Amplitude Analysis : steps in S-matrix theory

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These lectures are about the very basics of S-matrix theory. Formal definitions with all factors of "i" and "4π" are eschewed, in favor of the physical relevan of the ideas of analyticity, crossing and unitarity. These basic properties follow from causality, relativity and the conservation of probability as fundamental requirements of physical processes.

The S-matrix is a gigantic matrix that takes all initial states (at every possible momentum, with all possible initial spins) to every possible final state. Like the wavefunction of a single particle state in non-relativistic quantum mechanics, it is the modulus squared of its matrix elements that is related to the probability that any initial state is connected to some particular final state in a scattering process.

The matrix element for a particular initial and final state is related to what we call the "scattering amplitude". Its modulus squared is all we can measure in the spinless particle scattering, we consider in these lectures. If there is spin and the initial state is prepared in a defined spin state, one can measure interferences between scattering amplitudes in different spin states, as Suh-Urk Chung will describe in his parallel lectures.

The aim of Amplitude Analysis is to extract Amplitudes from experimental information. Such amplitudes encode the dynamics we wish to learn about: in hadron physics, the dynamics of QCD.

At low energies of a few GeV, scattering is dominated in general by resonances. At higher energies by smoother behavior related to crossed channel Regge exchanges. At near threshold energies in processes that involve π's and K's in the initial or final state chiral dynamics determines the interaction.

All such dynamics, resonance, chiral and Regge, are all embodied in the scattering amplitude, and so connected to each other.

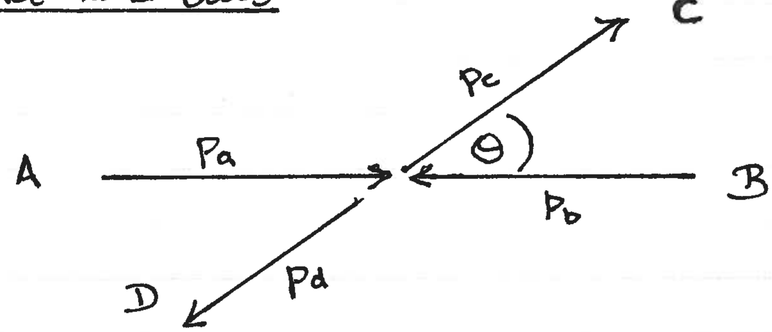
When resonances are distinct, and do not overlap, they are identified by enhancements in the cross-section. However, resonances not only have definite mass and width (defined later), but definite spin and parity, and isospin and strangeness and .... (as well as charge conjugation when this is appropriate). These translate into a characteristic angular distribution.

Amplitude analyses are about extracting such information.

In this first lecture, we define the kinematic variables relevant to a 4-body process. These have natural generalizations to N-body reactions, when $N > 4$.
For the example of spinless particle scattering we then consider how from perfect data on the differential cross-section, one can determine the underlying scattering amplitude. With 'imperfect' data, we can still get very close to the true amplitude at least in the kinematic region of the data, if not elsewhere.

s-channel kinematics

c.m. frame



$$p_a = (E_a, p, 0, 0) \quad , \quad p_b = (E_b, -p, 0, 0)$$

$$p_c = (E_c, q\cos\theta, q\sin\theta, 0), \; p_d = (E_d, -q\cos\theta, -q\sin\theta, 0)$$

$$s = (p_a + p_b)^2 = (p_c + p_d)^2$$

$$\therefore E_a + E_b = E_c + E_d = \sqrt{s}$$

$$p_a^2 = m_a^2 = E_a^2 - p^2 \, , \quad p_b^2 = m_b^2 = E_b^2 - p^2 \quad \text{etc}$$

$$\therefore E_{\substack{a\\b}} = \frac{\sqrt{s}}{2} \pm \frac{(m_a^2 - m_b^2)}{2\sqrt{s}} \, , \quad E_{\substack{c\\d}} = \frac{\sqrt{s}}{2} \pm \frac{(m_c^2 - m_d^2)}{2\sqrt{s}}$$

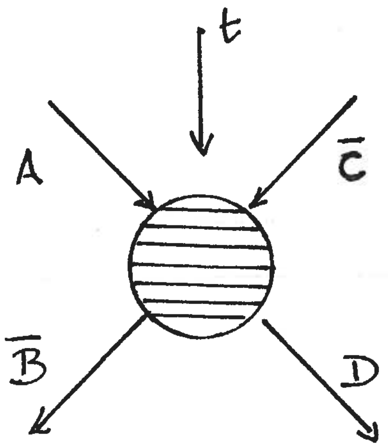
$$4p^2 s = s^2 - 2(m_a^2 + m_b^2)\, s + (m_a^2 - m_b^2)^2$$

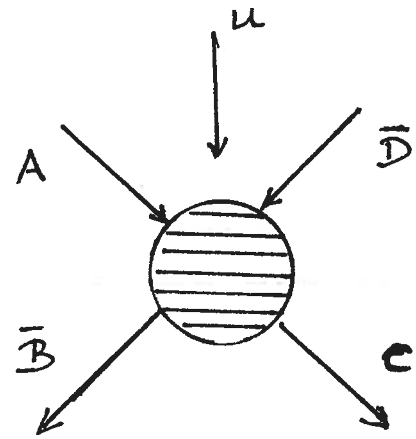
$$4q^2 s = s^2 - 2\,(m_c^2 + m_d^2)\, s + (m_c^2 - m_d^2)^2$$

s-channel physical region;

$$s \geq \max\{(m_a + m_b)^2, (m_c + m_d)^2\} \quad , \quad p^2, q^2 > 0$$

$$0 \leq \theta \leq \pi$$

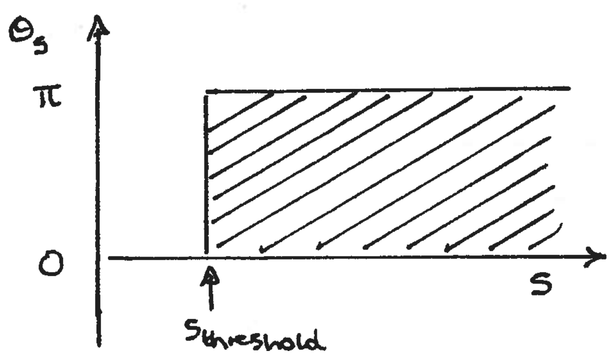


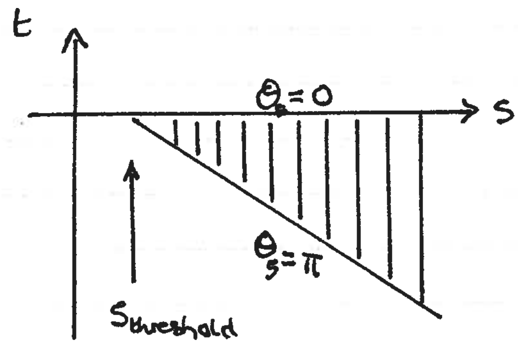


$$t = (p_a - p_c)^2 \quad , \quad u = (p_a - p_d)^2$$

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$$

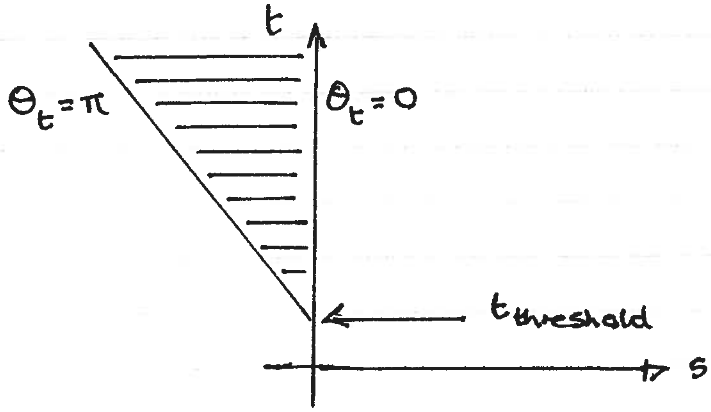
s-channel physical region





t-channel : $A\bar{C} \to \bar{B}D$ $\quad\quad \sqrt{t}$ = c.m. energy

s,u related to $\cos\theta_t$, θ_t t-channel cm scattering angle



Note define $\quad N_\kappa = \varepsilon_{\kappa\lambda\mu\nu}\, p_a^{\lambda}\, p_b^{\mu}\, p_c^{\nu}$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is the Levi-Civita symbol. Physically N_κ is the 4-vector normal to the scattering plane.

In the s-channel c.m. frame

$$N_3 = \sqrt{s}\, p\, q\, \sin\theta_s$$

in the physical region $p^2 > 0$, $q^2 > 0$, $0 \leqslant \sin\theta_s \leqslant 1$

On the boundary N_κ is a null-vector. That it is null is also true in t and u-channels. Thus the boundary of each physical region is specified by $N^2 = 0$.

Define the "Kibble function"

$$\phi(s,t,u) \equiv 4N^2 = 4 \begin{vmatrix} p_a^2 & p_a \cdot p_b & p_a \cdot p_c \\ p_b \cdot p_a & p_b^2 & p_b \cdot p_c \\ p_c \cdot p_a & p_c \cdot p_b & p_c^2 \end{vmatrix}$$

Boundary specified by $\phi(s,t,u) = 0$ for each physical region.

Examples :

1) all four masses equal

$$\varphi(s,t,u) = stu$$

then boundaries of 3 physical regions are defined by

$$s=0,\ t=0,\ u=0$$

2) $\pi N \to \pi N$ with $m_\pi \equiv \mu,\ m_N = M$

$$\varphi(s,t,u) = t\left[su - (M^2-\mu^2)^2\right]$$

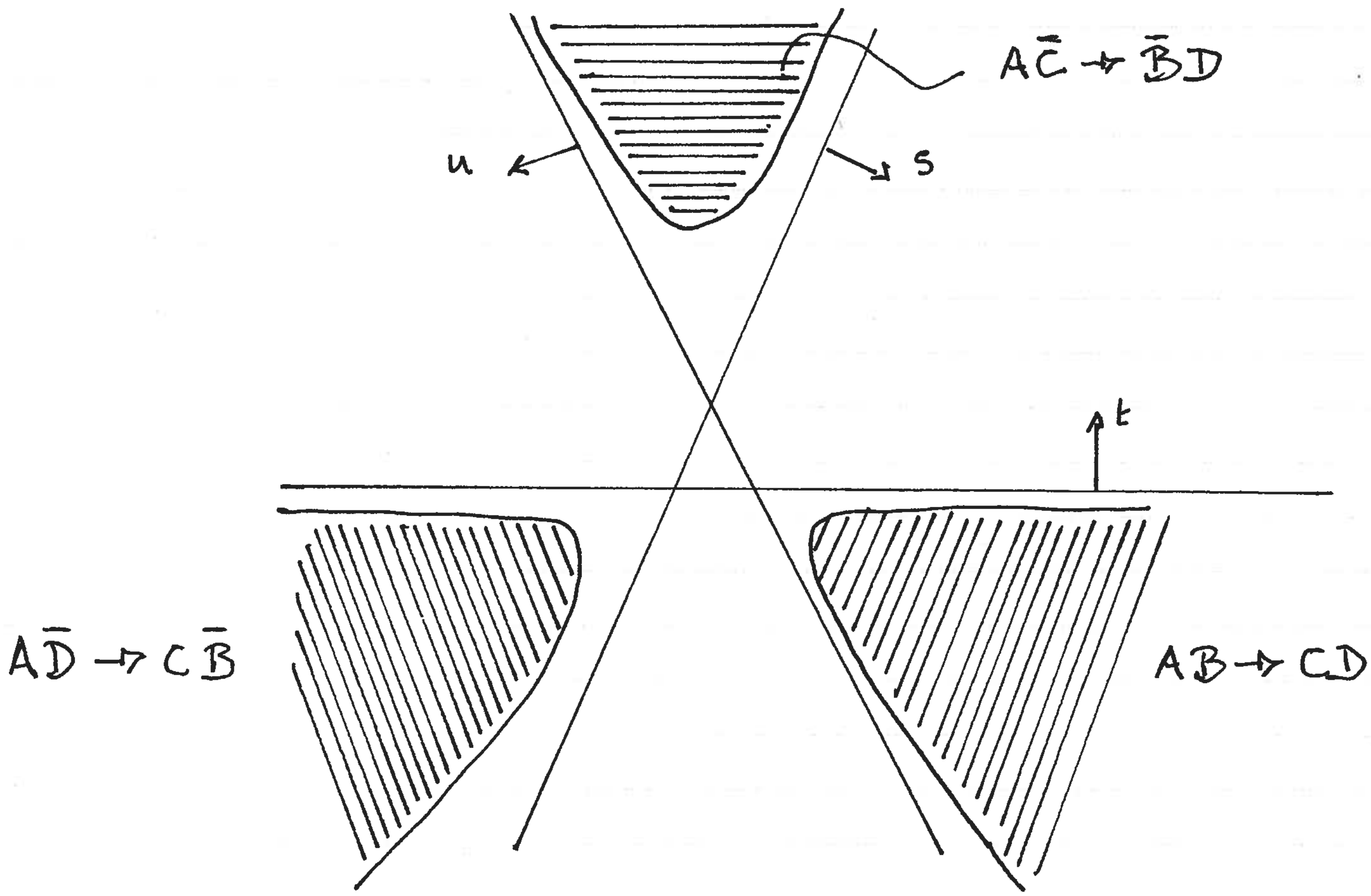
boundaries $t=0$, hyperbola $su = (M^2-\mu^2)^2$

3) $AB \to CD$ if $m_a > m_b + m_c + m_d$, then there is a 4th physical region corresponding to the decay $A \to \bar{B}CD$.

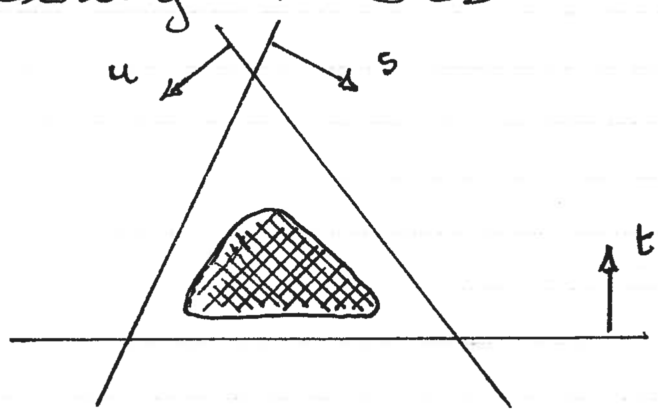
Find boundaries for $\eta \to 3\pi$ [see B. Kubis' lectures]

<u>Mandelstam plane</u> : $s + t + u = h$ (= height of central triangle)

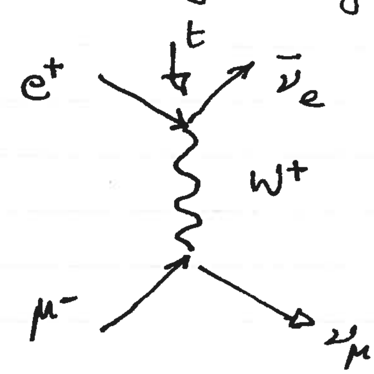
$$h = \sum_i m_i^2$$



If $m_a > m_b + m_c + m_d$, there is a fouth physical region describing $A \to \bar{B}CD$



The relativistic, analytic S-matrix has several remarkable properties. One is crossing. The amplitude for a 4-body process, in each of its 3 (or even 4) physical regions, is the boundary value of a single analytic function $F(s,t,u)$ evaluated in the appropriate physical region. In perturbation theory Feynman amplitudes simply embody this property. As an example, consider the process $e^+\mu^- \to \bar{\nu}_e \nu_\mu$ in the s-channel. At lowest order in G_F the amplitude is given by the Feynman graph with t-channel W-exchange

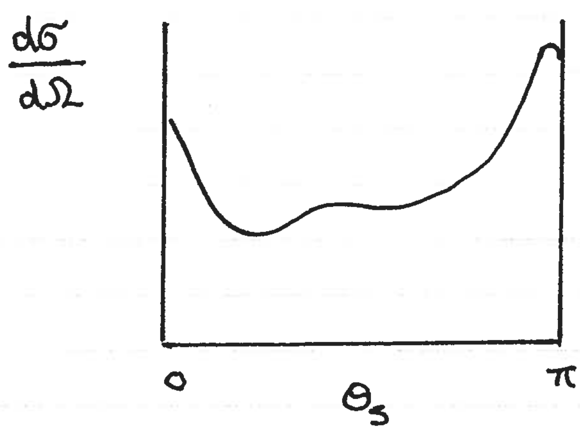


In the physical regions for $e^+\mu^- \to \bar{\nu}_e \nu_\mu$, $e^+ \bar{\nu}_\mu \to \mu^+ \bar{\nu}_e$ and $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, as well as $e^+ \nu_e \to \mu^+ \nu_\mu$, this same diagram describes each of these reactions evaluated in their different kinematic regimes. This is the property of crossing symmetry.

First steps in Amplitude Analysis

Determining the scattering amplitude is what Amplitude Analysis is about. The simplest illustration is for spinless particle scattering for $AB \to CD$.

At each energy, we measure the differential cross-section $d\sigma/d\Omega$ as a function of the c.m. scattering angle θ_s. Let us assume (unrealistically) that we have perfect data covering the full angular range $0 \lesssim \theta_s \lesssim \pi$.



An example of an angular distribution.

How do we find the scattering amplitude, in this ideal situation?

For $AB \to CD$: $$\frac{d\sigma}{d\Omega} = \frac{q}{64\pi^2 p s} |F(s,t)|^2$$

$$F(s,t) = F(s, z_s)$$

where $z_s = \cos\theta_s$.

At a given energy $\sqrt{s}$ measure angular distribution for spinless particle scattering (then no ϕ dependence)

$$\frac{d\sigma}{d\Omega} = \sum_{n=0}^{\infty} (2n+1) H_n(s) P_n(z_s)$$

Legendre functions are a complete basis for $-1 \leq z_s \leq 1$

As $$\int_{-1}^{1} dz\, P_m(z) P_n(z) = \frac{2\delta_{mn}}{(2m+1)}$$

$$\therefore H_n(s) = \frac{1}{2} \int_{-1}^{1} dz_s \frac{d\sigma}{d\Omega} P_n(z_s)$$

In practice, $H_n(s) \simeq 0$ for $n > N = 2L$.

$$\therefore \frac{d\sigma}{d\Omega} = \sum_{n=0}^{2L} (2n+1) H_n(s) P_n(z) = \frac{1}{64\pi^2 s} |F(s,z)|^2$$

for $AB \to AB$.

As n is truncated at $n = 2L$, expect $F(s,z)$ has dependence on z to z^L.

Expand $F(s,z)$ in terms of a complete set of angular momentum eigenfunctions, i.e

$$F(s,z) = \sum_{\ell=0} (2\ell+1)\, f_\ell(s)\, P_\ell(z)$$

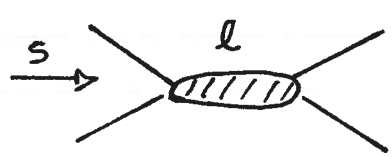
effectively $\ell \lesssim L$. [Later we will introduce a convenient factor of 16π in the definition of the partial waves, $f_\ell(s)$].

$\frac{d\sigma}{d\Omega}$ is represented by $(2L+1)$ real numbers H_n at each value of $\sqrt{s}$.

This is translated into $(L+1)$ complex numbers f_ℓ.

$f_\ell(s)$ (the partial wave amplitudes) encode the dynamics for a given orbital angular momentum, ℓ.

The overall phase of the amplitude is unknown, so $(2L+1)$ real numbers are fixed by the data on $d\sigma/d\Omega$.

$\sum_{\ell}$ 

represents the partial wave expansion.

Knowing $\frac{d\sigma}{d\Omega}$, how do we find $F(s,z)$?

Return to

$$\frac{d\sigma}{d\Omega} = \sum_{n=0}^{2L} (2n+1)\, H_n(s)\, P_n(z)$$

As each Legendre function $P_n(z)$ is a polynomial in z of $O(z^n)$,

$d\sigma/d\Omega$ is a polynomial in z of $O(z^{2L})$.

This can be rewritten as

$$\frac{d\sigma}{d\Omega} = h_0 + h_1 z + h_2 z^2 + \ldots + h_{2L} z^{2L}.$$

A polynomial of $O(z^{2L})$ has $2L$ roots.

As $d\sigma/d\Omega$ (and hence the h_n's) are real,

the roots are in complex conjugate pairs

[real roots would only happen at some 'odd' energy]

$\therefore$ we can write

$$\frac{d\sigma}{d\Omega} = h(s) \prod_{i=1}^{L} (z-z_i)(z-z_i^*)$$

$h(s)$ is real. Given physical meaning by rewriting the differential cross-section as

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{z=1} \prod_{i=1}^{L} \frac{(z-z_i)(z-z_i^*)}{(1-z_i)(1-z_i^*)}$$

$$= \frac{1}{64\pi^2 s} |F(s,z)|^2$$

where the z_i depend on s.

We can now factorize this equation as

$$F(s,z) = F(s,z=1) \prod_{i=1}^{L} \frac{(z-z_i)}{(1-z_i)}$$

The forward cross-section $\left.\frac{d\sigma}{d\Omega}\right|_{z=1}$ detemines

$|F(s,z=1)|$, but the phase of the forward amplitude is not known (or rather not fixed).

$$\therefore\ F(s,z) = |F(s,z=1)|\ e^{i\varphi} \prod_{i=1}^{L} \frac{(z-z_i(s))}{(1-z_i(s))}$$

Thus the $2L+1$ non-zero H_n are translated into

L complex zeros z_i and one overall normalization

$|F(s,z=1)|$.

But the factorization is not unique

1) there is undetermined forward phase ϕ, which yields a continuum ambiguity

2) for each zero z_i $(i=1,\dots,L)$, is this a zero of $F(s,z)$ or $F^*(s,z)$?

This yields a discrete ambiguity.

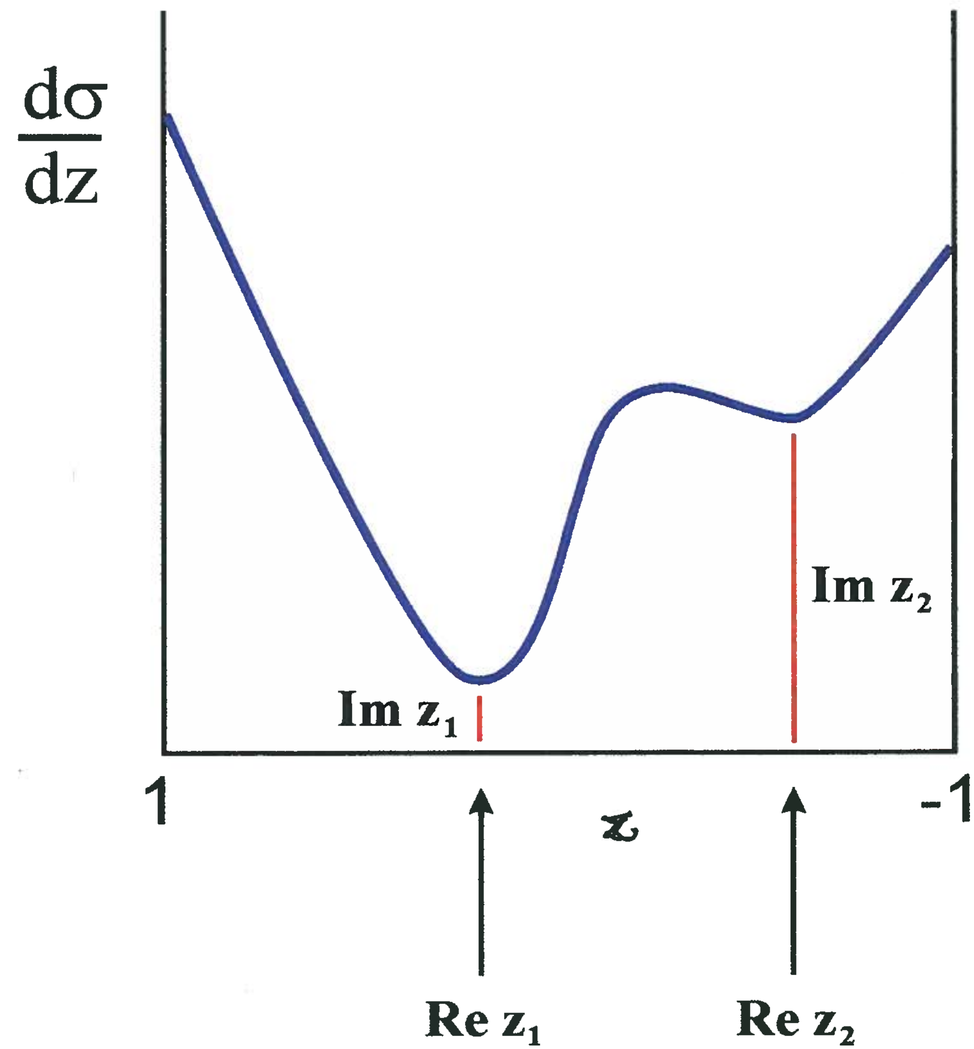
There are 2^L amplitudes describing exactly the same differential cross-section found by changing $z_i \rightarrow z_i^*$ for each $i=1,\dots,L$.

What are these zeros at $z=z_i(s)$

(or equally at $z=z_i^*(s)$) ?

For $-1 \leq \mathrm{Re}\, z_i \leq 1$, these correspond to dips in the angular distribution, see page 17.

$z \equiv \cos\theta_{c.m.}$



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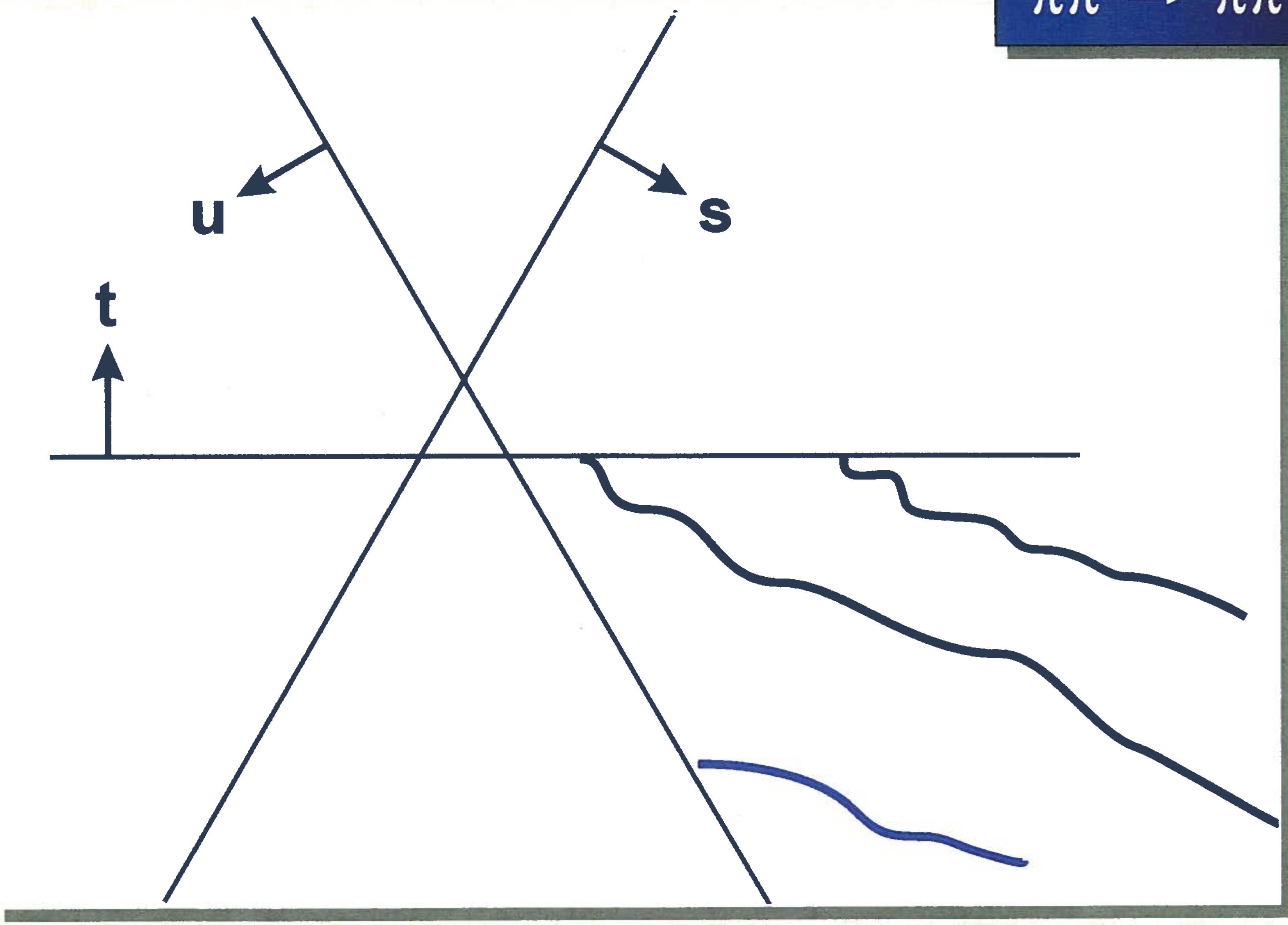
$\pi\pi \rightarrow \pi\pi$

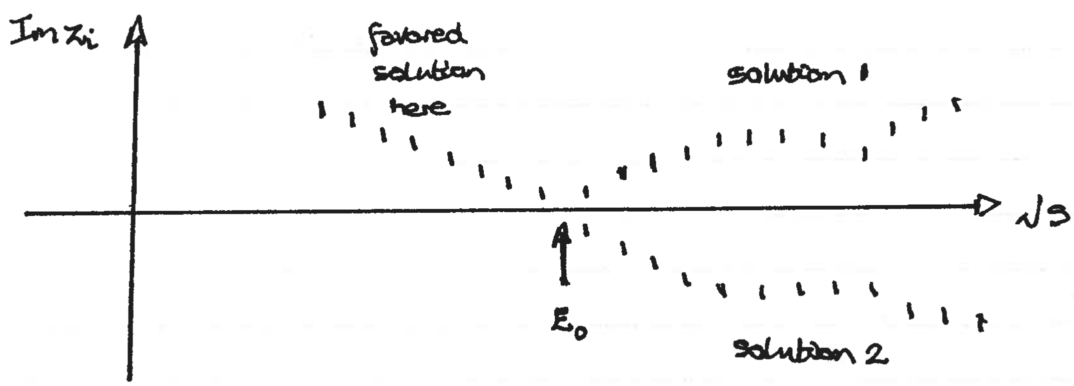


Indeed, while the position of a dip in z tells us the Real part of the position of a zero, the "height" of the cross-section at the dip is proportional to the Imaginary part of the position of the zero (in fact $(\mathrm{Im}\, z_i)^2$). This is illustrated in the Figure. Thus these zeros are "physical" objects. Moreover, their position is a continuous function of energy, $z_i(s)$. Their path can be mapped out as in Figure on page 18.

As the energy increases zeros enter the physical region either through the forward or backward directions (or both). While resonances appear at a fixed energy, zeros move with energy across the Mandelstam plane connecting the dynamics at one energy region to another.

While we see there are 2^L possible amplitudes describing exactly the same experimental data at each energy, because zeros are continuous the 2^L amplitudes are connected from one energy to another. Each will have different partial waves. Some of these will have configurations that will violate unitarity and/or analyticity, properties we discuss later. Because of the continuum ambiguity, the partial wave vectors can be rotated together (their relative orientation is fixed). As we will see in the case of elastic scattering like AB→AB, measurement of the total cross-section for AB→anything will fix the overall phase of the amplitude and so then the continuum ambiguity is resolved.

Continuity of zero trajectories means the discrete (or Barrelet) ambiguity is also restricted. Once one solution is determined to be correct (i.e. the physical one), an alternative only becomes possible if one of the zeros approaches the real axis (i.e. $\mathrm{Im}\, z_i \to 0$) and then it becomes possible for a choice of z_i or z_i^* being the zero of the amplitude, i.e. $\mathrm{Im}\, z_i$ or $-\mathrm{Im}\, z_i$.

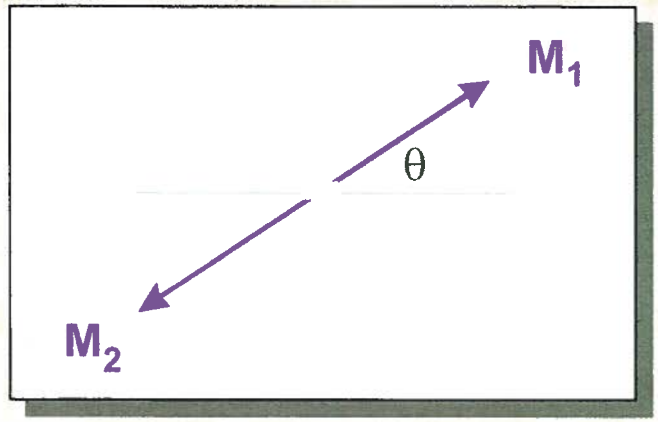


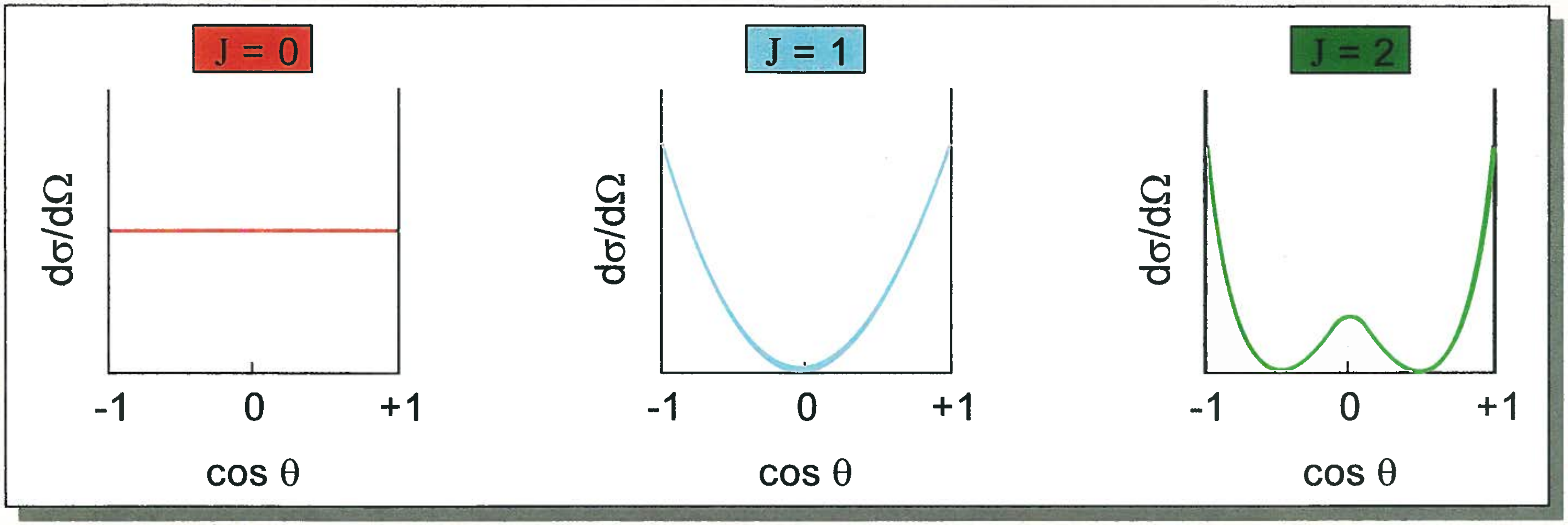
An example of $\mathrm{Im}\, z_i$ for the ith zero, below some energy E_0 there is one chosen solution — chosen by the partial waves satisfying unitarity. The lines at each energy, $\sqrt{s}$, depict experimental error bars. Near $\sqrt{s} = E_0$, $\mathrm{Im}\, z_i$ can change sign and for $\sqrt{s} > E_0$ the solutions bifurcate, i.e. there is a choice of $\mathrm{Im}\, z_i > 0$ or $\mathrm{Im}\, z_i < 0$. This leads to two different amplitudes.

What dynamics generates these zeros ?

In the simplest example of spinless particle scattering, with assumed well-separated resonances, there will be a sequence of peaks in the integrated cross-section. However, the angular distributions would have a distinctive shape, as seen in the Figure 23. If the resonance has spin 0, the differential cross-section would be flat. If spin 1, there would be one dip, if spin 2, two dips, and so on. Of course, this is only so in the idealized situation of a single dominant resonant partial wave in each energy region. Real data are, of course, more complicated. Nevertheless, the dips are generated by zeros of the underlying amplitude. They give the spin to the resonances. Thus spectroscopy is the interplay of poles and zeros traversing the Mandelstam plane.

Spin analysis





Spectroscopy: interplay of poles & zeros

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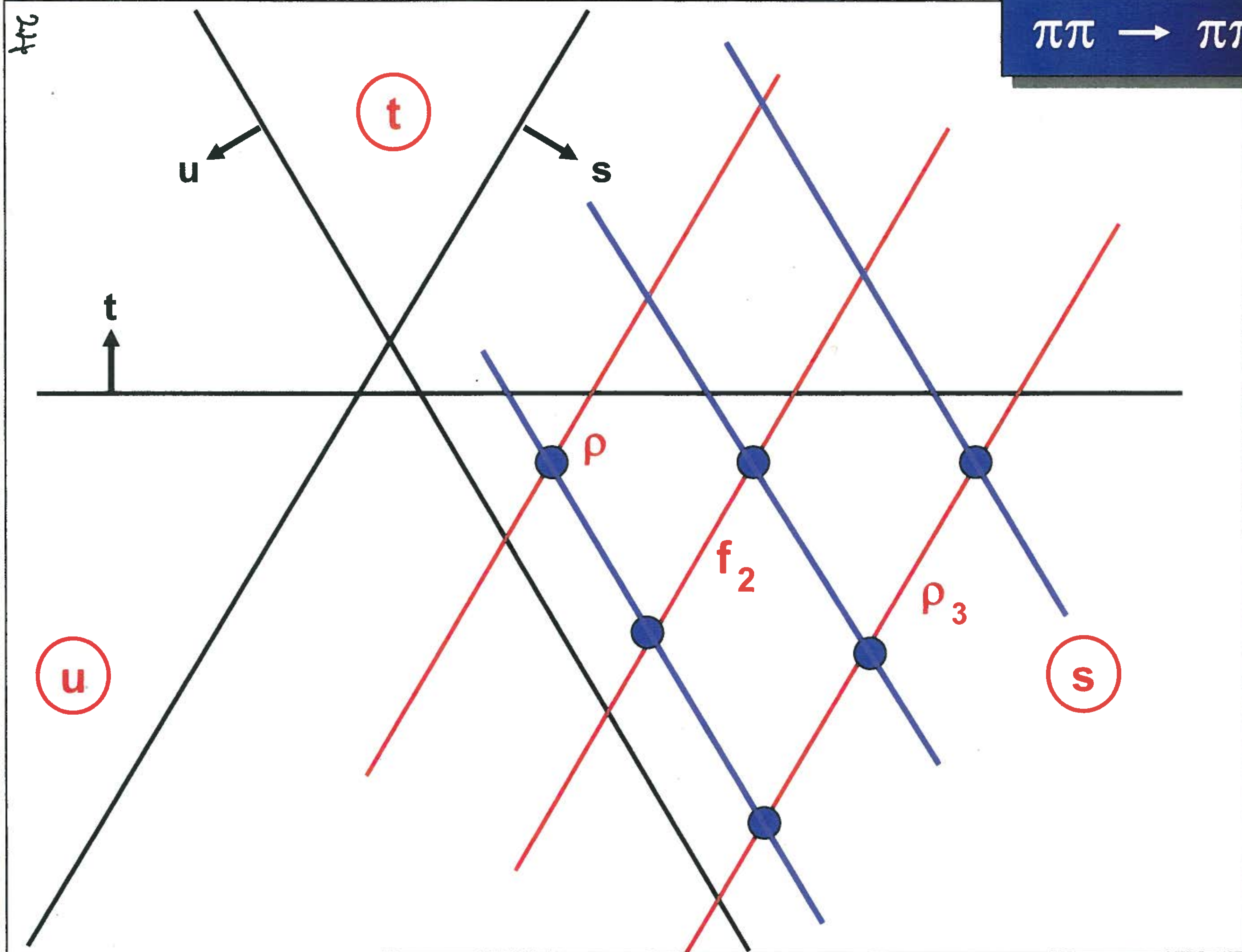
$\pi\pi \rightarrow \pi\pi$



This interplay is seen in an idealized world shown in Figure 24 for $\pi\pi$ scattering. The ρ has spin 1, so there is one zero in the physical region for $s \simeq m_\rho^2$. The $f_2(1270)$ has spin 2, so there are two zeros for $s \simeq m_{f_2}^2$. The $\rho_3(1680)$ has spin 3, so there are three zeros for $s \simeq m_{\rho_3}^2$, etc. Now zeros of a function of several complex variables are not isolated, they lie along contours. Thus the zeros that give spin to the resonances, known as Legendre zeros (as they coincide with the zeros of the appropriate Legendre polynomial) are connected from one energy to another. The Legendre zero of the ρ becomes one of the Legendre zeros of the f_2 and of the ρ_3, f_4 and so on. As the energy increases new zeros enter the physical region.

At higher energy those nearest the forward or backward direction become the zeros of the corresponding Regge residue. At very low energies for pseudoscalar meson scattering (eg $\pi\pi \to \pi\pi$, $\pi K \to \pi K$), the one zero in, or near, the physical region is a reflection of chiral dynamics, as we will discuss later. In the idealization illustrated in Fig. 24, motivated by the Veneziano model, the zeros follow straight lines. In reality they don't, but nevertheless zeros do connect chiral, resonance and Regge dynamics.

The formal definition of the S-matrix and the translation of its elements in to quantities defined in terms of observables can be found in many textbooks.

These all involve the definition of the volume element for a single particle state in momentum space known as Lips "Lorentz invariant phase space". This element is defined as

$$d\text{Lips}(P) = \frac{d^4P}{(2\pi)^3}\,\delta(P^2-m^2) = \frac{d^3\mathbf{p}}{2E(2\pi)^3}$$

for a particle of 4-momentum P^μ, 3-momentum $\mathbf{p}$.

The vector for a state of momentum p, $|p\rangle$ is then normalized so that

$$\int d\text{Lips}(P)\ \langle p'|p\rangle = 1.$$

For an n-particle state, Lips becomes

$$d\text{Lips}(P_1, P_2, \ldots, P_n) = (2\pi)^{-3n}\prod_{i=1}^{n}\frac{d^3\mathbf{p}_i}{2E_i}\ .$$

The S-matrix element S_{fi} describes the transformation of the initial state i into the final state f. [See some appropriate textbook for the definition of the vector spaces for i and f].

S_{fi} contains the transition matrix element T_{fi} from which the scattering amplitude is defined.

In the case when $i = f$, there is also the possibility that the particles come in and go out unchanged without interacting, and so is uninteresting.

Thus the definition of T_{fi} is given by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4\left(\sum_i P_i - \sum_f P_f\right) T_{fi} .$$

From the matrix element T_{fi}, the cross-section for $AB \to n$ (where initial state is AB, and the final state contains n-particles) is given by

$$\sigma(AB \to CDE \ldots) = \frac{1}{4p\sqrt{s}} \int d\mathrm{Lips}(Q_1, Q_2 \ldots Q_n)$$

$$\times (2\pi)^4 \, \delta^4 (p_a + p_b - \sum_i^n Q_i)$$

$$\times \, |\langle Q_1 \ldots Q_n, \mathrm{in} | T | p_a, p_b, \mathrm{in} \rangle|^2$$

Now conservation of probability means the S-matrix must be unitarity, i.e. $S^+S = SS^+ = 1$. In terms of T-matrix elements this means symbolicalli

$$i(T_{fi}^{+} - T_{fi}) = [T^+T]_{fi}$$

$$= \sum_n T_{fn}^{+} T_{ni}$$

where the sum over 'n' is the matrix multiplication. In the very special case that $i = f$, which means not just the initial and final state particles are the same, but that their momenta and spins are identical, even after interacting.

Then $\sum_n T^{+}_{fn} T_{ni} = \sum_n |T_{in}|^2$, so that

$$\mathrm{Im} \langle p_a, p_b, in | T | p_a, p_b, in \rangle$$

$$= \frac{1}{2} \sum_n \int d\mathrm{Lips}(Q_1, Q_2, \ldots Q_n) \quad (2\pi)^4 \delta^4(p_a + p_b - \sum_r^n Q_r$$

$$|\langle p_a, p_b, in | T | Q_1, Q_2, \ldots Q_n \rangle|^2$$

Now the right hand side is

$$2p\sqrt{s} \sum_n \sigma(AB \to n) = 2p\sqrt{s}\, \sigma_{total}(AB \to X)$$

where the total cross-section is the complete sum over all possible final states that are kinematically allowed.

Thus we have the optical theorem that

$$\sigma_{total}(s) = \frac{1}{2p\sqrt{s}} \mathrm{Im}\, F(s, \theta = 0)$$

where $F(s, \theta = 0)$ is the forward scattering amplitude for $AB \to AB$, and σ_{total} is for $AB \to X$