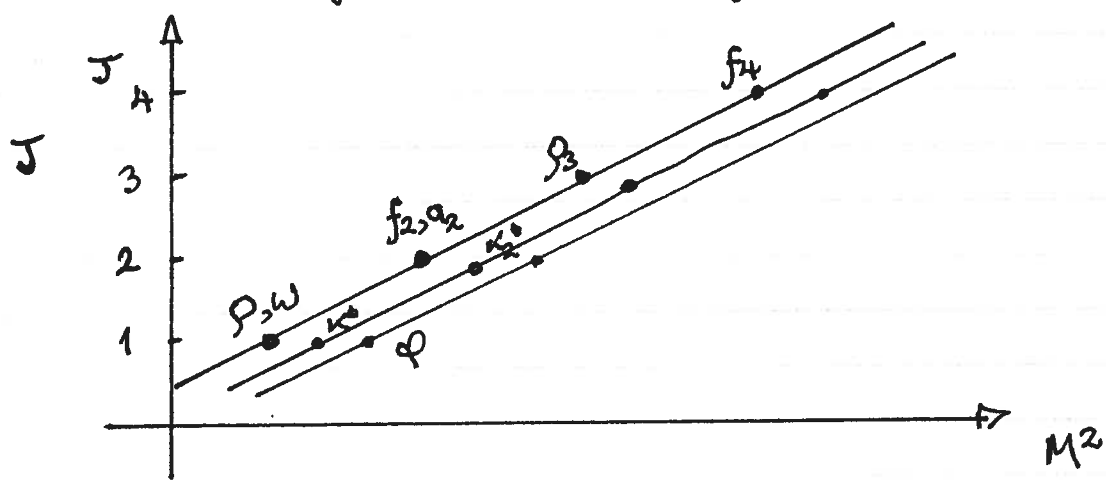
Lecture 4 : a quick tour of Regge theory

Consider the resonances we know of with increasing spin, we note that the maximum spin increases with mass. If we plot their spin, J, against mass^2 (strictly $\mathrm{Re}(M^2)$) for mesons we find



that the lie along a straight lines. Roughly parallel lines with different flavor quantum numbers.

We then have that leading "ρ-trajectory" is the line

$$\alpha(M^2) = 1 + \frac{(M^2 - m_\rho^2)}{(m_{f_2}^2 - m_\rho^2)}$$

$$\simeq 0.5 + \frac{M^2}{\mathrm{GeV}^2}$$

where $\alpha(M^2) = J$ at integer values.

Consider amplitude for a process such as $\pi\pi \to \pi\pi$, then in the t-channel if the process is dominated by resonances then

$$F(t, z_t) \simeq \sum_J b_J(t) \frac{P_J(z_t)}{\alpha(t) - J}$$

This amplitudes then has "resonances" with increasing spin as t increases,

with $J=1$ resonance at $t \sim m_\rho^2$

$J=2$ at $t \sim m_{f_2}^2$

$J=3$ at $t \sim m_{\rho_3}^2$, etc

where $\alpha(t) \simeq 0.48 + 0.9\, t \text{ GeV}^{-2}$.

Remarkably the same Regge trajectory controls the behavior of the amplitude in the s-chamel when s is Large $\gg \text{GeV}^2$, with t-fixed, $t<0$

74 but where $\alpha(t)$ is no longer on integer,

but is the continuation of the same Regge trajectory to $t \lesssim 0$. To see how, let us once again consider spinless particle scattering in the t-channel. It has the partial wave expansion

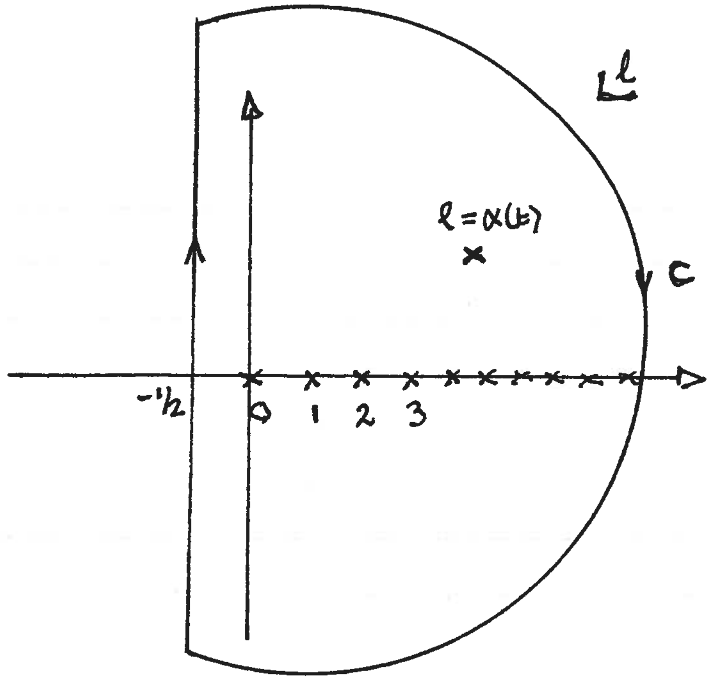
$$F(t, z_t) = 16\pi \sum_{L} (2L+1) f_L(t) P_L\left(z_t = 1 + \frac{2s}{t-4m^2}\right)$$

where we have seen $f_L(t) \sim \dfrac{1}{L-\alpha(t)}$. $L =$ integer.

Let us consider $L \to \ell$, where ℓ is not necessarily an integer and even complex. To start halfway to the answer, consider an integral in the complex ℓ-plane defined by

$$I(t, z_t) = \oint_C d\ell \,(2\ell+1) \frac{f(\ell,t)}{\sin \pi \ell} P_\ell(-z_t)$$

where the contour C runs up the line $\mathrm{Re}\,\ell = -1/2$ and is then closed at infinity in the right half plane.



Let us assume that integral round the contour at infinity vanishes then

$$I(t, z_t) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl \; (2l+1) \frac{f(l,t)}{\sin \pi l} \; P_l(-z_t)$$

but it equals, by the calculus of residues, to $2\pi i \times$ the sum of residues of poles enclosed by contour **C**. There are the explicit poles at $l = L$ integer generated by the zeros of $\sin \pi l$, and the poles implicit in $f(l,t)$ at $l = \alpha_n(t)$ in the complex plane., $n = 1, 2, \ldots$

$$\therefore \quad I(t, z_t) = -2\pi i \sum_{L=0}^{\infty} (2L+1) \frac{f(L,t)}{\pi} P_L(z_t)$$

$$- 2\pi i \sum_{n=1} (2\alpha_n(t)+1) \frac{\beta_n(t)}{\sin \pi \alpha_n(t)} P_{\alpha(t)}(-z_t)$$

Note 1) the minus sign comes from the contour being cyclic, not anticyclic.

2) the residue of each pole at $\ell = L$ (an integer) follow from

$$\frac{1}{\sin \pi \ell} \sim \frac{(-1)^L}{\pi(\ell - L)}$$

and $(-1)^L P_L(-z_t) = P_L(z_t)$

3) $\beta_n(t)$ is the residue of the nth pole at $\ell = \alpha_n(t)$ in $f(\ell, t)$.

We therefore have

$$16\pi \sum_L (2L+1) f_L(t) P_L(z_t)$$

$$= -16\pi^2 \sum_n \frac{(2\alpha_n(t)+1)\beta_n(t)}{\sin \pi \alpha_n(t)} P_{\alpha_n(t)}(-z_t)$$

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$$+ \; 8\pi i \int^{-\frac{1}{2}+i\infty} d\ell \, (2\ell+1) \, \underline{f_\ell(t)} \, P_\ell(-z_t)$$

This provides a representation that began as a partial wave series in the t-channel physical region to one applicable way outside when $|z_t| \to \infty$ at fixed t.

To be more exact, one should divide the t-channel amplitude into components that are s-u symmetric (L=even) and s-u antisymmetric (L=odd), so that we can appropriately relate $s \to \infty$ and $u \to \infty$ at fixed t. These two components have even and odd signature respectively, where signature is labelled by τ.

$$F^{\tau}(t, z_t) = \sum_n \bar{\beta}_n(t)\, P_{\alpha_n(t)}(-z_t)\, \frac{\left[1 + \tau e^{-i\pi\alpha_n(t)}\right]}{\sin\pi\alpha_n(t)}$$

$$+ \; 4\pi i \int_{-1/2-i\infty}^{-1/2+i\infty} dl \; (2l+1)\, f(l,t)\, \frac{P_l(-z_t) + \tau P_l(z_t)}{\sin\pi\alpha(t)}$$

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where $\bar{\beta}$ absorbs all π & factors of 2.

Now $\quad z_t = 1 + \dfrac{2s}{t-4m^2} \quad$ for equal mass scattering.

When $|z_t| \to \infty$, ie $|s| \to \infty$ at fixed t, $P_{\alpha(t)}(-z) \sim (-z)^{\alpha}$

[when L is an integer $P_L(z)$ is a polynomial of degree L; so when $z \to \infty$ $P_L(z) \sim z^L$. Remarkably, this generalises to non-integer ℓ, or even complex ℓ, as $P_\ell(z) \sim z^\ell + z^{-\ell-1}$ as $z \to \infty$

Thus the "background integral" at $\mathrm{Re}\,\ell = -1/2$ behaves like $(-s)^{-1/2}$ as $|s| \to \infty$. Thus the high energy limit for $F^{\tau}(t,s) \sim \beta(t)\left(-\frac{s}{s_0}\right)^{\alpha(t)}$

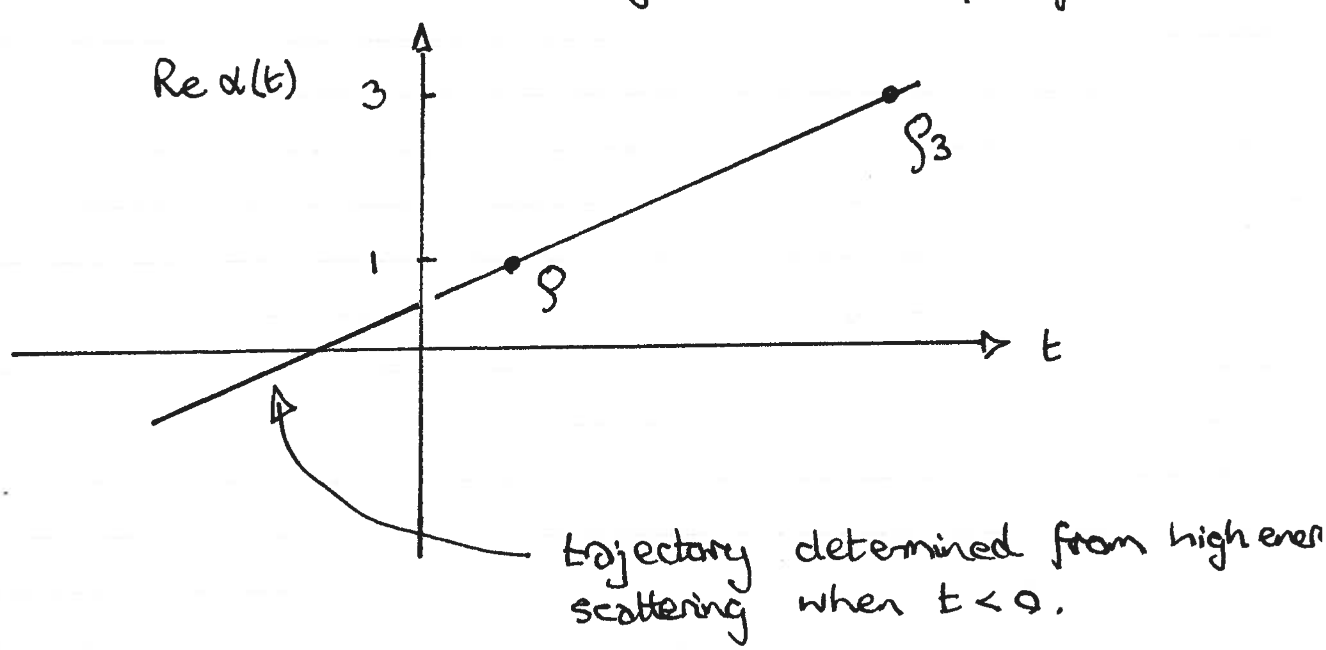
where s_0 is a scale that normalizes the behavior, typically chosen to be $1\ \mathrm{GeV}^2$ or $1/\alpha'$ the slope of the Regge trajectory. The leading Regge trajectory contains the particles with the quantum numbers of
79 exchanges in the crossed channel and controls the high energy behavior of the s & u-channel proce

A classic example is πN charge exchange, $\pi^- p \to \pi^0 n$ in the s-channel. The amplitude at high energy is controlled by the dominant Regge exchange in the t-channel, which being $\pi^- \pi^0 \to \bar{p} n$ has $\tau = -1$ and is the ρ-trajectory. The real and imaginary parts of the amplitude are given by

$$F(s,t) \sim \beta_\rho(t)\,(-\alpha' s)^{\alpha_\rho(t)} \frac{\left[1 - e^{-i\pi\alpha_\rho(t)}\right]}{\sin \pi \alpha_\rho(t)}$$

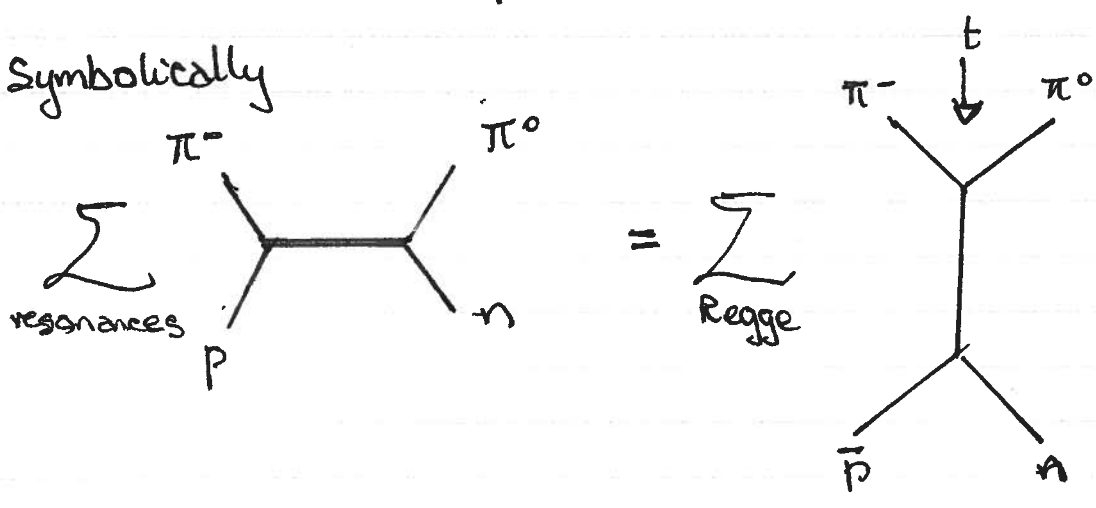
where $\alpha_\rho(t) \simeq 0.48 + 0.9t \text{ GeV}^{-2}$

with $\alpha' = 0.9 \text{ GeV}^{-2}$, the slope of the trajectc



Not only does a meson Regge exchange control the high energy ($s > 3$-4 GeV^2) behavior of the baryonic reaction $\pi^- p \to \pi^0 n$. The extrapolation of this Regge behavior to low energy actually "averages" the amplitude given by a sum of baryon resonances. Such averaging is defined by Finite Energy Sum Rules. There is a duality between s-channel resonances and t-channel Regge exchanges. Unlike Feynman graphs these are not added, but are alternative descriptions of the same physics. Which is more economic depends on the kinematic regime

Symbolically



The fact that unitarity imposes the constraint

$\rho \, \mathrm{Im}\, f_\ell(s) \lesssim 1$ for an elastic scattering process, limits the growth of the forward scattering amplitude and so by the optical theorem limits the high energy behavior of the total cross-section

$$\sigma_{total}(s) \lesssim \ln^2 s \,.$$

This is known as the Froissart bound. The scale of s is set by dynamics; it could be m_π^2, m_ρ^2 or $1/\alpha'$ as you wish.

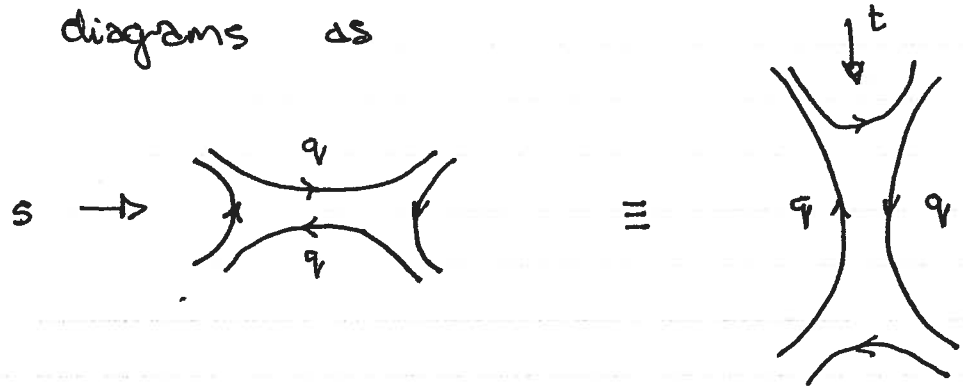
Regge behavior predicts $\sigma_{total} \sim (\alpha' s)^{\alpha(0)-1}$.

The Froissart bound (which is a truly asymptotic statement, while Regge behavior sets in when $s > 3$-$4\ \mathrm{GeV}^2$) requires that for the leading Regge trajectory that controls total cross-sections and

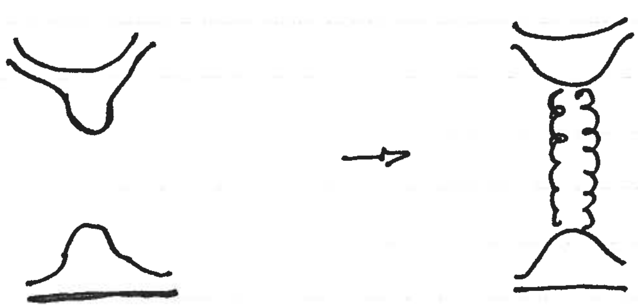
82 so carries vacuum quantum numbers

$$\alpha(0) \lesssim 1 \,.$$

The behavior of total cross-sections at currently accessible energies gives $\alpha(0)$ effectively just a little above 1. This corresponding Regge exchange is known as the "pomeron" named after Pomeranchuk. Its t-dependence has a slope of $\sim 0.25\ \mathrm{GeV}^{-2}$, much flatter than trajectories with non-zero quantum numbers. While the duality relation we introduced above can be graphically expressed in terms of quark line diagrams as



Pomeron exchange involves no quark exchange

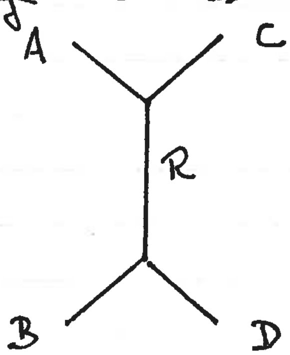


gluon exchange

but rather gluon exchange. If particles lie on the pomeron trajectory at $J = 2, 4, \ldots$ etc, then these are predicted to be glueballs. The $J = 2$ state would have mass $\sim 2\,\mathrm{GeV}$.

Searches for such a state started long ago with no definite conclusion among several possible candidates. Precision searches are continuing with COMPASS @ CERN and in the near future with GlueX @ JLab.

An important consequence of the concept of a single dominant Regge exchange with any particular quantum numbers is that the Regge residues (or couplings) should factorize



$$\sim b_{A\bar{C}R}(t)\, b_{\bar{B}DR}(t)\, (-\alpha' s)^{\alpha_R(}$$

so that the Regge couplings for $\pi\pi \to \pi\pi$, $\pi N \to \pi N$ and $NN \to NN$ scattering are related, for example

$$\left[b_{\pi\pi\rho}(t)\right]^2 \left[b_{N\bar{N}\rho}(t)\right]^2 = \left[b_{\pi\pi\rho}(t)\, b_{N\bar{N}\rho}(t)\right]^2 .$$

Such relations appear to hold for the pomeron as much as for ρ-exchange.

Whilst the idea of continuing angular momentum to complex values began with potential scattering, an intrinsically non-relativistic process, it reaches its full potential in S-matrix theory where relativistic scattering provides a physical relation between s, t and u-channel reactions. It is remarkable that high energy s-channel scattering is controlled by the exchanges in the crossed channel, where t & u are continued from positive to negative domains.