Lecture 2: Scattering and partial waves: Unitarity

For elastic scattering $AB \to AB$ $\quad p = q$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \, |F(s,t)|^2$$

Then the cross-section, called $\sigma_{elastic}$, is given by

$$\sigma_{elastic} = \int d\Omega \, \frac{d\sigma}{d\Omega}$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} dz \, \frac{1}{64\pi^2 s} \, |F(s,z)|^2$$

where $z = \cos\theta_s$. To perform the angular integration make a partial wave expansion of $F(s,z)$, the spinless particle scattering amplitude:

$$F(s,z) = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1) \, f_\ell(s) \, P_\ell(z)$$

$$\therefore \sigma_{elastic}(s) = \frac{2\pi}{64\pi^2 s} \int_{-1}^{1} dz \, (16\pi)^2 \sum_\ell (2\ell+1) \, f_\ell^*(s) \, P_\ell(z) \, \times$$

$$\times \sum_{\ell'} (2\ell'+1) \, f_{\ell'}(s) \, P_{\ell'}(z)$$

$$= \frac{16\pi}{s} \, \frac{1}{2} \int_{-1}^{1} dz \sum_\ell (2\ell+1) \, f_\ell^*(s) \, P_\ell(z) \, \times$$

$$\sum_{\ell'} (2\ell'+1) \, f_{\ell'}(s) \, P_{\ell'}(z)$$

& $\sigma_{elastic}(s) = \frac{16\pi}{s} \sum_{\ell} \sum_{\ell'} (2\ell+1)(2\ell'+1) f_{\ell}^{*}(s) f_{\ell'}(s) \times$

$$\frac{1}{2} \int_{-1}^{1} dz \, P_{\ell}(z) \, P_{\ell'}(z')$$

$$= \frac{16\pi}{s} \sum_{\ell} (2\ell+1) \, |f_{\ell}(s)|^{2}$$

The optical theorem gives

$$\sigma_{tot}(s) = \frac{1}{2p\sqrt{s}} \operatorname{Im} F(s, z=1)$$

$$= \frac{16\pi}{2p\sqrt{s}} \sum_{\ell} (2\ell+1) \operatorname{Im} f_{\ell}(s)$$

as $P_{\ell}(z=1) = 1$.

At low energy in what is called the "elastic region" there is only the AB final state, when AB is the initial scattering. Then

$$\sigma_{tot}(s) = \sigma_{elastic}(s)$$

In this region of elastic unitarity

$$\frac{16\pi}{2p\sqrt{s}} \sum_{\ell} (2\ell+1) \operatorname{Im} f_\ell(s) = \frac{16\pi}{s} \sum_{\ell} (2\ell+1) |f_\ell(s)|^2 .$$

It is then plausible that for each partial wave

$$\operatorname{Im} f_\ell(s) = \frac{2p}{\sqrt{s}} |f_\ell(s)|^2$$

– indeed this can be rigorously proved.

Above the inelastic threshold, $\sigma_{tot}(s) \geqslant \sigma_{elastic}(s)$

then

$$\sigma(AB \to X) = \sigma(AB \to AB) + \sigma(AB \to X \text{ (not } AB)),$$

so for each partial wave for $AB \to AB$ scattering

$$\operatorname{Im} f_\ell(s) \geqslant \frac{2p}{\sqrt{s}} |f_\ell(s)|^2$$

since cross-sections are positive.

Note that at the threshold for $AB \to AB$, the amplitude $F(s,z)$ has an imaginary part most simply expressed in terms of its partial waves so that for

$s > (m_A + m_B)^2 \equiv s_t$ s at threshold

$$\text{Im}\, f_\ell(s) = \frac{2p}{\sqrt{s}} \, |f_\ell(s)|^2$$

as we have just seen. Indeed

If we consider equal mass scattering $AA \to AA$ with $m_A = m$, then the factor ("phase space", ρ)

$$\rho \equiv \frac{2p}{\sqrt{s}} = \frac{\sqrt{s - s_t}}{\sqrt{s}} \qquad \text{where} \quad s_t = 4m^2.$$

To understand the structure of an amplitude that has an imaginary part proportional to ρ, let us first consider the function

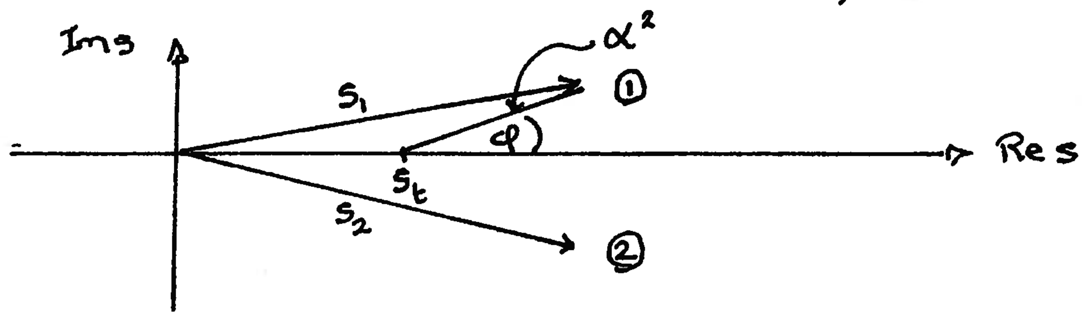
$$g(s) \equiv \sqrt{s_t - s}$$

Clearly for real $s < s_t$, $g(s)$ is real.

But when $s > s_t$ and s is real $g(s)$ is imaginary.

Let us consider its value at $s_1 = s_t + \alpha^2 e^{i\varphi}$

and then at $s_2 = s_t + \alpha^2 e^{i(2\pi-\varphi)}$, with α real.



we consider the phase $\varphi \to 2\pi - \varphi$ (rather than $\varphi \to -\varphi$)

so that we pass from the region where $\mathrm{Im}\, s > 0$ to where $\mathrm{Im}\, s < 0$ by going across the real s axis where $g(s)$ is real.

At $s = s_1$: $g(s_1) = \sqrt{s_t - s_1} = \sqrt{-\alpha^2 e^{i\varphi}}$

We choose $-1 = e^{i\pi}$, then

$$g(s_1) = \alpha\, e^{i(\pi+\varphi)/2} = i\alpha\, e^{i\varphi/2}$$

At $s = s_2$: $g(s_2) = \sqrt{-\alpha^2 e^{i(2\pi-\varphi)}}$

$$= \alpha\, e^{i(3\pi-\varphi)/2} = -i\alpha\, e^{-i\varphi/2}$$

We see that even if $\varphi \to 0$, $g(s_1) \neq g(s_2)$

The function $g(s)$ has a discontinuity along the real axis, when $s > s_t$.

$$\text{disc } g(s) = g(s+i\varepsilon) - g(s-i\varepsilon)$$

$$= 2i\sqrt{s-s_t} \qquad [\varepsilon \sim \varphi].$$

$$= 2i \ \text{Im} g(s)$$

The function $g(s)$ is analytic, namely it can be represented by its Taylor series in neighborhood of almost every value of s (except where it has a discontinuity)

Since $g(s)$ is real for s real $< s_t$, it is then called a "real analytic" function, which can be proved to satisfy the Schwarz reflection property

$$g(s^*) = g^*(s).$$

 Indeed we saw $g(s_2) = g^*(s_1)$ and $s_2 = s_1^*$.

So far we have considered $g(s)$ on the so called "first sheet". We chose $-1 = e^{i\pi}$.

But, of course, we could have chosen $-1 = e^{-i\pi}$,

then $g(s_1) = -i\alpha e^{i\varphi/2}$, $g(s_2) = i\alpha e^{-i\varphi/2}$

It is the negative of the function on sheet 1. It is $g(s)$ on sheet 2. Every square root has a plus or minus sign.

To understand the scattering amplitude and its analytic structure, we need to consider a slightly different function

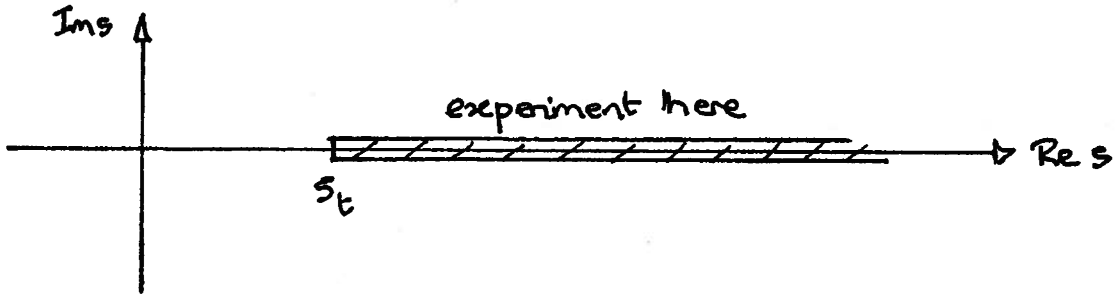
$$h(s) = \sqrt{\frac{s_t - s}{s}}$$

It is easy to see now that on sheet 1

$$h(s+i\varepsilon) = i\sqrt{\frac{s-s_t}{s}} \quad , \quad h(s-i\varepsilon) = -i\sqrt{\frac{s-s_t}{s}}$$

when $\varepsilon \to 0+$ and $s > s_t$.

As expected the function $h(s)$ has a discontinuity along the real axis for $s > s_t$. This is called a "cut".



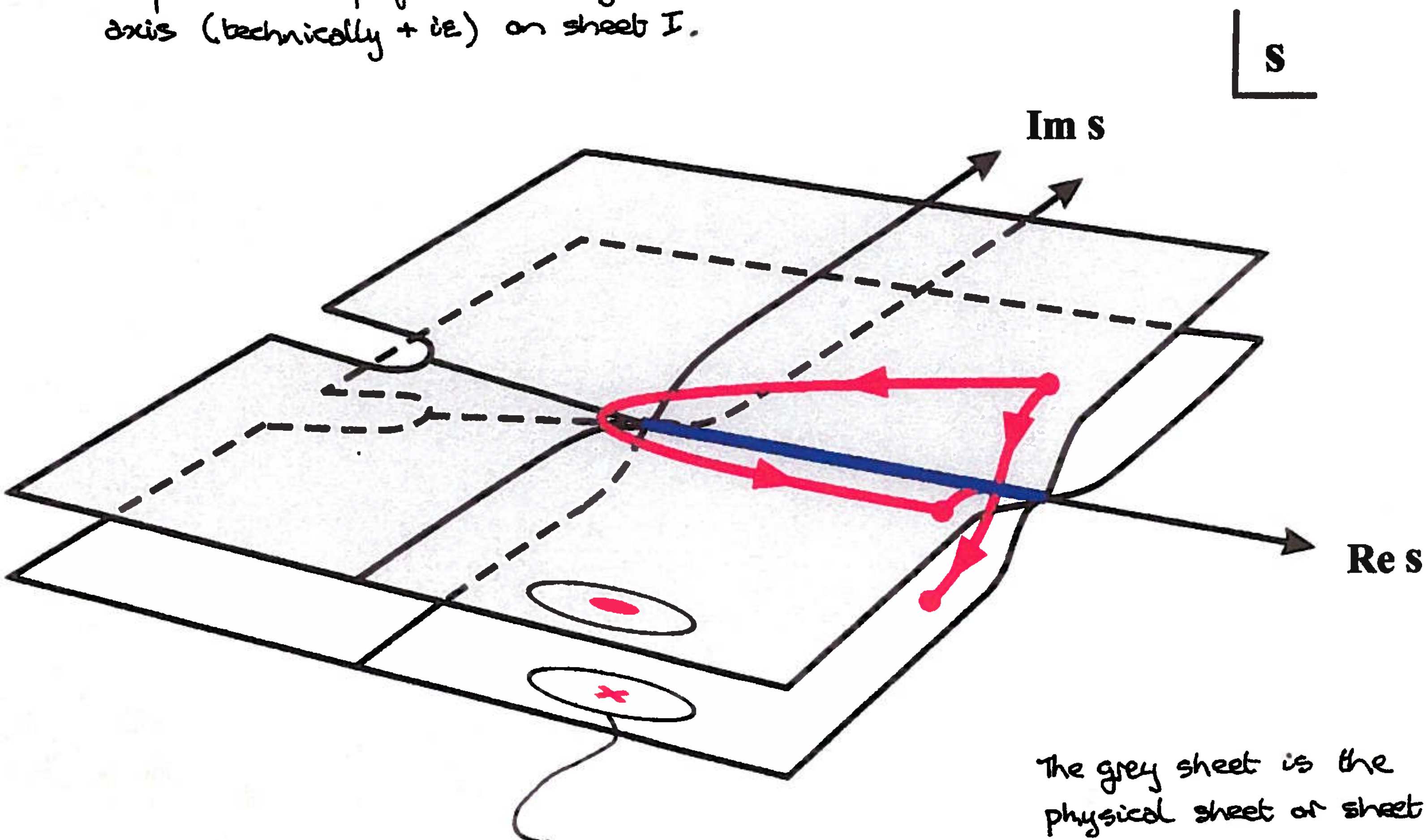
Now on sheet 2 when $\varepsilon \to 0+$, $s > s_t$

$$h(s+i\varepsilon) = -i\sqrt{\frac{s-s_t}{s}}, \quad h(s-i\varepsilon) = i\sqrt{\frac{s-s_t}{s}}$$

See the figure on page 35

While $h(s)$ has a cut, with $h(s+i\varepsilon) \neq h(s-i\varepsilon)$ on either sheet, we see $h_I(s+i\varepsilon) = h_{II}(s-i\varepsilon)$ where the roman numeral labels the sheet. This explains the continuation routes indicated in the figure, page 35.

The cut structure at a single threshold.
Experiment is performed along the real axis (technically $+ i\epsilon$) on sheet I.



resonance pole on sheet II.
On sheet I there is a corresponding zero of the S-matrix.

The grey sheet is the physical sheet or sheet I.
The white sheet is the unphysical sheet or sheet II where resonance poles lie.

For $AB \to AB$ in the region of elastic unitarity each partial wave satisfies

$$\mathrm{Im}\, f_\ell(s) = \rho(s)\, |f_\ell(s)|^2 .$$

The solution of this equation can be represented by

$$f_\ell(s) = \frac{1}{\rho(s)}\, e^{i\delta_\ell(s)} \sin\delta_\ell(s)$$

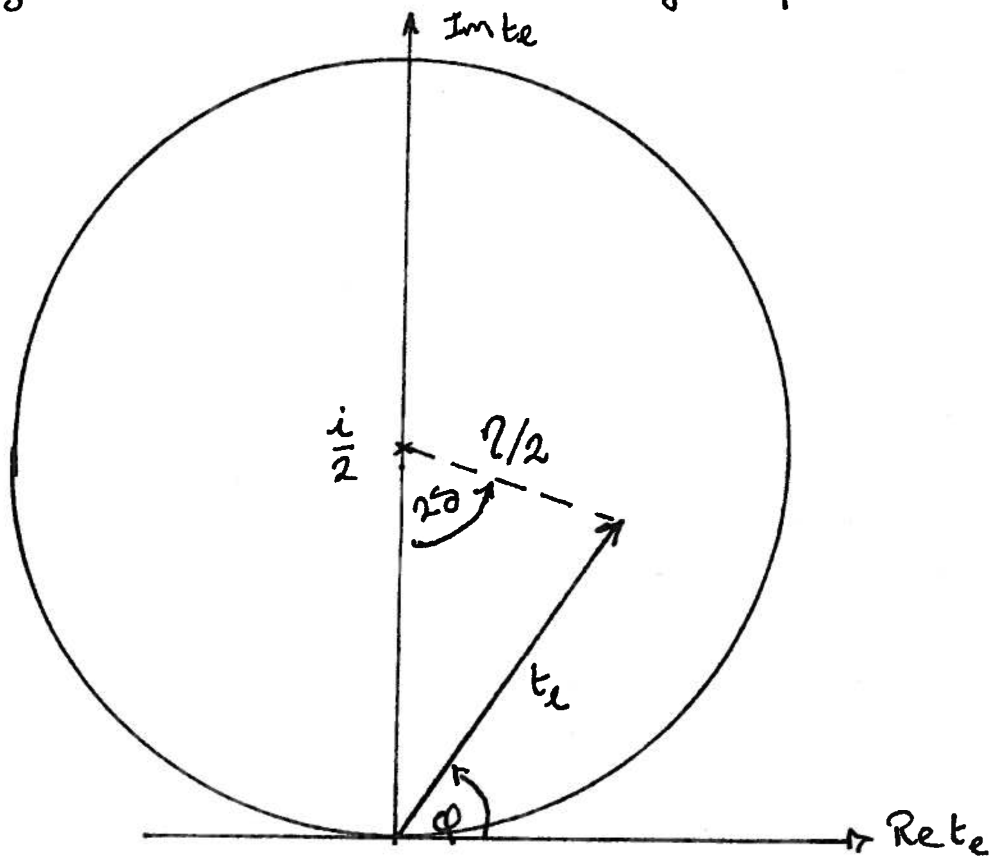
where δ_ℓ is the phase-shift. Importantly, the unitarity equation is non-linear and so it fixes the magnitude of $f_\ell(s)$ in terms of it's phase. Above the inelastic threshold, the partial wave amplitude for $AB \to AB$ is represented by

$$f_\ell(s) = \frac{1}{\rho(s)}\, \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2i}$$

where $\eta_\ell = 1$ for elastic unitarity, so η_ℓ is the "inelasticity"

$$\mathrm{Im}\, f_\ell - \rho\, |f_\ell|^2 = \frac{1}{\rho}\, \frac{1}{4}\,(1-\eta_\ell^2)$$ is proportional to the inelastic cross-section.

Defining $t_\ell \equiv \rho f_\ell$, then t_ℓ can be conveniently displayed as a vector on an Argand plot



In the elastic region, t_ℓ runs around the circle then $\eta = 1$, and the phase $\phi = \delta$.

Above the inelastic threshold, $AB \to CD$ becomes possible and unitarity relates the partial waves for $AB \to AB$, $AB \to CD$ and $CD \to CD$. Let us label the initial and final states by 1 and 2, where $1 \equiv AB$, $2 = CD$. Then writing the hadronic amplitudes as T_{11}, T_{12}, T_{22} for each ℓ.

Time-reversal invariance implies $T_{12} = T_{21}$.

Then two-channel unitarity requires

$$\operatorname{Im} T_{11} = \rho_1 |T_{11}|^2 + \rho_2 |T_{12}|^2$$
$$\operatorname{Im} T_{12} = \rho_1 T_{11}^* T_{12} + \rho_2 T_{12}^* T_{22}$$
$$\operatorname{Im} T_{22} = \rho_1 |T_{12}|^2 + \rho_2 |T_{22}|^2$$

The generalization to any number of channels is obvious

Now how do we solve this set of equations.

Let us first return to the single channel case

$$\text{Im}\, T_{11} = \rho_1 \, |T_{11}|^2 .$$

Note that

$$\text{Im}\,\frac{1}{T_{11}} = \text{Im}\,\frac{T_{11}^*}{|T_{11}|^2} = -\frac{\text{Im}\, T_{11}}{|T_{11}|^2} = -\rho_1$$

so unitarity fixes the imaginary part of $1/T_{11}$, leaving it's real part to be fixed by experiment and/or dynamics. Let us define

$$\text{Re}\,\frac{1}{T_{11}} = \frac{1}{K_{11}} \qquad \text{where } K_{11} \text{ is real}$$

$$\therefore \frac{1}{T_{11}} = \frac{1}{K_{11}} - i\rho_1 \quad \Rightarrow \quad T_{11} = \frac{K_{11}}{1 - i\rho_1 K_{11}}$$

In terms of the phase-shift representation

$$K_{11} = \frac{1}{\rho_1} \tan\delta_{11}$$

Whilst any real function K_{11}, with no right hand cut,

43 satisfies elastic unitarity, an example is to model

$$K_{11} = \frac{g_1^2}{M^2 - s},$$

then

$$T_{11} = \frac{g_1^2}{M^2 - s - i\varsigma_1 g_1^2}$$

which is a Breit-Wigner form. As we will emphasize again later, such a form is an approximation valid only in the region near the pole at $s = M^2 - i\varsigma_1 g_1^2$. This does NOT mean it is a good approximation for real $s \simeq M^2$, particularly if g_1^2 is large (as for instance for the σ and κ). In the single channel case $g_1^2 = M\Gamma$, where Γ is the width.

Now the K-matrix provides a generalization to any number of coupled channels that can be understood by returning to the two-channel case.

From the 3 unitarity equations given above one can readily show

$$\operatorname{Im} T^{-1} = -\rho$$

where T is a 2×2 matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$

but recall $T_{21} = T_{12}$, and

ρ is a diagonal matrix of phase-space element

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$

Then

$$T^{-1} = K^{-1} - i\rho$$

writing out this 2×2 equation we have

$$T_{11} = \frac{K_{11} - i\rho_2 \det K}{\Delta}$$

$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\rho_1 \det K}{\Delta}$$

where each T-matrix element has a common denominator

$$\Delta = 1 - i\rho_1 K_{11} - i\rho_2 K_{22} - \rho_1\rho_2 \det K$$

where $\det K = K_{11} K_{22} - K_{12}^2$.

It is when $\Delta = 0$ at complex s (on the nearby unphysical sheet) that determines the poles of the S-matrix, and hence the spectrum of hadrons.

One may worry that it is necessary to understand the effect of every threshold, even if the channel is not yet open. It is straightforward to see this is not the case. To see this, consider the two channel solution below the 2nd threshold, then $i\rho_2$ is real. If in

$$T_{11} = \frac{K_{11} - i\rho_2 \det K}{1 - i\rho_1 K_{11} - i\rho_2 K_{22} - \rho_1\rho_2 \det K}$$

we write $$K = \frac{K_{11} - i\rho_2 \det K}{1 - i\rho_2 K_{22}}$$, which is real,

then

$$T_{11} = \frac{K}{1 - i\rho_1 K}$$, the single channel case.

Thus as each threshold is crossed, the meaning of K-matrix elements of the previously open channels has to be re-interpreted in terms of the increased number of channels, but the parametrizations are equivalent.

While parametrizing the K-matrix is a convenient representation of coupled-channel unitarity, it doesn't have the correct analytic structure. It has no left hand cut (see later), or even the correct right hand cut. This latter defect can be remedied by noting that $\quad \mathrm{Im}\, T^{-1} = -\rho \quad$ does not mean we know nothing about the real part.

Indeed in the case of S-wave interactions:

$$T^{-1} = \bar{K}^{-1} + \frac{\rho}{\pi} \ln\left(\frac{\rho+1}{\rho-1}\right)$$

where $\bar{K}$ is a modified K-matrix, and the 2nd term on the right hand side is the "Chew-Mandelstam function in matrix form

$$\begin{pmatrix} \frac{\rho_1}{\pi} \ln\left(\frac{\rho_1+1}{\rho_1-1}\right) & 0 \\ 0 & \frac{\rho_2}{\pi} \ln\left(\frac{\rho_2+1}{\rho_2-1}\right) \end{pmatrix}.$$

Above the threshold 'n'

$$\frac{\rho_n}{\pi} \ln\left(\frac{\rho_n+1}{\rho_n-1}\right) = \frac{\rho_n}{\pi} \ln\left(\frac{1+\rho_n}{1-\rho_n}\right) - i\rho_n .$$

Fitting data for baryonic channels incorporating the Chew-Mandelstam function has been used by the

48 SAID group as Ron Workman will discuss, and more recently by the Bonn-Gatchina group in 2011