Dalitz Plot Analysis of Heavy Quark Mesons Decays (2).

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Uncertain Amplitudes.

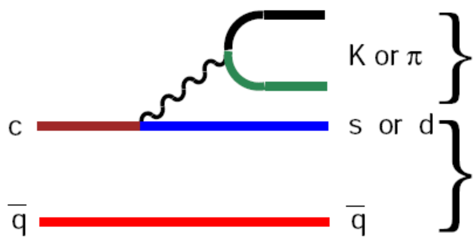
□ Dalitz analyses have to deal heavily with light meson spectroscopy where there are regions having controversial properties and parameterizations.

□ Since the aim is to have a very accurate description of the Dalitz plot, these uncertain regions need to be discussed.

□ From past hadronic experiments we gained information on the structure of the light mesons.

□ With the availibility of large samples of charmed and B mesons decays, we can obtain new information, which complement or supersede past measurements.

□ In particular, D mesons decays are more coupled to $u\bar{u}$, $d\bar{d}$ states, while D_s^+ and B_s mesons are more coupled to $s\bar{s}$ mesons.



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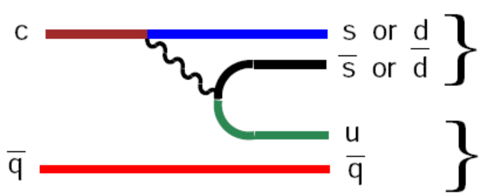

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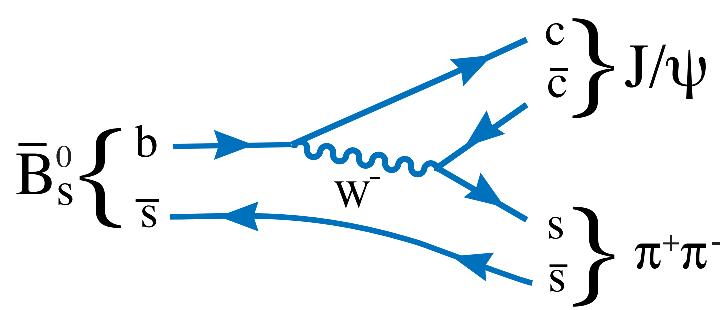


Scalar mesons.

- This is the sector where many problems still exist.
- The table of the scalar mesons.

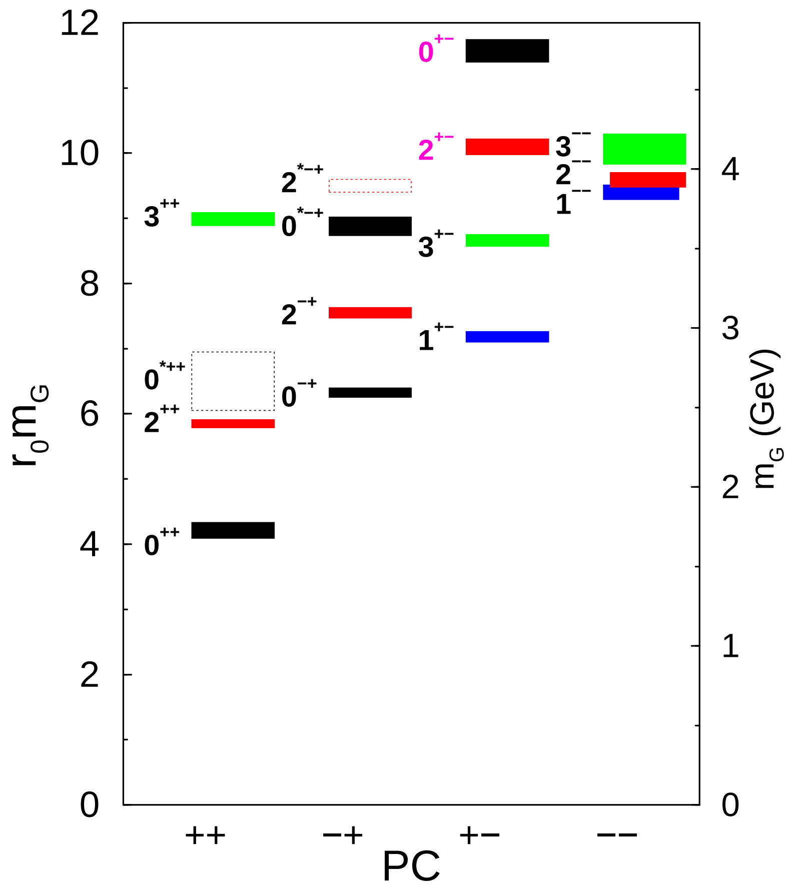
I = 1/2	I = 1	I = 0
$k(800)$		σ
	$a_0(980)$	$f_0(980)$
		$f_0(1370)$
$K_0^*(1430)$	$a_0(1490)$	$f_0(1500)$
		$f_0(1700)$
$K_0^*(1950)$		

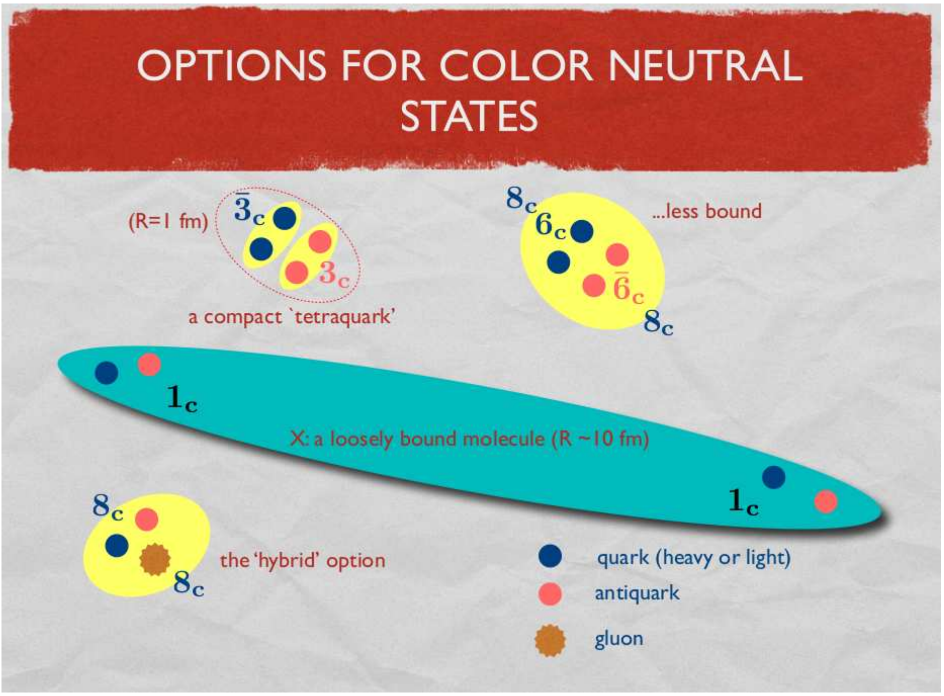
- Two nonets? 4-quark states? Gluonium?
- Are the $k(800)$ and σ true resonances?

Exotic mesons.

□ Within these states may hide exotic mesons such as gluonium or molecular states.

□ Glueballs Spectrum frpm Lattice QCD (see arXiv:0708.4016,hep-ph/0601110) and possible exotic mesons from A. Polosa (CHARM2012).



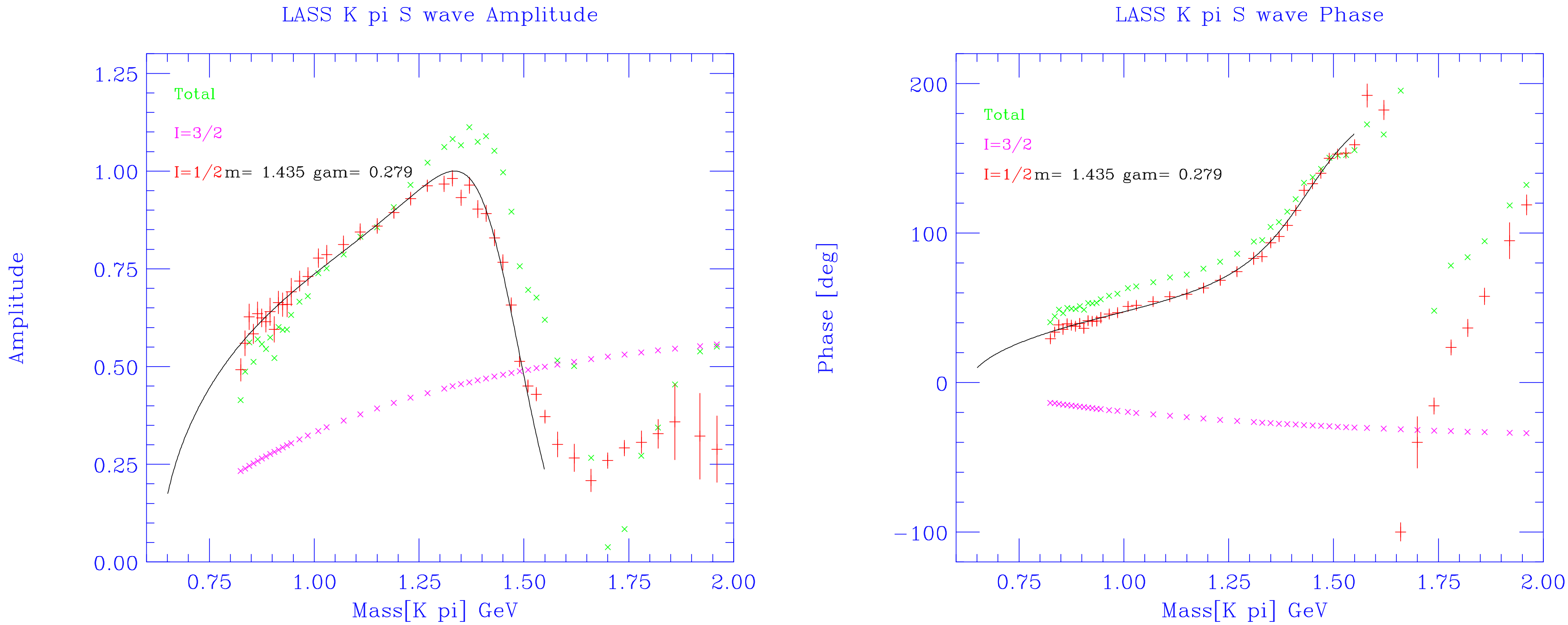


The $K\pi$ S-wave amplitude.

- The $K\pi$ S-wave amplitude and phase has been studied by LASS experiment in the reaction:

$$K^- p \to K^- \pi^+ n$$

at 11 GeV/c (see for example (SLAC - PUB - 5236)).



- The figure also evidences the I=3/2 contribution.
- This is the only existing accurate measurement.
- It is understood in terms of a scattering length contribution+the $K_0^*(1430)$ resonance.

The $K\pi$ S-wave amplitude.

□ This S-wave contribution is described using the $I = 1/2$ amplitude for S-wave $K^-\pi^+$ elastic scattering.(arXiv:0811.0564)

□ For $m_{K\pi^-} > 1.5$ GeV, $|A_S|$ is obtained by interpolation from the measured values. For lower mass values, A_S is a pure-elastic amplitude (within error) and is parameterized as

$$A_S = \frac{1}{\cot\delta_B - i} + e^{2i\delta_B}\left(\frac{1}{\cot\delta_R - i}\right)$$

where the first term is non-resonant, and the second is a resonant term rotated by $2\delta_B$ in order to maintain elastic unitarity.

$$q\cot\delta_B = \frac{1}{a} + \frac{1}{2}rq^2$$

with $a = 1.94$ GeV^{-1} and $r = 1.76$ GeV^{-1}

$$\cot\delta_R = \frac{m_S^2 - m_{K\pi}^2}{m_S\Gamma_S}$$

with

$$\Gamma_S = \Gamma_S^0\left(\frac{q}{q_S}\right)\frac{m_S}{m_{K\pi}}$$

m_S (= 1.435 GeV) is the mass of the $K_0^*(1430)$ resonance and Γ_S^0 (= 0.279 GeV) is its width.

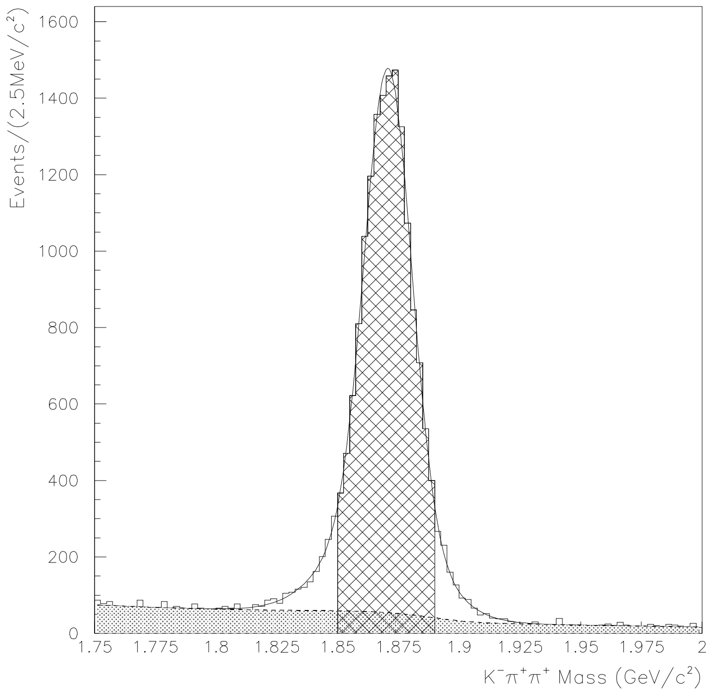
□ This parameterization corresponds also to a K-matrix approach describing a rapid phase shift coming from the resonant term and a slow rising phase shift governed by the non-resonant term.

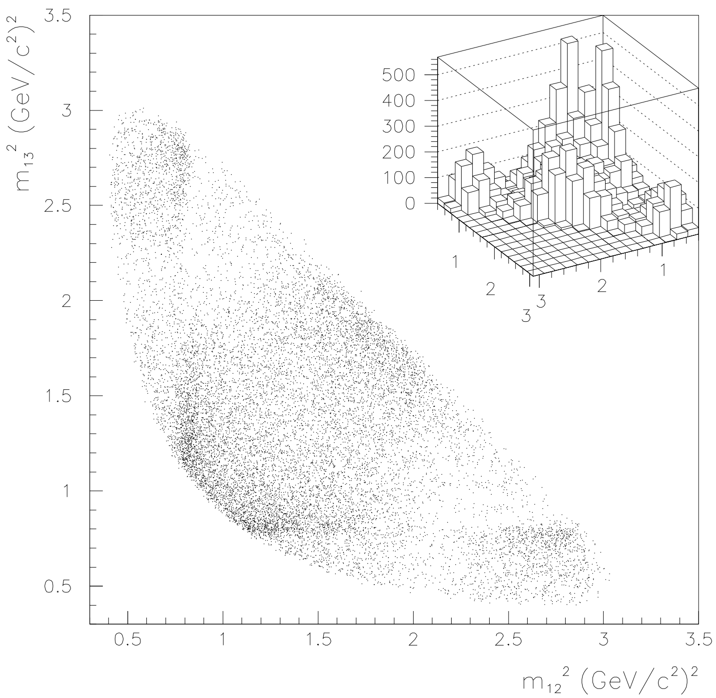
The evidence for $\kappa(800)$ from E791.

□ New information came from experiment E791 at Fermilab (hep-ex/0204018) which studied ≈ 15000 events from the decay:

$$D^+ \to K^- \pi^+ \pi^+$$

□ Mass spectrum and Dalitz plot.





□ Symmetrized Dalitz plot. The plot evidences strong S-P interferences and large scalar contribution.

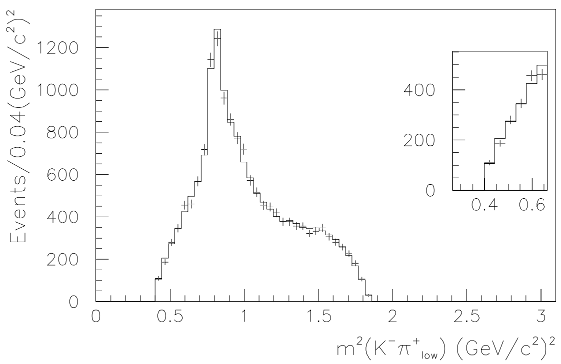
The evidence for $\kappa(800)$ from E791.

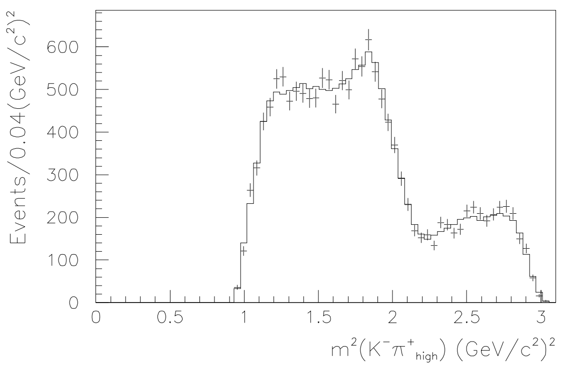
- In order to fit the Dalitz plot (isobar model) a large Non Resonant contribution is needed.
- This is rather unusual for charm decays.

Mode	**Model A**	**Model B**	**Model C**
NR	90.9 ± 2.6	89.5 ± 16.1	$13.0 \pm 5.8 \pm 4.4$
	1.0 (fixed)	2.72 ± 0.55	$1.03 \pm 0.30 \pm 0.16$
	0° (fixed)	$(-49 \pm 3)^\circ$	$(-11 \pm 14 \pm 8)^\circ$
	–	–	$47.8 \pm 12.1 \pm 5.3$
$\kappa\pi^+$	–	–	$1.97 \pm 0.35 \pm 0.11$
	–	–	$(187 \pm 8 \pm 18)^\circ$
	13.8 ± 0.5	12.1 ± 3.3	$12.3 \pm 1.0 \pm 0.9$
$\bar{K}^*(892)\pi^+$	0.39 ± 0.01	1.0 (fixed)	1.0 (fixed)
	$(54 \pm 2)^\circ$	0° (fixed)	0° (fixed)
	30.6 ± 1.6	28.7 ± 10.2	$12.5 \pm 1.4 \pm 0.5$
$\bar{K}_0^*(1430)\pi^+$	0.58 ± 0.01	1.54 ± 0.75	$1.01 \pm 0.10 \pm 0.08$
	$(54 \pm 2)^\circ$	$(6 \pm 12)^\circ$	$(48 \pm 7 \pm 10)^\circ$
	0.4 ± 0.1	0.5 ± 0.3	$0.5 \pm 0.1 \pm 0.2$
$\bar{K}_2^*(1430)\pi^+$	0.07 ± 0.01	0.21 ± 0.18	$0.20 \pm 0.05 \pm 0.04$
	$(33 \pm 8)^\circ$	$(-3 \pm 26)^\circ$	$(-54 \pm 8 \pm 7)^\circ$
	3.2 ± 0.3	3.7 ± 1.9	$2.5 \pm 0.7 \pm 0.3$
$\bar{K}^*(1680)\pi^+$	0.19 ± 0.01	0.56 ± 0.48	$0.45 \pm 0.16 \pm 0.02$
	$(66 \pm 3)^\circ$	$(36 \pm 25)^\circ$	$(28 \pm 13 \pm 15)^\circ$
χ^2/ν	167/63	126/63	46/63

The evidence for $\kappa(800)$ from E791.

□ A better fit is obtained introducing a new low mass $K\pi$ scalar resonance: $\kappa(800)$.





$$m = 797 \pm 19 \pm 42 \quad MeV, \qquad \Gamma = 410 \pm 43 \pm 85 \quad MeV$$

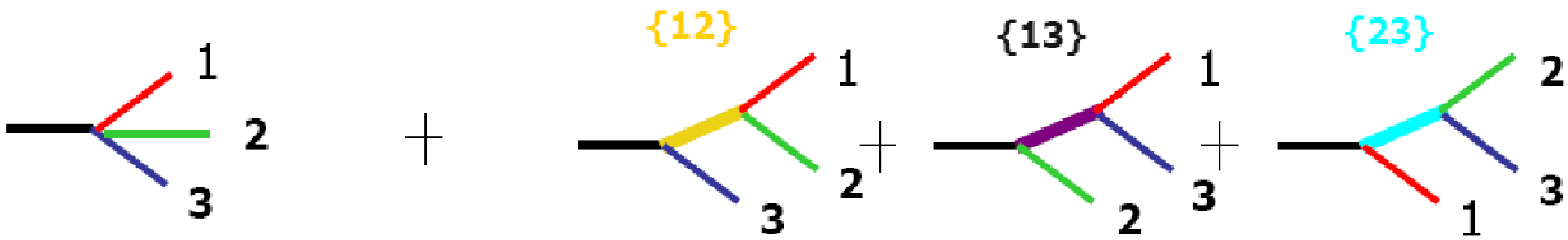
□ The analysis by E791 achieved good agreement with their data by including a low-mass $K^-\pi^+$ scalar resonance κ that significantly redistributed all fit fractions (FF) observed by earlier experiments.

□ This particular model, even though it is based on the largest data set, greatly disagrees with previous analyses.

Model Independent Amplitude Analysis (MIPWA).

□ Charmed mesons decay to light hadrons, therefore a fundamental laboratory for studying light meson spectroscopy, especially for spin 0 and spin 1 mesons.

□ The method assumes an isobar model: the dacay proceedes through a flat Non Resonant contribution + intermediate resonance production:



□ In some cases some of the decay channels can be switched off by physics.

□ From a Brian Meadows idea: extract the amplitude and phase of complicated waves from the Dalitz analysis of charm decays (hep-ex/0507099).

□ Examples are $D^+ \to K^- \pi^+ \pi^+$ and $D_s^+ \to \pi^- \pi^+ \pi^+$.

□ Two identical particles, therefore one open decay channel only but combinatorial issue.

Model Independent Amplitude analysis.

□ For $D^+ \to K^- \pi^+ \pi^+$, the two identical pions in the final state should obey Bose symmetry. Assuming that the three-body decay is dominated by two-body intermediate states, there would be two identical $K^- \pi^+$ waves interfering with each other.

□ This two-fold symmetry significantly reduces the degrees of freedom in the regular Dalitz plot analysis and allows the application of a model-independent partial wave analysis.

□ We would also expect a small contribution from the isospin-two $\pi^+ \pi^+$ S wave, which exhibits nontrivial dynamics as observed in scattering experiments.

□ In this case only one channel is open: $(K^- \pi^+)$.

- **Method:**

□ The scalar contribution is left free in the Dalitz plot analysis in terms of a complex number:

$$c_{m(K\pi)} e^{i\phi_{m(K\pi)}}$$

over a $K^- \pi^+$ mass greed which is related to the number of available events.

□ The fit measures amplitude and phase in each bin of the $K\pi$ mass.

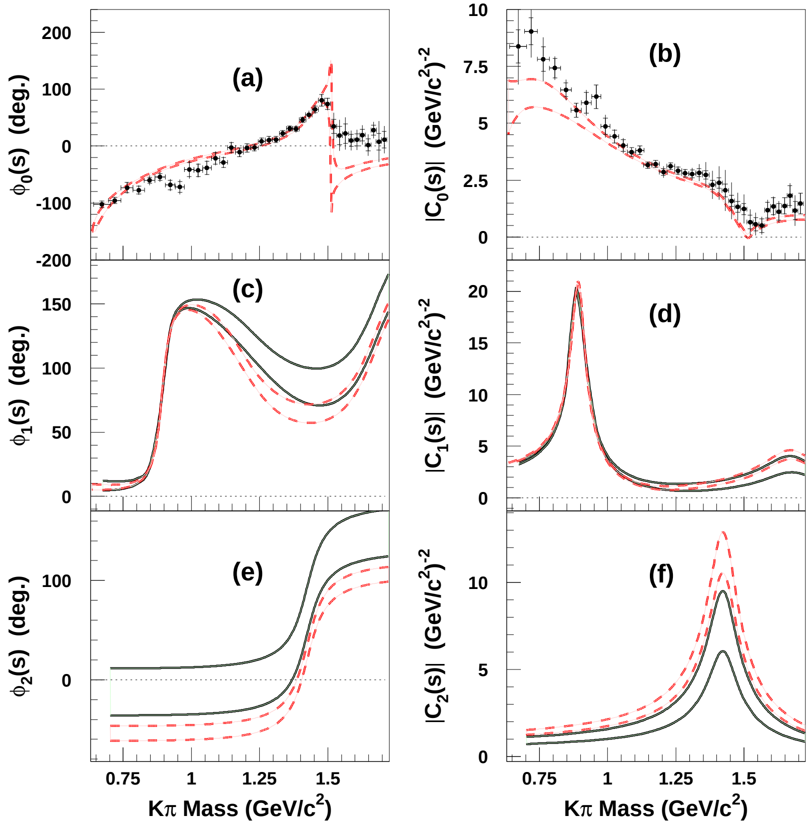
□ The P-wave is described as the sum of a Breit-Wigner propagator term for the $K^*(892)$ resonance and the $K_1^*(1680)$.

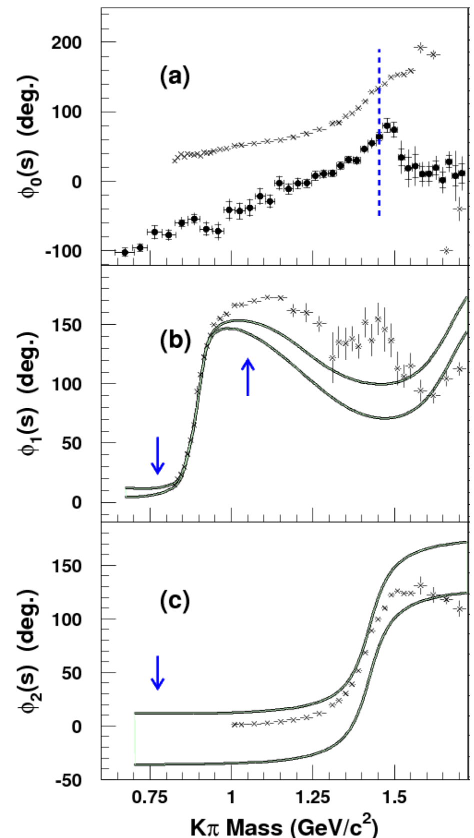
□ The D-wave is described by a single Breit-Wigner term for the $K_2^*(1430)$ resonance, with a further complex coefficient.

Results.

□ Resulting S- amplitude and phase. Comparison with LASS elastic data.

□ Dashed curves are from the isobar model.





□ Compared with LASS elastic scattering phase, an overall shift in phase of $(-74.4 \pm 1.8 \pm 1.0)^{\circ}$ relative to the P-wave is still required.

□ These results do not conform, exactly, to the expectations of the Watson theorem which would require phases in each wave to match, apart from an overall shift, those for $K^{-}\pi^{+}$ scattering for invariant masses below $K\eta'$ threshold.

The problem of the multiple solutions

□ It is evident that both MIPWA and isobar fits are good and that no statistically significant distinction between these two descriptions of the data can be drawn with a sample of this size.

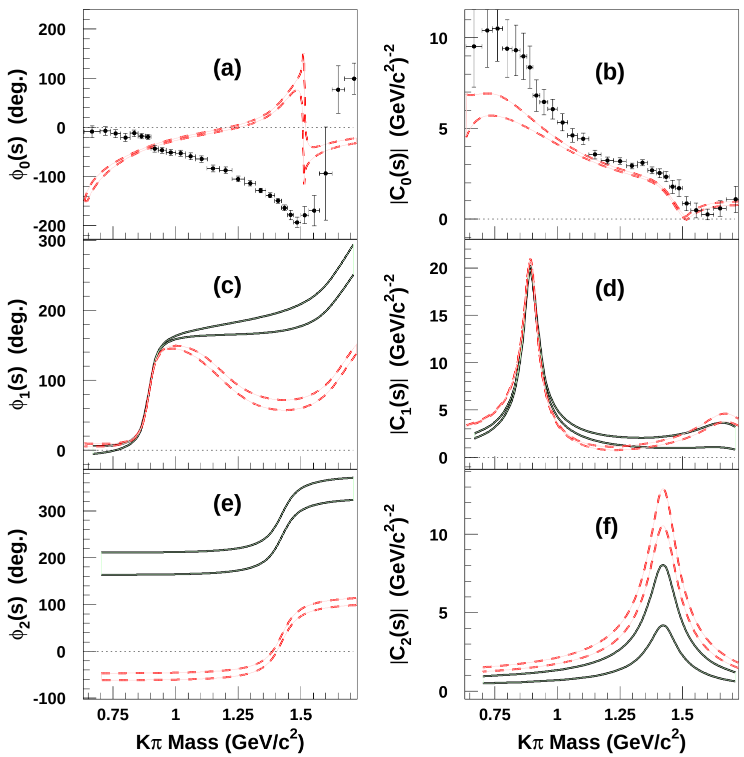
□ The fitting procedure allows a great deal of freedom to the S-wave amplitude. Consequently, ambiguities in solutions are anticipated. To study possible ambiguities in the MIPWA solution, fits with random starting values for the c_i, γ_i parameters, and also with different $K^-\pi^+$ mass slices are made.

□ One other local maximum in the likelihood is found, and this is labelled solution B. Solution A is the only one with an acceptable χ^2/NDF and has the greatest likelihood value. So it is emphasized that solutions A is, in fact, unique.

□ Solution B provides a qualitatively reasonable description of the distribution of the data on the Dalitz plot.

□ However, this solution clearly exhibits retrograde motion around the unitarity circle as $K^-\pi^+$ invariant mass increases.

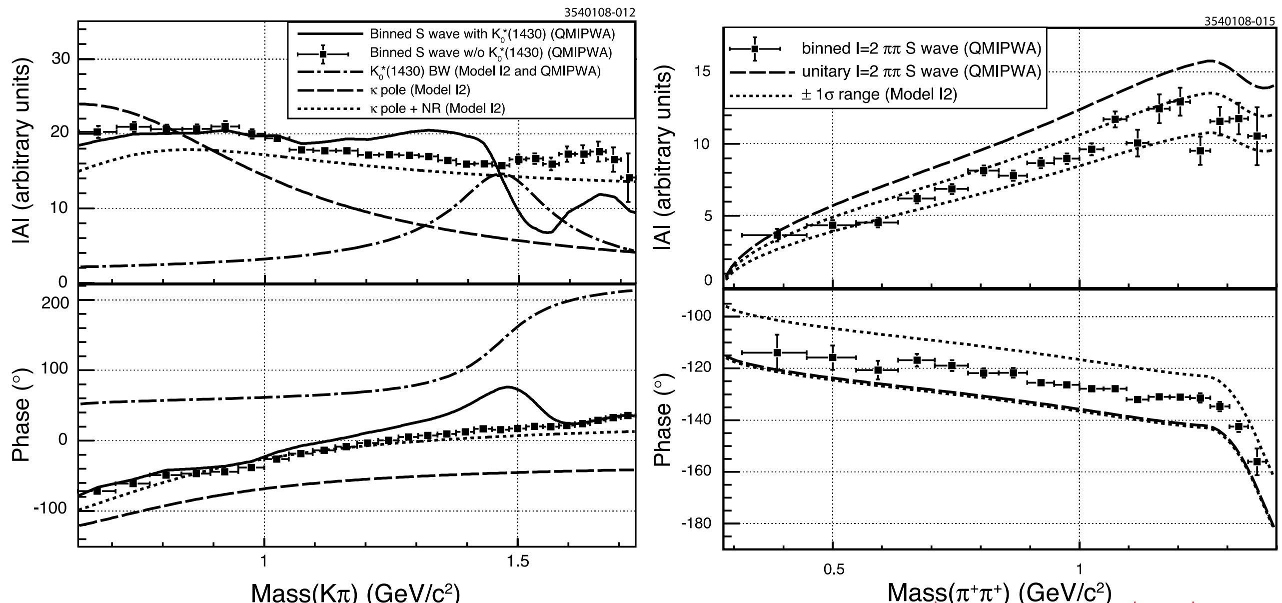
□ This violates the Wigner causality principle, thus eliminating it from further consideration.



CLEO Analysis.

□ CLEO selects 140793 $D^+ \to K^-\pi^+\pi^+$ candidates for the Dalitz plot analysis obtaining a very clean sample with a background fraction of about 1.1% and 9 times larger than the data set used by E791 (arXiv:0802.4214).

□ They perform a slightly modified version of the MIPWA analysis.



□ They find that the total observed S wave magnitude in the $D^+ \to K^-\pi^+\pi^+$ decay is essentially constant from $K\pi$ production threshold to 1.4 GeV$/c^2$.

□ The phase shows smooth variation from -80° to 40° in the same range.

□ At higher invariant mass $m_{K\pi} > 1.4$ GeV$/c^2$, the S wave behavior is dominated by the $K_0^*(1430)$ resonance.

□ They find that the P wave contribution is dominated by $K^*(892)$ and $K^*(1680)$ Breit-Wigner resonances, and the D wave has only a contribution from $K_2^*(1430)$.

Mode	Parameter	Model I2 (B-W for κ)	Model I2	QMIPWA
$\overline{K}^*(892)\pi^+$	a	1 – fixed	1 – fixed	1 – fixed
	FF (%) 2×	5.15±0.24	5.27±0.08±0.15	4.94±0.23
	m (MeV/c^2)	895.4±0.2	895.7±0.2±0.3	895.7 – fixed
	Γ (MeV/c^2)	44.5±0.7	45.3±0.5±0.6	45.3 – fixed
$\overline{K}^*(1680)\pi^+$	a	4.45±0.23	3.38±0.16±0.78	2.88±0.84
	FF (%) 2×	0.238±0.024	0.144±0.013±0.12	0.098±0.059
$\overline{K}_2^*(1430)\pi^+$	a	0.866±0.030	0.915±0.025±0.04	0.794±0.073
	FF (%) 2×	0.124±0.011	0.145±0.009±0.03	0.102±0.020
$\overline{K}_0^*(1430)\pi^+$	a	3.97±0.15	3.74±0.02±0.06	3.74 – fixed
	FF (%) 2×	7.53±0.65	7.05±0.14±0.55	6.65±0.31
	m (MeV/c^2)	1461.1±1.0	1466.6±0.7±3.4	1466.6 – fixed
	Γ (MeV/c^2)	177.9±3.1	174.2±1.9±3.2	174.2 – fixed
$\kappa\pi^+$	a	5.69±0.17	10.80±0.05±0.35	0
	FF (%) 2×	8.5±0.5	21.6±0.3±3.2	0
Pole	$\Re m_0$ (MeV/c^2)		706.0±1.8±22.8	
	$\Im m_0$ (MeV/c^2)		–319.4±2.2±20.2	
Breit-Wigner	m (MeV/c^2)	888.0±1.9		
	Γ (MeV/c^2)	550.4±11.8		
NR	a	17.1±0.4	23.3±0.1±1.6	0
	FF (%)	38.0±1.9	73.8±0.8±9.6	0
Binned $K^-\pi^+$ S wave	FF (%) 2×	0	0	41.9±1.9
$I = 2$ $\pi^+\pi^+$ S wave	a	30.3±2.7	25.5±0.3±2.9	33.1±2.6
	FF (%)	13.4±2.3	9.8±0.2±2.0	15.5±2.8
	$\sum$ FF$_i$ (%)	94.4	152.0	122.8
Goodness	χ^2/ν	426/385	416/385	359/347
	Probability (%)	7.4%	13.2%	31.5%

FOCUS Analysis.

□ Using data collected by the high energy photoproduction experiment FOCUS at Fermilab has performed a Dalitz plot analysis of the decay $D^+ \to K^-\pi^+\pi^+$ using 53653 Dalitz-plot events with a signal fraction of $\sim 97\%$ (arXiv:0705.2248) (arXiv:0905.4846).

□ Within the K-matrix approach, they present the determination in D-decays of the two separate isospin contributions, $I = 1/2$ and $I = 3/2$, for the S-wave $K\pi$ system.

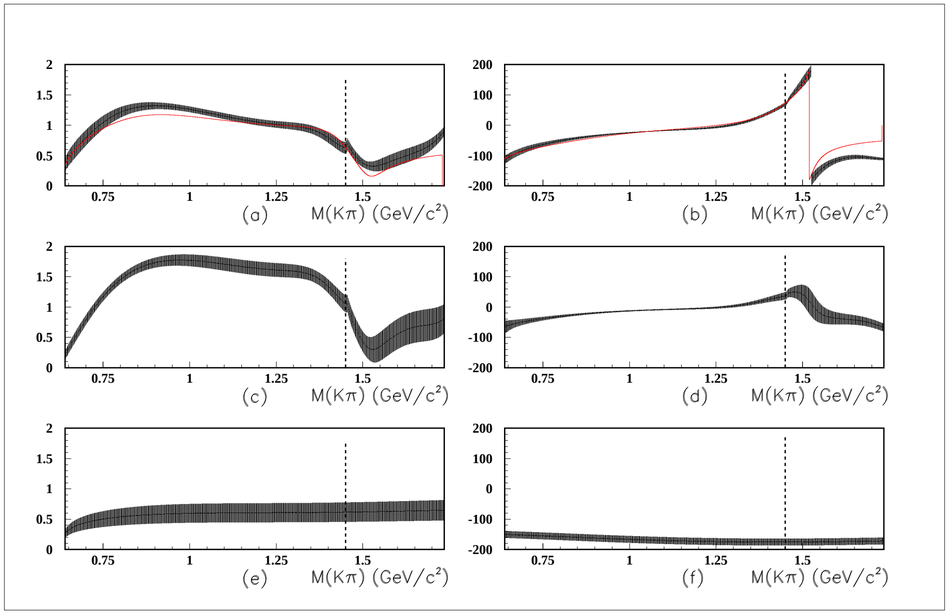
□ I=1/2, Total and I=3/2 contributions.

□ The isobar model with its Breit–Wigner representation for all states requires both a κ and a $K_0^*(1430)$ whose parameters are not what elastic scattering would require.

□ However, such Breit–Wigner parameters are effective rather than genuine pole positions.

□ In contrast, the *K-matrix* fit has built in consistency with $K\pi$ scattering.

□ These results indicate close consistency with $K\pi$ scattering data, and consequently with Watson's theorem predictions for two-body $K\pi$ interactions in the low $K\pi$ mass region.

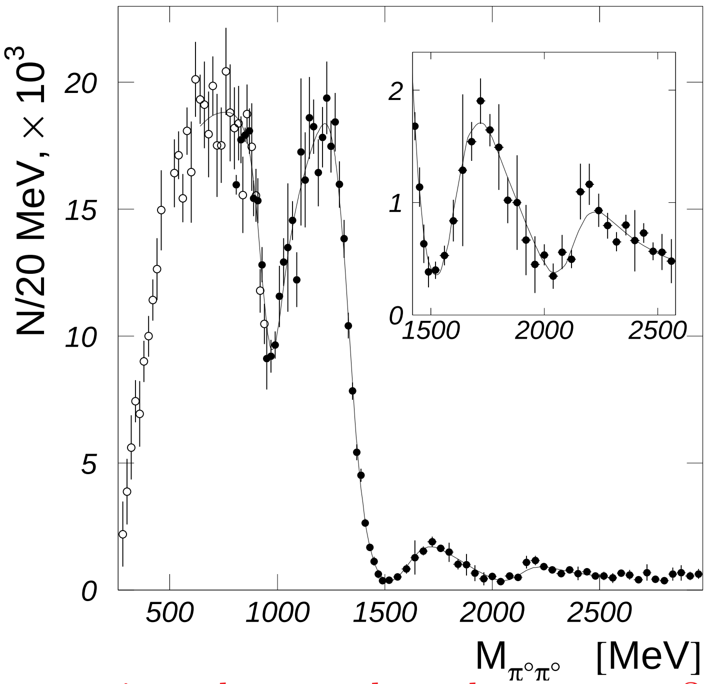


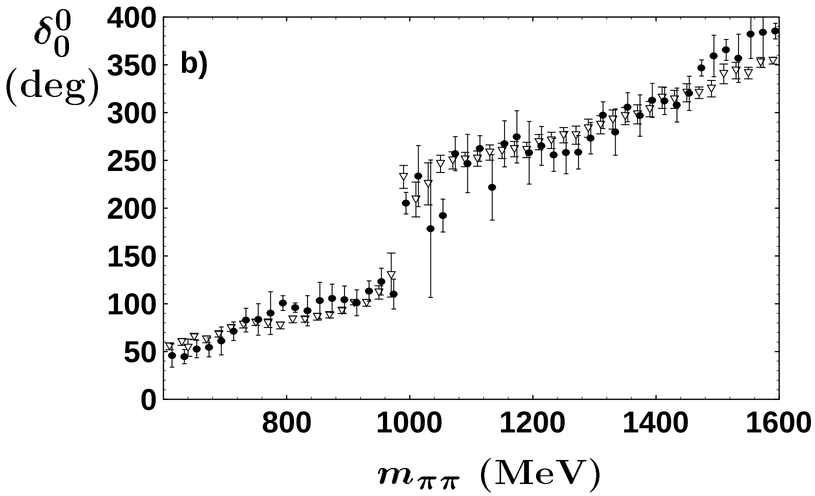
Scalar mesons: The σ.

□ The σ is a very wide amplitude extending from the $\pi\pi$ threshold up to 1.5 GeV. (arXiv:0708.4016)

□ The $\pi\pi$ amplitude and phase has been measured in:

$$\pi^- p \longrightarrow \pi\pi n$$





□ Slowly moving phase: a broad resonance? $\sigma(500)$?

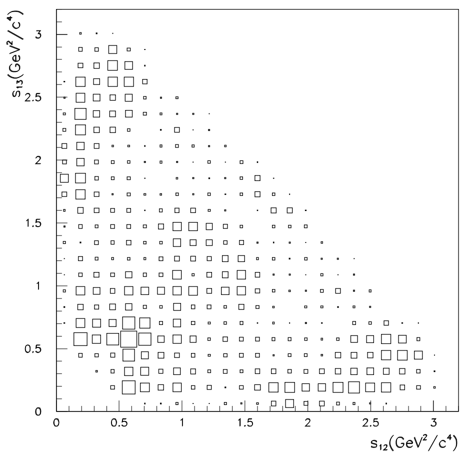
□ The spectrum can be understood in terms of a slowly moving phase with the presence of a narrow $f_0(980)$ resonance and a broader $f_0(1400)$ resonance.

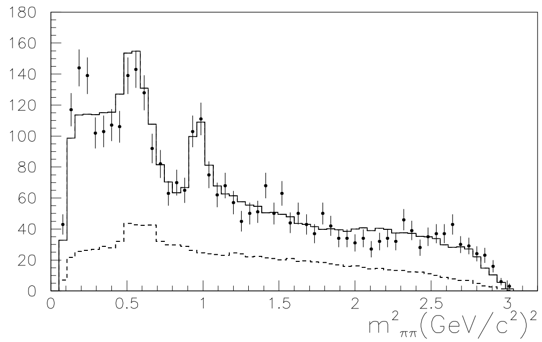
□ Alternative proposal: The $\sigma(500)$ identified as the scalar glueball (**Red Dragon**) (hep-ph/9811518).

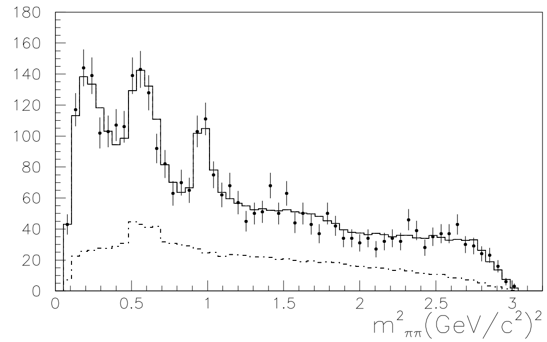
The evidence for $\sigma(500)$.

□ The existence of the $\sigma(500)$ has been recently triggered again by the Dalitz Plot analysis of $D^+ \to \pi^+\pi^+\pi^-$ (E791) ($\approx$ 1200 events) (hep-ex/0007028).

□ Dalitz plot. Fit without and with a $\sigma(500)$ resonance.







□ Amplitudes symmetrized because of the presence of two identical pions.

$$\mathcal{A}_n = \mathcal{A}_n[(\mathbf{12})\mathbf{3}] + \mathcal{A}_n[(\mathbf{13})\mathbf{2}]$$

□ In order to obtain a good fit of the Dalitz plot they need to introduce a scalar resonance close to threshold.

□ And they extract the following σ parameters.

$$m = 478 \pm 24 \pm 17 \quad MeV$$
$$\Gamma = 324 \pm 41 \pm 21 \quad MeV$$

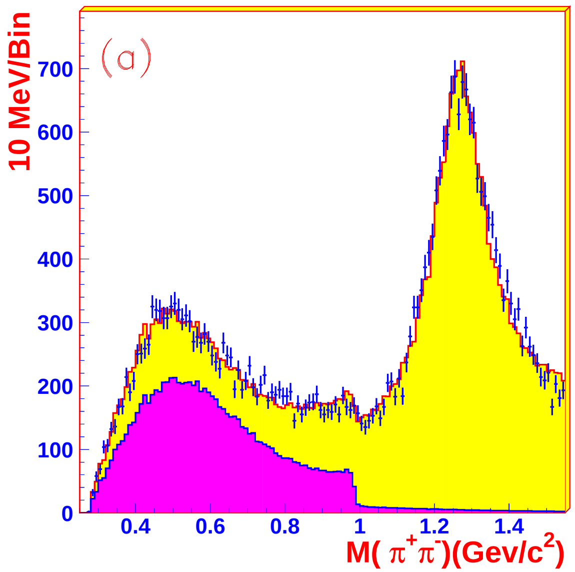
□ Results from the Dalitz analysis.

mode	Fit 1 Fraction(%) Magnitude Phase	Fit 2 Fraction(%) Magnitude Phase
$\sigma\pi^+$	–	$46.3 \pm 9.0 \pm 2.1$
	–	$1.17 \pm 0.13 \pm 0.06$
	–	$(205.7 \pm 8.0 \pm 5.2)^\circ$
$\rho^0(770)\pi^+$	20.8 ± 2.4	$33.6 \pm 3.2 \pm 2.2$
	1(fixed)	1(fixed)
	0(fixed)	0(fixed)
NR	38.6 ± 9.7	$7.8 \pm 6.0 \pm 2.7$
	1.36 ± 0.20	$0.48 \pm 0.18 \pm 0.09$
	$(150.1 \pm 11.5)^\circ$	$(57.3 \pm 19.5 \pm 5.7)^\circ$
$f_0(980)\pi^+$	7.4 ± 1.4	$6.2 \pm 1.3 \pm 0.4$
	0.60 ± 0.07	$0.43 \pm 0.05 \pm 0.02$
	$(151.8 \pm 16.0)^\circ$	$(165.0 \pm 10.9 \pm 3.4)^\circ$
$f_2(1270)\pi^+$	6.3 ± 1.9	$19.4 \pm 2.5 \pm 0.4$
	0.55 ± 0.08	$0.76 \pm 0.06 \pm 0.03$
	$(102.6 \pm 16.0)^\circ$	$(57.3 \pm 7.5 \pm 2.9)^\circ$
$f_0(1370)\pi^+$	10.7 ± 3.1	$2.3 \pm 1.5 \pm 0.8$
	0.72 ± 0.12	$0.26 \pm 0.09 \pm 0.03$
	$(143.2 \pm 9.7)^\circ$	$(105.4 \pm 17.8 \pm 0.6)^\circ$
$\rho^0(1450)\pi^+$	22.6 ± 3.7	$0.7 \pm 0.7 \pm 0.3$
	1.04 ± 0.12	$0.14 \pm 0.07 \pm 0.02$
	$(45.8 \pm 14.9)^\circ$	$(319.1 \pm 39.0 \pm 10.9)^\circ$

The evidence for $\sigma(500)$.

- □ BES: study of $J/\psi \to \omega\pi^+\pi^-$ (hep-ex/0406038).
- □ Large threshold scalar enhancement.
- □ Several fitting models. From the mean of six analyses they obtain a pole position at:

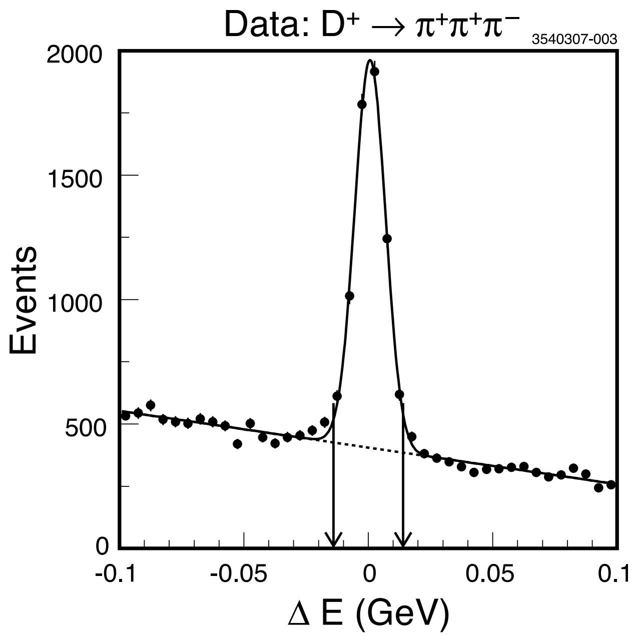
$$m = (541 \pm 39) - i(252 \pm 42) MeV$$

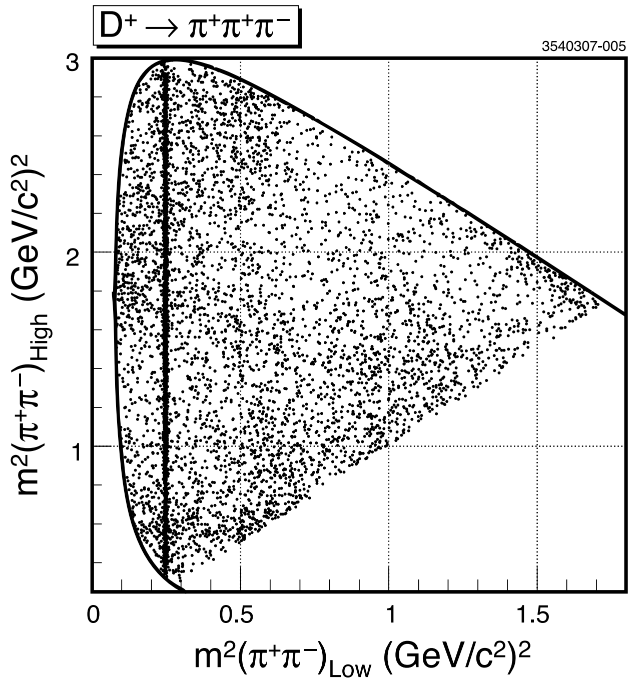


Dalitz analysis of $D^+ \to \pi^+\pi^+\pi^-$ from CLEO.

□ Data collected at the $\psi(3770) \to D^+D^-$ resonance (arXiv:0704.3954).

□ Presence of $K_S^0 \to \pi^+\pi^-$ removed by a mass cut. $\approx$ 2600 events.



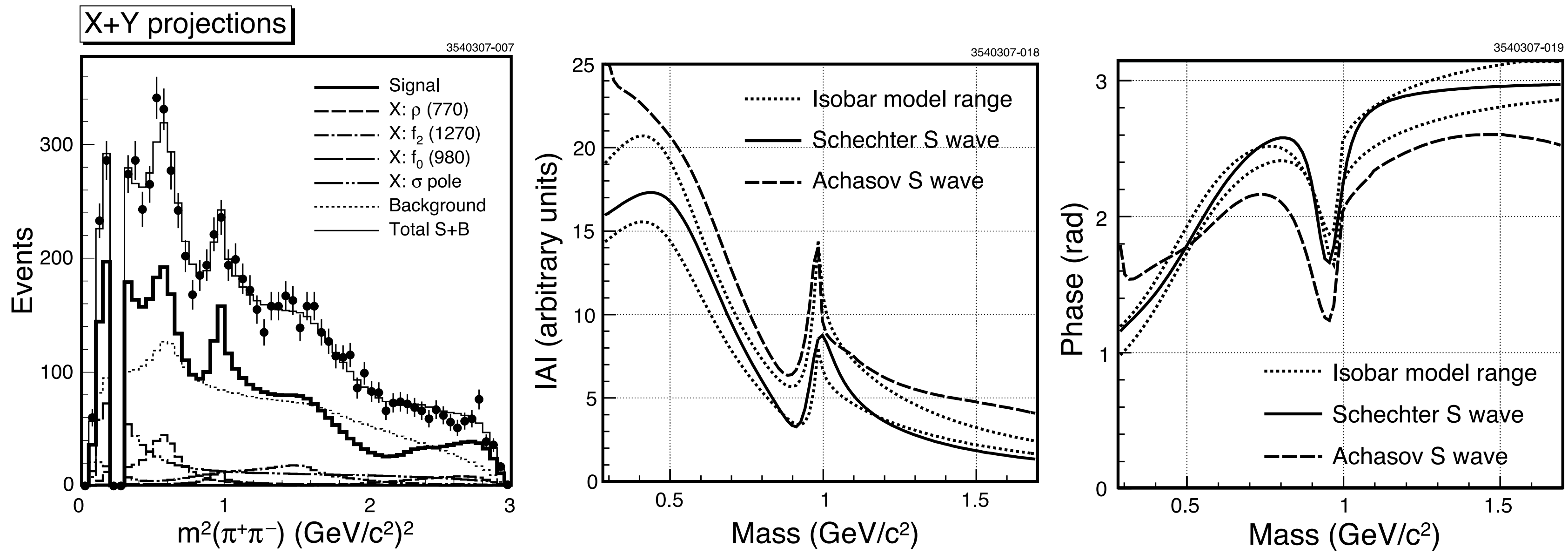


□ They perform a Dalitz analysis using three different models: isobar, Schechter and Achasov.

□ The isobar model includes the best description of the σ and the Flatté parameterization for the threshold effects on the $f_0(980)$.

Dalitz analysis of $D^+ \to \pi^+\pi^+\pi^-$ from CLEO.

- Comparison between three different models for the $\pi\pi$ S-wave.



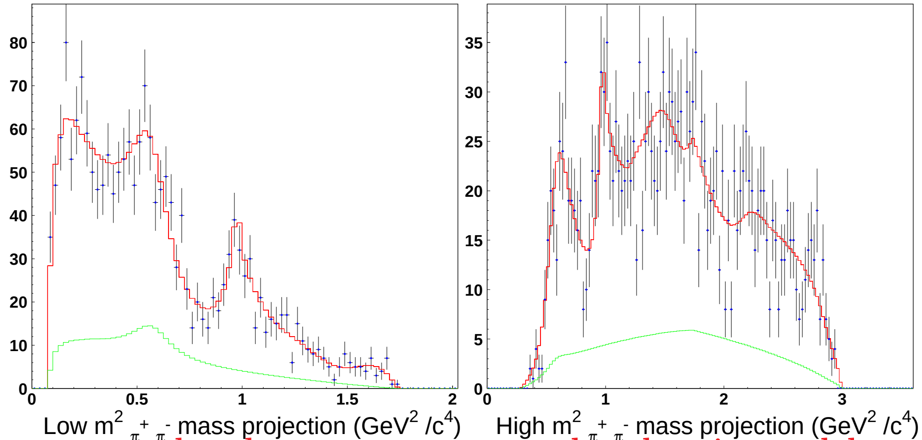
- Results of the isobar model analysis.

Mode	Amplitude (a.u.)	Phase (°)	Fit fraction (%)
$\rho(770)\pi^+$	1(fixed)	0(fixed)	20.0±2.3±0.9
$f_0(980)\pi^+$	1.4±0.2±0.2	12±10±5	4.1±0.9±0.3
$f_2(1270)\pi^+$	2.1±0.2±0.1	−123±6±3	18.2±2.6±0.7
$f_0(1370)\pi^+$	1.3±0.4±0.2	−21±15±14	2.6±1.8±0.6
$f_0(1500)\pi^+$	1.1±0.3±0.2	−44±13±16	3.4±1.0±0.8
σ pole	3.7±0.3±0.2	−3±4±2	41.8±1.4±2.5

- The sum of all fit fractions is 90.1%, and the fit probability is ≃28% for 90 degrees of freedom.

K-matrix fits to $D^+ \to \pi^+\pi^+\pi^-$ Dalitz plot.

- FOCUS: Analysis of $D^+ \to \pi^+\pi^+\pi^-$. 1527 ± 51 signal events (hep-ex/0312040).
- Use of the K-matrix formalism for the description of the $\pi\pi$ S-wave.
- The *K-matrix* treatment of the S-wave component of the decay amplitude modeled in terms of the five virtual channels considered: $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ and 4π.



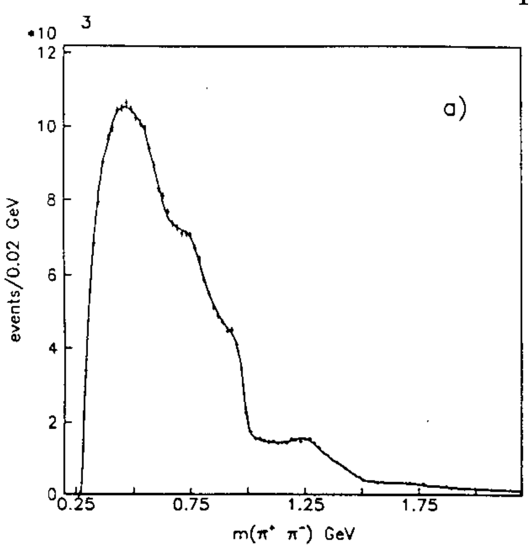
- Beside the S-wave component, the decay appears to be dominated by the $\rho^0(770)$ plus a $f_2(1270)$ component.
- The direct three-body non-resonant component was not necessary since the S-wave could reproduce the entire non-resonant portion of the Dalitz plot.

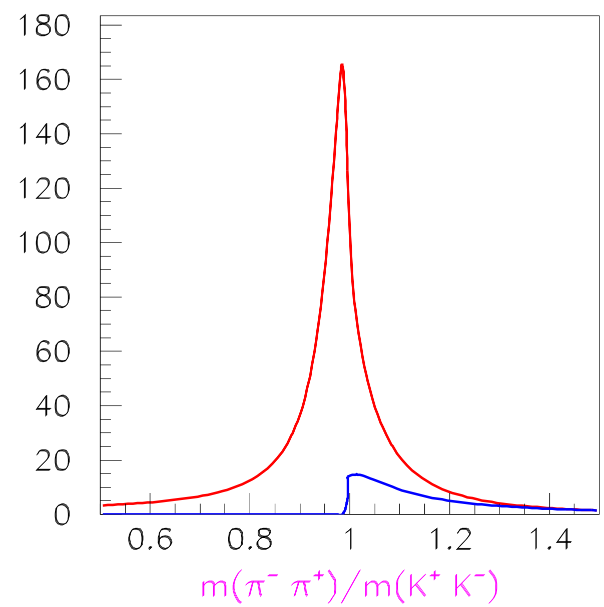
decay channel	fit fraction (%)	phase (deg)	amplitude coefficient
$(S\text{-wave})\,\pi^+$	$56.00 \pm 3.24 \pm 2.08$	0 (fixed)	1 (fixed)
$f_2(1270)\,\pi^+$	$11.74 \pm 1.90 \pm 0.23$	$-47.5 \pm 18.7 \pm 11.7$	$1.147 \pm 0.291 \pm 0.047$
$\rho^0(770)\,\pi^+$	$30.82 \pm 3.14 \pm 2.29$	$-139.4 \pm 16.5 \pm 9.9$	$1.858 \pm 0.505 \pm 0.033$
Fit C.L.	7.7 %		

- Good description of the data. No need for additional resonances.

The $f_0(980)$ resonance.

- ☐ The $f_0(980)$ resonance has been discovered many years ago but has still uncertain parameters and interpretations because is just sitting at the $K\bar{K}$ threshold and strongly coupled to the $\pi\pi$ and $K\bar{K}$ final states.
- ☐ Many good data exist on its $\pi\pi$ projection.
- ☐ A few good data on its the $K\bar{K}$ projection, complicated by the presence of the $a_0(980)$ resonance.
- ☐ The $f_0(980)$ lineshape is determined by:
  - Its coupling to the $K\bar{K}$ final state;
  - The interference with other wide scalar resonances;
  - Resolution effects related to both the relatively narrow width and the presence of thresholds;



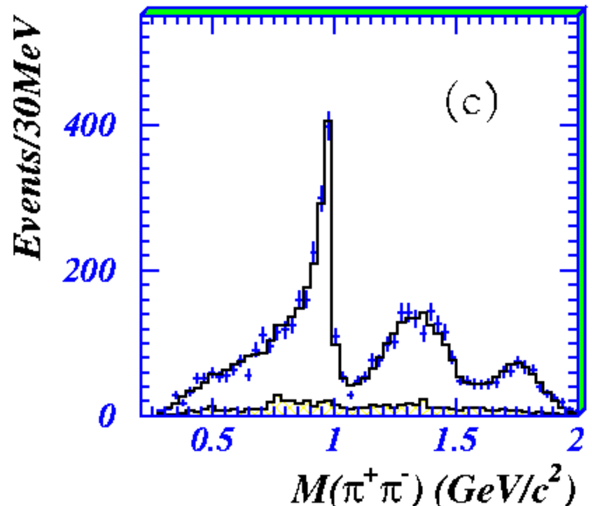


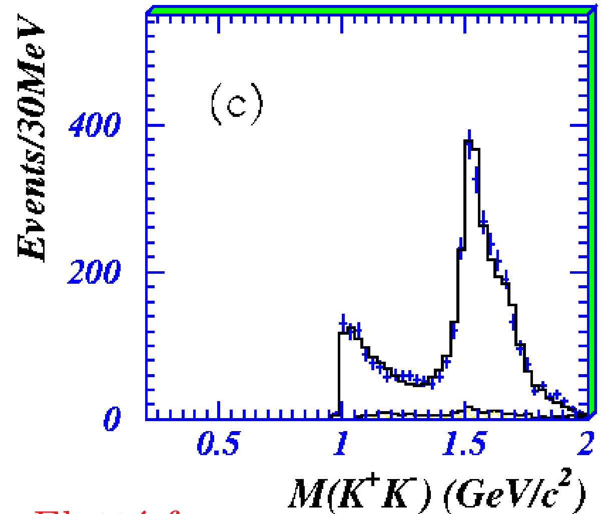
- ☐ $f_0(980)$ as a sharp drop in $pp \to p_f(\pi^+\pi^-)p_s$ at 300 GeV/c.

□ First analyses performed by MarkIII and DM2. Recent analysis from BES (hep-ex/0411001).

□ Study of:

$$J/\psi \to \phi\pi^+\pi^- \qquad and \qquad J/\psi \to \phi K^+K^-$$





□ Here the $f_0(980)$ amplitude has been fitted to the Flatté form:

$$f = \frac{1}{M^2 - s - im_0(g_1\rho_{\pi\pi} + g_2\rho_{K\bar{K}})}. \tag{1}$$

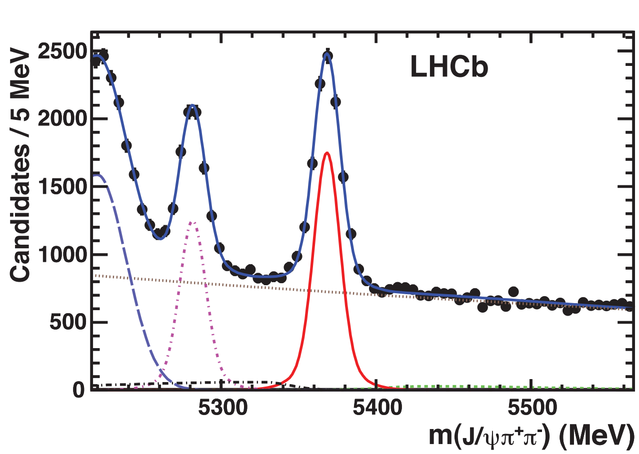
□ ρ is Lorentz invariant phase space, $2k/\sqrt{s}$, where k refers to the π or K momentum in the rest frame of the resonance.

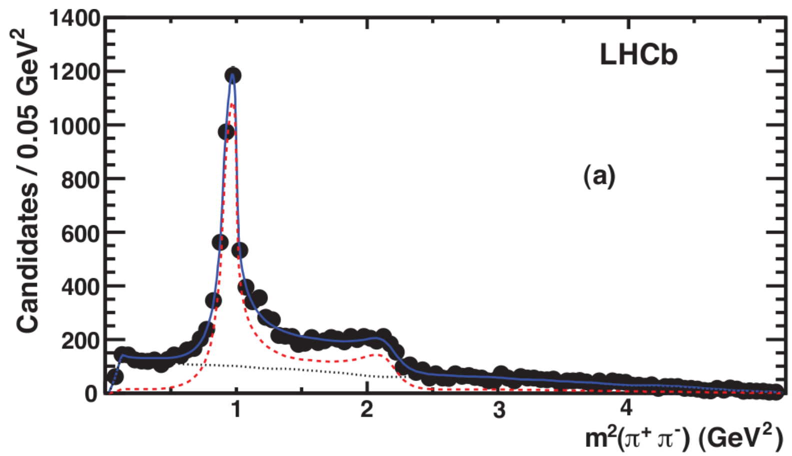
□ Fitted $f_0(980)$ parameters.

$M = 965 \pm 8 \pm 6 MeV/c^2, \quad g_1 = 165 \pm 10 \pm 15 MeV/c^2, \quad g_2/g_1 = 4.21 \pm 0.25 \pm 0.21$

Extracting the $f_0(980)$ parameters.

□ Study of $B_s \to J/\psi\pi^+\pi^-$ at LHCb:





□ The largest component is the $f_0(980)$ that is described by a Flatté function. The data are best described by adding the $f_0(1370)$, the $f_2(1270)$ and a non-resonance contribution.

Components	3R+NR	3R+NR+ρ	3R+NR+$f_0(1500)$	3R+NR+$f_0(600)$
$f_0(980)$	107.09 ± 3.51	104.84 ± 3.91	72.99 ± 5.82	115.24 ± 5.32
$f_0(1370)$	32.57 ± 4.10	32.30 ± 3.72	113.67 ± 13.57	34.47 ± 3.98
$f_0(1500)$	-	-	15.00 ± 4.83	-
$f_0(600)$	-	-	-	4.68 ± 2.46
NR	12.84 ± 2.32	12.16 ± 2.22	10.66 ± 2.06	23.65 ± 3.59
$f_2(1270)$, $\lambda = 0$	0.76 ± 0.25	0.77 ± 0.25	1.07 ± 0.37	0.90 ± 0.31
$f_2(1270)$, $\|\lambda\| = 1$	0.33 ± 1.00	0.26 ± 1.12	1.02 ± 0.83	0.61 ± 0.87
Sum	153.6 ± 6.0	151.1 ± 6.0	214.4 ± 15.7	179.6 ± 8.0
Probability(%)	8.41	7.05	7.57	9.61

The $f_0(1370)$.

□ Due to its interference with the broad $\pi\pi$ S-wave and $f_0(1500)$, the $f_0(1370)$ (and can appear shifted in invariant mass spectra.

□ Therefore, the application of simple Breit-Wigner forms arrive at slightly different resonance masses.

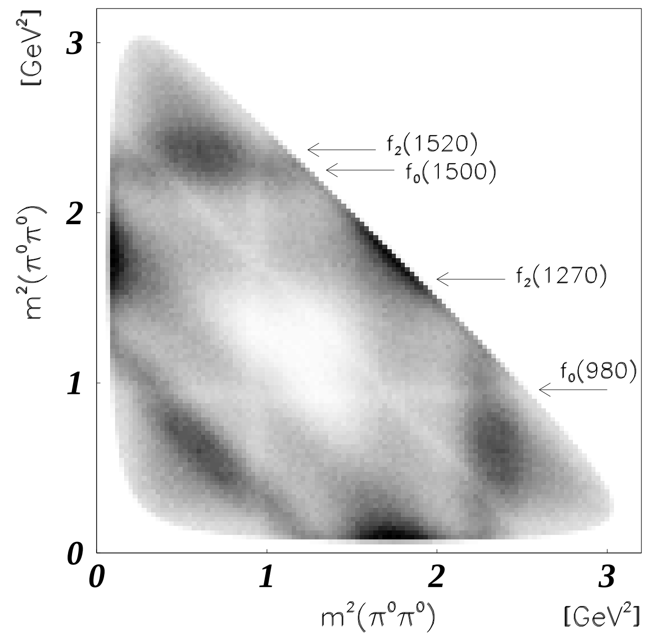
□ Since it does not show up prominently in the $\pi\pi$ spectra, its mass and width are difficult to determine.

□ Multichannel analyses of hadronically produced two- and three-body final states agree on a mass between 1300 MeV and 1400 MeV and a narrow $f_0(1500)$, but arrive at a somewhat smaller width for $f_0(1370)$.

The $f_0(1500)$.

□ The $f_0(1500)$ (M=1509±10,$\Gamma = 116 \pm 17$ MeV) was discovered by Crystal Barrel in $\bar{p}p$ annihilations at rest.

$$\bar{p}p \to \pi^0\pi^0\pi^0, \bar{p}p \to \eta\eta\pi^0, \bar{p}p \to \eta'\eta\pi^0, \bar{p}p \to K_L^0 K_L^0 \pi^0$$



□ Rates:

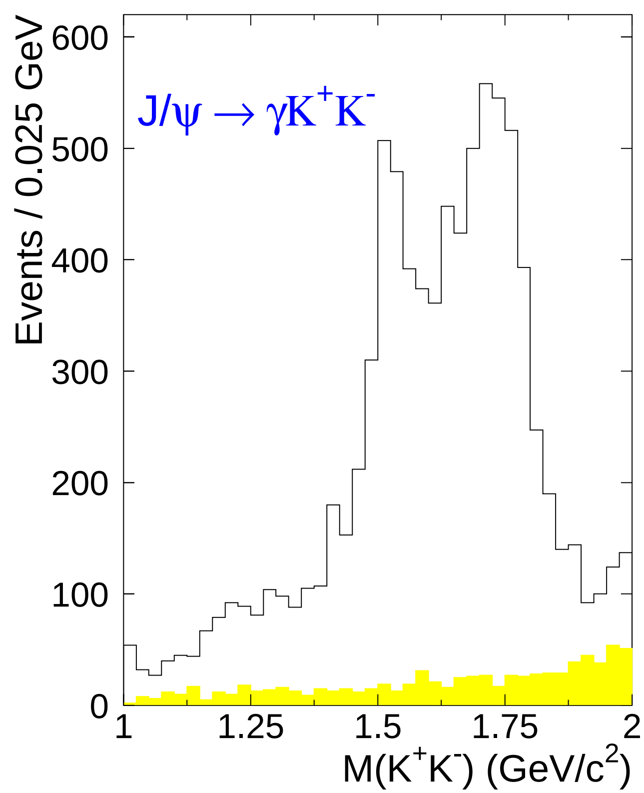
$$\pi\pi : K\bar{K} : \eta\eta : \eta\eta' = (5.1 \pm 2.0) : (0.71 \pm 0.21) : (1.0) : (1.3 \pm 0.5)$$

□ At moment little evidence for $f_0(1500)$ production in heavy flavors decays.

The $f_0(1700)$

□ The $f_0(1700)$ has been discovered in radiative $J/\psi \to \gamma K^+ K^-$ although its spin has been controversial for some time between $J^P = 0^+$ and $J^P = 2^+$.

□ The K^+K^- mass spectrum from BES (arXiv:hep-ex/0209031), (hep-ex/0307058). Signals are due to $f_2'(1525)$ and $f_0(1700)$.



□ BES measured parameters:

$m = 1740 \pm 4^{+10}_{25}, \qquad \Gamma = 166^{+5}_{-8}{}^{+15}_{-10}\ MeV$

□ At moment little evidence for $f_0(1700)$ production in heavy flavors decays.

Dalitz analysis of $D^0 \to \bar{K}^0\pi^+\pi^-$ (Belle).

□ The question of the scalar mesons came out again in the framework of the measurement of γ using the Dalitz analysis of D^0 decays.

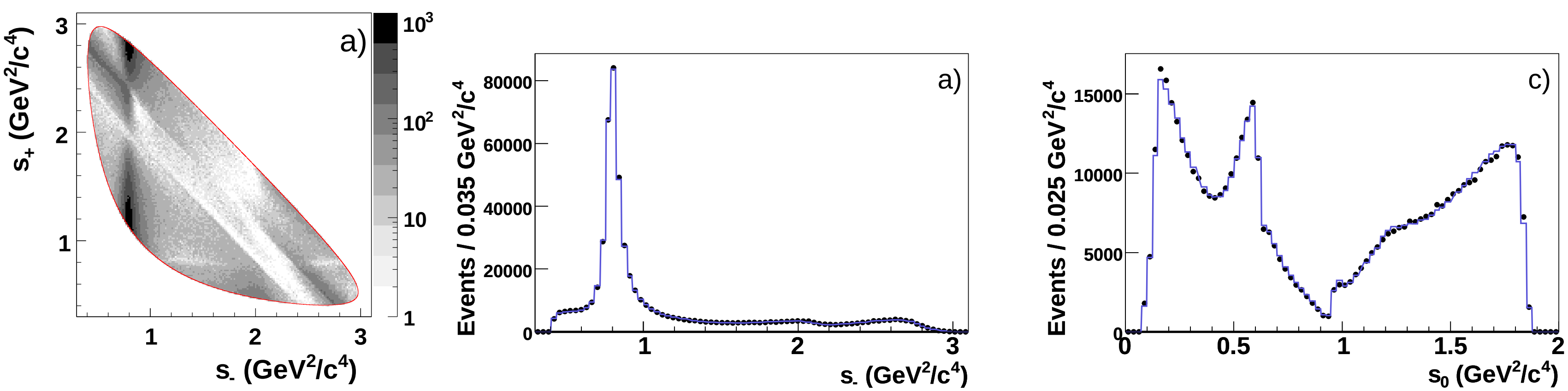
□ To obtain a good fit it was necessary to introduce two scalar mesons, labeled as σ_1 and σ_2.

Intermediate state	Amplitude	Phase ($^\circ$)	Fit fraction
$K_S^0\sigma_1$	1.43 ± 0.07	212 ± 3	9.8%
$K_S^0\rho^0$	1.0 (fixed)	0 (fixed)	21.6%
$K_S^0\omega$	0.0314 ± 0.0008	110.8 ± 1.6	0.4%
$K_S^0 f_0(980)$	0.365 ± 0.006	201.9 ± 1.9	4.9%
$K_S^0\sigma_2$	0.23 ± 0.02	237 ± 11	0.6%
$K_S^0 f_2(1270)$	1.32 ± 0.04	348 ± 2	1.5%
$K_S^0 f_0(1370)$	1.44 ± 0.10	82 ± 6	1.1%
$K_S^0\rho^0(1450)$	0.66 ± 0.07	9 ± 8	0.4%
$K^*(892)^+\pi^-$	1.644 ± 0.010	132.1 ± 0.5	61.2%
$K^*(892)^-\pi^+$	0.144 ± 0.004	320.3 ± 1.5	0.55%
$K^*(1410)^+\pi^-$	0.61 ± 0.06	113 ± 4	0.05%
$K^*(1410)^-\pi^+$	0.45 ± 0.04	254 ± 5	0.14%
$K_0^*(1430)^+\pi^-$	2.15 ± 0.04	353.6 ± 1.2	7.4%
$K_0^*(1430)^-\pi^+$	0.47 ± 0.04	88 ± 4	0.43%
$K_2^*(1430)^+\pi^-$	0.88 ± 0.03	318.7 ± 1.9	2.2%
$K_2^*(1430)^-\pi^+$	0.25 ± 0.02	265 ± 6	0.09%
$K^*(1680)^+\pi^-$	1.39 ± 0.27	103 ± 12	0.36%
$K^*(1680)^-\pi^+$	1.2 ± 0.2	118 ± 11	0.11%
non-resonant	3.0 ± 0.3	164 ± 5	9.7%

Dalitz analysis of $D^0 \to \bar{K}^0 \pi^+ \pi^-$ (BaBar).

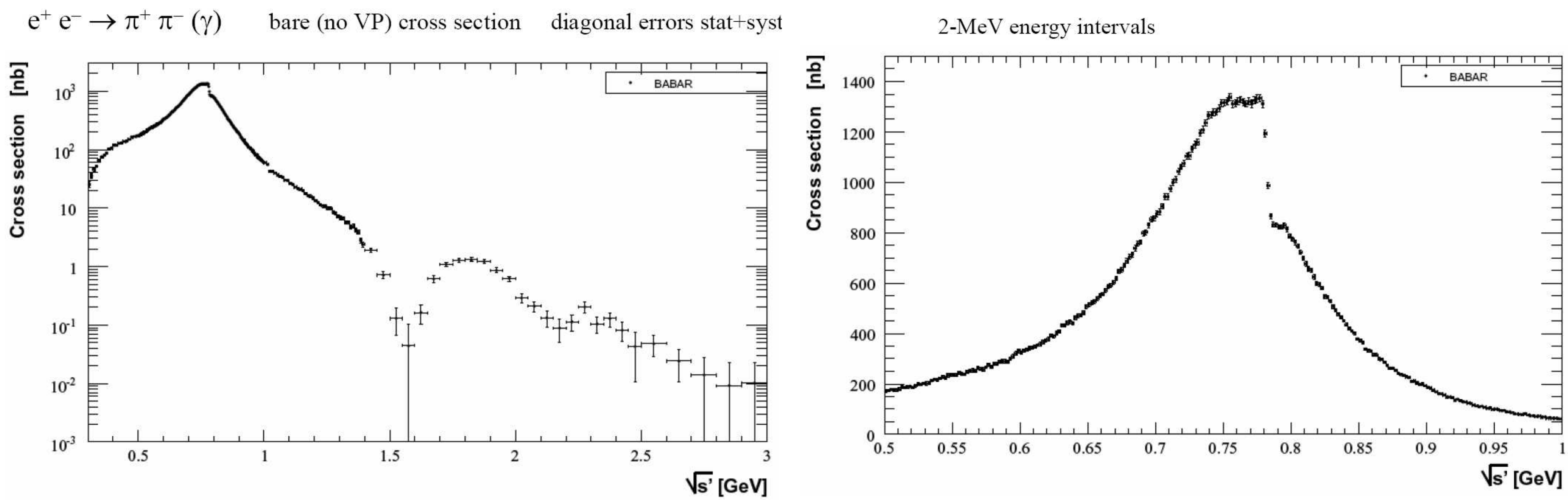
□ BaBar followed a different method: K-matrix description of the $\pi\pi$ S-wave.

Component	Amplitude	Phase (rad)	Fit fraction (%)
$K^*(892)^-$	1.735 ± 0.005	2.331 ± 0.004	57.0
$\rho(770)^0$	1	0	21.1
$K_0^*(1430)^-$	2.650 ± 0.015	1.497 ± 0.007	6.1
$K_2^*(1430)^-$	1.303 ± 0.013	2.498 ± 0.012	1.9
$\omega(782)$	0.0420 ± 0.0006	2.046 ± 0.014	0.6
$K^*(892)^+$	0.164 ± 0.003	-0.768 ± 0.019	0.6
$K^*(1680)^-$	0.90 ± 0.03	-2.97 ± 0.04	0.3
$f_2(1270)$	0.410 ± 0.013	2.88 ± 0.03	0.3
$K_0^*(1430)^+$	0.145 ± 0.014	1.78 ± 0.10	< 0.1
$K_2^*(1430)^+$	0.115 ± 0.013	2.69 ± 0.11	< 0.1
$\pi\pi$ S-wave			15.4



The $\rho(770)$ description.

□ The description of the $\rho(770)$ has to take into account the presence of broad ρ' high mass resonances and the presence of the ρ/ω interference.

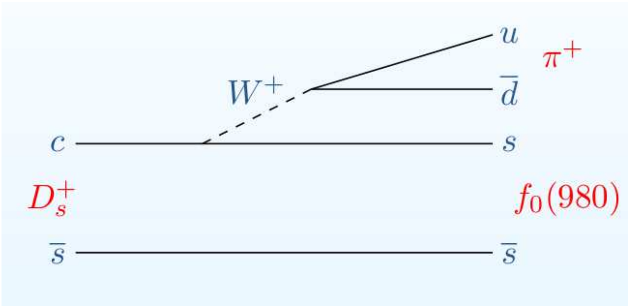


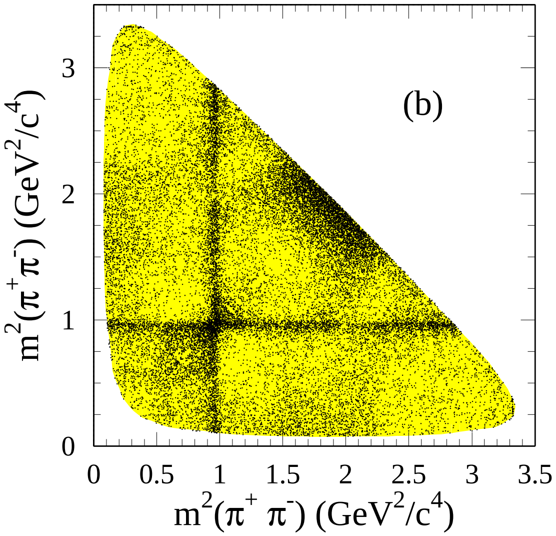
□ For $\pi\pi$ vector resonances ($\rho(770)$ and $\rho(1450)$) standard method is the use Gounaris-Sakurai (GS) Breit-Wigner parametrization (G.J. Gounaris, J.J. Sakurai, Phys. Rev. Lett. **21**, 24 (1968).

Dalitz Plot Analysis of three-body D_s^+ decays $D_s^+ \to \pi^+\pi^-\pi^+$

□ BaBar has performed a Dalitz plot analysis of $D_s^+ \to \pi^+\pi^-\pi^+$ (arXiv:0808.0971).

□ D_s^+ mesons are strongly coupled to $s\bar{s}$ mesons, therefore it is possible to extract the $\pi\pi$ S-wave coupled to $s\bar{s}$.





□ The resulting D_s^+ signal region contains 13179 events with a purity of 80%.

□ Dalitz plot, symmetrized along the two axes.

□ We observe a clear $f_0(980)$ signal, evidenced by the two narrow crossing bands. We also observe a broad accumulation of events in the 1.9 GeV region.

Dalitz Plot Analysis of three-body D_s^+ decays $D_s^+ \to \pi^+\pi^-\pi^+$

□ The efficiency is found to be almost uniform as a function of the $\pi^+\pi^-$ invariant mass with an average value of ≈ 1.6 %.

□ In the Dalitz plot analysis spin-1 and spin-2 resonances are described by relativistic Breit-Wigner.

□ For the $\pi^+\pi^-$ $\mathcal{S}$-wave amplitude, a different approach is used because:

- Scalar resonances have large uncertainties. In addition, the existence of some states needs confirmation.
- Modelling the $\mathcal{S}$-wave as a superposition of Breit-Wigners is unphysical since it leads to a violation of unitarity when broad resonances overlap.

□ To overcome these problems, the Model-Independent Partial Wave Analysis has been used.

□ Instead of including the $\mathcal{S}$-wave amplitude as a superposition of relativistic Breit-Wigner functions, the $\pi^+\pi^-$ mass spectrum is divided into 29 slices and the $\mathcal{S}$-wave is parametrized by an interpolation between the 30 endpoints in the complex plane:

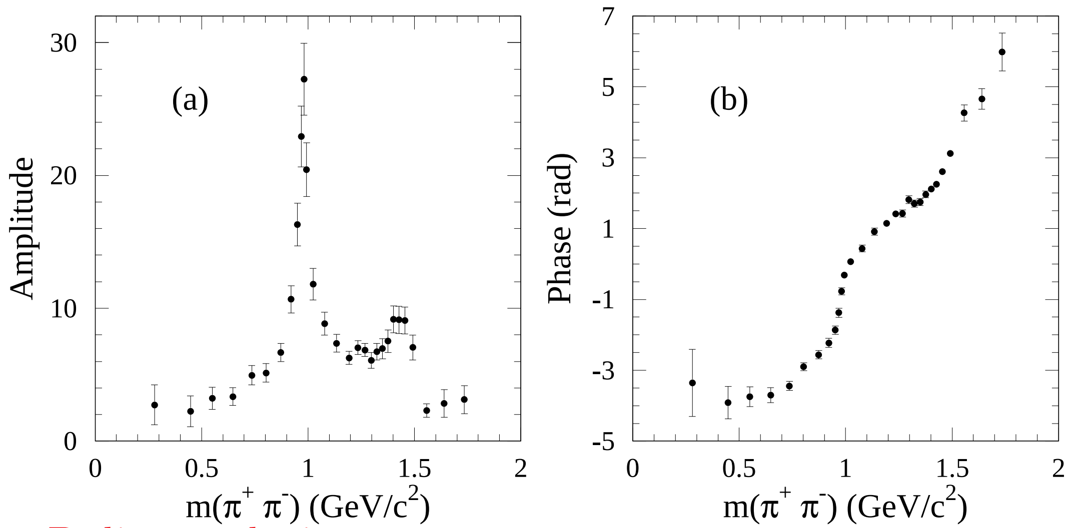
$$A_{\mathcal{S}-\mathrm{wave}}(m_{\pi\pi}) = \mathrm{Interp}(c_k(m_{\pi\pi})e^{i\phi_k(m_{\pi\pi})})_{k=1,..,30}. \tag{2}$$

□ The amplitude and phase of each endpoint are free parameters. The width of each slice is tuned to get approximately the same number of $\pi^+\pi^-$ combinations ($\simeq 13179 \times 2/29$).

□ Interpolation is implemented by a Relaxed Cubic Spline. The phase is not constrained in a specific range in order to allow the spline to be a continuous function.

Dalitz Plot Analysis of $D_s^+ \to \pi^+\pi^-\pi^+$

□ Resulting $\mathcal{S}$-wave $\pi^+\pi^-$ amplitude and phase.

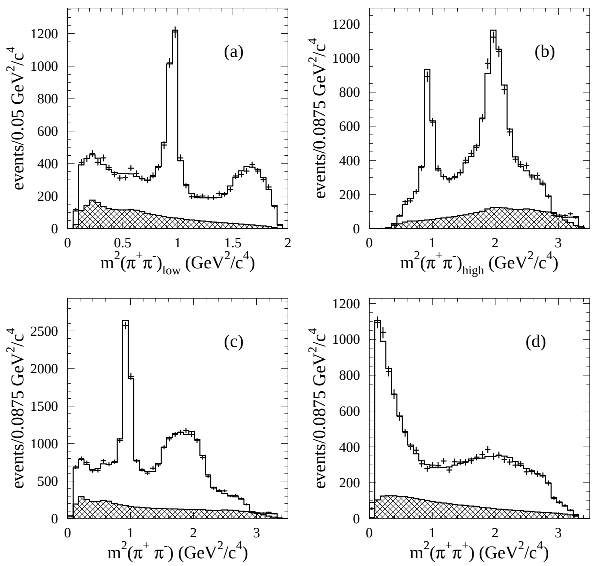


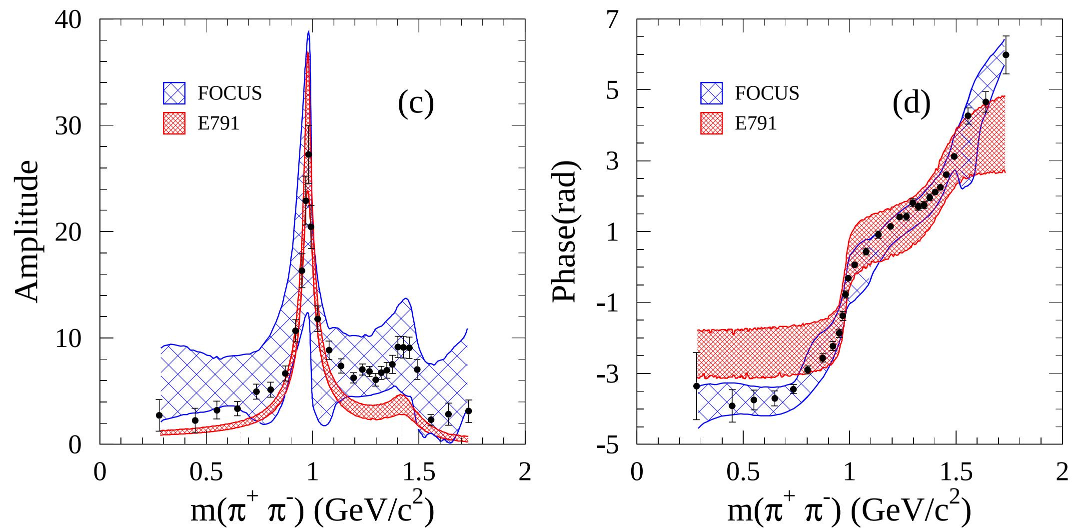
□ The results from the Dalitz analysis.

Decay Mode	Decay fraction(%)	Amplitude	Phase(rad)
$f_2(1270)\pi^+$	$10.1 \pm 1.5 \pm 1.1$	1.(Fixed)	0.(Fixed)
$\rho(770)\pi^+$	$1.8 \pm 0.5 \pm 1.0$	$0.19 \pm 0.02 \pm 0.12$	$1.1 \pm 0.1 \pm 0.2$
$\rho(1450)\pi^+$	$2.3 \pm 0.8 \pm 1.7$	$1.2 \pm 0.3 \pm 1.0$	$4.1 \pm 0.2 \pm 0.5$
$\mathcal{S}$-wave	$83.0 \pm 0.9 \pm 1.9$		
Total	$97.2 \pm 3.7 \pm 3.8$		
χ^2/NDF	$\frac{437}{422-64} = 1.2$		

Dalitz Plot Analysis of three-body D_s^+ decays $D_s^+ \to \pi^+\pi^-\pi^+$

- Dalitz plot projections together with the fit results.
- The hatched histograms show the background distribution.

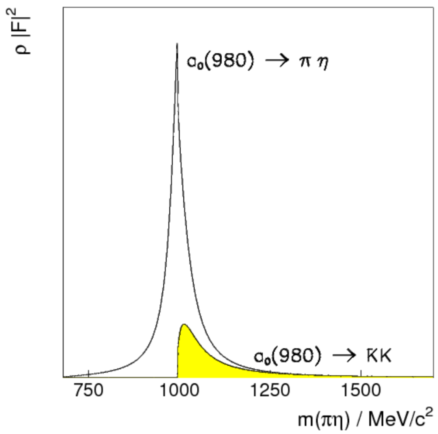


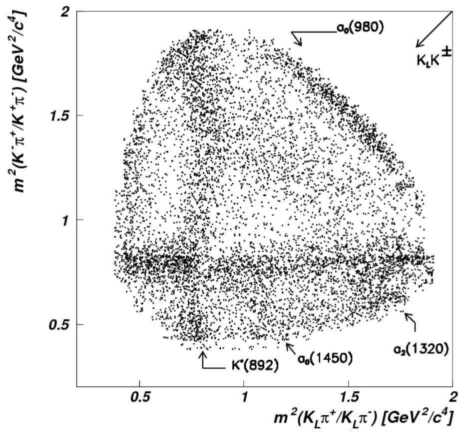


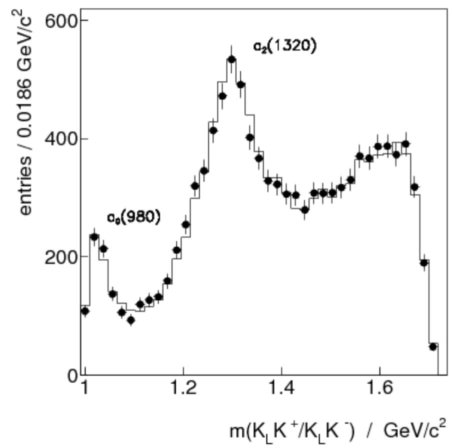
- $\mathcal{S}$-wave amplitude and phase compared to the FOCUS (K-matrix) and E791 (isobar) amplitudes.

$a_0(980)/f_0(980) \rightarrow K\bar{K}$.

- $a_0(980)$ and $f_0(980)$ resonances are close to the $K\bar{K}$ threshold.
- The $a_0(980)$ decays mostly to $\eta\pi$.







- The best measurements of the $a_0(980)$ parameters come from Crystal Barrel experiment in in $\bar{p}p \rightarrow K^0_L K^+\pi^-$ annihilations (http://www-meg.phys.cmu.edu/cb/papers/KlKpi.ps.gz).
- It has been described by a coupled channel Breit Wigner of the form:

$$BW_{ch}(a_0)(m) = \frac{g_{\bar{K}K}}{m_0^2 - m^2 - i(\rho_{\eta\pi}\, g^2_{\eta\pi} + \rho_{\bar{K}K}\, g^2_{\bar{K}K})}$$

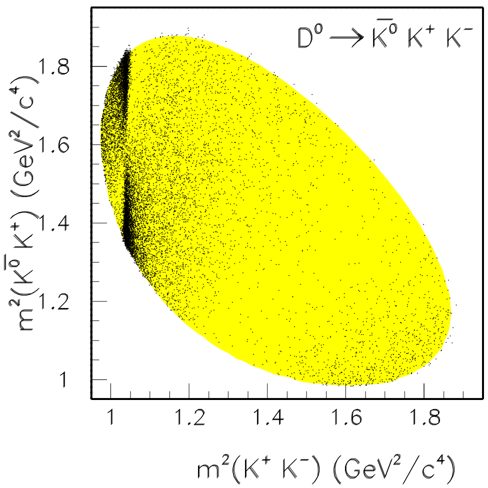
where $\rho(m) = 2q/m$ while $g_{\eta\pi}$ and $g_{\bar{K}K}$ describe the $a_0(980)$ couplings to the $\eta\pi$ and $\bar{K}K$ systems respectively.

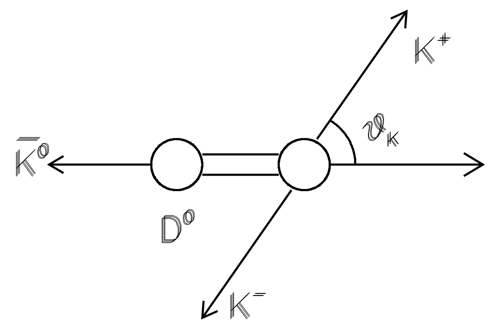
- They obtain:

$$m_0{=}999{\pm}2\ \mathrm{MeV}/c^2,\ g_{\eta\pi}{=}324{\pm}15\ (\mathrm{MeV})^{1/2},\ \frac{g^2_{\eta\pi}}{g^2_{\bar{K}K}}{=}1.03{\pm}0.14,\ \ g_{\bar{K}K} = 329 \pm 27\ (\mathrm{MeV})^{1/2}$$

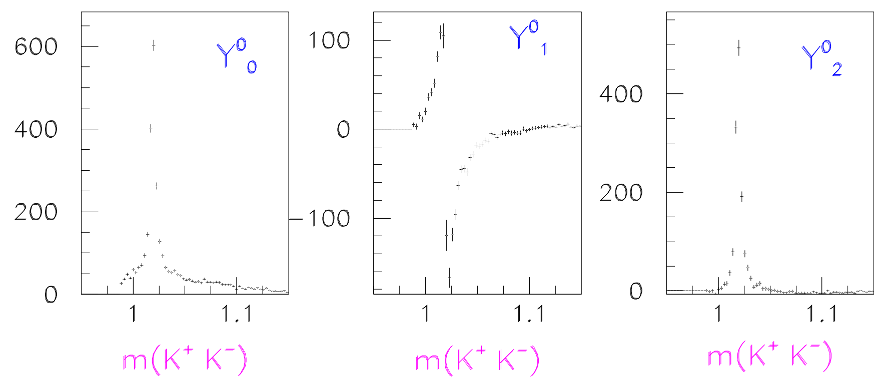
Partial Wave Analysis of the K^+K^- system.

- BaBar has performed a Dalitz analysis of $D^0 \to \bar{K}^0 K^+ K^-$ (arXiv:hep-ex/0507026).
- Due to the uncertain parameters of the scalar mesons close to threshold, where both $a_0(980)/f_0(980)$ can contribute, their lineshape has been extracted directly from the data.
- Assume, in the K^+K^- threshold region, a diagram:





- Efficiency corrected Y^0_L moments:



Partial Wave Analysis of the K^+K^- system.

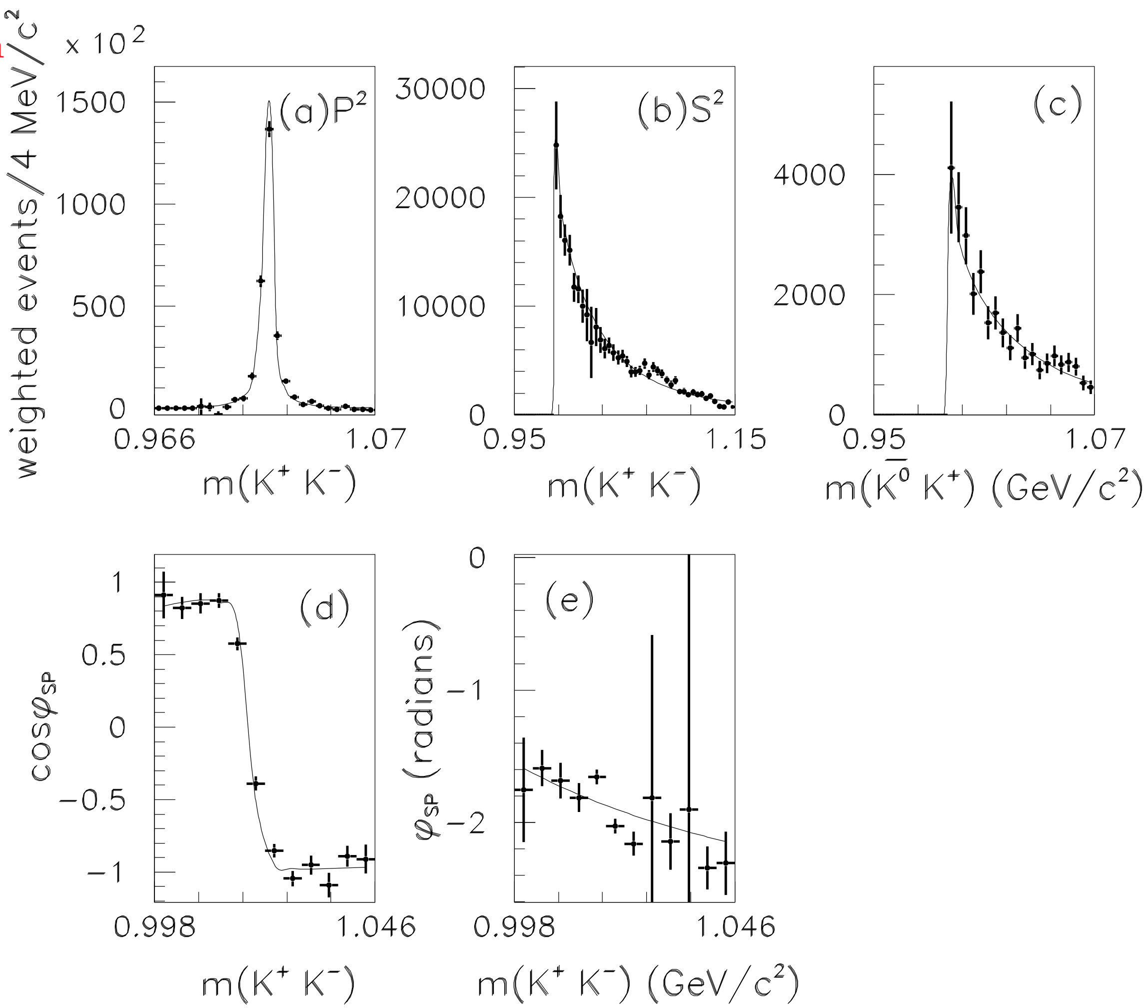
□ S, P waves and relative phase can be extracted using:

$\sqrt{4\pi}Y_0^0 = S^2 + P^2$

$\sqrt{4\pi}Y_1^0 = 2SPcos\phi$

$\sqrt{4\pi}Y_2^0 = 0.894P^2$

□ Correcting for phase space a simultaneous fit has been performed using also the $\bar{K}^0K^+$ projection:



Partial Wave Analysis of the K^+K^- system.

□ The distributions have been fitted using the following model:

- The P-wave is entirely due to the $\phi(1020)$ meson.
- The scalar contribution in the K^+K^- mass projection is entirely due to the $a_0(980)^0$.
- The $\overline{K}^0K^+$ mass distribution is entirely due to $a_0(980)^+$.
- The angle ϕ_{SP} is obtained fitting the S, P waves and $\cos\phi_{SP}$ with:

$$c_{a_0}BW_{a_0} + c_\phi BW_\phi e^{i\alpha}$$

Here BW_{a_0} and BW_ϕ are the Breit-Wigner describing the $a_0(980)$ and $\phi(1020)$ resonances.

□ The $a_0(980)$ has been described by the coupled channel Breit Wigner.

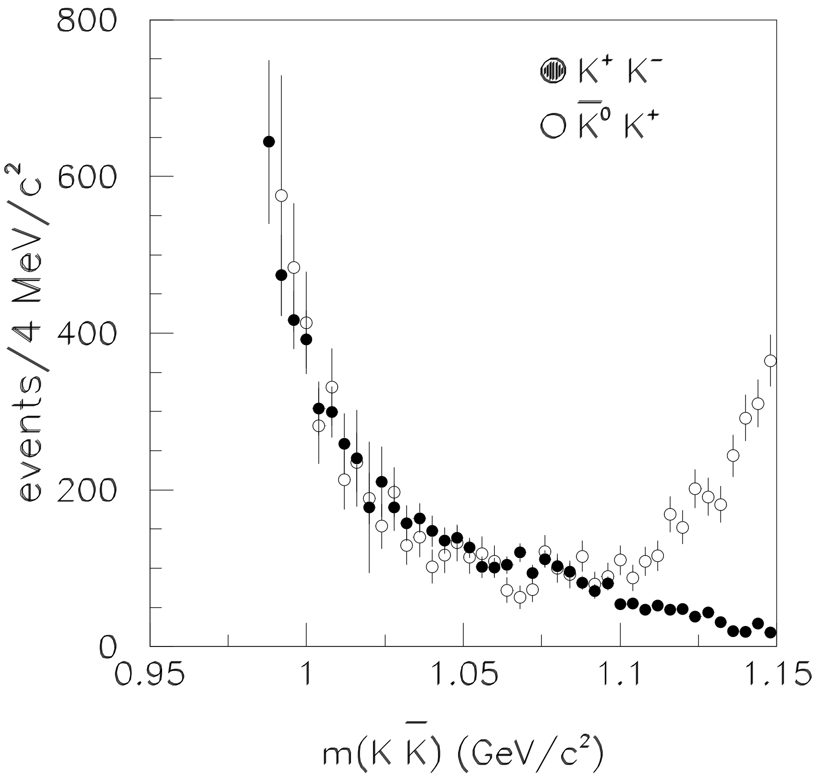
□ Fixing m_0 and $g_{\eta\pi}$ to the Crystal Barrel measurements it is possible to measure:

$$g_{\bar{K}K} = 464 \pm 29 \ (\mathrm{MeV})^{1/2}.$$

□ The figure shows also the residual $a_0(980)$ phase, obtained by first computing ϕ_{SP} in the range $(0,\pi)$ and then subtracting the known phase motion due to the $\phi(1020)$ resonance. It is almost constant, as expected for the tail of a Breit-Wigner.

Little $f_0(980)$ contribution.

- Since $f_0(980)$ has I=0, it cannot decay to $\bar{K}^0 K^+$.
- Therefore the $\bar{K}^0 K^+$ projection contains only $a_0(980)^+$.
- Superposition of the two normalized projections, phase space corrected:

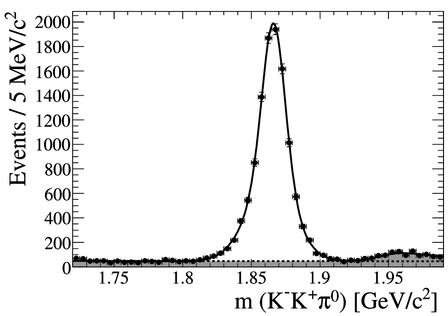


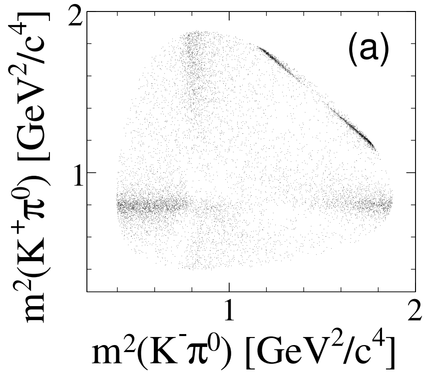
- Cosistent with little $f_0(980)$ contribution.

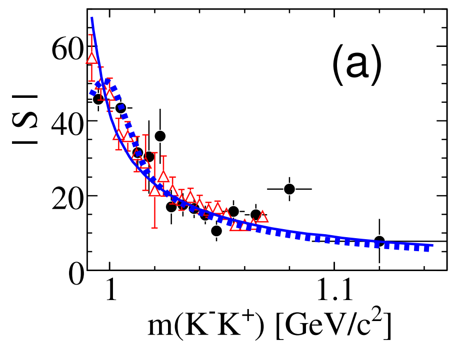
Dalitz Analysis of the $D^0 \to K^+K^-\pi^0$ system.

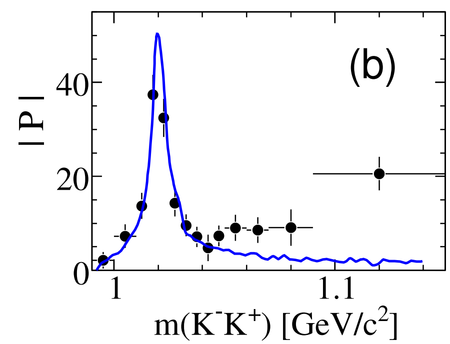
□ Using 385 fb^{-1} of e^+e^- collisions, *BABAR* has performed a Dalitz analysis of the singly Cabibbo-suppressed decay $D^0 \to K^-K^+\pi^0$ (arXiv:0704.3593).

□ Similar PWA performed in the threshold K^+K^- region.







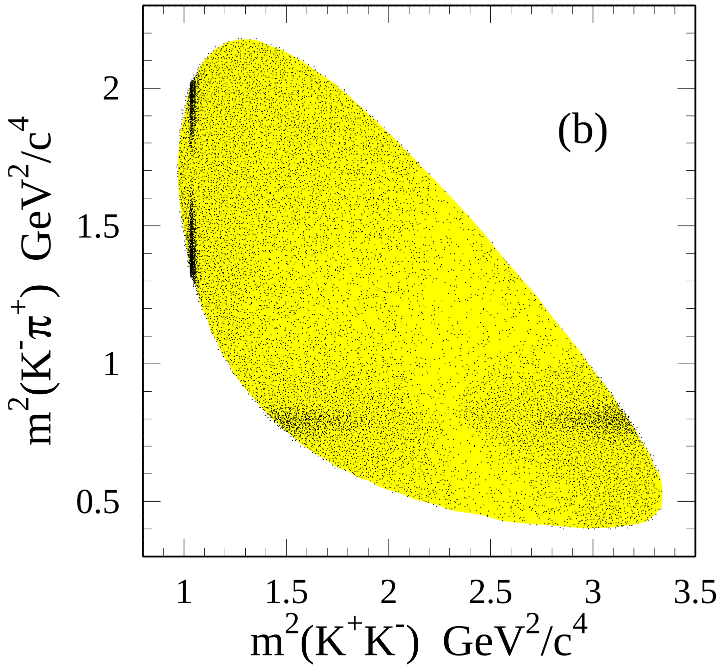


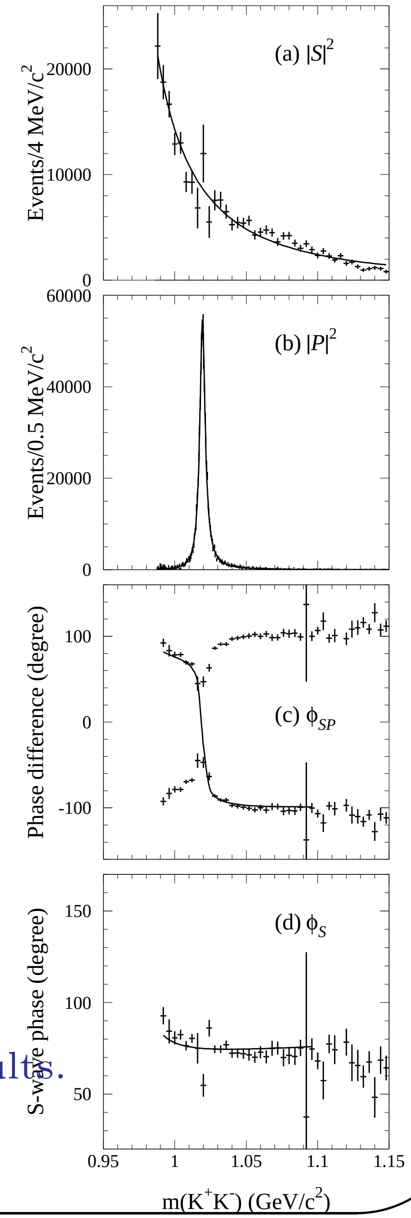
□ Similar results as for the $D^0 \to \bar{K}^0 K^+K^-$ decay.

Dalitz Analysis of $D_s^+ \to K^+K^-\pi^+$

□ *BaBar* has performed a Dalitz analysis of $D_s^+ \to K^+K^-\pi^+$ decays (arXiv:1011.4190).

□ Dalitz plot.





□ To extract the scalar contribution close to the K^+K^- threshold, a PWA has been performed as for the previous channels, obtaining similar results.

Partial Wave Analysis of the K^+K^- threshold region for $D_s^+ \to K^+K^-\pi^+$

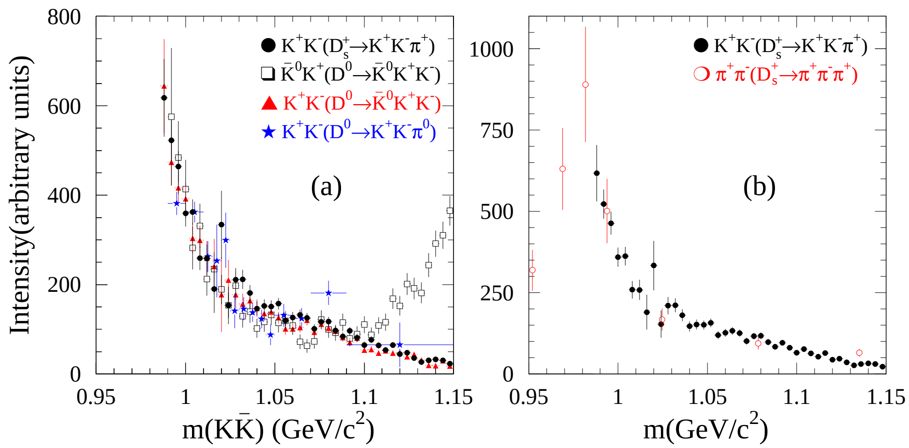
□ The $\mathcal{S}$-wave profile from this analysis is compared with the $\mathcal{S}$-wave intensity values extracted for Dalitz plot analyses of $D^0 \to \overline{K}^0K^+K^-$ and $D^0 \to K^+K^-\pi^0$.

□ The four distributions are normalized in the region from threshold up to 1.05 GeV and show a substantial agreement.

□ As the $a_0(980)$ and $f_0(980)$ mesons couple mainly to the $u\bar{u}/d\bar{d}$ and $s\bar{s}$ systems respectively, the former is favoured in $D^0 \to \overline{K}^0K^+K^-$ and the latter in $D_s^+ \to K^+K^-\pi^+$. Both resonances can contribute in $D^0 \to K^+K^-\pi^0$. We conclude that the $\mathcal{S}$-wave projections in the $K\overline{K}$ system for both resonances are consistent in shape.

□ *It has been suggested that this feature supports the hypothesis that the $a_0(980)$ and $f_0(980)$ are 4-quark states.* (arXiv:hep-ph/0703272)

□ Comparison between the $\mathcal{S}$-wave profile from this analysis with the $\pi^+\pi^-$ $\mathcal{S}$-wave profile extracted from BABAR data in a Dalitz plot analysis of $D_s^+ \to \pi^+\pi^-\pi^+$. The observed agreement supports the argument that only the $f_0(980)$ is present in this limited mass region.



Four-Body charm decays

□ Four-Body Dalitz analyses of charm decays have been performed for the first time by MarkIII experiment at SLAC (SLAC-PUB-5447, Sept. 1991).

□ Much later FOCUS collaboration made an amplitude analysis of $D^0 \to K^+ K^- \pi^+ \pi^-$ (2669 events) without separating D^0 from $\bar{D}^0$ (hep-ex/0411031).

□ Recently, CLEO collaboration performed a similar analysis, separating D^0 from $\bar{D}^0$ using ≈ 3000 events (arXiv:1201.5716.

□ The spin formalism can be derived either using the Zemach tensors or using the Lorentz-invariant helicity amplitudes.

□ The decay chain, in the isobar formalism can go through intermediate one-resonance or two-resonances.

Four-Body charm decays

□ We are studying the decay:

$$D^0 \to K^+ K^- \pi^+ \pi^-$$

□ We define with $p_{K+}, p_{K-}, p_{\pi+}, p_{\pi-}$ the momenta of the four particles in the D^0 center of mass system.

The method is the following:

- We use a symmetric and traceless tensor of rank-L made with p^i to describe orbital angular momenta.
- We use a symmetric and traceless tensor of rank-S made with t^i to describe the spin of intermediate resonances. For a resonace decaying as $R \to a + b$, the t^i are defined as:

$$t_R^i = p_a^i - p_b^i - (p_a^i + p_b^i) \frac{m_a^2 - m_b^2}{m_{ab}^2}$$

- The tensors are then contracted to obtain a scalar, the spin of the D^0.

$\phi(1020)\rho(770)$

□ This final state can occur in S, P, or D wave.

1) S-wave.

$t^i_{K^+K^-}$ and $t^i_{\pi^+\pi^-}$ describe the spin of the $\phi(1020)$ and $\rho(770)$ respectively.

$t^i_{K^+K^-} = p^i_{K^+} - p^i_{K^-}$

$t^i_{\pi^+\pi^-} = p^i_{\pi^+} - p^i_{\pi^-}$

The amplitude is the following:

$$A_1 = BW_\phi BW_\rho(\mathbf{t}_{K^+K^-} \cdot \mathbf{t}_{\pi^+\pi^-})$$

2) P-wave

$p^i_{K^+K^-}$ describes the angular momentum L=1 between the K^+K^- and $\pi^+\pi^-$.

$p^i_{K^+K^-} = p^i_{K^+} + p^i_{K^-}$

The amplitude is the following:

$$A_2 = BW_\phi BW_\rho(\mathbf{t}_{K^+K^-} \times \mathbf{p}_{K^+K^-}) \cdot \mathbf{t}_{\pi^+\pi^-}$$

$\phi(1020)\rho(770)$

3) D-wave

$p^{ij}_{K^+K^-}$ describes the angular momentum L=2 between the K^+K^- and $\pi^+\pi^-$.

$p^{ij}_{K^+K^-} = p^i_{K^+K^-} p^j_{K^+K^-} - \delta^{ij} p^2_{K^+K^-}/3$

The amplitude is the following:

$$A_3 = BW_\phi BW_\rho (t^i_{K^+K^-} p^{ij}_{K^+K^-}) t^j_{\pi^+\pi^-}$$

where indices contraction is implied.

$K^*(890)^0\overline{K}^*(890)^0$

□ This final state can occur in S, P, or D wave.

1) S-wave.

$t^i_{K^+\pi^-}$ and $t^i_{K^-\pi^+}$ describe the spin of the $K^*(890)^0$ and $\overline{K}^*(890)^0$ respectively.

$t^i_{K^+\pi^-} = p^i_{K^+} - p^i_{\pi^-} - (p^i_{K^+} + p^i_{\pi^-})\frac{m_k^2 - m_\pi^2}{m^2_{K^+\pi^-}}$

$t^i_{K^-\pi^+} = p^i_{K^-} - p^i_{\pi^+} - (p^i_{K^-} + p^i_{\pi^+})\frac{m_k^2 - m_\pi^2}{m^2_{K^-\pi^+}}$

The amplitude is the following:

$$A_4 = BW_{K^*} BW_{\bar{K}^*}(\mathbf{t}_{K^+\pi^-} \cdot \mathbf{t}_{K^-\pi^+})$$

2) P-wave

$p^i_{K^+\pi^-}$ describes the angular momentum L=1 between the $K^+\pi^-$ and $K^-\pi^+$.

$p^i_{K^+\pi^-} = p^i_{K^+} + p^i_{\pi^-}$

The amplitude is the following:

$$A_5 = BW_{K^*} BW_{\bar{K}^*}(\mathbf{t}_{K^+\pi^-} \times \mathbf{p}_{K^+\pi^-}) \cdot \mathbf{t}_{K^-\pi^+}$$

3) D-wave

$p^{ij}_{K^+\pi^-}$ describes the angular momentum L=2 between the $K^+\pi^-$ and $K^-\pi^+$.

$p^{ij}_{K^+\pi^-} = p^i_{K^+\pi^-} p^j_{K^+\pi^-} - \delta^{ij} p^2_{K^+\pi^-}/3$

The amplitude is the following:

$$A_6 = BW_{K^*} BW_{\bar{K}^*} (t^i_{K^+\pi^-} p^{ij}_{K^+\pi^-}) t^j_{K^-\pi^+}$$

$K_1^+(1270)K^-$

- The $K_1^+(1270)$ is described by a simple Breit-Wigner.
- First we describe the $K_1^+(1270)$ decay which is a spin-1 particle decaying to $K^+\pi^+\pi^-$.
- We label with p' the momenta of the particles in the $K^+\pi^+\pi^-$ rest frame.

1) $K_1^+(1270) \rightarrow K^*(890)^0\pi^+$

$t'^i_{K^+\pi^-}$ describes the K^* spin:

$t'^i_{K^+\pi^-} = p'^i_{K^+} - p'^i_{\pi^-} - (p'^i_{K^+} + p'^i_{\pi^-})\frac{m_k^2 - m_\pi^2}{m^2_{K^+\pi^-}}$

This $t'^i_{K^+\pi^-}$ vector has to be combined with $p^i_{K^-}$ to obtain a spin 0.

The amplitude is the following:

$$A_7 = BW_{K_1(1270)} BW_{K^*} (\mathbf{t'}_{K^+\pi^-} \cdot \mathbf{p}_{K^-})$$

2) $K_1^+(1270) \rightarrow K^+\rho(770)$

$t'^i_{\pi^+\pi^-}$ describes the $\rho(770)$ spin:

$t'^i_{\pi^+\pi^-} = p'^i_{\pi^+} - p'^i_{\pi^-}$

The amplitude is the following:

$$A_8 = BW_{K_1(1270)} BW_\rho (\mathbf{t'}_{\pi^+\pi^-} \cdot \mathbf{p}_{K^-})$$

3) $K_1^+(1270) \to K^+\omega(770)$

$$A_9 = BW_{K_1(1270)} BW_\omega (\mathbf{t'}_{\pi^+\pi^-} \cdot \mathbf{p}_{K^-})$$

4) $K_1^+(1270) \to K_0^*(1410)^0\pi^+$

This decay is described by $\mathbf{p'}_{\pi^+}$. The amplitude is the following:

$$A_{10} = BW_{K_1(1270)} BW_{K_0^*} (\mathbf{p'}_{\pi^+} \cdot \mathbf{p}_{K^-})$$

$K_1^+(1400)K^-$

We use only the $K^*(890)$ mode and just replace the Breit-Wigner.

$K_1^+(1400) \to K^*(890)^0\pi^+$

$$A_{15} = BW_{K_1(1400)} BW_{K^*} (\mathbf{t'}_{K^+\pi^-} \cdot \mathbf{p}_{K^-})$$

$\phi(\pi^+\pi^-)_{S-wave}$

In this case we have L=1, therefore the amplitude is:

$$A_{17} = BW_{\phi}(\mathbf{t}_{K^+K^-} \cdot \mathbf{p}_{\pi^+\pi^-})$$

$\rho(770)(K^+K^-)_{S-wave}$

In this case we have L=1, therefore the amplitude is:

$$A_{18} = BW_{\rho}(\mathbf{t}_{\pi^+\pi^-} \cdot \mathbf{p}_{K^+K^-})$$

$\omega(770)(K^+K^-)_{S-wave}$

In this case we have L=1, therefore the amplitude is:

$$A_{19} = BW_{\omega}(\mathbf{t}_{\pi^+\pi^-} \cdot \mathbf{p}_{K^+K^-})$$

$K^*(890)(K^-\pi^+)_{S-wave}$

In this case we have L=1, therefore the amplitude is:

$$A_{20} = BW_{K^*}(\mathbf{t}_{K^+\pi^-} \cdot \mathbf{p}_{K^+\pi^-})$$

$\bar{K}^*(890)(K^+\pi^-)_{S-wave}$

In this case we have L=1, therefore the amplitude is:

$$A_{21} = BW_{\bar{K}^*}(\mathbf{t}_{K^-\pi^+} \cdot \mathbf{p}_{K^-\pi^+})$$

$K_1^*(1410)K$

The $K_1^*(1410)$ is described by a simple Breit-Wigner.

$K_1^{*+}(1410)K^-$.

$$A_{22} = BW_{K_1^*(1410)} BW_{K^*} (\mathbf{t'}_{K^+\pi^-} \times \mathbf{p'}_{\pi^+}) \cdot \mathbf{p}_{K^-})$$

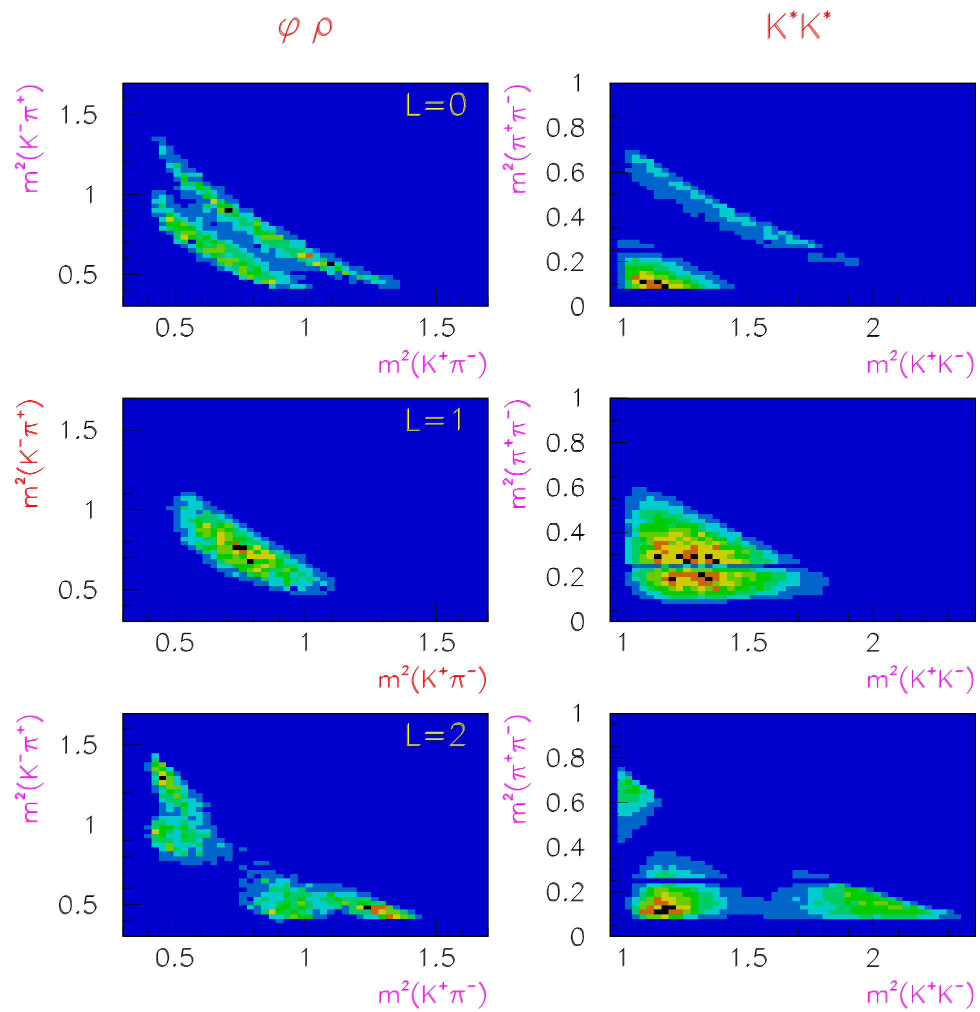
Table of the Zemach tensors amplitudes

Channel	L	Zemach tensor
1) $\phi(1020)\rho(770)$	0	$BW_\phi BW_\rho(\mathbf{t}_{K^+K^-}\cdot\mathbf{t}_{\pi^+\pi^-})$
2) $\phi(1020)\rho(770)$	1	$BW_\phi BW_\rho(\mathbf{t}_{K^+K^-}\times\mathbf{p}_{K^+K^-})\cdot\mathbf{t}_{\pi^+\pi^-}$
3) $\phi(1020)\rho(770)$	2	$BW_\phi BW_\rho(t^i_{K^+K^-}p^{ij}_{K^+K^-})t^j_{\pi^+\pi^-}$
4) $K^*(890)^0\overline{K}^*(890)^0$	0	$BW_{K^*}BW_{\bar{K}^*}(\mathbf{t}_{K^+\pi^-}\cdot\mathbf{t}_{K^-\pi^+})$
5) $K^*(890)^0\overline{K}^*(890)^0$	1	$BW_{K^*}BW_{\bar{K}^*}(\mathbf{t}_{K^+\pi^-}\times\mathbf{p}_{K^+\pi^-})\cdot\mathbf{t}_{K^-\pi^+}$
6) $K^*(890)^0\overline{K}^*(890)^0$	2	$BW_{K^*}BW_{\bar{K}^*}(t^i_{K^+\pi^-}p^{ij}_{K^+\pi^-})t^j_{K^-\pi^+}$
7) $K_1^+(1270)\to K^*(890)^0\pi^+$		$BW_{K_1(1270)}BW_{K^*}(\mathbf{t'}_{K^+\pi^-}\cdot\mathbf{p}_{K^-})$
8) $K_1^+(1270)\to K^+\rho(770)$		$BW_{K_1(1270)}BW_\rho(\mathbf{t'}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^-})$
9) $K_1^+(1270)\to K^+\omega(770)$		$BW_{K_1(1270)}BW_\omega(\mathbf{t'}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^-})$
10) $K_1^+(1270)\to K_0^*(1410)^0\pi^+$		$BW_{K_1(1270)}BW_{K_0^*}(\mathbf{p'}_{\pi^+}\cdot\mathbf{p}_{K^-})$
11) $K_1^-(1270)\to\bar{K}^*(890)^0\pi^-$		$BW_{K_1(1270)}BW_{\bar{K}^*}(\mathbf{t'}_{K^-\pi^+}\cdot\mathbf{p}_{K^+})$
12) $K_1^-(1270)\to K^-\rho(770)$		$BW_{K_1(1270)}BW_\rho(\mathbf{t'}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^+})$
13) $K_1^-(1270)\to K^-\omega(770)$		$BW_{K_1(1270)}BW_\omega(\mathbf{t'}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^+})$
14) $K_1^-(1270)\to\bar{K}_0^*(1410)^0\pi^-$		$BW_{K_1(1270)}BW_{\bar{K}_0^*}(\mathbf{p'}_{\pi^-}\cdot\mathbf{p}_{K^+})$
15) $K_1^+(1400)\to K^*(890)^0\pi^+$		$BW_{K_1(1400)}BW_{K^*}(\mathbf{t'}_{K^+\pi^-}\cdot\mathbf{p}_{K^-})$
16) $K_1^-(1400)\to K^*(890)^0\pi^-$		$BW_{K_1(1400)}BW_{\bar{K}^*}(\mathbf{t'}_{K^-\pi^+}\cdot\mathbf{p}_{K^+})$
17) $\phi(\pi^+\pi^-)_{S-wave}$	1	$BW_\phi(\mathbf{t}_{K^+K^-}\cdot\mathbf{p}_{\pi^+\pi^-})$
18) $\rho(770)(K^+K^-)_{S-wave}$	1	$BW_\rho(\mathbf{t}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^+K^-})$
19) $\omega(770)(K^+K^-)_{S-wave}$	1	$BW_\omega(\mathbf{t}_{\pi^+\pi^-}\cdot\mathbf{p}_{K^+K^-})$
20) $K^*(890)(K^-\pi^+)_{S-wave}$	1	$BW_{K^*}(\mathbf{t}_{K^+\pi^-}\cdot\mathbf{p}_{K^+\pi^-})$
21) $\bar{K}^*(890)(K^+\pi^-)_{S-wave}$	1	$BW_{\bar{K}^*}(\mathbf{t}_{K^-\pi^+}\cdot\mathbf{p}_{K^-\pi^+})$
22) $K_1^*(1410)^+\to K^*(890)^0\pi^+$		$BW_{K_1^*(1410)}BW_{K^*}(\mathbf{t'}_{K^+\pi^-}\times\mathbf{p'}_{\pi^+})\cdot\mathbf{p}_{K^-}$
23) $K_1^*(1410)^-\to\bar{K}^*(890)^0\pi^-$		$BW^*_{K_1(1410)}BW_{\bar{K}^*}(\mathbf{t'}_{K^-\pi^+}\times\mathbf{p'}_{\pi^-})\cdot\mathbf{p}_{K^+}$

The Lorentz invariant helicity amplitudes

Channel	L	Amplitude
$D \to V_1 V_2,\ V_1 \to P_1 P_2,$ $V_2 \to P_3 P_4$	0	$q^{\mu}_{V_1}(g^{\mu\nu} - p^{\mu}_{V_1} p^{\nu}_{V_1}/M^2_{V_1})(g^{\nu\sigma} - p^{\nu}_{V_1} p^{\sigma}_{V_1}/M^2_{V_2}) q^{\sigma}_{V_2}$
$D \to V_1 V_2,\ V_1 \to P_1 P_2,$ $V_2 \to P_3 P_4$	1	$\epsilon_{\alpha\beta\gamma\delta} p^{\alpha}_{D} q^{\beta}_{D} q^{\gamma}_{V_1} q^{\delta}_{V_2}$
$D \to V_1 V_2,\ V_1 \to P_1 P_2,$ $V_2 \to P_3 P_4$	2	$q^{\mu}_{V_1}(g^{\mu\nu} - p^{\mu}_{V_1} p^{\nu}_{V_1}/M^2_{V_1}) p^{\nu}_{V_2} \times q^{\mu}_{V_2}(g^{\mu\nu} - p^{\mu}_{V_2} p^{\nu}_{V_2}/M^2_{V_2}) p^{\nu}_{V_1}$
$D \to A P_1,\ A \to V P_2,$ $V \to P_3 P_4$	1	$p^{\mu}_{P_1}(g^{\mu\nu} - p^{\mu}_{A} p^{\nu}_{A}/M^2_{A})(g^{\nu\sigma} - p^{\nu}_{V} p^{\sigma}_{V}/M^2_{V}) q^{\sigma}_{V}$
$D \to A P_1,\ A \to S P_2,$ $S \to P_3 P_4$	1	$p^{\mu}_{P_1}(g^{\mu\nu} - p^{\mu}_{A} p^{\nu}_{A}/M^2_{A}) q^{\nu}_{A}$
$D \to P P_1,\ P \to V P_2,$ $V \to P_3 P_4$	0	$p^{\mu}_{P_2}(g^{\mu\nu} - p^{\mu}_{V} p^{\nu}_{V}/M^2_{V}) q^{\nu}_{V}$
$D \to V S,\ V \to P_1 P_2,$ $S \to P_3 P_4$	1	$p^{\mu}_{S}(g^{\mu\nu} - p^{\mu}_{V} p^{\nu}_{V}/M^2_{V}) q^{\nu}_{V}$
$D \to V_1 P_1,\ V_1 \to V_2 P_2,$ $V_2 \to P_3 P_4$	1	$\epsilon_{\alpha\beta\gamma\delta} p^{\alpha}_{V_1} D q^{\beta}_{V_1} q^{\gamma}_{P_1} q^{\delta}_{V_2}$
$D \to T P_1,\ T \to V P_2,$ $V \to P_3 P_4$	2	$((p_{P_1} \cdot q_T) - (p_{P_1} \cdot p_T)(q_T \cdot p_T)/M^2_T) \times \epsilon_{\alpha\beta\gamma\delta} p^{\alpha}_{T} D q^{\beta}_{T} q^{\gamma}_{V} p^{\delta}_{P_1}$
Three-body NR		Replace $1/p^2$ for $1/M^2$
Four-body NR		Constant

An example of a few amplitudes projections



Results from CLEO

□ Amplitude, phase, and fit fraction for each component.

Amplitude	$\lvert a_i \rvert$	ϕ_i (rad)	Fit Fraction (%)
$K_1(1270)^+(K^{*0}\pi^+)K^-$	1.0	0.0	$7.3 \pm 0.8 \pm 1.9$
$K_1(1270)^-(\overline{K^{*0}}\pi^-)K^+$	$0.35 \pm 0.06 \pm 0.03$	$1.10 \pm 0.22 \pm 0.23$	$0.9 \pm 0.3 \pm 0.4$
$K_1(1270)^+(\rho^0 K^+)K^-$	$5.86 \pm 0.77 \pm 2.03$	$0.80 \pm 0.13 \pm 0.08$	$4.7 \pm 0.7 \pm 0.8$
$K_1(1270)^-(\rho^0 K^-)K^+$	$6.90 \pm 0.59 \pm 3.07$	$0.03 \pm 0.16 \pm 0.23$	$6.0 \pm 0.8 \pm 0.6$
$K^*(1410)^+(K^{*0}\pi^+)K^-$	$6.18 \pm 0.64 \pm 0.75$	$0.73 \pm 0.11 \pm 0.33$	$4.2 \pm 0.7 \pm 0.8$
$K^*(1410)^-(\overline{K^{*0}}\pi^-)K^+$	$6.78 \pm 0.65 \pm 1.25$	$1.18 \pm 0.13 \pm 0.48$	$4.7 \pm 0.7 \pm 0.7$
$K^{*0}\overline{K^{*0}}$ S wave	$0.34 \pm 0.04 \pm 0.14$	$0.39 \pm 0.12 \pm 0.18$	$6.1 \pm 0.8 \pm 0.9$
$\phi\rho^0$ S wave	$1.04 \pm 0.10 \pm 0.31$	$1.89 \pm 0.14 \pm 0.35$	$38.3 \pm 2.5 \pm 3.8$
$\phi\rho^0$ D wave	$1.44 \pm 0.19 \pm 0.38$	$1.43 \pm 0.22 \pm 0.48$	$3.4 \pm 0.7 \pm 0.6$
$\phi\left\{\pi^+\pi^-\right\}_S$	$6.17 \pm 0.52 \pm 1.58$	$1.85 \pm 0.13 \pm 0.37$	$10.3 \pm 1.0 \pm 0.8$
$\left\{K^-\pi^+\right\}_P\left\{K^+\pi^-\right\}_S$	$83.4 \pm 6.8 \pm 29.3$	$0.14 \pm 0.12 \pm 0.28$	$10.9 \pm 1.2 \pm 1.7$

Results from CLEO

□ Two-body fit projections.