Amplitude Analysis
An Experimentalists View

Lectures at the “Extracting Physics from Precision Experiments
Techniques of Amplitude Analysis”

Klaus Peters
GSI Darmstadt and GU Frankfurt
Williamsburg, June 2012
The Course

4 Lectures

Introduction
Mission / Amplitude Analysis Concepts and Procedures / Use Cases

Kinematics and more
Dalitz-Plots / Observables / Coordinate Systems / Examples

K-Matrix
Derivation / Examples / Properties / Fitting / Interpretation

Experiments
Selection / Background / Numerical Issues / Goodness-of-Fit / Computers
The Course

6 Lectures

Introduction
Mission / Amplitude Analysis Concepts and Procedures / Use Cases
Kinematics and more
Dalitz-Plots / Observables / Coordinate Systems / Examples
Spin
...
Dynamics
...
K-Matrix
Derivation / Examples / Properties / Fitting / Interpretation
Experiments
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Books

Dynamics

J.M. Blatt & V.F. Weisskopf – *Theoretical Nuclear Physics*
J.R. Taylor – *Scattering Theory*
M.L. Goldberger & K.M. Watson – *Collision Theory*
J. Gillespie – *Final State Interactions*
H. Burkhardt – *Dispersion Relation Dynamics*

Kinematics and more

M. Nikolic – *Kinematics and Multiparticle Systems*
E. Byckling & K. Kajantie – *Particle Kinematics*

Spin

M.E. Rose – *Elementary Theory of Angular Momentum*

Overview

D.V. Bugg (Edtitor) – Nato School on *Hadron Spectroscopy and the Confinement Problem*
Reviews and Articles

General
R.S. Longacre – *Techniques in Meson Spectroscopy*, BNL 49445

Spin
S.U. Chung – *Spin Formalisms*, CERN Yellow Report 71-8
V. Filippini et al. – *Covariant Spin Tensors in Meson Spectroscopy*, PRD 51(1995) 2247

Dynamics
Amplitude Analysis
An Experimentalists View

K. Peters

Part I

Introduction
Introduction

Mission
Concepts
Procedures
Use Cases
What is the mission?

Particle physics at small distances is quite well understood
  One Boson Exchange, Heavy Quark Limits
This is not true at large distances
  Hadronization, Light mesons
  are barely understood compared to their abundance
Understanding interaction/dynamics of light hadrons will
  improve our knowledge about non-perturbative QCD
  parameterizations will give provide
toolkit to analyze heavy quark processes
  thus an important tool also for precise standard model tests

We need
  Appropriate parameterizations for the multi-particle phase space
  A translation from the parameterizations to effective degrees of
  freedom for a deeper understanding of QCD
Many states may contribute to a final state not only ones with well defined (already measured) properties not only expected ones. Many mixing parameters are poorly known \(K\)-phases, \(SU(3)\) phases in addition also \(D/S\) mixing \((b_1, a_1\) decays).
**The Need for Partial Wave Analysis**

*Example:* Consider the reaction $\bar{p}p \rightarrow K^+ K^- \pi^0$

What *really* happened...

What you see is always the same ...

... etc.

PWA = technique to find out what happens in between
Goal

For whatever you need the parameterization of the $n$-Particle phase space

It contains the static properties of the unstable (resonant) particles within the decay chain like

mass
width
spin and parities

as well as properties of the initial state
and some constraints from the experimental setup/measurement

The main problem is, you don’t need just a good description, you need the right one

Many solutions may look alike, but only one is right
$n$-Particle Phase space, $n=3$
But...

the mission is way more general,

...there are many more questions, which can only be answered with a correct phase space description

whenever states mix and an need to be unambiguously disentangled

the focus then moves away from masses and line shapes to yields and phases
example: $D^0\bar{D}^0$-Mixing and CPV

$D^0 \rightarrow h^+ h^- \pi^0$, $h=K, \pi$

$D^0 \rightarrow \pi^- \pi^+ \pi^0$

$D^0 \rightarrow K^- K^+ \pi^0$

Data from BaBar

search for asymmetry in production cross section or in branching fractions
example: $D^0\bar{D}^0$-Mixing and CPV

$D^0 \to \pi^-\pi^+\pi^0$

$D^0 \to K^-K^+\pi^0$

$\chi^2$-distribution shows: no observed CP-violation

not enough statistics to verify SM prediction
example: CKM Angle $\gamma$ in $B^- \rightarrow D^0 K^- (+c.c.)$

Direct $CP$ violation in interference between $b \rightarrow c\bar{u}s, u\bar{c}s$

$$B^- \rightarrow \tilde{D}^{(*)0} K^{(*)-}$$

Interference, if

$$D^0 \rightarrow f \leftarrow \bar{D}^0$$

$$\frac{A(B^- \rightarrow \bar{D}^0 K^-)}{A(B^- \rightarrow D^0 K^-)} = r_B e^{i(\delta_B - \gamma)}, \quad \frac{A(B^+ \rightarrow D^0 K^+)}{A(B^+ \rightarrow \bar{D}^0 K^+)} = r_B e^{i(\delta_B + \gamma)}$$

$r_B$ Ratio of magnitudes of amplitudes, small

$\delta_B$ $CP$ invariant strong phase

Most sensitive channel to date: $\tilde{D}^0 \rightarrow K_S^{0} \pi^+ \pi^-$:


Requires a detailed understanding of the $D^0$ decay as input
Quality

High Quality is needed

and achievable...

this lecture is basically about how to model the input for such fits
to reveal all the physics of a multi-particle reaction
How to obtain this in an effective way?

Important aspects...

General considerations
Course of action
Phase space
Observables
Hypotheses
Background
Fitting
Mathematical problems
Quality Assurance
Experimental Techniques

Scattering Experiments

$\pi N - N^*$ measurement
$\pi N$ - meson spectroscopy
  E818, E852 @ AGS, GAMS
  Compass, VES
$pp$ meson threshold production
  WASA @ Celsius, COSY
$pp$ or $\pi p$ in the central region
  WA76, WA91, WA102
$\gamma N$ – photo production
  Cebaf, Mami, Elsa, Graal

“At-rest” Experiments

$\bar{p}N @$ rest at LEAR
  Asterix, Obelix, Crystal Barrel
  PANDA
$J/\psi$ decays
  MarkIII, DM2, BES, CLEO-c
$\phi(1020)$ decays
  Kloe @ Dafne, VEPP
$D$ and $D_s$ decays
  FNAL, Babar, Belle, Belle-II
Experimental Techniques

partial waves decomposition → via moment analysis
systematic studies to limit #waves
dynamics appear as amplitude variations
resonance parameters from fits to amplitudes

ad-hoc introduction of waves
ad-hoc introduction of dynamic amplitudes ("resonances")
systematic studies to limit #waves and #resonances
resonance parameters appear as fit parameters
Experimental Techniques

Scattering Experiments

- exchange model needed
- ad-hoc intermediate resonances
  → parameters fixed for wave decomposition

“At-rest” Experiments

- independent of production model
- intermediate resonances treated
  → identically to final state resonances
- crossing bands may provide high resolution interferometer
Momentum Analysis in a Dalitz Plot

see M. Pappagallo, Charm06

\[
\begin{align*}
\sqrt{4\pi} \langle Y_0^0 \rangle &= S^2 + P^2 \\
\sqrt{4\pi} \langle Y_1^0 \rangle &= 2 |S| |P| \cos \varphi_{SP} \\
\sqrt{4\pi} \langle Y_2^0 \rangle &= \frac{2}{\sqrt{5}} P^2
\end{align*}
\]

In some cases it’s possible if no sharp bands overlap
General considerations (I)

Which processes take place?

Interactions?
Basic processes – scattering vs. decay – which scattering
(Physics of) Initial State – recoils – inclusive/exclusive
Physics background
Leading effects

Scales?
Dynamics – range parameters
Approximations – low energy or threshold expansions

do scales differ for different sub-processes?
factorization of dynamics, like in open-charm decays
General considerations (II)

What are conserved properties?

**kinematics**
- energy/momentum conservation
- kinematically fitted data?

**quantum numbers**
- quark/isospin conservation/symmetries
- good and bad quantum numbers (isospin, parity, CP)
- impact on spin formalisms
- interferences of Feynman graphs
- phase space
- full set of observables?
- integrate over part of the phase-space
### Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Helicity</th>
<th>Transversity</th>
<th>Canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>partial wave expansion</td>
<td>simple</td>
<td>complicated</td>
<td>complicated</td>
</tr>
<tr>
<td>parity conservation</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>crossing relation</td>
<td>no</td>
<td>good</td>
<td>bad</td>
</tr>
<tr>
<td>specification of kinematical constraints</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>
Example: Isospin Dependence

\( \bar{p}p \) initial states differ in isospin

\[
\begin{align*}
{^1S_0} & \quad I^G(J^P) = 1^- (0^{--}) \\
{^3S_1} & \quad I^G(J^P) = 0^+ (1^{--})
\end{align*}
\]

Calculate isospin Clebsch-Gordan

\[
\begin{align*}
\rho^0\pi^0 & \rightarrow (1010|00) = -\sqrt{\frac{1}{3}} \\
\rho^0\pi^0 & \rightarrow (1010|10) = 0 \\
\rho^{\pm}\pi^{\mp} & \rightarrow (1(\pm1) 1(\mp1)|00) = \sqrt{\frac{1}{3}} \\
\rho^{\pm}\pi^{\mp} & \rightarrow (1(\pm1) 1(\mp1)|10) = \pm\sqrt{\frac{1}{2}}
\end{align*}
\]

\( {^1S_0} \) destructive interferences

\( {^3S_1} \rho^0\pi^0 \) forbidden
What are the relevant parameters?

Order of magnitude
relevant for coding?
leading terms?

[[ Parameter too small, different formulation ]]
[[ Examples only smallest L ]] 

Relations
are the parameters related to each other? (D/S, phases, ...)
which one is the master and which the slave?
Normalization/Constraints

[[ Example relations]]
[[ Example couplings normalized to 1]]
General considerations (IV)

Can the process be factorized or simplified?

Whole tree needed? or is a leave sufficient

**Rules**
which rules/conditions can be used to formulate the model
which rules/conditions have to be applied during the fit
e.g. what is fixed by definitions
Course of action

Data Analysis

Modelling

Fitting

Quality Assurance

Review and Publication
Course of action (I)

Data analysis

Data
extract relevant data set(s) with appropriate
statistics, high purity and high efficiency

MC
signal MC, may be mixed due to experimental conditions

Background
extract from data and/or generate via Monte Carlo data sets
from potential background channels

Representation
represent the data in $n$-tupels of relevant (transformed?)
observables for the fit and the visualization
Course of action (II)

Modelling

Data
Visual inspection of the data !!

Physics
create list of hypotheses (incl. production, spins, dynamics and if so, background)

Mathematics
optimize the mathematical form
may improve speed and may reduce numerical instabilities
Reduced Amplitudes (I)

# of parameters explodes with increasing number of initial and final states

- forget about the tree, reduce amplitude to the final state of interest
- feed by many initial and several intermediate states
- spin density matrix $\rho_{mn}$
Reduced Amplitudes (II)

Caveat:
Two channels may interfere in one tree, but may not in another
thus measured rate I has different formalisms

Coherent

\[ I = |A + e^{i\phi}B|^2 = |A|^2 + |B|^2 + 2[\text{Re}(AB^*)\sin \phi + \text{Im}(AB^*)\cos \phi] \]

Incoherent

\[ I = |A|^2 + |B|^2 \]

Effective coherence

\[ I = |A|^2 + |B|^2 + C [\text{Re}(AB^*)\sin \phi + \text{Im}(AB^*)\cos \phi] \]

(C = -2, ..., 2)
Course of action (III)

Fitting

fit model(s) to the data
likelihood definition, what is to be minimized
(max. Likelihood, Chi2,...)

needs a strategy to find the best solution
systematic studies for a variety of hypotheses
vary initial stats, resonances, parameterizations

need a strategy for each fit
optimizer (gradient/random/genetic)
sequence (different optimizers, fixation and release of parameters)
criteria for convergence and termination
Quality Assurance

Documentation
excellent documentation! is the key
what was done? formulae!
(intermediate) results!

Validation
validation of the result (for example with toy MC)

Significance
scrutinize the significance of new findings
check various methods to investigate the goodness-of-fit

Errors
determination of statistical and systematic errors
Course of action (V)

Review and Publication

review
process which may lead to a reanalysis at various entry points

publication
publish only things you are confident about
there is an undefined border where
the experiment ends and the theoretical bias starts
Phase space

visual inspection of the phase space distribution

are the structures?
structures from signal or background?
are there strong interferences, threshold effects, potential resonances?

\[ \phi \rightarrow K^+K^- \]

\[ K^*(892) \rightarrow K^-\pi^+ \]
Kinematical Reflections

Kinematic situation can produce mass peaks not being true resonances called Reflections

Example:
Dalitz plot of

\[ D_s^+ \rightarrow K^+K^-\pi^0 \]

in this case „fakes“ are simple to spot...
... but it can be much less obvious!

Example: \( D^+_s \rightarrow K_S^0 \pi^+ \pi^- \)
Observables should be aligned with the problem/process
is polarization relevant?
is dynamics present in all particle pairs?
are there isolated structures or regions with strong correlations?

Typical observables are

\[ m^2(s) \]
invariant mass square, Mandelstam \( s \)

\[ T \]
kinetic energy \( \cos \theta \)
decay angle of resonances \( \cos \psi \)
angle between decay places,
..... a.m.o.m.
are there symmetries in the phase space?

unique assignment of phase space coordinates is important to avoid double counting

transformation necessary?

Most Dalitz plots are symmetric:

Problem: sharing of events

Possible solution: transform DP
Hypotheses

**Select basic model**
- usually isobar model, is not appropriate in all cases
- rescattering, t-channel and Deck effects may lead to artifacts

**Select formalism to handle the spin**
- select basis (helicity reflectivity, canonical....)
- or tensors (Zemach, covariant or Lorentz-invariant)
- depends on the process and the goals

**Select set of dynamical functions**
- which resonances and thresholds are known
- which do you guess from inspection
- how much freedom is needed,
- how well do I know the processes involved
- analysis of angular moments might be helpful as a start

**Selection of parameters and optimization**
**First results** may indicate that the assumptions are wrong and one has to start over
Isobar Model

Generalization

construct any many-body system as a tree of subsequent two-body decays
the overall process is dominated by two-body processes
the two-body systems behave identical in each reaction
different initial states may interfere

We need

need two-body “spin”-algebra
various formalisms
need two-body scattering formalism
final state interaction, e.g. Breit-Wigner
Interference problem

PWA

The phase space diagram in hadron physics shows a pattern due to interference and spin effects.

This is the unbiased measurement

What has to be determined?

Analogy Optics ⇔ PWA

# lamps ⇔ # level
# slits ⇔ # resonances
positions of slits ⇔ masses
sizes of slits ⇔ widths

but only if spins are properly assigned

bias due to hypothetical spin-parity assumption

Optics

\[ I(x) = \left| A_1(x) + A_2(x)e^{i\varphi} \right|^2 \]

Dalitz plot

\[ I(m) = \left| A_1(m) + A_2(m)e^{i\varphi} \right|^2 \]
Use Cases #1 of 4: Hadron Decays

Reactions (examples)

\[ \frac{J/\psi}{\rightarrow} 3 \text{ hadrons (e.g. } \pi\pi\pi) \]
\[ D_s \rightarrow 3 \text{ hadrons (e.g. } K\bar{K}\pi) \]

Initial state has a well defined \( J^{PC} \)

\[ \begin{align*}
J/\psi & \quad J^{PC} = 1^{--} \\
D_s & \quad J^{PC} = 0^{-+}
\end{align*} \]

Focus/Mission/Goal

Properties of intermediate resonances (\( J^{PC} \), mass, width, decay ratios)
spin-parity of the decaying particle

Typical Experiments

Tau-Charm Factories, B-Factories
Use Cases #1 of 4: Hadron Decays (cont’d)

Procedure
Formulate spin-dependent amplitude using all available constraints
hadronics decays conserve P, weak decays don’t
Formulate dynamics (resonance cocktail) with guesses for the properties (if not sufficiently well known)

Fit the parameters of the model to the data

Repeat for various hypothesis and identify the „best solution“
Use Cases #2 of 4: \( p\bar{p} \) annihilation

Reactions (examples)

\[ p\bar{p} \rightarrow 3 \text{ hadrons (at rest)} \]
\[ p\bar{p} \rightarrow 3 \text{ hadrons (in flight)} \]

Initial state is a mixture of well defined \( J^{PC} \)'s

Focus/Mission/Goal
Properties of intermediate resonances (\( J^{PC} \), mass, width, decay ratios)

Typical Experiments
Asterix and Crystal Barrel @ LEAR
Proton-Antiproton Annihilation @ Rest

Atomic initial system
formation at high \( n, l \) \((n \sim 30)\)
slow radiative transitions
de-excitation through collisions
(Auger effect)
Stark mixing of \( l \)-levels
(Day, Snow, Sucher, 1960)

Advantages
\( J^{PC} \) varies with target density
isospin varies with \( n \) (d) or p target
incoherent initial states
unambiguous PWA possible

Disadvantages
phase space very limited
small kaon yield
Use Cases #2 of 4: p̅p annihilation (cont’d)

Very similar to hadron decays for p̅p at rest

Procedure
Formulate spin-dependent amplitude using all available constraints but for a couple of possible initial states!
hadronics decays conserve P, weak decays don’t
Formulate dynamics (resonance cocktail) with guesses for the properties (if not sufficiently well known)

Fit the parameters of the model to the data

Repeat for various hypothesis and identify the „best solution“
Proton-Antiproton Annihilation in Flight

Annihilation in flight

scattering process:
no well defined initial state
maximum angular momentum
rises with energy

Advantages

larger phase space
formation experiments

Disadvantages

many waves interfere with each other
many waves due to large phase space

\[ \sigma_{\text{ann}} = \sum \sigma_l \]

\[ \sigma_l(p) = (2l+1) \pi \left[ 1 - \exp(-\chi_l(p)) \right] / p^2 \]

\[ \chi_l(p) = N(p) \exp\left(-3(l+1)/4p R^2\right) \]

with \( R^2 = \langle r^2 \rangle \) (Baryon)

\[ \text{ang. mom. } l \sim p_{\text{cms}} / 0.2 \text{ GeV/c} \]
Scattering Amplitudes in $\bar{p}p$ in Flight (II)

\[ H^J_{\nu_1\nu_2} = \sum_{\nu L S} \frac{\sqrt{2L+1}}{\sqrt{2J+1}} (L0S\nu|J\nu)(s_1\nu_1s_2 - \nu_2S\nu)(JMLS|JM) \]

using all constraints leads to 4 incoherent sets of coherent amplitudes

<table>
<thead>
<tr>
<th>Singlet even $L$</th>
<th>$J^{PC}$</th>
<th>$L$</th>
<th>$S$</th>
<th>$H_{++}$</th>
<th>$H_{+-}$</th>
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and only 2 of them contribute to a particular exclusive final state
Use Cases #3 of 4: Diffractive Production

Reactions (examples)

\[ \pi p \rightarrow 3 \text{ hadrons } p, n \]

unknown initial state
scattering process
expressed in terms of moments

Typical Graphs

Focus/Mission/Goal
Properties of X (\( J^{PC} \), mass, width, decay ratios)

Typical Experiments
E852 at AGS (BNL) or COMPASS at CERN
Procedure

- $\pi^-$-beam and $t$-exchange produce $X$
  - Partitioning in bins of $m(3\pi)$
  - Potentially also binning in $t$.
- Wave $\sim t^m e^{-bt} \rightarrow$ background as function of $t$
- Formulate spin-dependent amplitude and dynamics for the intermediate resonance cocktail
  - with best guesses for the properties

Mass dependent fits

- performed for a fixed $m(3\pi)$ (and may be $t$) and all fits result in moments (and errors) per bin, which are then translated into wave content

The result of one slice is then input (start value) for the next slice to be fitted.
Use Cases #4 of 4: Photo-production

Reactions (examples)
\[ \gamma p \rightarrow 3 \text{ hadrons } p,n \]
known initial states
scattering process
finally expressed in terms of moments
\[ \rightarrow \text{ very similar to diffractive production for high energies and multi-hadron final states} \]

Focus/Mission/Goal
Properties of X (J^{PC}, mass, width, decay ratios)

Typical Experiments
Crystal Barrel at ELSA or Gluex/Clas12 @ Jlab12
Use cases are different

There **is** a common aspect

and for each decay we need a proper formulation in our model

and we always have to fit our model to the data
THANK YOU

for today