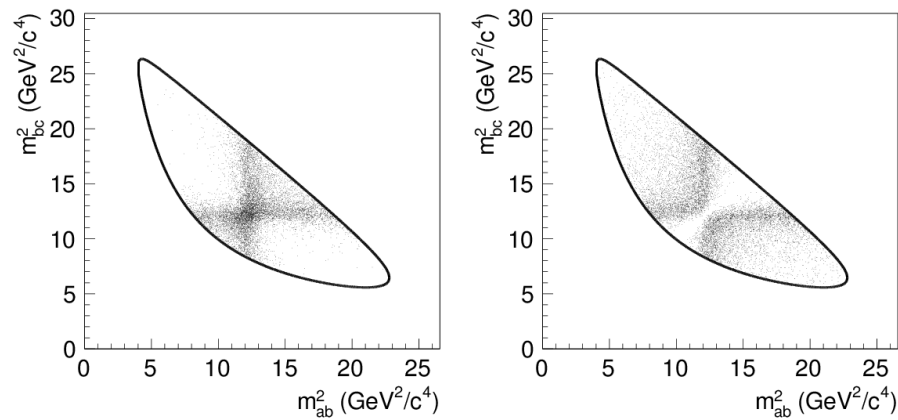


□ Angular distributions for the vector and tensor intermediate states introduce the characteristic non-uniformity of the event density.



□ Finally, the region where the amplitudes of two resonances overlap is sensitive to the phase difference between the two amplitudes.

Two scalars, $\Delta\phi = 0$, Two scalars, $\Delta\phi = \pi$



Zemach tensors

- Non-relativistic Zemach tensors.
- Suppose we are studying the decay:

$$D^0 \rightarrow K_S^0 \pi^+ \pi^-$$

We define with $p_1^{K_S^0}$, $p_2^{\pi^+}$, $p_3^{\pi^-}$ the momenta of the three particles in the D^0 center of mass system.

- The method is the following:

- We use a symmetric and traceless tensor of rank-L made with p^i to describe orbital angular momenta.
- We use a symmetric and traceless tensor of rank-S made with t^i to describe the spin of intermediate resonances. For a resonance decaying as $R \rightarrow a + b$, the t^i are defined as:

$$t_R^i = p_a^i - p_b^i - (p_a^i + p_b^i) \frac{m_a^2 - m_b^2}{m_{ab}^2}$$

- The tensors are then contracted to obtain the spin of the D^0 , in this case 0.

Zemach tensors

- If we consider the decay:

$$D^0 \rightarrow RK_S^0$$

- For a spin 1 resonance R in the $\pi^+\pi^-$ system: L=1, S=1.
- Dot product of two vectors.

$$W_1 = \mathbf{t}_1 \cdot \mathbf{p}_1$$

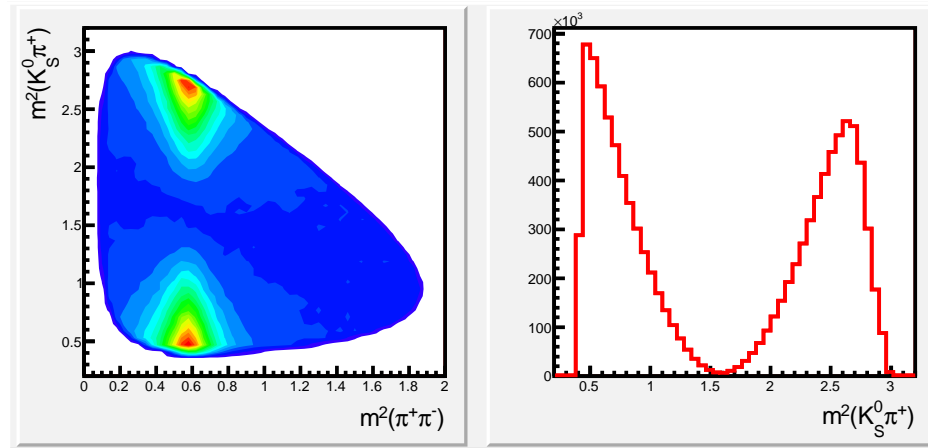
- For a spin 2 resonance R in the $\pi^+\pi^-$ system: L=2, S=2.

- Dot product of two rank-2 tensors.

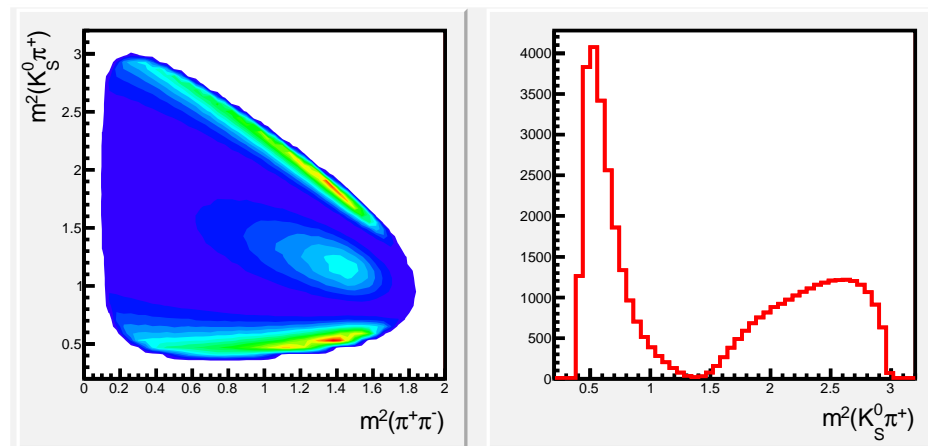
$$W_2 = [p_1^i p_1^j - \frac{1}{3} p_1^2 \delta^{ij}] [t_1^i t_1^j - \frac{1}{3} t_1^2 \delta^{ij}]$$

MC simulations of the two decays.

□ Amplitude for $D^0 \rightarrow K_S^0 \rho^0(770)$. Identical to that from the helicity formalism.



□ Amplitude for $D^0 \rightarrow K_S^0 f_2(1270)$. Identical to that from the helicity formalism.



Example: the ω spin.

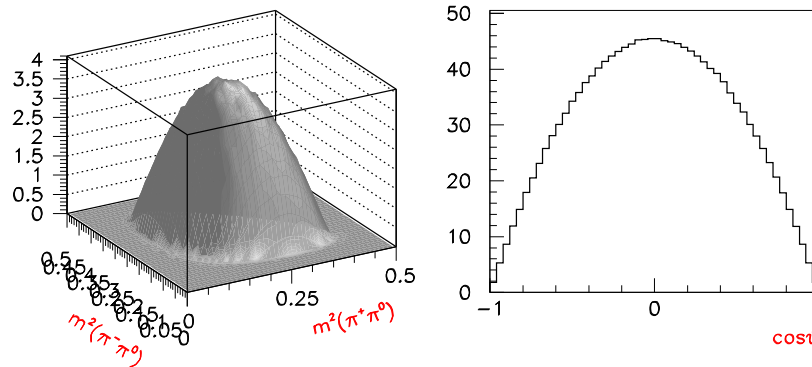
- The $\omega(770)$ meson is a spin-1 resonance:

$$\omega(770) \rightarrow \pi^+ \pi^- \pi^0$$

- Using the Zemach formalism, the matrix element describing the ω decay is:

$$W_1 = |\mathbf{p}_{\pi^+} \times \mathbf{p}_{\pi^-}|^2$$

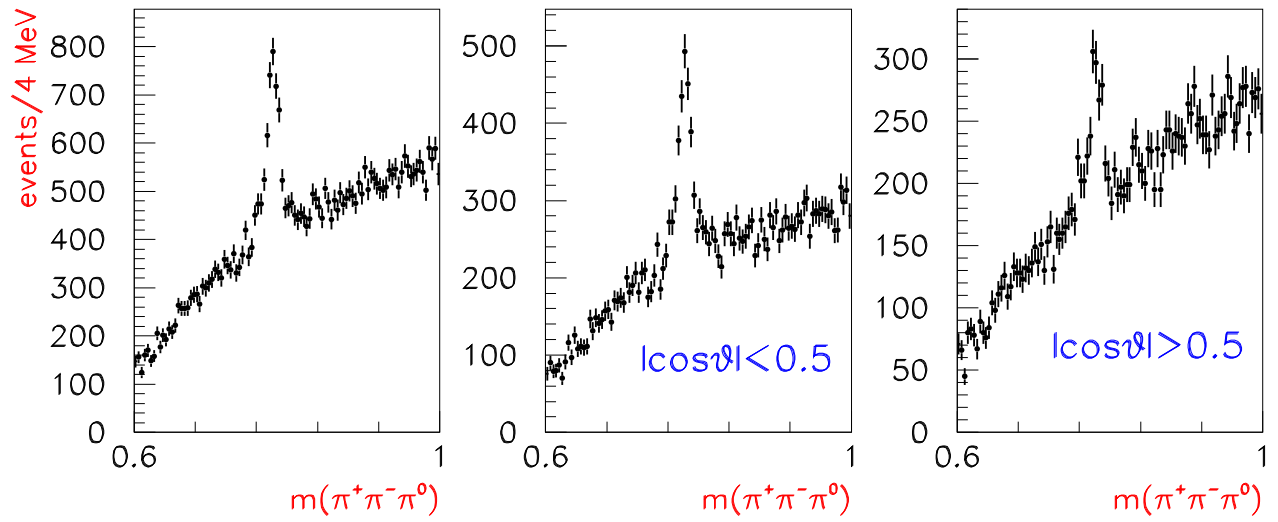
- Monte Carlo simulation of the ω decay:



- We define θ as the angle formed by the π^+ with respect to the π^0 in the $\pi^+\pi^-$ rest frame.
- The distribution is of the form: $(1 - \cos^2\theta)$.

Example: the ω spin.

- This distribution can be used to select events with an enhanced ω signal.
- A test on inclusive BaBar data.

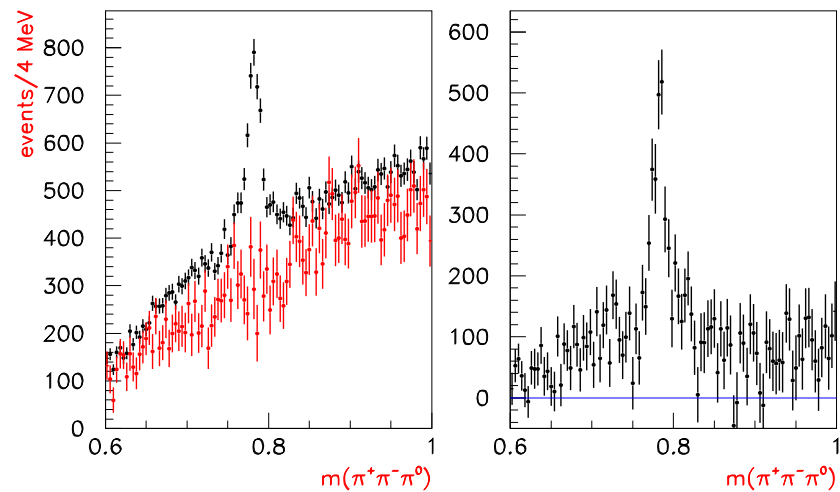


Example: the ω spin.

- Another way to use this information is to weight the events by the expression (Y_2^0 moment):

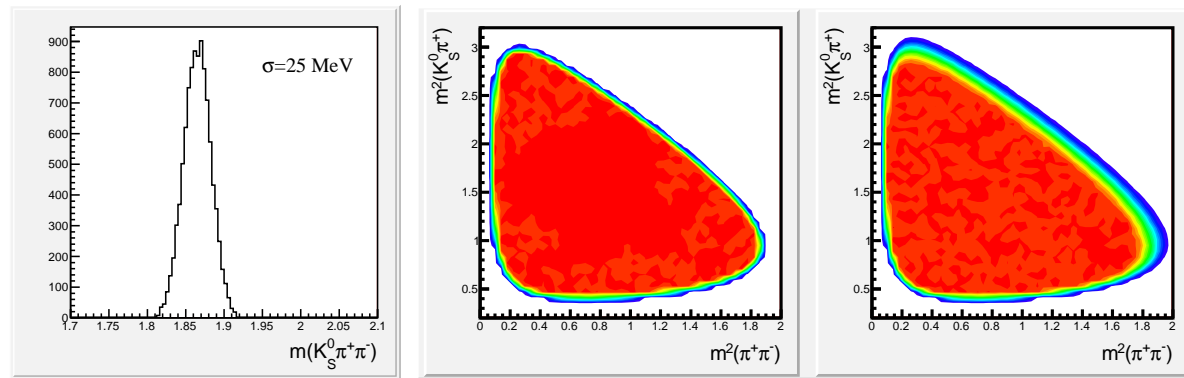
$$w(\cos\theta) = \frac{5}{2}(1 - 3 \cdot \cos^2\theta)$$

- The weighted distribution should project all the ω events and weight to zero all the background.
- A test from inclusive BaBar data.

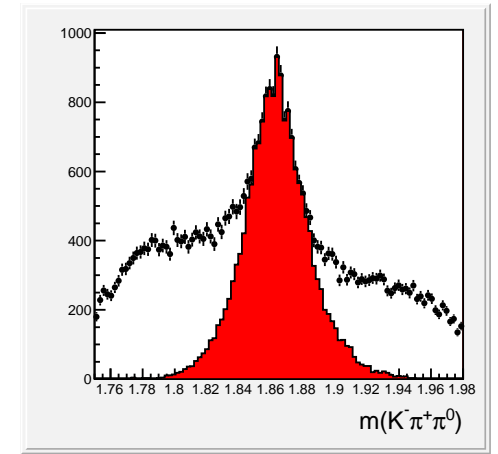


Resolution Smearing.

- The experimental resolution produces a smearing of the signal, therefore the edge of the Dalitz plot becomes smeared. Example of a $D^0 \rightarrow K_S \pi^+ \pi^-$ signal (somewhat exaggerated).
- Imposing a mass constraint solves the problem. However in presence of important background sidebands may be lost.

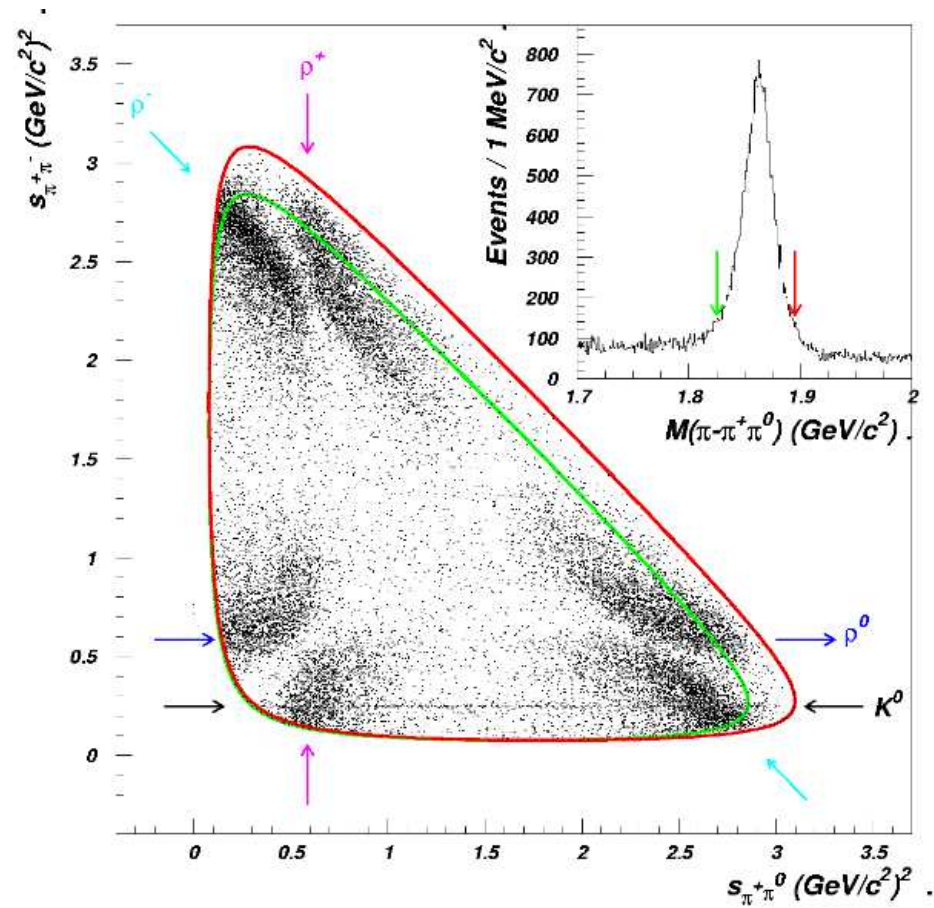


- Example from BaBar data. $D^0 \rightarrow K^- \pi^+ \pi^0$ signal. In red are events which fit kinematically the D^0 hypothesis.



Resolution Smearing.

□ $D^0 \rightarrow \pi^+ \pi^- \pi^0$ Dalitz plot with overlaid contours with the D^0 spread in mass.

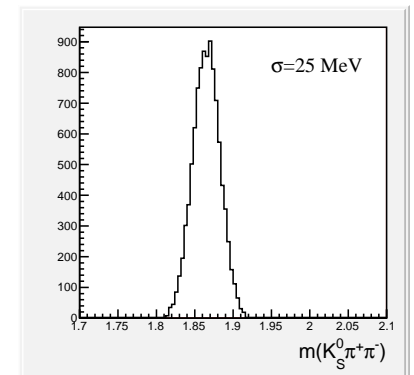


Resolution Smearing.

- Effect of the bad handling of the resolution.
- Events generated with a D^0 having a 25 MeV mass resolution but fitted without taking into account the smearing.

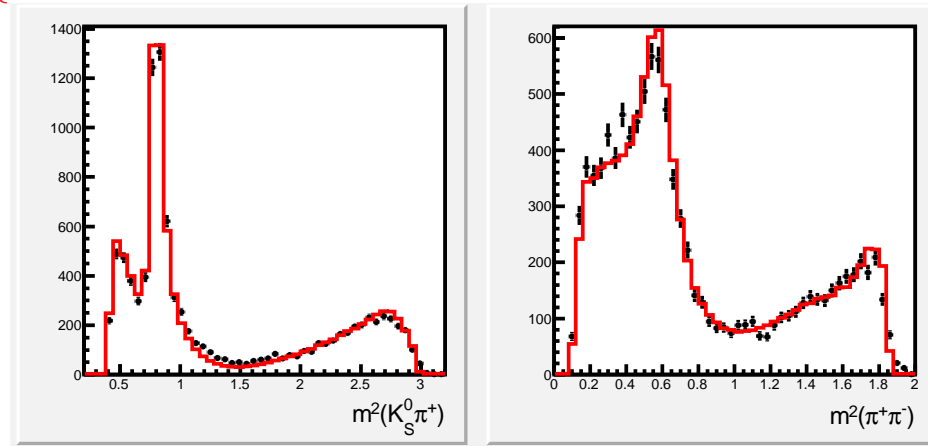
Amplitude	Generated	Wrong Smearing	Right Smearing
$K_S^0 \sigma$	9.3	4.8	9.3 ± 0.4
$K_S^0 \rho(770)$	42.9	49.3	44.4 ± 1.2
$K_S^0 f_2(1270)$	16.9	19.4	16.6 ± 1.6
$\pi^- K^*(890)$	45.2	46.1	43.3 ± 1.1
χ^2/NDF		2.09	1.19

- The problem is solved at level of the normalization of the amplitudes.
- The amplitudes are normalized generating a large number of MC events via “Phase Space” (with a uniform distribution on the Dalitz plot).
- Normalization integrals need to be computed on a MC simulation having the same resolution as for the data.
- Example of a generated D^0 mass.

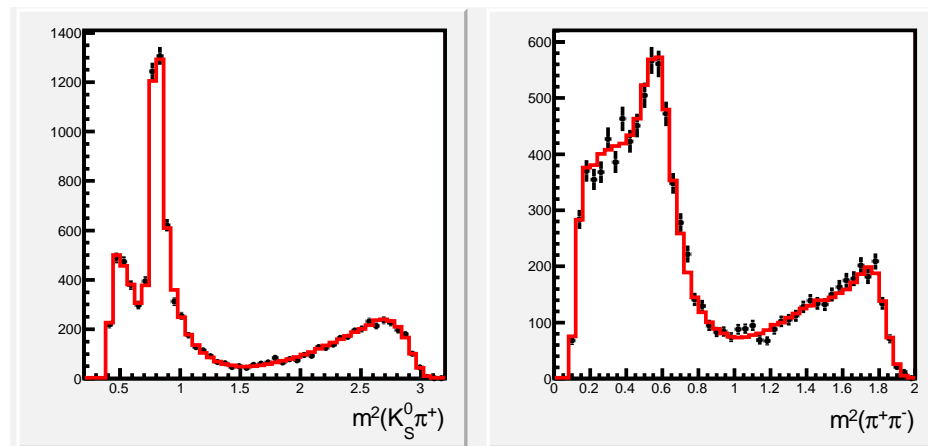


Resolution Smearing.

- Fit projections on a Monte Carlo simulation and Dalitz analysis without considering the effect of the experimental resolution.



- Fit projections obtained taking into account the experimental resolution.



Efficiency.

- Two ways of handling the efficiency:
 - In case of a limited sample of Monte Carlo events: fit a 2-D efficiency on the Dalitz plot;
 - Use a large sample of Full Detector simulation signal events.

- The first method can only be used in a simple three-body Dalitz analysis.
 - Generate a “Phase Space” large sample of MC events. Produce the “Generated” Dalitz plot.
 - Reconstruct the events and apply all the selection criteria as for the real data and produce a “Reconstructed” Dalitz plot.
 - Obtain the efficiency using the ratio of the two 2-D distributions.

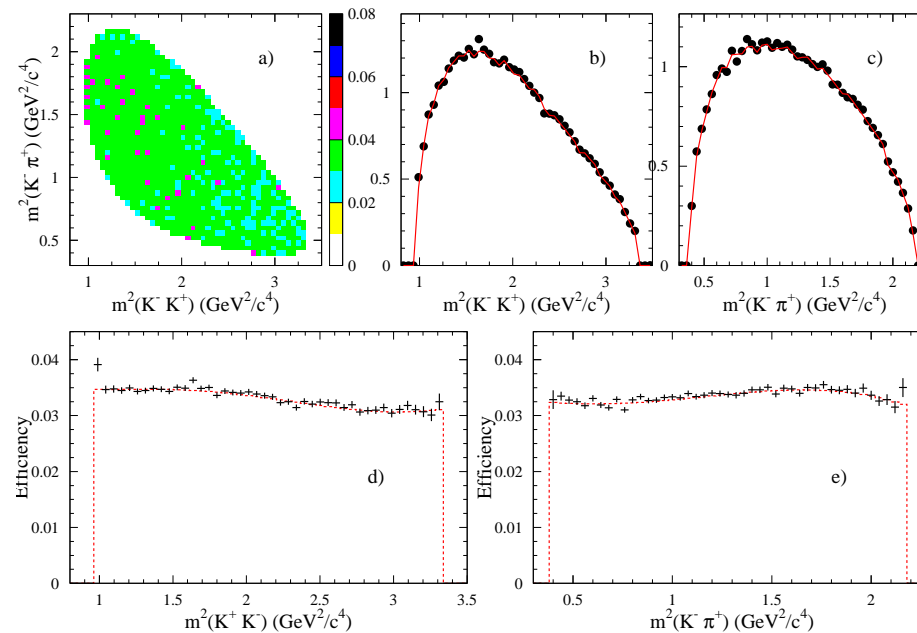
Efficiency fitting.

□ The standard way to fit the efficiency is to use the expression:

$$\eta(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2 \quad (6)$$

where $x = m^2(ab)$, $y = m^2(bc)$.

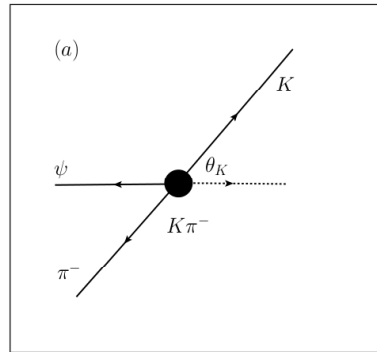
□ Example from BaBar: $D_s^+ \rightarrow K^- K^+ \pi^+$.



□ Fit projections obtained without and with correction for the Dalitz plot phase space.

Efficiency fitting.

- A detailed efficiency fitting has been performed in the search for Z^+ resonances (see later).
(arXiv:0811.0564)
- The efficiency for $B \rightarrow J/\psi K \pi$ has been parameterized as a function of $(m(K\pi), \cos\theta)$.



where θ is the K helicity angle.

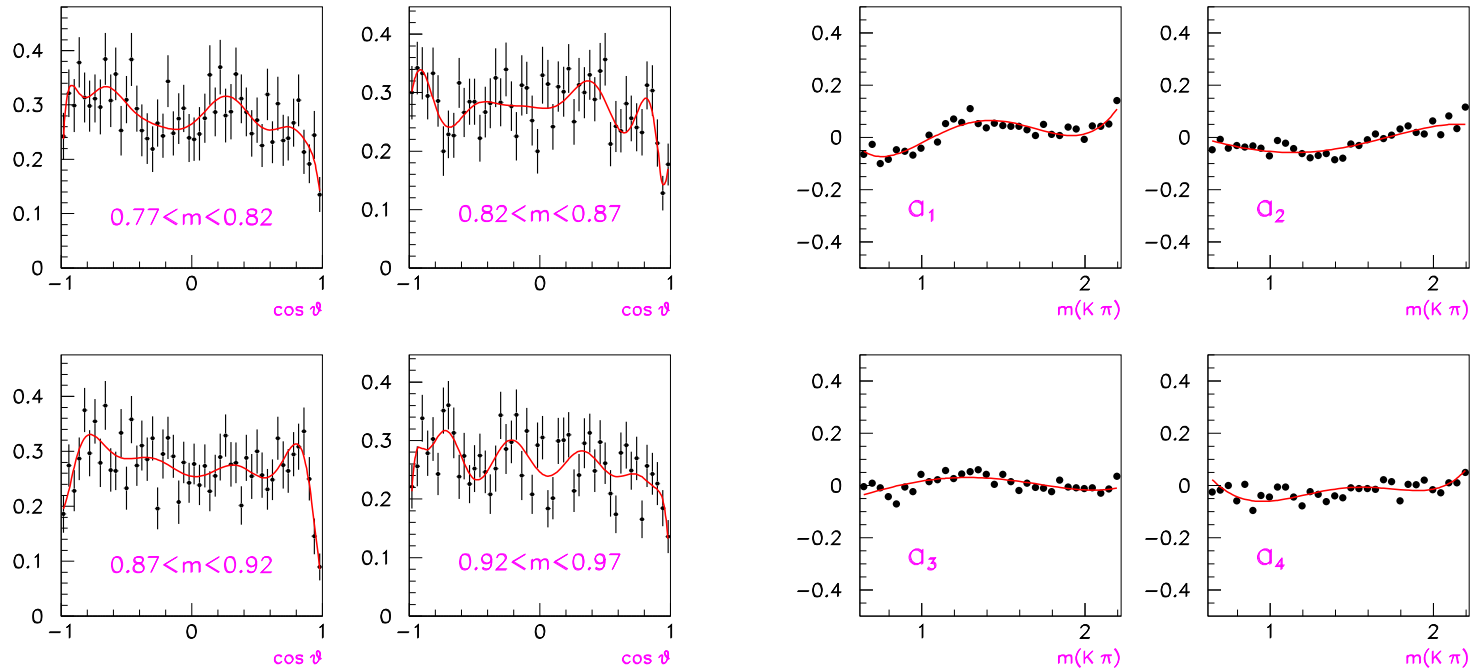
- We first fit the efficiency as a function of $\cos\theta$ in separate 50 MeV intervals of $m(K\pi)$, in terms of Legendre polynomials up to $L = 12$:

$$\epsilon(\cos\theta) = \sum_{L=0}^{12} a_L Y_L^0(\cos\theta)$$

Efficiency fitting.

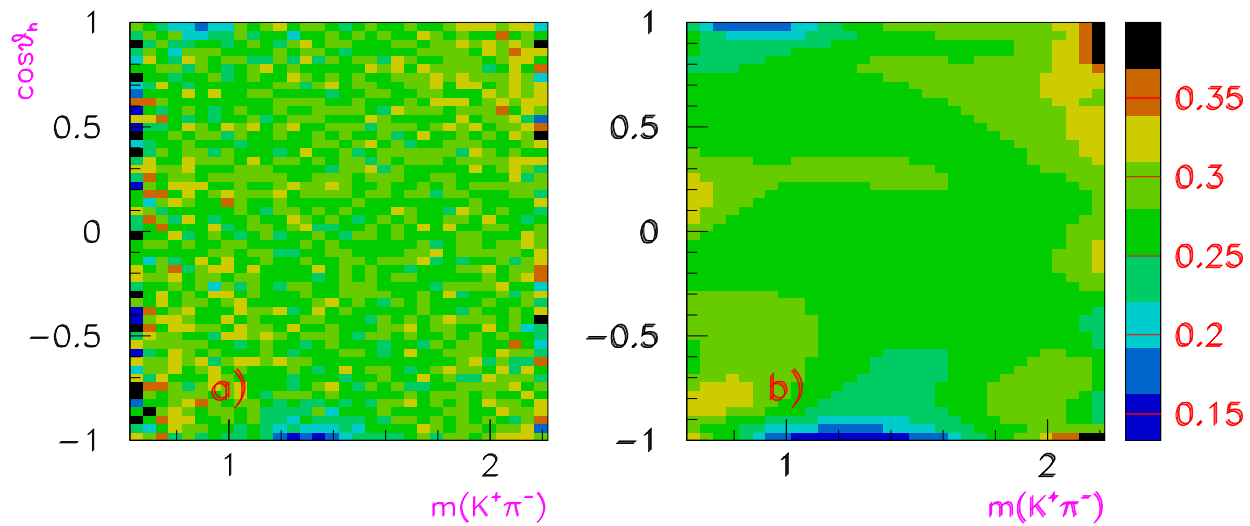
□ For each value of L , we fit the $a_L(m)$ as a function of $m(K\pi)$ using a 6th-order polynomial in $m(K\pi)$.

□ Examples of a few fits.



Efficiency fitting.

- Plots of the raw and the fitted efficiency.



- The events are weighted by:

$$w(m(K\pi), \cos\theta) = 1/\epsilon(m(K\pi), \cos\theta)$$

Background.

□ To parametrize the background Dalitz distribution, a possible way is to use the data sample in the sideband regions. An unbinned maximum likelihood is performed using the function:

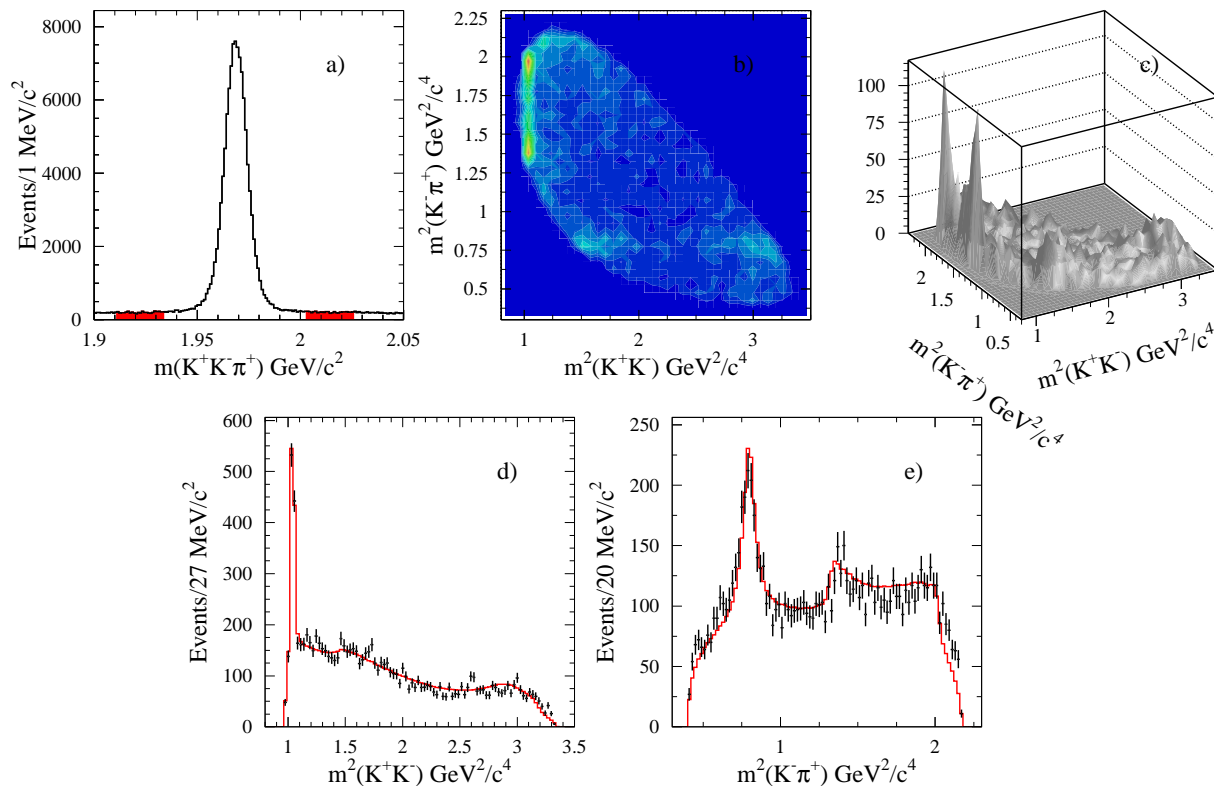
$$\mathcal{L} = \prod_{events} \left[\frac{\sum_i k_i^2 B_i B_i^*}{\sum_i k_i^2 I_{B_i B_i^*}} \right]$$

where:

- k_i 's are real coefficients floated in the fit;
- B_i 's are the dynamical functions of background.
- I_i 's are the normalization integrals to be computed in the sidebands regions.
- The total amplitude is assumed an incoherent sum of intermediate processes.

Background.

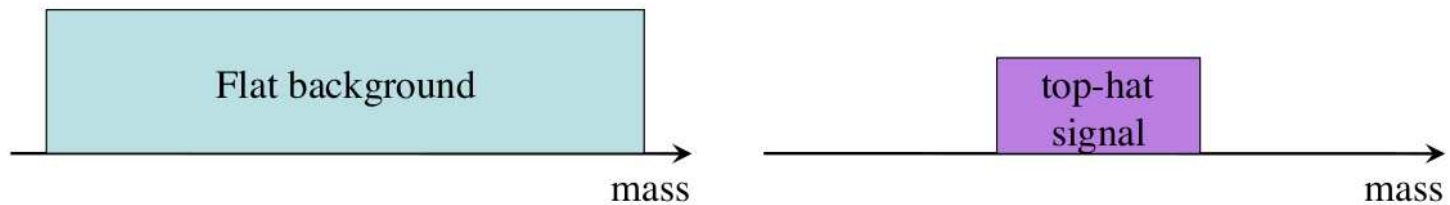
- Example from the $D_s^+ \rightarrow K^+ K^- \pi^+$ Dalitz analysis.
- The Dalitz plot of sidebands events shows the presence of $\phi(1020)$ and $\bar{K}^*(892)^0$ contamination in the background. The fit returns the presence of a large scalar contribution due mainly to a non-resonant contribution.



- Ad-hoc resonances have been included to obtain a good description of the sidebands.

Background: sFit method.

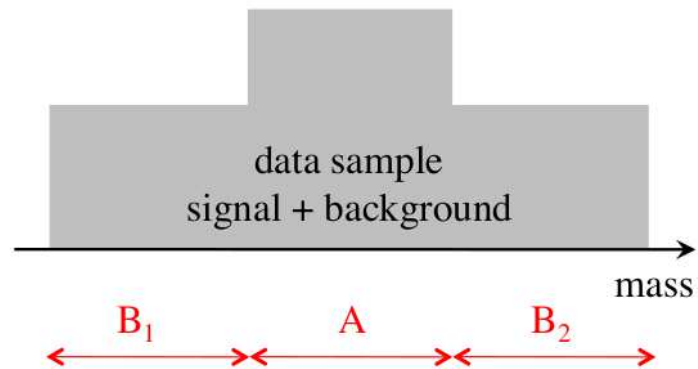
- A new method has been recently developed ([arXiv:0905.0724v1](https://arxiv.org/abs/0905.0724v1)) which avoids to obtain full information from the sidebands regions.



A = signal region containing N_A candidates

B_1 & B_2 = sidebands containing $N_{B_1} + N_{B_2}$ candidates

Assume regions A, B_1 , B_2 all have the same width

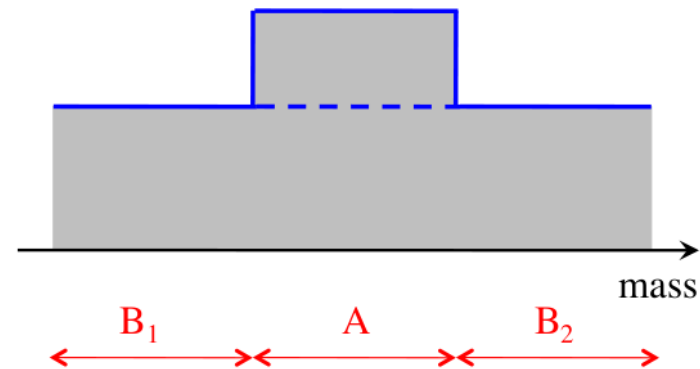


Background: sFit method.

Fit distribution with
top-hat signal + flat background

Plain curve = total function

Dashed curve = background contribution



→ the math of the sPlot technique (see NIM A, 555 (2005) 356-369)
allows the direct calculation of the sWeights for the signal:

sWeight = +1 in the signal region (A)

sWeight = -0.5 in the sidebands (B₁ and B₂)

sum of sWeights = $N_A - 0.5 (N_{B_1} + N_{B_2})$ = signal yield

NB: looks like a simple sideband subtraction !

Define:

$$N_{\text{bkg}} = 0.5 (N_{B_1} + N_{B_2})$$

$$N_{\text{sig}} = N_A - N_{\text{bkg}}$$

□ Possible bias introduced by the method require to be studied.

The Likelihood function.

□ An unbinned maximum likelihood fit is performed using the function:

$$\mathcal{L} = \prod_{events} \left[x(M_D) \cdot \eta(x, y) \frac{\sum_{i,j} c_i c_j^* A_i(x, y) A_j^*(x, y)}{\sum_{i,j} c_i c_j^* I_{A_i A_j^*}} + [1 - x(M_D)] \frac{\sum_i k_i^2 B_i(x, y) B_i^*(x, y)}{\sum_i k_i^2 I_{B_i B_i^*}} \right]$$

where:

- x is the fraction of signal, depending on the $h^+ h^- h^+$ invariant mass (M_D) before the mass constraint fit. It is defined as:

$$x(M_D) = \frac{G(M_D)}{G(M_D) + P(M_D)} \quad (7)$$

where G and P are the Gaussian and the linear polynomial used for fitting the final mass spectrum.

- $\eta(x, y)$ is the efficiency, parametrized by 2-D 3rd order polynomial;
- $A_i(x, y), B_i(x, y)$ describe signal and background amplitude contributions respectively;
- k_i is the real amplitude of the i -th component for the background. They are computed by fitting sideband regions;

The Likelihood function.

- $I_{A_i A_j^*} = \int A_i(x, y) A_j^*(x, y) \eta(x, y) dx dy$ and $I_{B_i B_i^*} = \int |B_i(x, y)|^2 dx dy$ are the normalization integrals. The amplitudes are normalized on a phase space Monte Carlo weighted by the polynomial fitted efficiency. The events have been generated using a Phase-Space Monte-Carlo program;
 - c_i is the complex amplitude of the i -th component for the signal. They are allowed to float during the fit process.
- One of the coefficients c_i is set to 1 (reference amplitude) and its phase is set to zero.

Normalization integrals.

□ It is easy to show that, having two complex numbers $a = c_1 A_1 e^{i\phi_1}$ and $b = c_2 A_2 e^{i\phi_2}$, we can write:

$$|a + b|^2 = |c_1 A_1 e^{i\phi_1} + c_2 A_2 e^{i\phi_2}|^2 = c_1^2 A_1^2 + c_2^2 A_2^2 \\ + c_1 \cdot c_2 \cdot [\cos(\phi_2 - \phi_1) 2 \operatorname{Re}(A_1 \cdot A_1^*) + \sin(\phi_2 - \phi_1) \operatorname{Im}(A_1 \cdot A_1^*)]$$

□ The normalization is performed separately on the terms:

$$A_1^2, A_2^2, \operatorname{Re}(A_1 \cdot A_1^*), \operatorname{Im}(A_1 \cdot A_1^*)$$

□ Technically this can be performed using different methods.

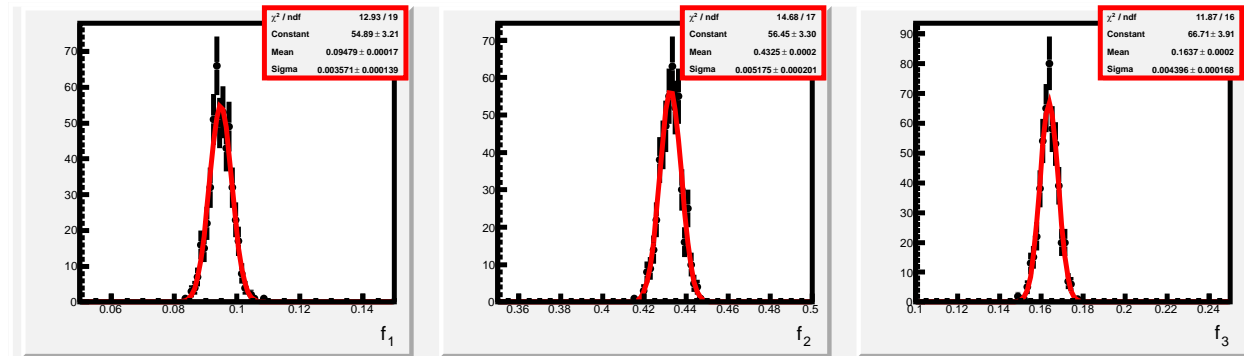
- Generate a large number of phase-space Monte-Carlo events weighted by the efficiency on the Dalitz plot. The integration is performed numerically. This method works well when the integrals are computed only once.
- Analytically, when the integrals need to be computed at each step of the minimization process.

Fractions.

- Fit fractions for the different amplitudes are computed as.

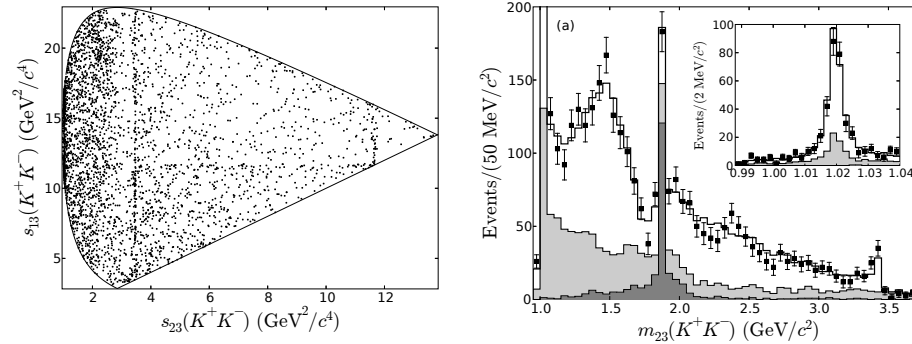
$$\text{Fit Fraction} = \frac{\int |a_r e^{i\phi_r} \mathcal{A}(ABC|r)|^2 d\mathcal{DP}}{\int |\sum_j a_j e^{i\phi_j} \mathcal{A}(ABC|j)|^2 d\mathcal{DP}}. \quad (8)$$

- The errors on the fractions can be computed using the above expression through a Monte Carlo technique.
- Random numbers are generated using the fitted amplitudes and phases and the fit covariance matrix.
- For each generation we histogram the resulting fractions.
- Example from a MC simulation. The distributions are Gaussian and the σ give the error on the fraction.



Fractions.

- The sum of the fractions does not necessarily add to 100%, because of interferences.
- This effect becomes critical in presence of broad overlapping scalar resonances.
- Example from the study of $B^+ \rightarrow K^+ K^- K^+$ from BaBar.



Comp.	ρ	ϕ (rad)	F (%)
$\phi(1020)$	1.66 ± 0.06	$2.99 \pm 0.20 \pm 0.06$	$11.8 \pm 0.9 \pm 0.8$
$f_0(980)$	5.2 ± 1.0	$0.48 \pm 0.16 \pm 0.08$	$19 \pm 7 \pm 4$
$X_0(1550)$	8.2 ± 1.1	$1.29 \pm 0.10 \pm 0.04$	$121 \pm 19 \pm 6$
$f_0(1710)$	1.22 ± 0.34	$-0.59 \pm 0.25 \pm 0.11$	$4.8 \pm 2.7 \pm 0.8$
χ_{c0}^I	0.437 ± 0.039	$-1.02 \pm 0.23 \pm 0.10$	$3.1 \pm 0.6 \pm 0.2$
χ_{c0}^{II}	0.604 ± 0.034	0.29 ± 0.20	6.0 ± 0.7
NR	13.2 ± 1.4	0	$141 \pm 16 \pm 9$

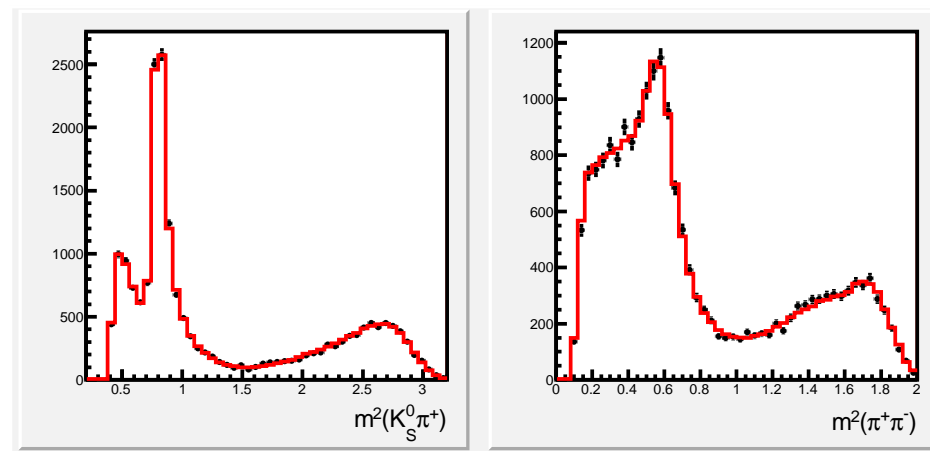
Fit quality.

□ Several methods have been developed to test the fit quality.

- χ^2 on individual Dalitz plot projections;
- 2-D χ^2 on the Dalitz plot;
- 2-D χ^2 on the Dalitz plot with adaptive binning;
- Likelihood test;
- χ^2 on Legendre polynomial moments;

Fit quality, χ^2 .

- After fitting, a Monte Carlo simulation is performed using the fit results using a large sample of simulated events.
 - The MC data are normalized to the real data sample.
 - The MC data are superimposed to the data and a χ^2 can be computed on each distribution. □
- Example from simulated data (20 000 events).

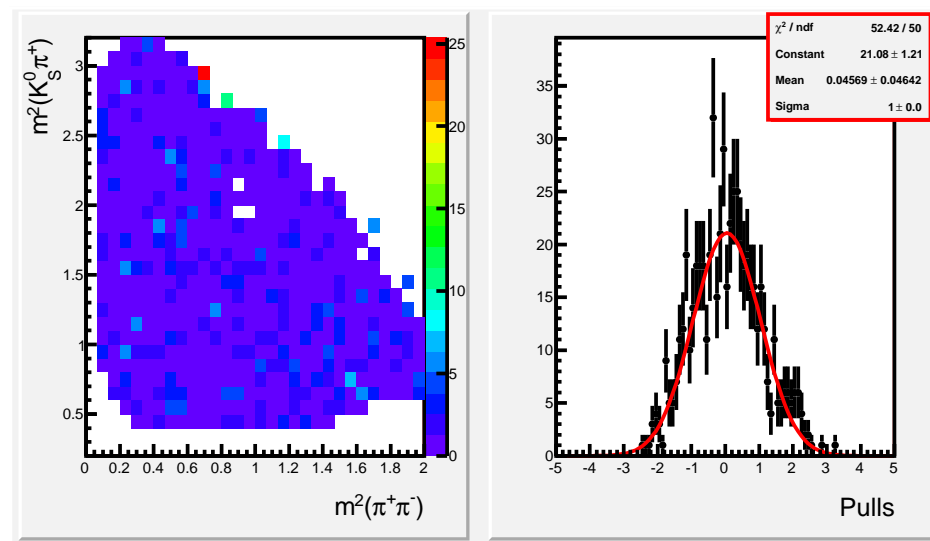


- In this example χ^2/NDF values are 40/46 and 45/47 for the two distributions, respectively.

Fit quality, 2-D χ^2 .

A better estimator of the fit quality is the χ^2 on the Dalitz plot.

□ Using the same simulated data and using uniformly distributed 30×30 cells we obtain:



□ The 2-D χ^2 / NDF in this case is 620/577.

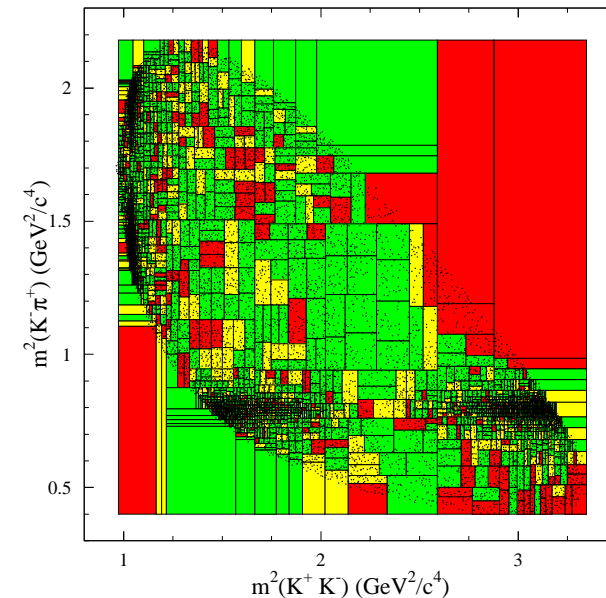
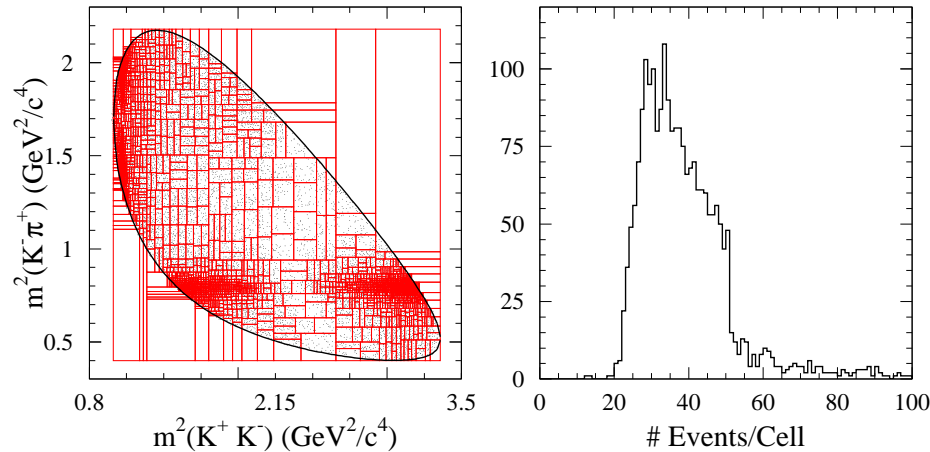
□ The figure also shows the Pulls distribution. For each Dalitz plot cell j :

$$Pull(j) = (data(j) - mc(j)) / \sqrt{mc(j)}$$

□ For a good fit the distribution of the Pulls should be centered at zero and have $\sigma = 1$.

Fit quality, 2-D χ^2 adaptive binning.

- Rather often, the distribution of events on the Dalitz plot has regions of strong accumulations of events and depleted regions.
- For this reason, a better 2-D χ^2 can be obtained using the method of “Adaptive Binning” where the cells have variable sizes in order to have an almost uniform events distribution.
- Example from the $D_s^+ \rightarrow K^+ K^- \pi^+$ Dalitz plot analysis. 2305 cells for 101 K events.



- χ^2 distribution with adaptive binning. Green: $\chi^2 < 1$, Yellow: $1 < \chi^2 < 3$, Red: $\chi^2 > 3$
- $\chi^2/NDF = 2843/2291 = 1.2$.

Fit quality, Likelihood.

□ A confidence level can be calculated directly from the likelihood function by utilizing the best fit parameters. This idea was described by ARGUS (ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. **B 308**, 435 (1993)), and (arXiv:hep-ex/0011065) and is a direct application of the Central Limit Theorem from statistics.

□ Assuming the candidates are truly distributed according to the likelihood function which gives the best fit, the average value is:

$$\mu = \frac{1}{N} \sum_{i=1}^N (-2 \ln \mathcal{L}) \approx \int \mathcal{L}(-2 \ln \mathcal{L}) d\mathcal{DP}$$

where N is the number of candidates.

• The variance is given by:

$$\sigma_{\mu}^2 = \frac{1}{N} \sum_{i=1}^N (-2 \ln \mathcal{L} - \mu)^2 \approx \int \mathcal{L}(-2 \ln \mathcal{L})^2 d\mathcal{DP} - \mu^2$$

• Because we have a large number of candidates distributed according to this function, the Central Limit Theorem tells us that the mean should follow a normal distribution.

• The sum of minus log likelihoods, which is the value minimized in the fit, has a mean of $N\mu$ and follows a normal distribution with a variance of $N\sigma_{\mu}^2$. Thus, the minimal value will come from a normal distribution with mean:

$$\langle -2 \sum \ln \mathcal{L} \rangle = N \int \mathcal{L}(-2 \ln \mathcal{L}) d\mathcal{DP} - n$$

and standard deviation

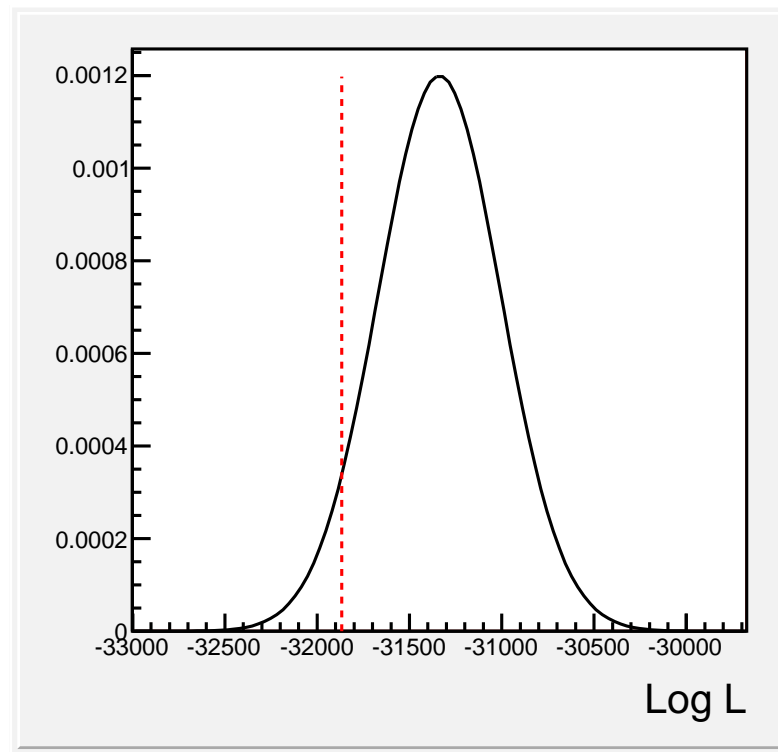
$$\sigma_{\langle -2 \sum \ln \mathcal{L} \rangle} = \sqrt{N \int \mathcal{L}(-2 \ln \mathcal{L})^2 d\mathcal{DP} - N\mu^2}$$

where n is the number of parameters extracted from the fit.

Fit quality, Likelihood.

- The confidence level for the fit is then just the area of a Gaussian with the above mean and width which lies above the value obtained in our fit.

- Example from simulated events.



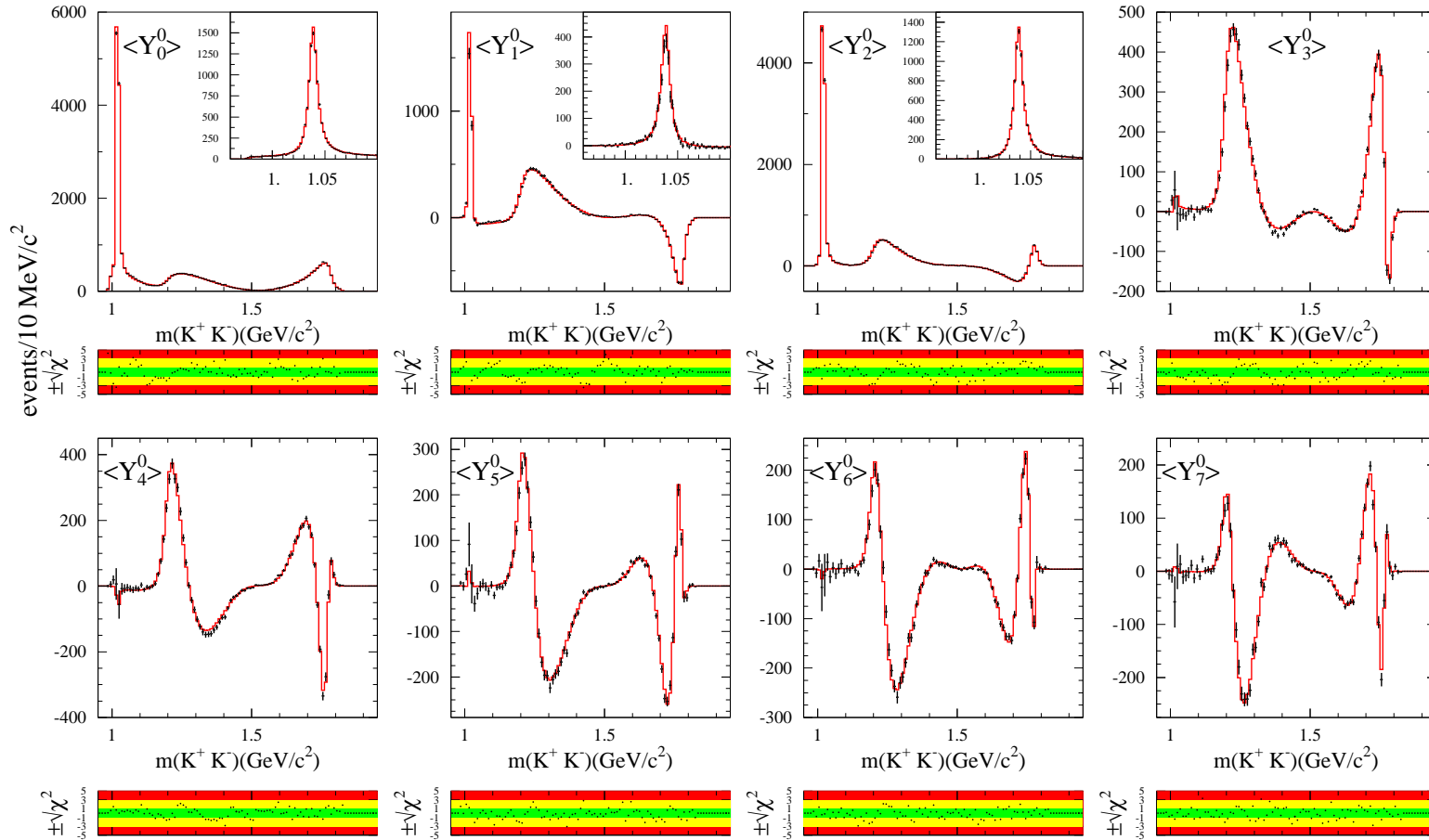
- Fit probability: 7%.

Legendre Polynomials Moments.

- **E** very important test makes use of the Legendre Polynomials Moments.
- Moments can be computed as functions of all the available three mass combinations.
- **Moments are related to the resonances amplitudes and phases contributing to the decay.**
- On a given mass projection we have a superposition of true physics along this mass as well as reflections from the other mass combinations.
- **Therefore usually it is not easy, except for particular cases, to give a physical significance to the observed structures.**
- On the other hand these distributions go deeply inside the angular distributions and interference effects.
- **Therefore it is important to obtain a good description of all the Moments along all the mass combinations.**
- Fits to the Dalitz plot can give good descriptions of the mass projections but bad descriptions of the angular distributions.

Legendre Polynomials Moments.

□ Example from the $D_s^+ \rightarrow K^+ K^- \pi^+$ Dalitz plot analysis.



□ Pulls distributions are below each figure.

Legendre Polynomials Moments.

□ The meaning of the moments can be understood by the following relations to the S, P and D waves (spin 0, 1 and 2 resonances).

$$N = S_0^2 + P_0^2 + D_0^2$$

$$Y_1^0 = S_0 P_0 \cos(\delta_{S_0} - \delta_{P_0}) + 2\sqrt{\frac{2}{5}} P_0 D_0 \cos(\delta_{P_0} - \delta_{D_0})$$

$$Y_2^0 = \sqrt{\frac{2}{5}} P_0^2 + \frac{\sqrt{10}}{7} D_0^2 + \{\sqrt{2} S_0 D_0 \cos(\delta_{S_0} - \delta_{D_0})$$

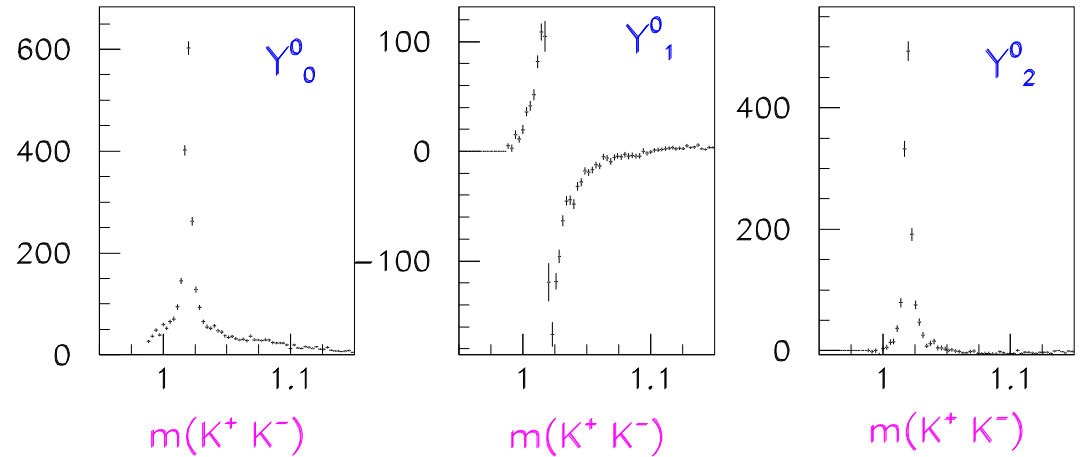
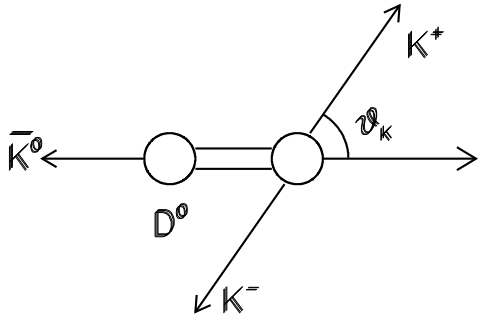
$$Y_3^0 = 3\sqrt{\frac{6}{35}} P_0 D_0 \cos(\delta_{P_0} - \delta_{D_0})$$

$$Y_4^0 = \frac{3\sqrt{2}}{7} D_0^2$$

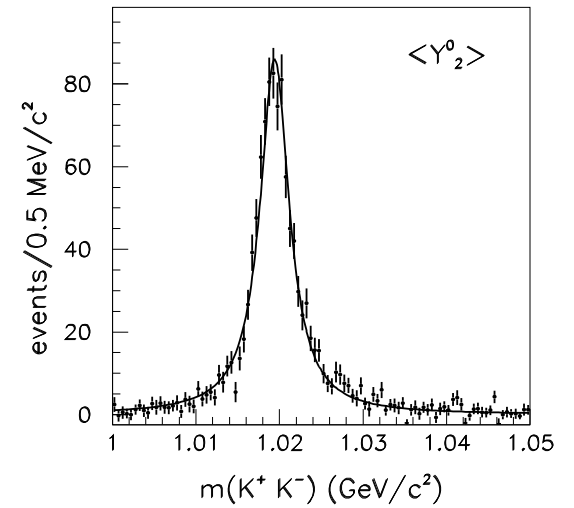
□ The S_i , P_i , and D_i are amplitude magnitudes and i denotes the relevant helicity; the corresponding phase angles are denoted by δ with the appropriate subscript.

Legendre Polynomials Moments. Examples.

□ $D^0 \rightarrow \bar{K}^0 K^+ K^-$. Definition of the helicity angle.

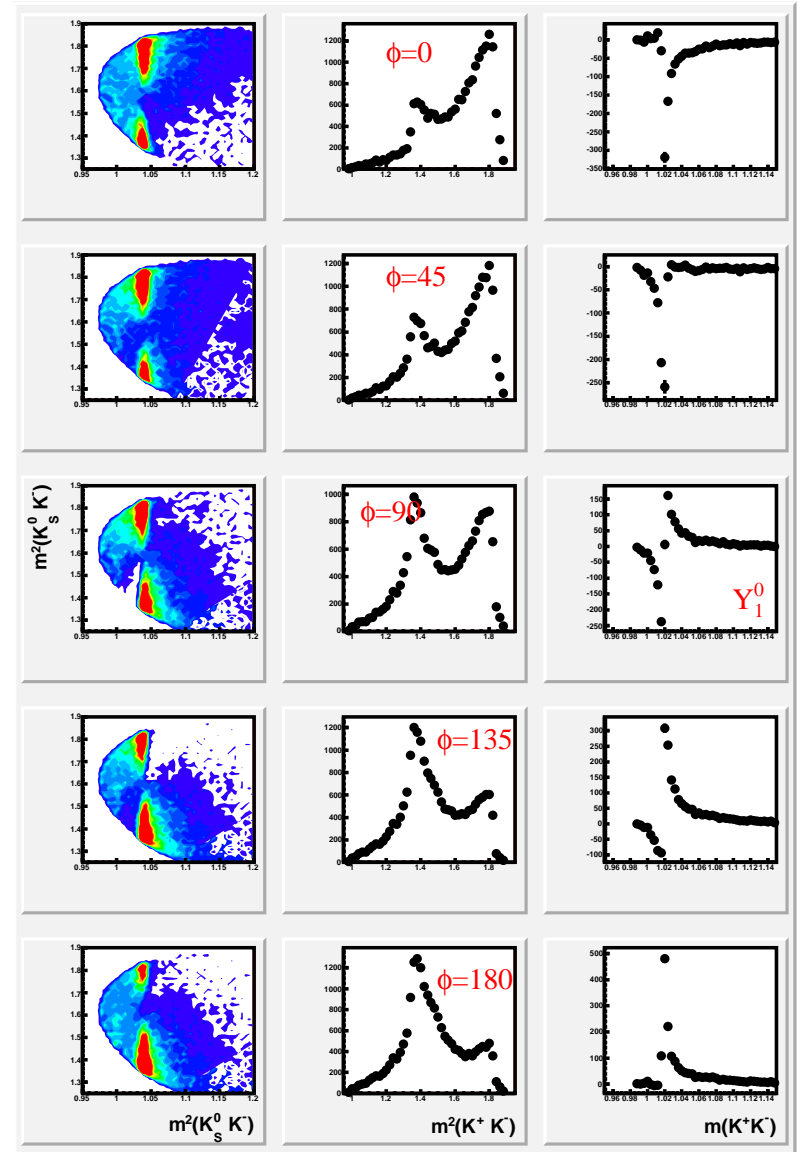


- Y_1^0 shows the S-P interference pattern.
- Y_2^0 shows the Spin 1 $\phi(1020)$ signal with no background.
- Fit to the Y_2^0 moment to obtain the $\phi(1020)$ parameters.



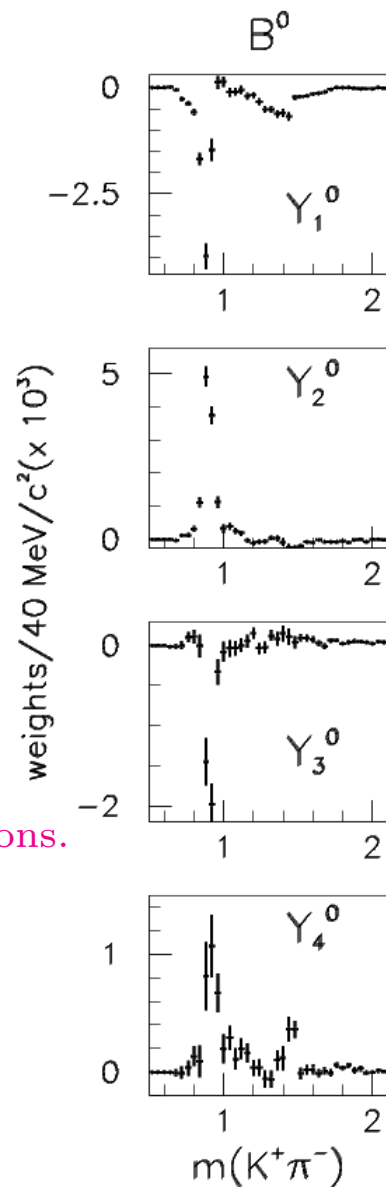
Interference effects on the Dalitz plot.

- Monte Carlo simulation of $D^0 \rightarrow \bar{K}^0 K^+ K^-$.
- Dalitz plot, $K^0 K^-$ mass spectrum and Y_1^0 moment for different value of the $\phi/a_0(980)$ phase.



Legendre Polynomials Moments. Examples.

□ From $B^0 \rightarrow J/\psi K^- \pi^+$.



□ $K^*(890)$ present in several distributions.

□ Y_4^0 shows the P ($K^*(890)$) and D waves ($K_2^*(1430)$).

Measurement of resonances parameters.

- Some resonances parameters need to be extracted directly from the Dalitz analysis.
- The analysis does not change. However the complication comes from the fact that normalization integrals need to be re-computed at each minimization step.
- For large data samples and long list of parameters the fit can take very long.
- Example. For a model independent Dalitz analysis (see later) ≈ 80 parameters and 500K events: 2.5 days.
- Methods are being developed for parallel computing.

Fit strategy.

- Insert amplitudes one by one and observe the likelihood increase and the χ^2/NDF decrease.
- Study all the possible fit possibilities. Example from the $D_s^+ \rightarrow K^+ K^- \pi^+$ Dalitz analysis.

	Fit1	Fit2	Fit3	Fit4	Fit5	Fit6	Fit7	Fit8	Fit9
	Decay Fractions(%)								
#Res	5	6	6	6	6	6	6	6	6
$f_0(980)$	16.55	20.64	16.43	16.15	15.21	14.60	14.99	18.18	17.34
$f_0(1370)$	—	—	1.06	—	—	—	—	—	—
$f_0(1500)$	—	—	—	0.21	—	—	—	—	—
$f_0(1710)$	1.69	0.60	1.15	2.27	1.82	1.81	1.75	1.74	1.63
$\phi(1020)$	39.65	41.35	41.37	41.02	41.05	41.20	41.14	40.11	40.68
$\phi(1680)$	—	—	—	—	—	—	—	—	—
$f_2(1270)$	—	—	—	—	0.03	—	—	—	—
$f_2(1525)$	—	—	—	—	—	0.01	—	—	—
$f_2(1710)$	—	—	—	—	—	—	0.16	—	—
$K_0^*(1430)$	1.90	0.60	2.36	1.57	1.93	1.63	1.66	0.95	1.03
LASS	—	—	—	—	—	—	—	—	—
$K^*(892)$	46.82	48.49	47.85	47.20	47.04	47.32	46.77	47.16	46.48
$K_1^*(1410)$	—	—	—	—	—	—	—	0.29	—
$K_1^*(1680)$	—	—	—	—	—	—	—	—	0.28
$K_2^*(1430)$	—	—	—	—	—	—	—	—	—
NR	—	3.80	—	—	—	—	—	—	—
TOT	106.60	115.48	110.22	108.41	107.08	106.56	106.46	108.43	107.43
Likelihood	-383412.	-383884.	-384344.	-383765.	-383678.	-383655.	-383710.	-383787.	-383741.
χ^2/NDF	1.62	1.43	1.24	1.49	1.53	1.53	1.52	1.49	1.51

Results from a Dalitz analysis.

- Results from the Dalitz analysis of $B_s \rightarrow J/\psi\pi^+\pi^-$.
- χ^2/ndf and $-\ln\mathcal{L}$ of different resonance models.

Resonance model	$-\ln\mathcal{L}$	χ^2/ndf	Probability (%)
Single R	59269	1956/1352	0
2R	59001	1498/1348	0.25
3R	58973	1455/1345	1.88
3R+NR (preferred)	58945	1415/1343	8.41
3R+NR (alternate)	58946	1414/1343	8.70
3R+NR + $\rho(770)$ (preferred)	58945	1418/1341	7.05
3R+NR + $\rho(770)$ (alternate)	58944	1416/1341	7.57
3R+NR + $f_0(1500)$ (preferred)	58943	1416/1341	7.57
3R+NR + $f_0(1500)$ (alternate)	58941	1407/1341	10.26
3R+NR + $f_0(600)$ (preferred)	58935	1409/1341	9.60
3R+NR + $f_0(600)$ (alternate)	58937	1412/1341	8.69