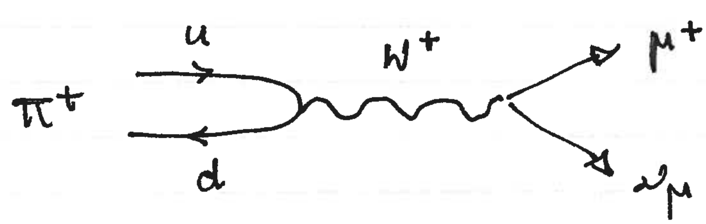
# Lecture 3: chiral dynamics and analyticity: an example

At low energies, $\pi$'s and $K$'s are the Goldstone bosons of chiral symmetry breaking, as described in the lectures of Bastian Kubis.

Pions, in particular, are very light. It is $m_\pi^2$ that is proportional to the current masses of $u$, $d$ quarks. Though pseudoscalar particles they couple through an axial vector current. This was the basis of much of current algebra. Nowadays, we see the decay $\pi^+ \to \mu^+ \nu_\mu$, as a $u$ quark turning into a $d$ quark and a $W^+$ boson that then couples to the two leptons




The $W$ with a mass of 80 GeV is way off its mass-shell. Only at $p^2 = m_W^2$ is the $W$ a vector boson. At $p^2 = m_\pi^2$, the pion has an axial vector coupling to this off-shell boson.

If pions were massless, and indeed $m_\pi^2 \ll m_\rho^2, m_N^2$, the axial vector current would be conserved. Conserved currents lead to low energy theorems.

For massless $\pi\pi$ scattering, this low energy theorem requires the $\pi\pi$ amplitude to be zero at threshold. Thus a strong hadronic interaction is weak at low enough momenta. This is the basis of Chiral Perturbation Theory.

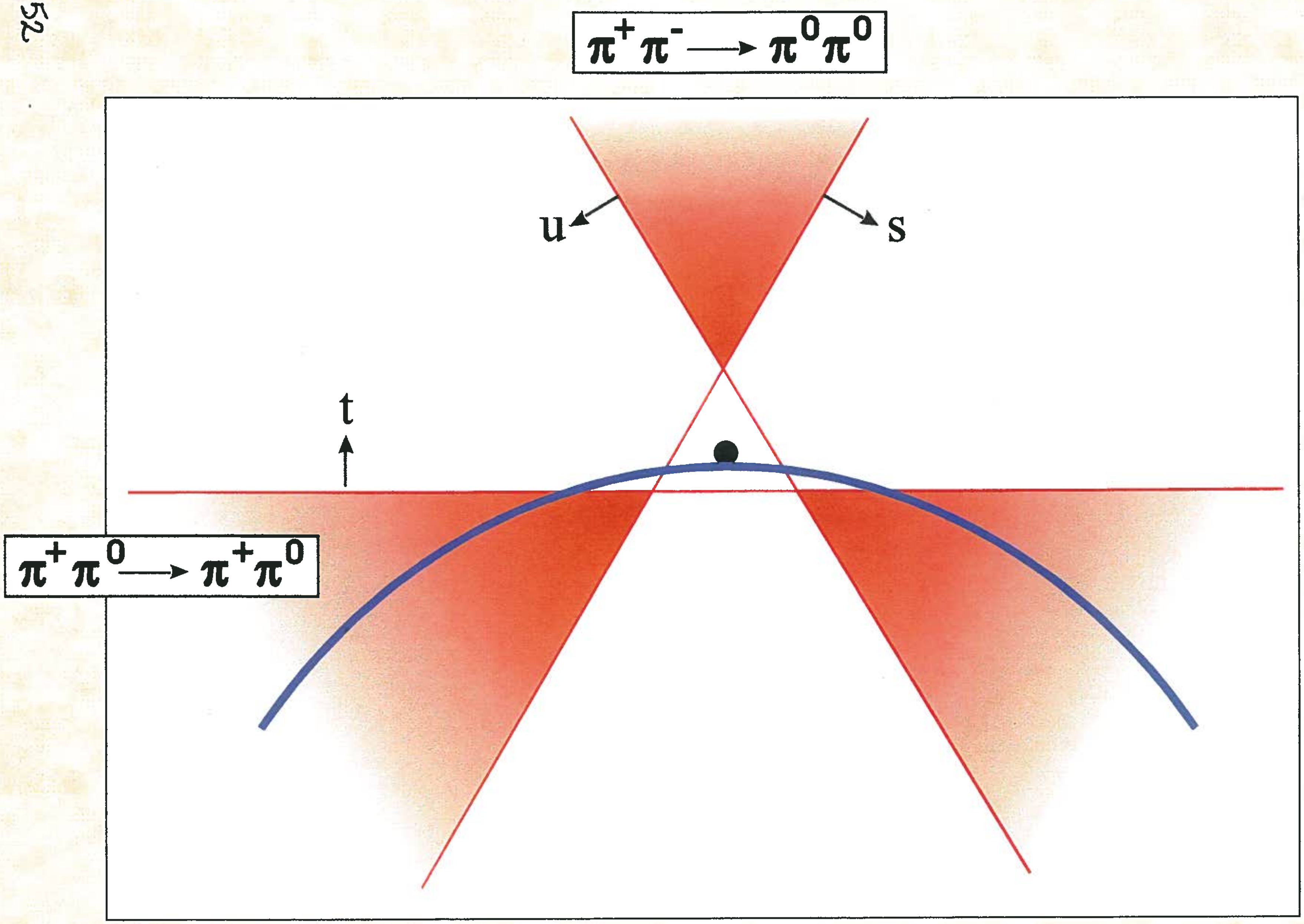
Remarkably, Adler showed that for $\pi\pi \to \pi\pi$ scattering with one massless pion (and the other 3 having their physical values) is zero when the momentum of the massless pion is zero. Thus at $s = t = u = m_\pi^2$, the $\pi\pi$ amplitude vanishes. This world of one massless pion is very close to the real world
50 since corrections are $O(m_\pi^2/m_\rho^2)$ or $O(m_\pi^2/32\pi f_\pi^2)$ where $f_\pi$ is the pion decay constant $\sim 0.1$ GeV.

Thus for physical mass $\pi\pi$ scattering, we expect the Adler condition to impose a zero of the amplitude somewhere inside the Mandelstam triangle.

As mentioned several times, an analytic function of several complex variables does not have an isolated zero, but must have a line of zeros.

In fact, this Adler zero is connected to the Legendre zero of the $\rho$ in $\pi\pi$ scattering, and to the Legendre zero of the $K^*$ in $\pi K$ scattering.

The presence of such a low energy (in fact subthreshold) zero imposes a remarkeable requirement on the high energy behavior of the scattering amplitude. This is a consequence of the amplitude being an analytic function, as we now show.



π+π- → π0π0
u
s
t
π+π0 → π+π0

The full scattering amplitude has, in general, a right hand cut generated by each threshold in the s-channel, and a left hand cut created by each threshold in the crossed channel. For $F(s,t)$ at fixed-$t$, this crossed channel is the u-channel. If we consider the amplitude for $\pi^-\pi^0 \to \pi^-\pi^0$ scattering in the s-channel, the u-channel is identical, while the t-channe is $\pi^-\pi^+ \to \pi^0\pi^0$.

For $0 \leq t < 4m^2$ (where $m = m_\pi$) the discontinuity across the right hand cut is $\mathrm{Im}\,F(s,t)$, which is real. So IF $|F(s,t)| \to 0$ as $|s| \to \infty$, $F(s,t)$ satisfies an unsubtracted dispersion relation at fixed-$t$ : $t \in [0, 4m^2)$ given by

$$F(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\mathrm{Im}\,F(s',t)}{s'-s} + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \, \frac{\mathrm{Im}\,F(u',t)}{u'-u} .$$

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As $F(s,t)$ is s-u symmetric in this case, one can write this as

$$F(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left( \frac{1}{s'-s} + \frac{1}{s'-u} \right) \operatorname{Im} F(s',t) .$$

The assumption that $|F(s,t)| \to 0$ as $|s| \to \infty$ ensures the circular integral at infinity on $|s|$ vanishes in the Cauchy representation.

Now $F(s,t)$ is the amplitude for a physical process, so at $t=0$ $\operatorname{Im} F(s,t=0) = 2p\sqrt{s}\,\sigma_{\text{total}} > 0$.

Moreover for $0 < t < 4m^2$

$$\operatorname{Im} F(s,t) = 16\pi \sum_{\ell} (2\ell+1) \operatorname{Im} f_\ell(s)\, P_\ell\left(1 + \frac{2t}{s-4m^2}\right)$$

$$> 16\pi \sum_{\ell} (2\ell+1) \operatorname{Im} f_\ell(s) > 0$$

as $P_\ell(x) > P_\ell(1) = 1$ for $x > 1$.

Indeed the growth in the $P_\ell(x)$ for $x > 1$ (remember they are polynomials of degree $x^\ell$) means the partial wave series for $\operatorname{Im} F(s,t)$ only converges for $t < 4m^2$

Since $\text{Im}\, F(s',t) > 0$, then the factors $(s'-s)$ & $(s'-u)$ will also be positive if $0 < s, u < 4m^2$.

Therefore the amplitude $F(s,t)$, from its dispersive integral must be positive if $s,t,u$ are inside the Mandelstam triangle.

As discussed by Bastian Kubis, chiral dynamics imposes a zero on the $\pi\pi$ scattering amplitude (here for $\pi^-\pi^0 \to \pi^-\pi^0$ on the s-channel) inside the Mandelstam triangle. We have seen this isn't possible.

$\therefore$ Chiral dynamics is incompatible with the amplitude decreasing at infinity. However, if the amplitude grows asymptotically (which experiment indicates it does) we have no such inconsistency.

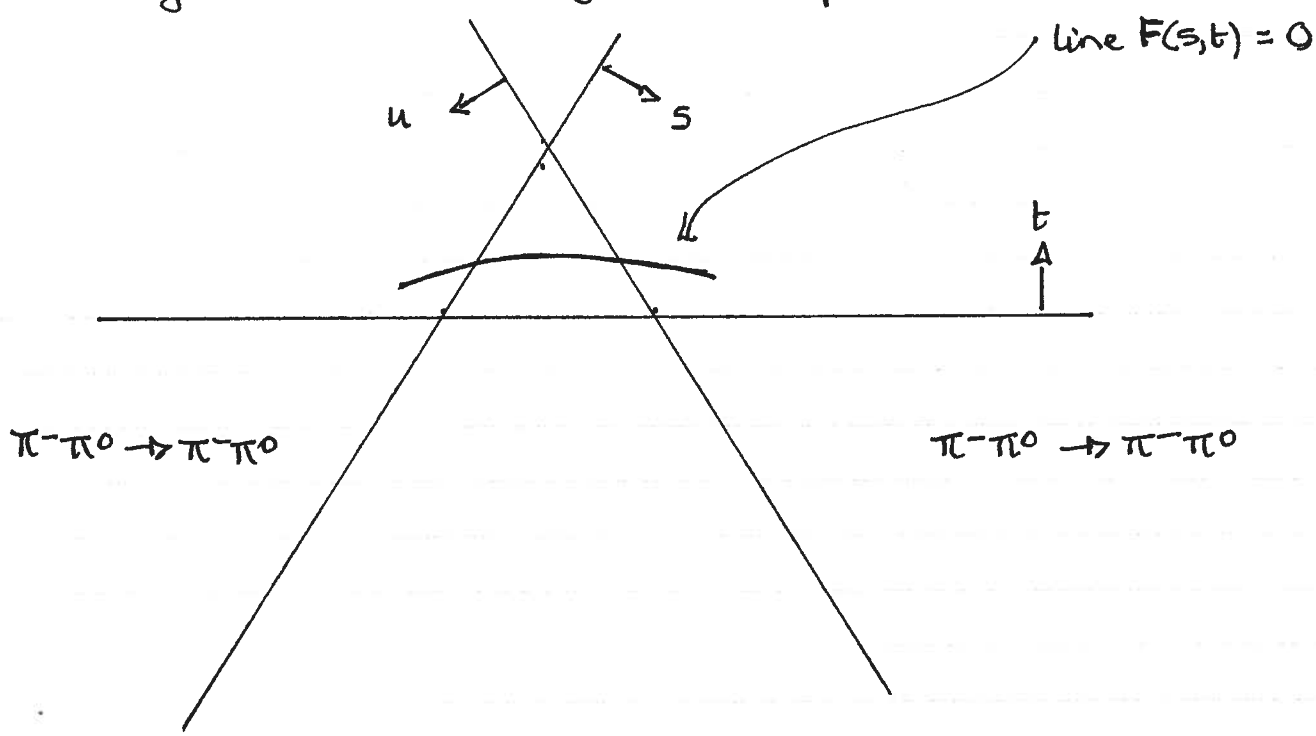
Instead provided $|F(s,t)| < |s|^2$ as $|s| \to \infty$ then positivity requires inside the right half of the

Mandelstam triangle

$$F(s,t) - F(s = 2m^2 - \tfrac{t}{2}, t) > 0$$

for $4m^2 > s > u > 0$, and by symmetry for $4m^2 > u > s > 0$.

This just requires that along a line at fixed-$t$ the amplitude grows as one moves away from the axis of symmetry $s = u$. Now there is no problem having a zero of the amplitude inside the Mandelstam triangle as chiral dynamics requires.



Very few hadrons are stable. Protons seem to live forever, certainly on the timescale of hadronic interactions, viz $\sim 10^{-23}$ s. The propagator in momentum space of a stable particle is $\dfrac{1}{M^2 - p^2}$ where $M$ is its mass and $p^\mu$ its 4-momentum.

$\pi$'s, $K$'s decay be weak interaction (or electromagnetically for the $\pi^0$) and so too have propagators like that of the nucleon $\sim \dfrac{1}{m^2 - p^2}$.

However, there is ample evidence that at low energies scattering of hadrons leads to resonant states: particles that exist for a short time and then decay, like excited spectral lines of atoms. These resonances have a propagator with a complex mass, described by a Breit-Wigner
57 form. It is important to realize that this is a mere approximation and not always very exact.

Breit-Wigner form :

The non-relativistic wavefunction for a particle of mass M has a time dependence

$$\sim e^{-iMt}$$

Its Fourier transform into energy space is then

$$\int dt\, e^{iEt} \quad e^{-iMt} \quad \sim \quad \frac{1}{M-E}$$

This is the non-relativistic propagator for a stable particle.

If the particle is unstable its wavefunction decays

$$\sim e^{-iMt} \; e^{-\Gamma t/2}$$

where $\Gamma$ is inversely proportional to its lifetime

Then the Fourier transform becomes

$$\sim \frac{1}{M - E - i\Gamma/2}$$

Relativistically this takes a form

$$\sim \frac{1}{M^2 - s - iM\Gamma}$$

where $s = p^2$ (its 4-momentum squared).

This is the Breit-Wigner form.

Unitarity in the single channel case tells us that $\Gamma$ contains the phase space factor $\rho$, and so one often redefines $\Gamma$ to take that into account and writes the amplitude for the relevant partial wave as

$$f_\ell(s) \sim \frac{1}{M^2 - s - i\rho m\Gamma}$$

When the resonance decays to several open channels then $\rho m \Gamma$ is replaced by $\sum_i \rho_i m \Gamma_i$ where the sum is over open channels and $\Gamma_i$ is the partial width to channel '$i$'.

As discussed in the lectures of SU Chung and Klaus Peters, $\Gamma$ is really a function of $s$ and contains centrifugal barrier factors for each channel depending on the angular momentum involved.

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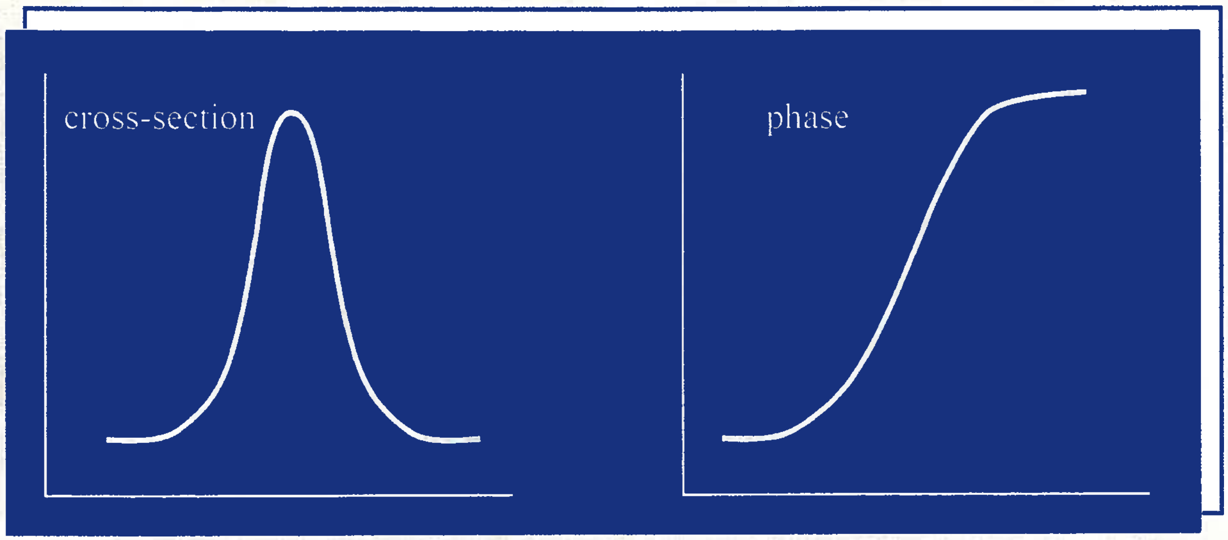
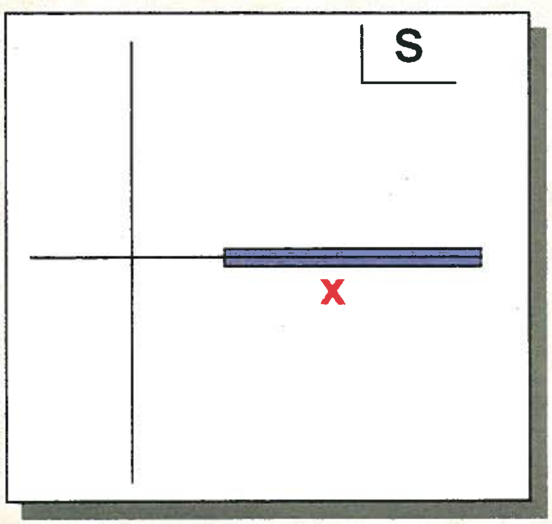
# Hadron States




**Breit-Wigner**

$$\frac{1}{M^2 - s - iM\Gamma}$$

For an isolated resonance, isolated from other resonances, and away from strongly coupled thresholds, the resonance will give a peak in a cross-section at an energy ~ mass of the resonance, with a width inversely proportional to its lifetime, as in Figure page .
The phase of the corresponding amplitude will rise by ~ 180° if the resonance is elastic, much less if it is very inelastic. This is embodied in the Breit-Wigner description which has a pole in the complex s-plane on the nearby unphysical sheet, as shown on page . When resonances overlap, and overlap with strongly coupled thresholds, then only the presence of a pole in the complex s-plane is proof that there is a state in the spectrum.
It is the position of this pole that is universal, appearing at the same place in every process in which

the resonance can couple. The appearance on the real energy axis, where experiments are performed, may well be highly process dependent. Thus the $f_0(980)$ appears as a dip in $\pi\pi \to \pi\pi$, yet a peak in $\psi \to \phi\pi\pi$. For the $I=J=0$ $\pi\pi$ cross-section see Figure page .

The low mass enhancement is related to the presence of a pole, the $\sigma$ deep in the complex plane. For a typical Breit-Wigner-like resonance such as the $\rho(770)$, the pole is sufficiently nearby that the phase change of the $I=J=1$ $\pi\pi$ amplitude along the real axis is not so different from that in the complex plane nearer the $\rho$-pole see Figures pages . In contrast, for the $\sigma$ the phase change for $\mathrm{Im}\,E = -250$ MeV and $\mathrm{Im}\,E = +1$ MeV are dramatically different. Near the pole the phase of the $I=J=0$ $\pi\pi$ amplitude changes by

$180^\circ$, while along the real axis it varies much more slowly. It is this that made the $\sigma$ very difficult to uncover until the right analytic tools, which embodied crossing symmetry, were applied (viz the Roy equations) and data very close to threshold on $K \to e\nu\pi\pi$ reached their present precision, see Figures pages .

Analyticity tells us that if $\Gamma$ is a function of $s$, then so is $M^2$: real and imaginary parts are related. Consequently the Breit-Wigner form is an approximation only valid near the pole itself when $M, \Gamma$ are close to constant. For a very short-lived state the amplitude may change dramatically between the pole position and the real energy axis where experiment is performed as we have seen.

Moreover, since the imaginary part of the propagator is generated by the loop of hadrons to which the resonance decays, analyticity requires the real part of its mass function must also be a function of $s$. With the propagator given by $1/[M^2(s) - s]$, the real and imaginary parts of $M^2(s)$ are shown in Figures pages .

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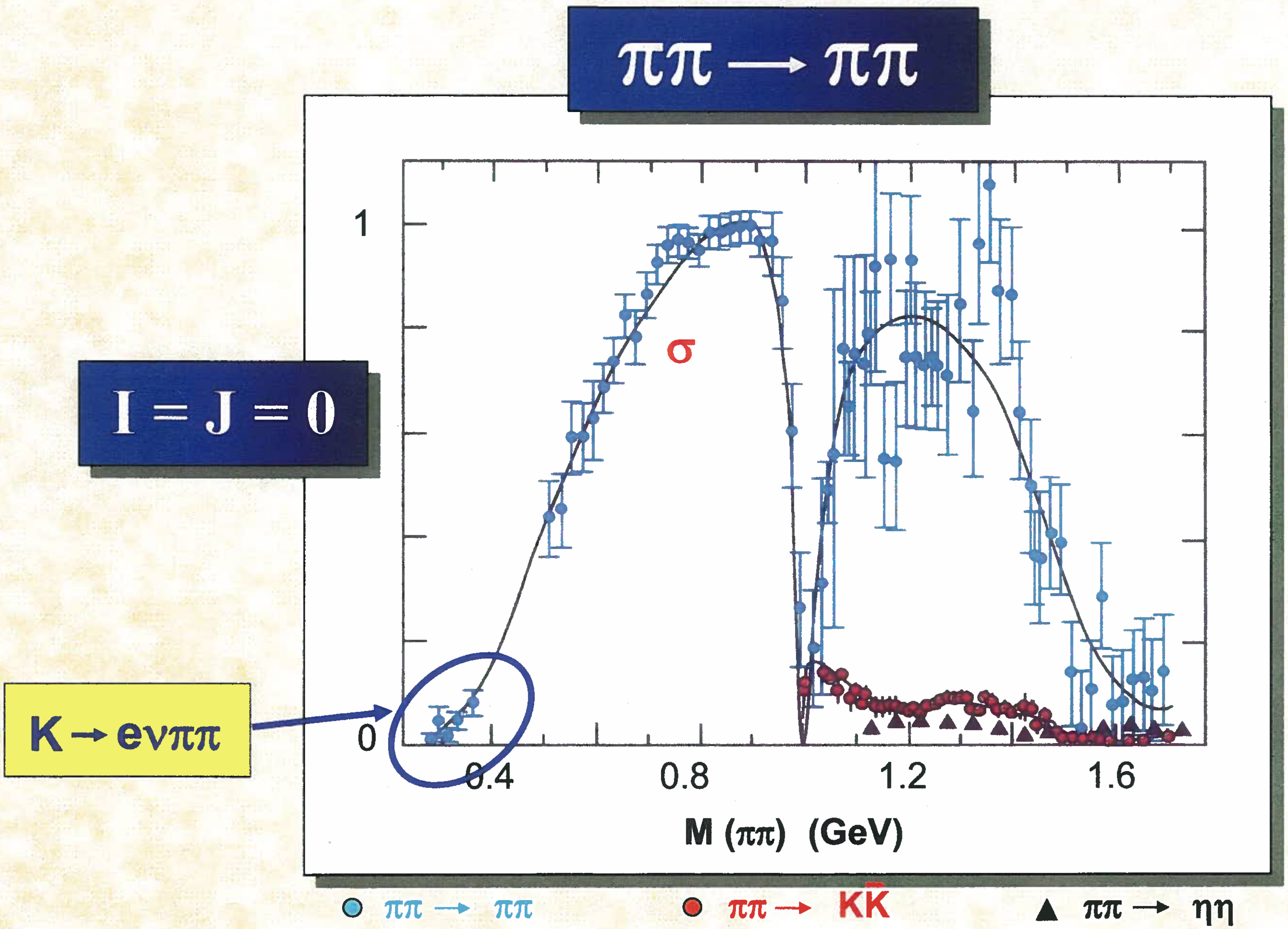
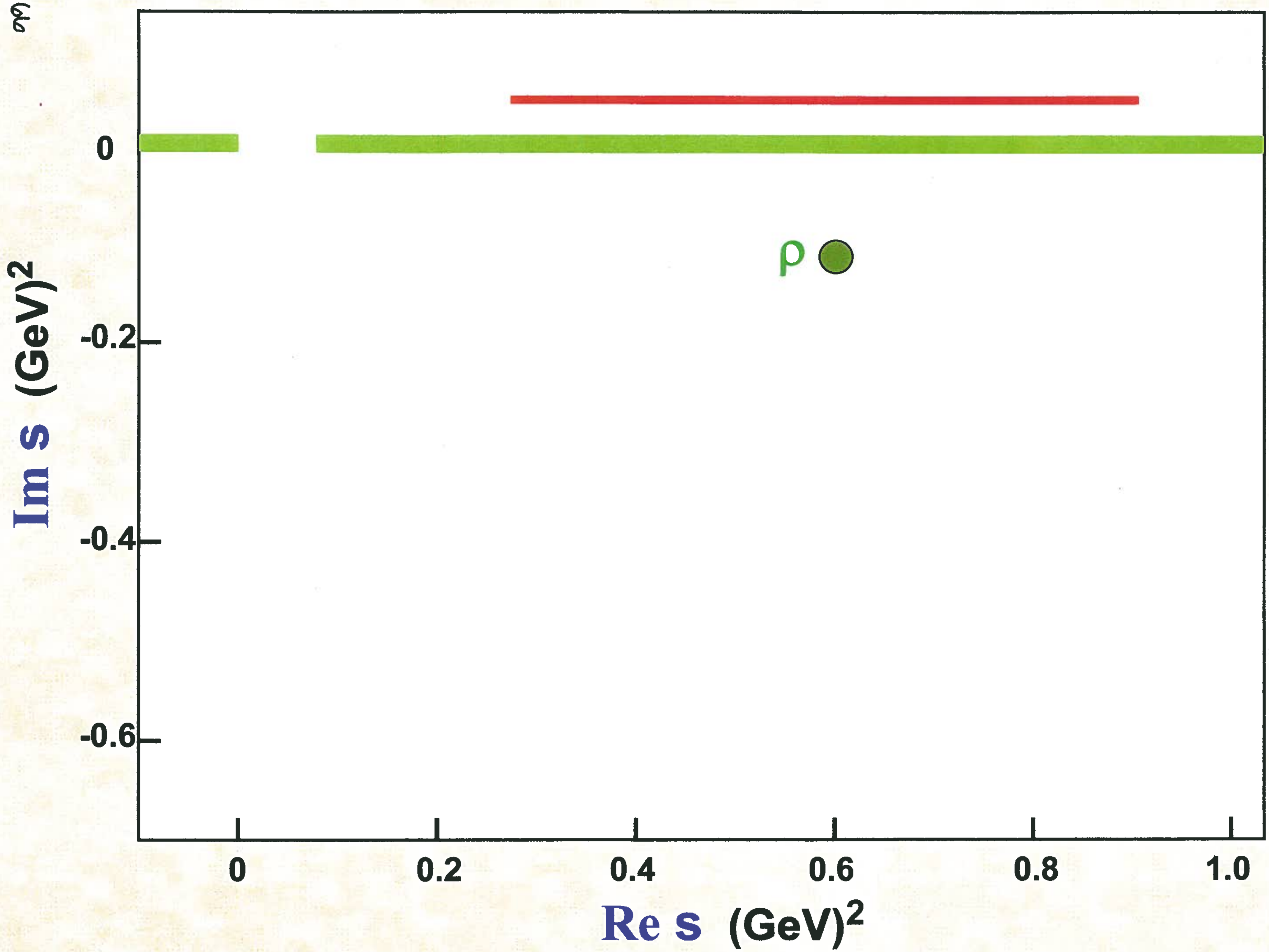
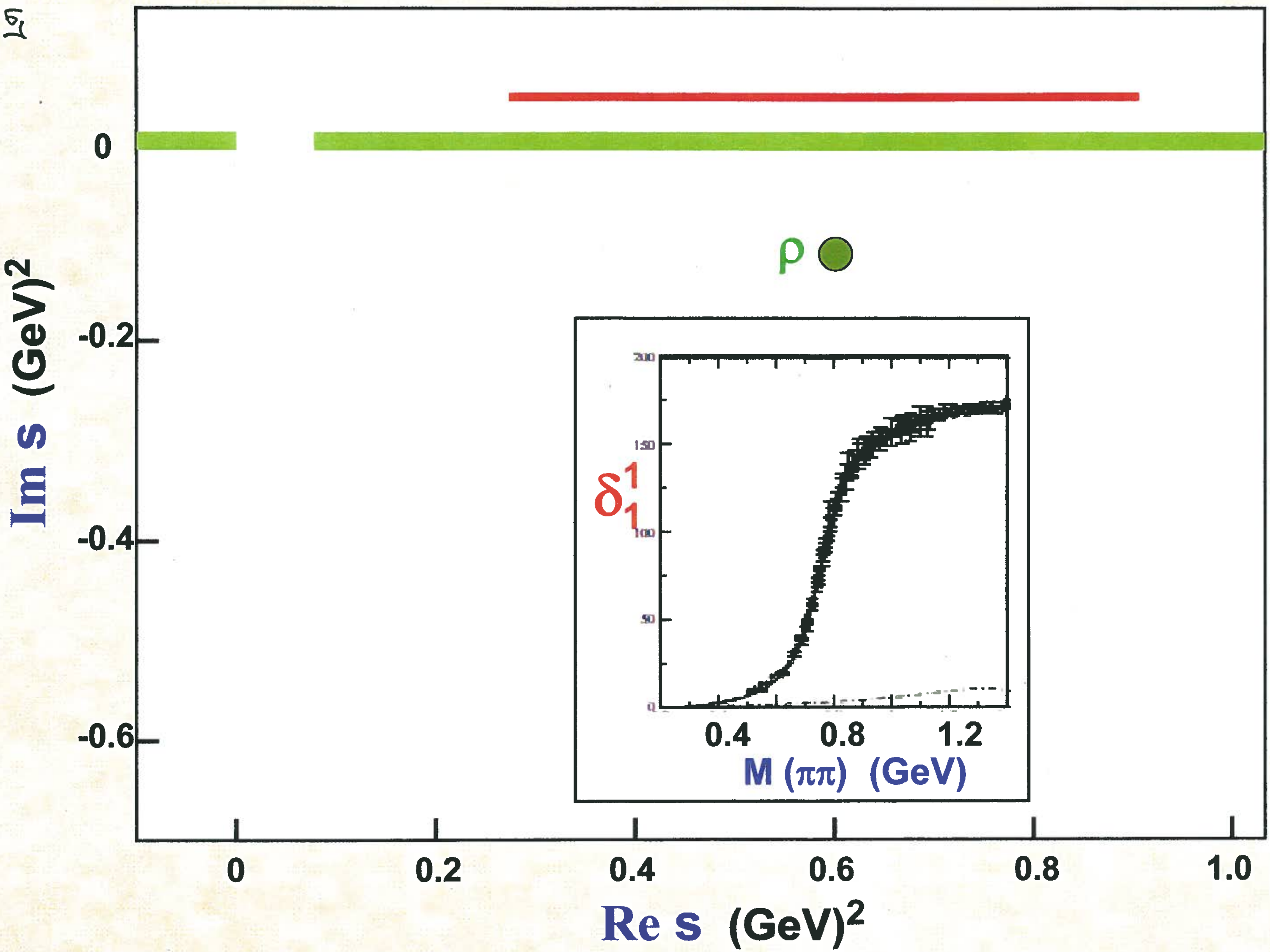
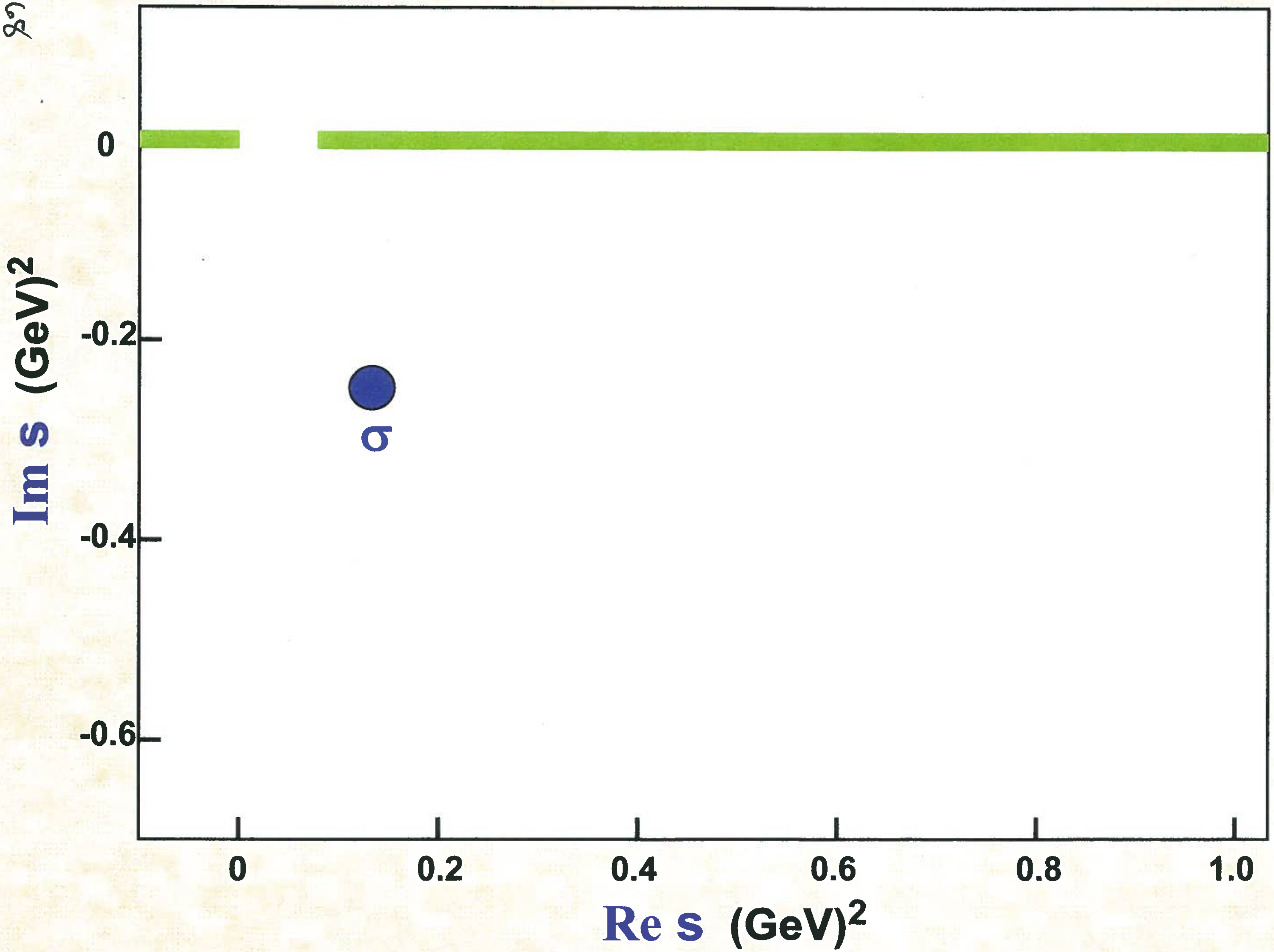


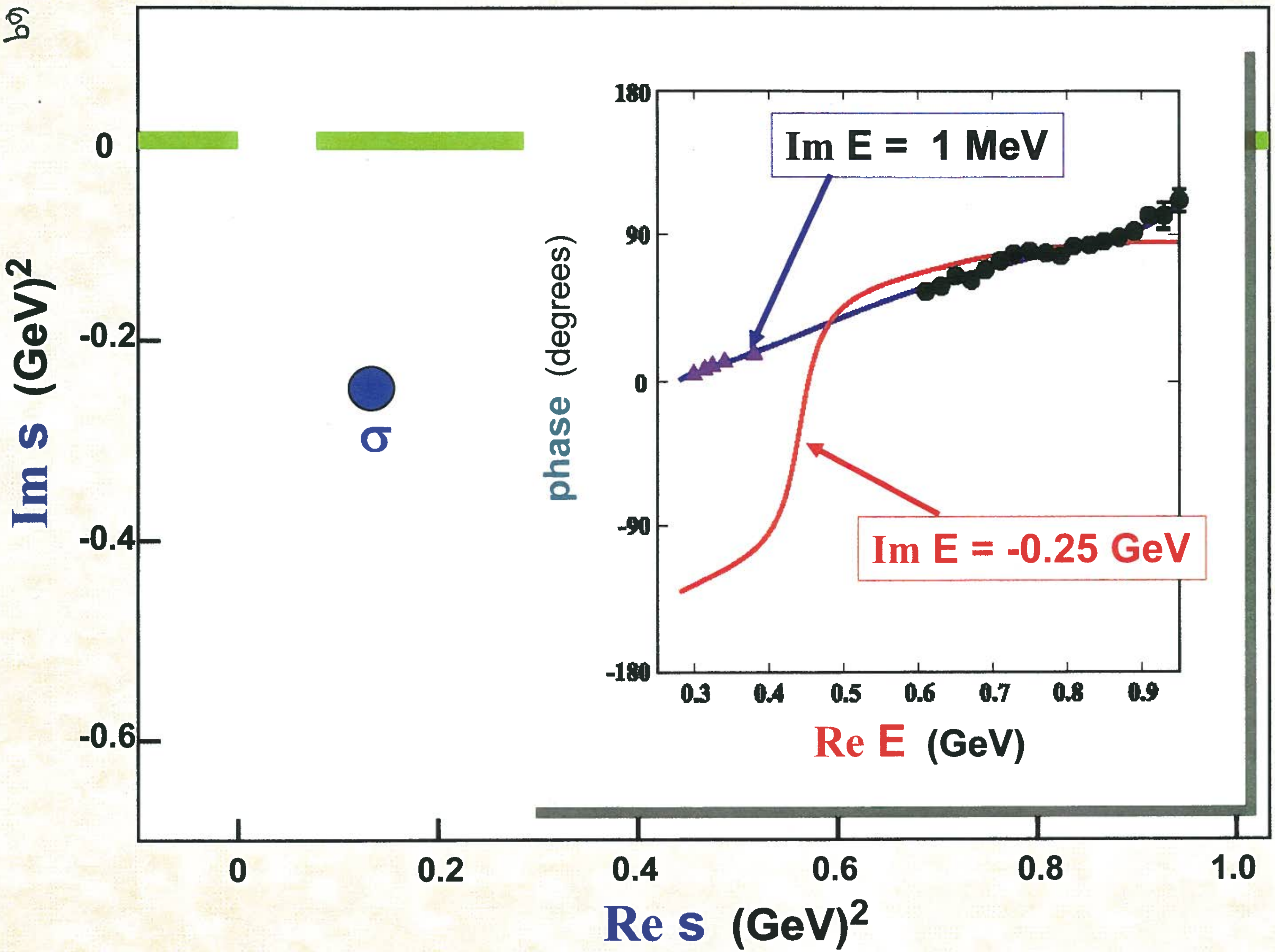


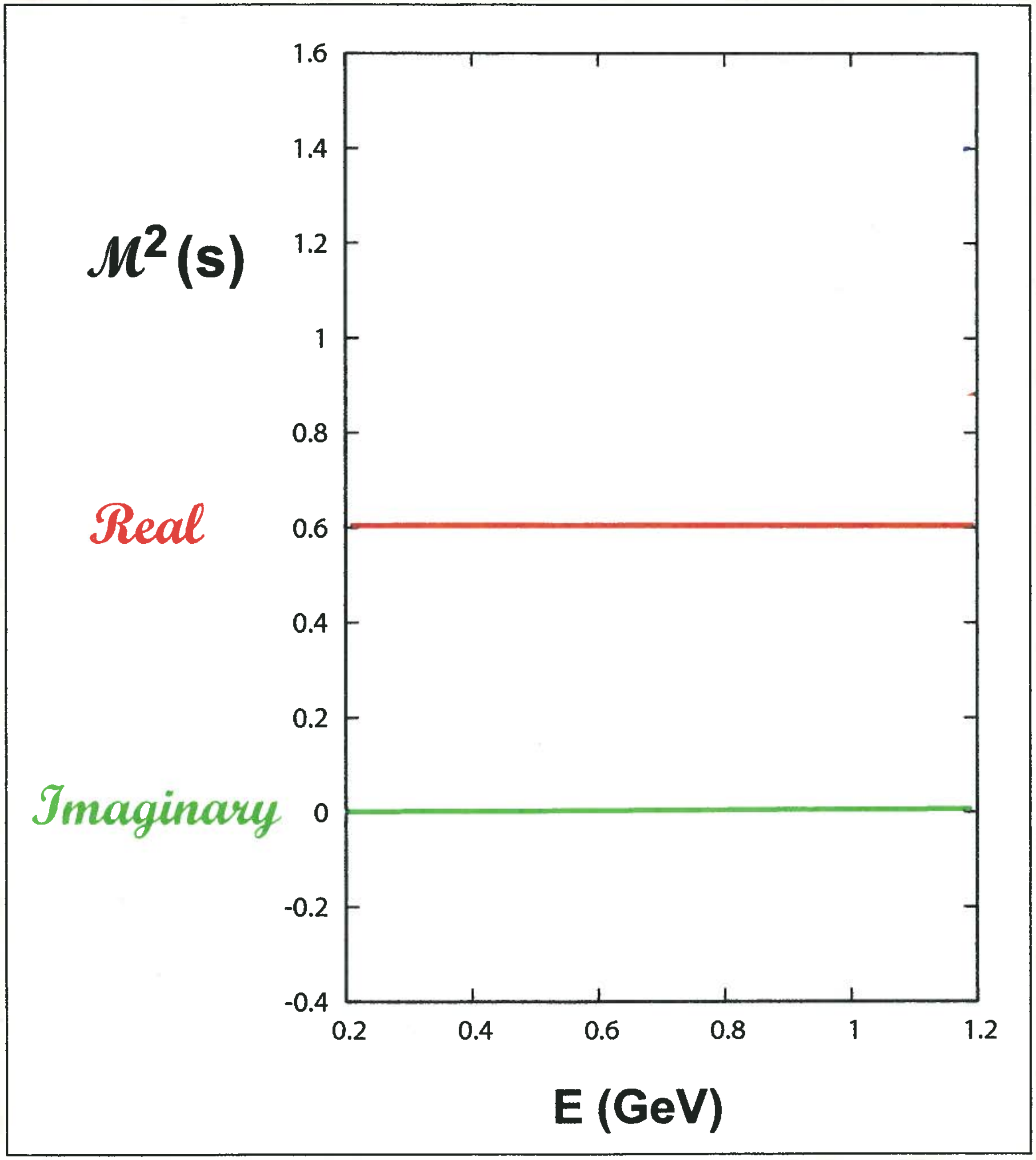
0
-0.2
-0.4
-0.6
Im s (GeV)$^2$
ρ
0
0.2
0.4
0.6
0.8
1.0
Re s (GeV)$^2$



ρ
$\delta_1^1$
0.4
0.8
1.2
M (ππ) (GeV)
0
-0.2
-0.4
-0.6
Im s (GeV)$^2$
0
0.2
0.4
0.6
0.8
1.0
Re s (GeV)$^2$


0
-0.2
-0.4
-0.6
Im s (GeV)^2
σ
0
0.2
0.4
0.6
0.8
1.0
Re s (GeV)^2

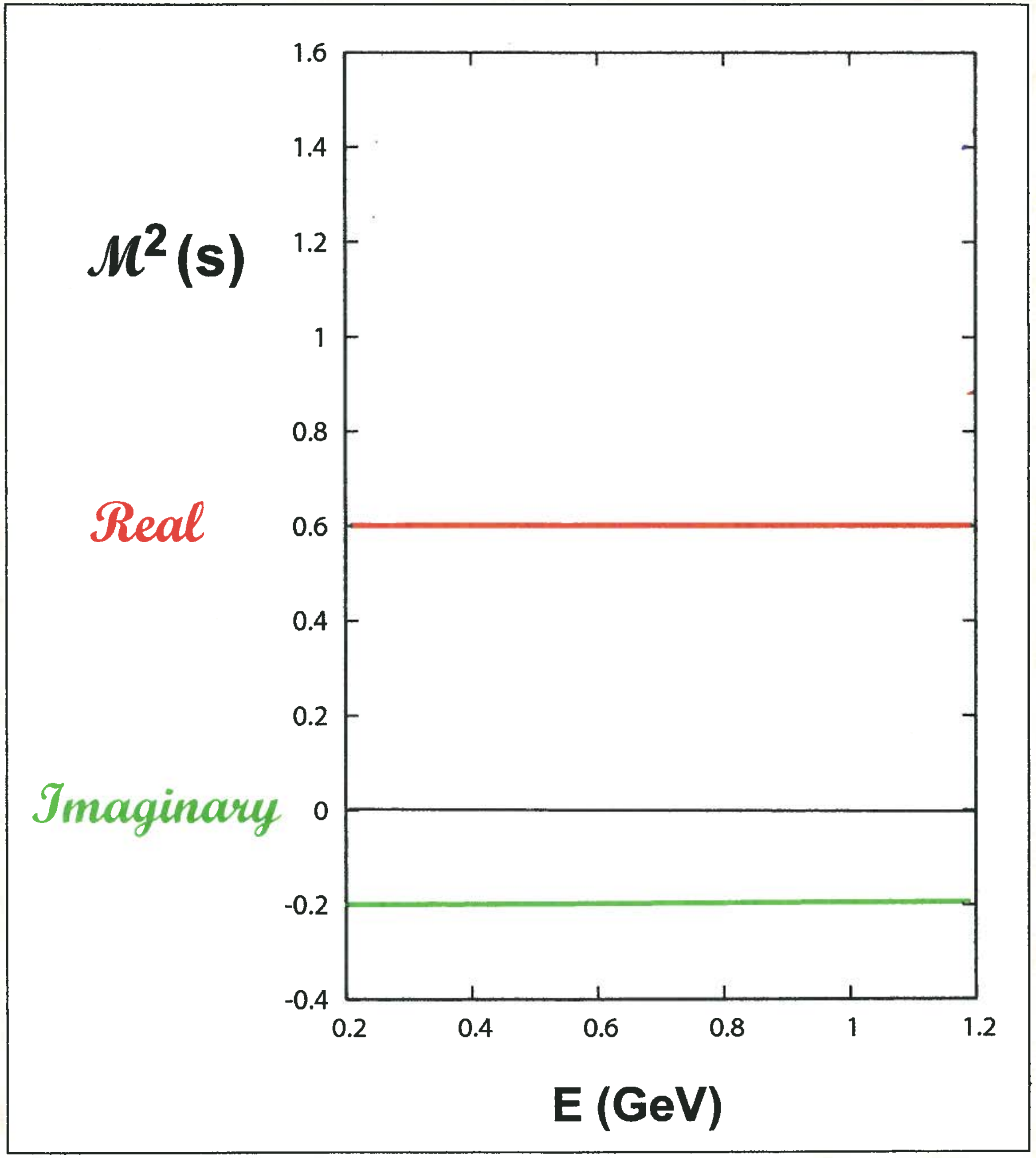

Im s (GeV)$^2$
0
-0.2
-0.4
-0.6
σ
Re s (GeV)$^2$
0
0.2
0.4
0.6
0.8
1.0
phase (degrees)
180
90
0
-90
-180
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Re E (GeV)
Im E = 1 MeV
Im E = -0.25 GeV




Function $\mathcal{M}^2(s)$ of the particle propagator

$$\frac{1}{\mathcal{M}^2(s)-s}$$

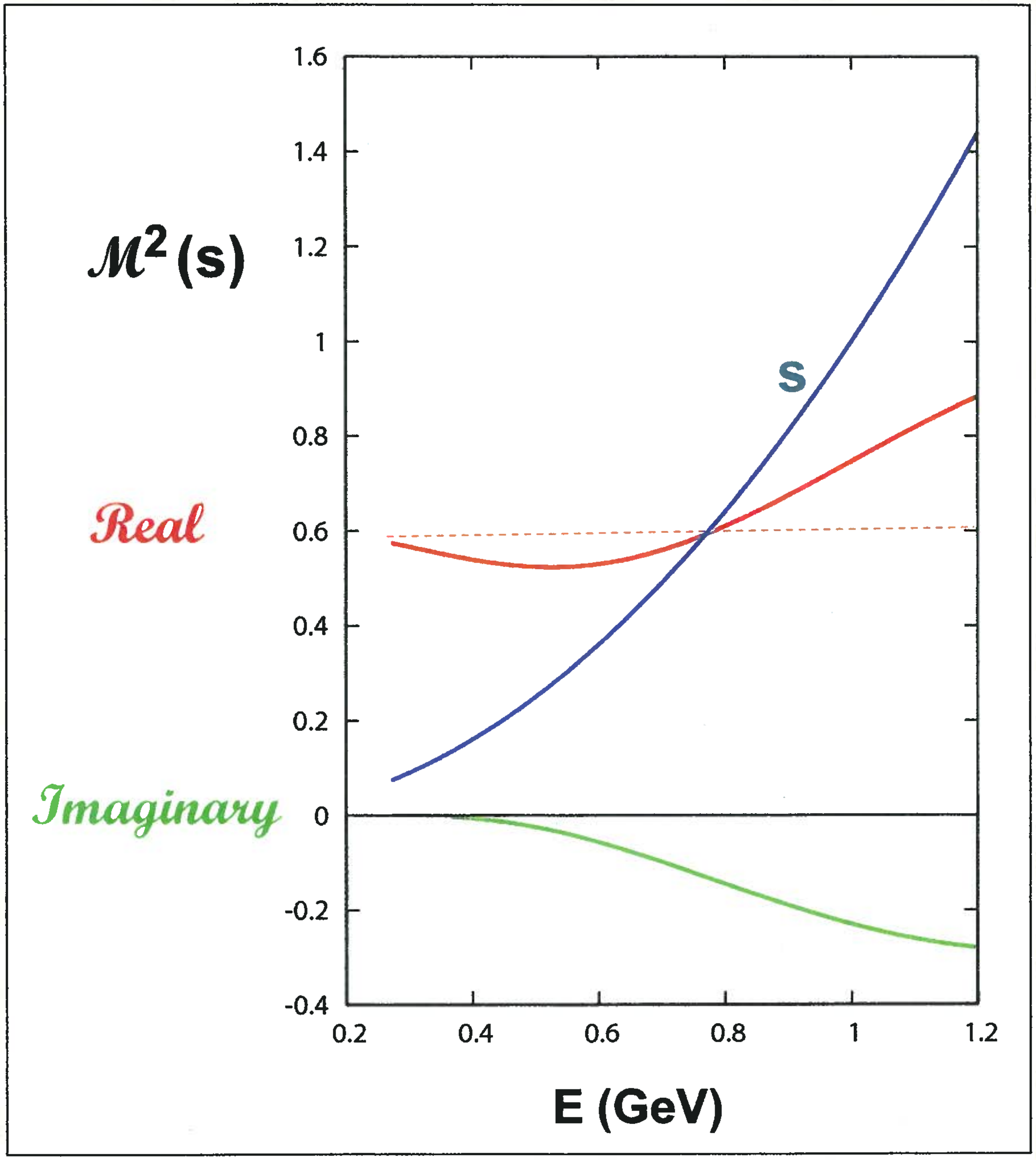
for a stable particle.




Function $\mathcal{M}^2(s)$ of the particle propagator

$$\frac{1}{\mathcal{M}^2(s) - s}$$

for the Breit-Wigner approximation to the $\rho$.




Function $M^2(s)$ for the particle propagator $\frac{1}{M^2(s) - s}$ for a $\rho$ resonance as required by pion loops, illustrating how approximate the Breit-Wigner form is!