Amplitude Analysis
An Experimentalists View

K. Peters

Part II

Kinematics and More

Kinematics and More

Phasespace

Dalitz-Plots

Observables

Spin in a nutshell

Examples

Goal

For whatever you need the parameterization of the *n*-Particle phase space

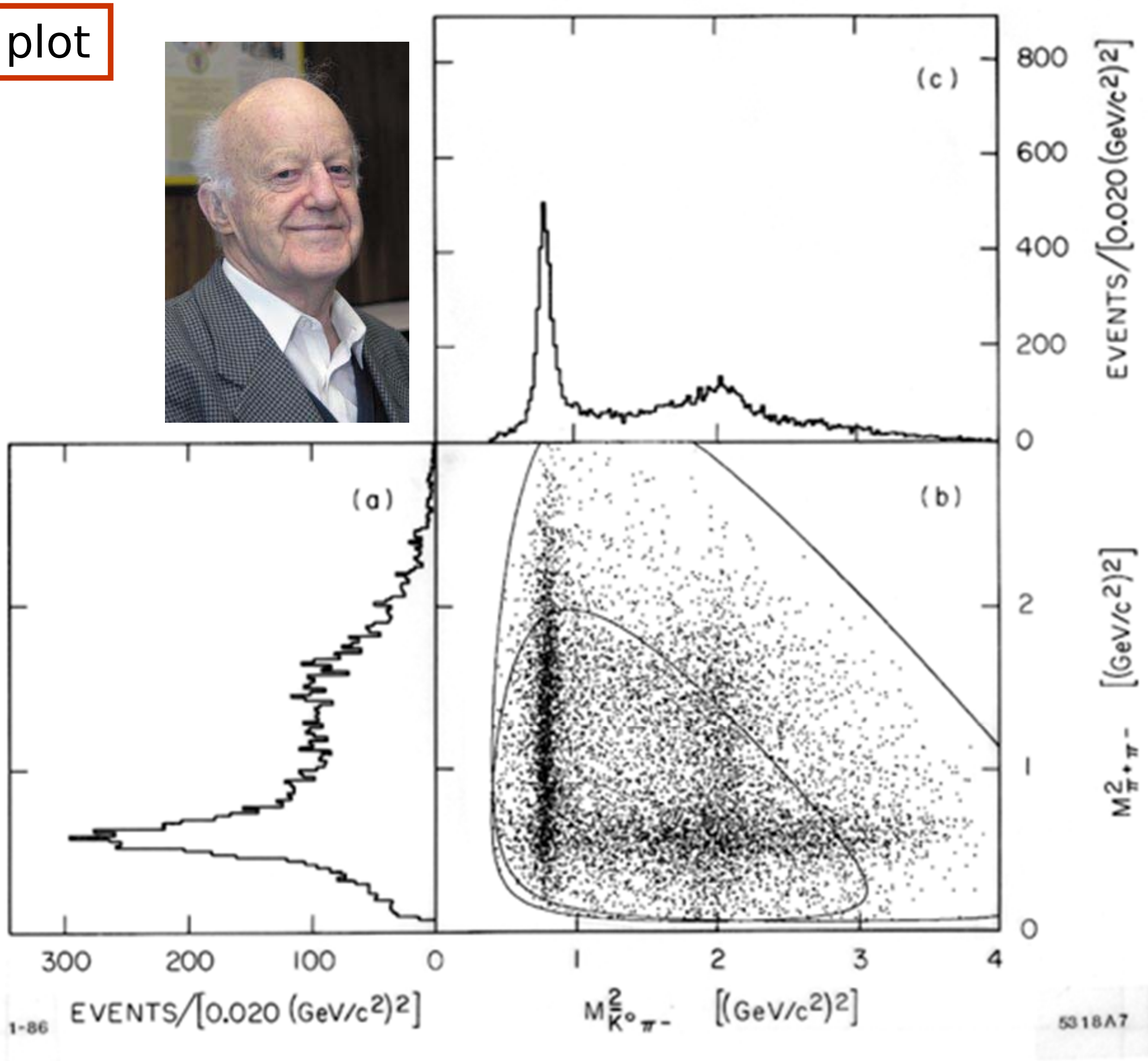
- It contains the static properties of the unstable (resonant) particles within the decay chain like
  - mass
  - width
  - spin and parities
- as well as properties of the initial state
- and some constraints from the experimental setup/measurement

The main problem is, you don‘t need just a good description, you need the right one

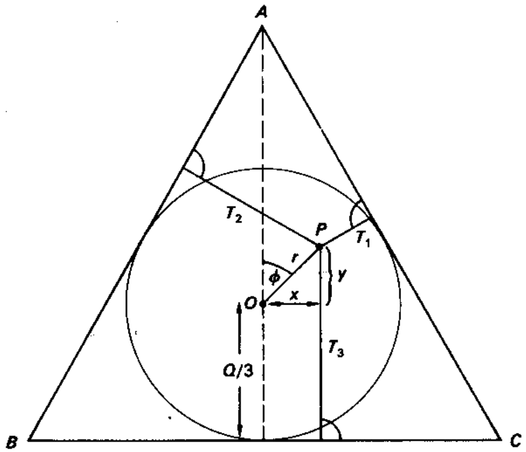
- Many solutions may look alike, but only one is right

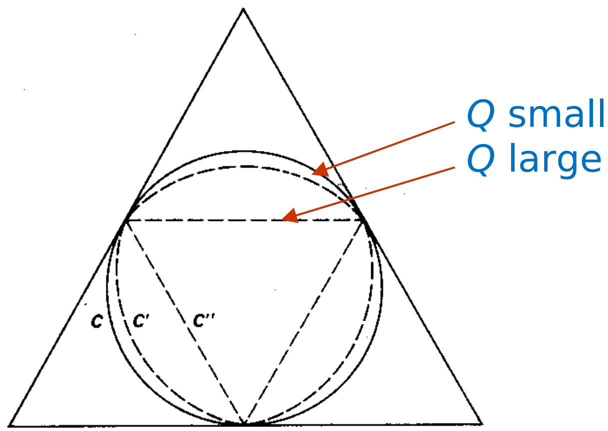
n-Particle Phase space, n=3

Dalitz plot



Phase Space Plot - Dalitz Plot





$dN \sim (E_1 dE_1)\ (E_2 dE_2)\ (E_3 dE_3)/(E_1 E_2 E_3)$

Energy conservation $E_3 = E_{tot}\text{-}E_1\text{-}E_2$

Phase space density $\rho = dN/dE_{tot} \sim dE_1\, dE_2$

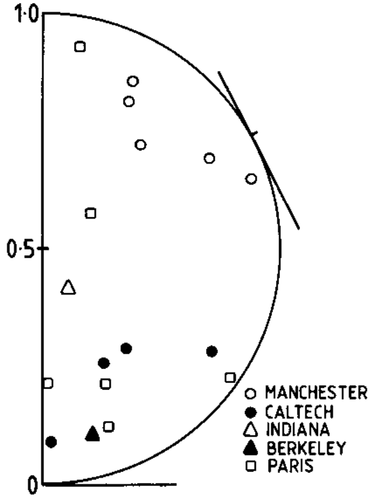
Kinetic energies $Q=T_1+T_2+T_3$

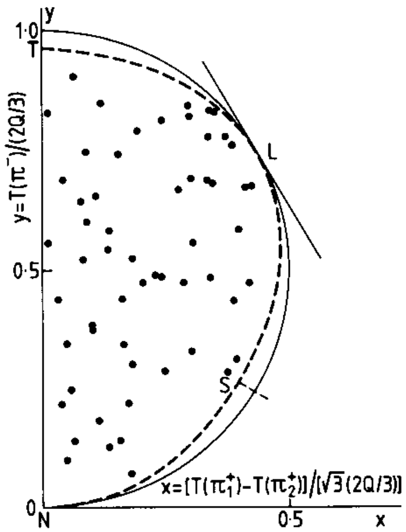
Plot $x=(T_2\text{-}T_1)/\sqrt{3}$

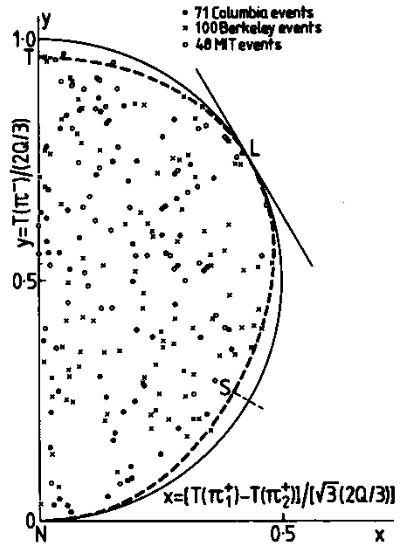
$y=T_3\text{-}Q/3$

Flat, if no dynamics is involved

The first plots → τ/θ-Puzzle





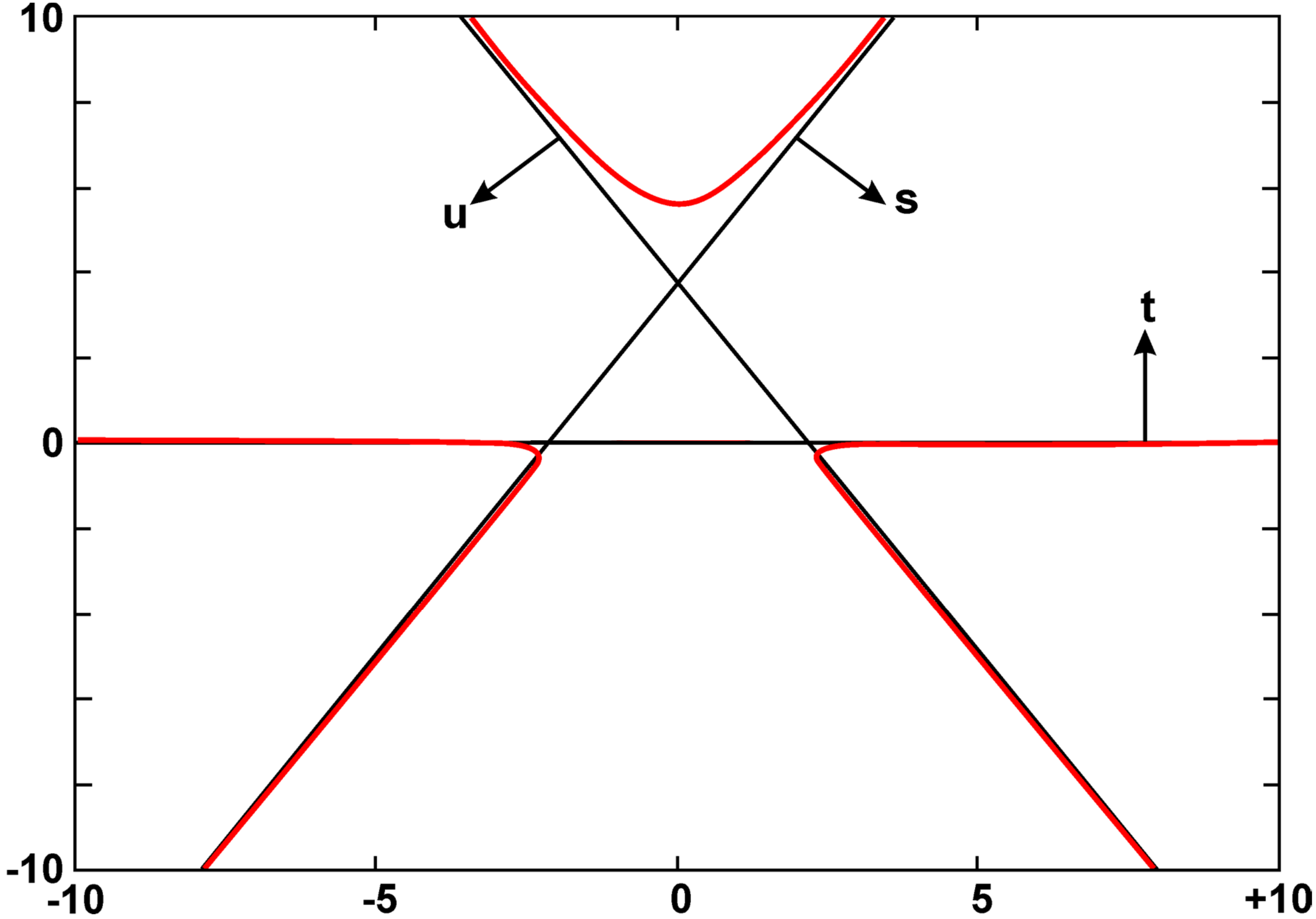


Dalitz applied it first to K_L-decays

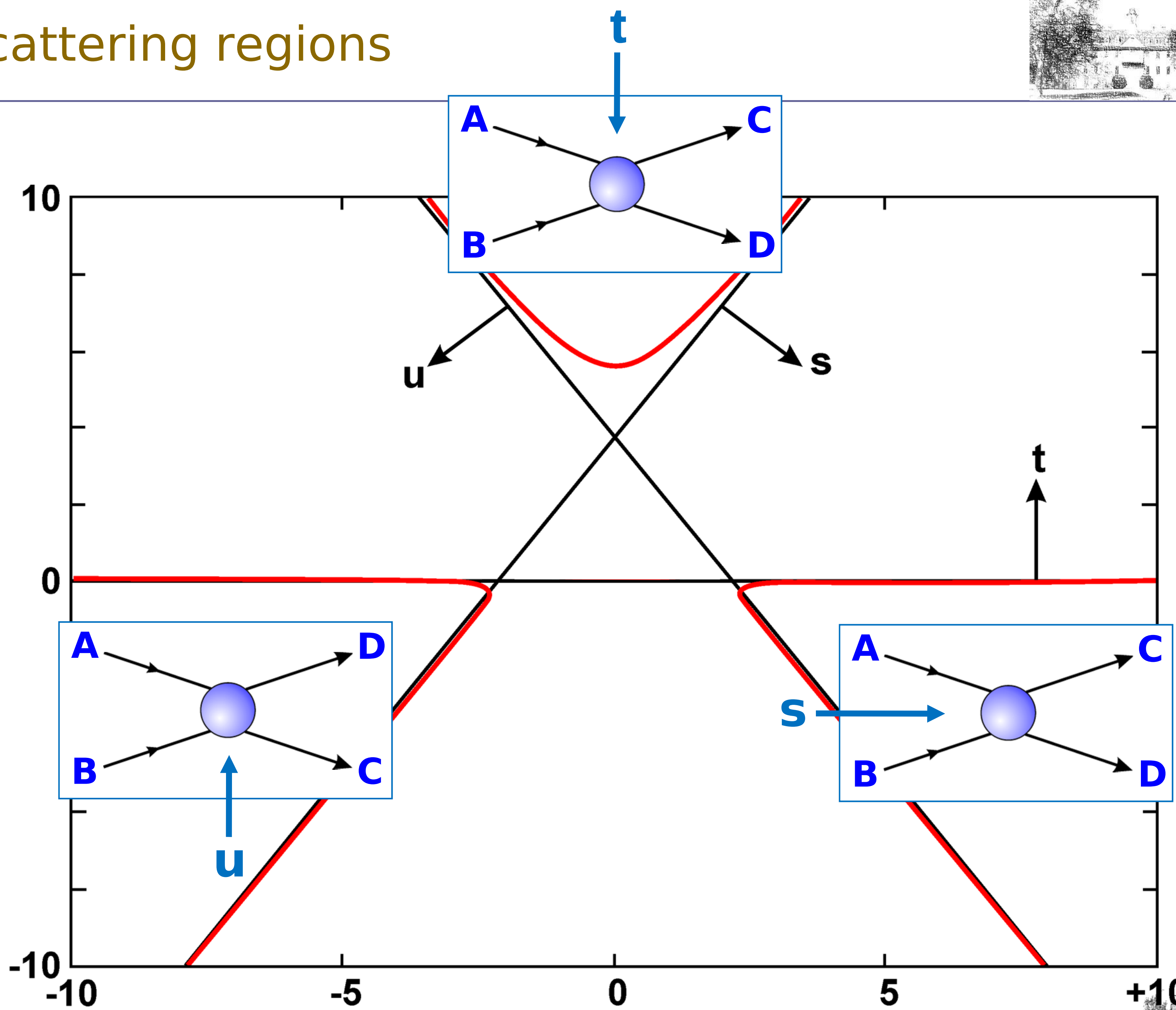
- The former τ/θ puzzle with only a few events
- goal was to determine spin and parity

And he never called them Dalitz plots

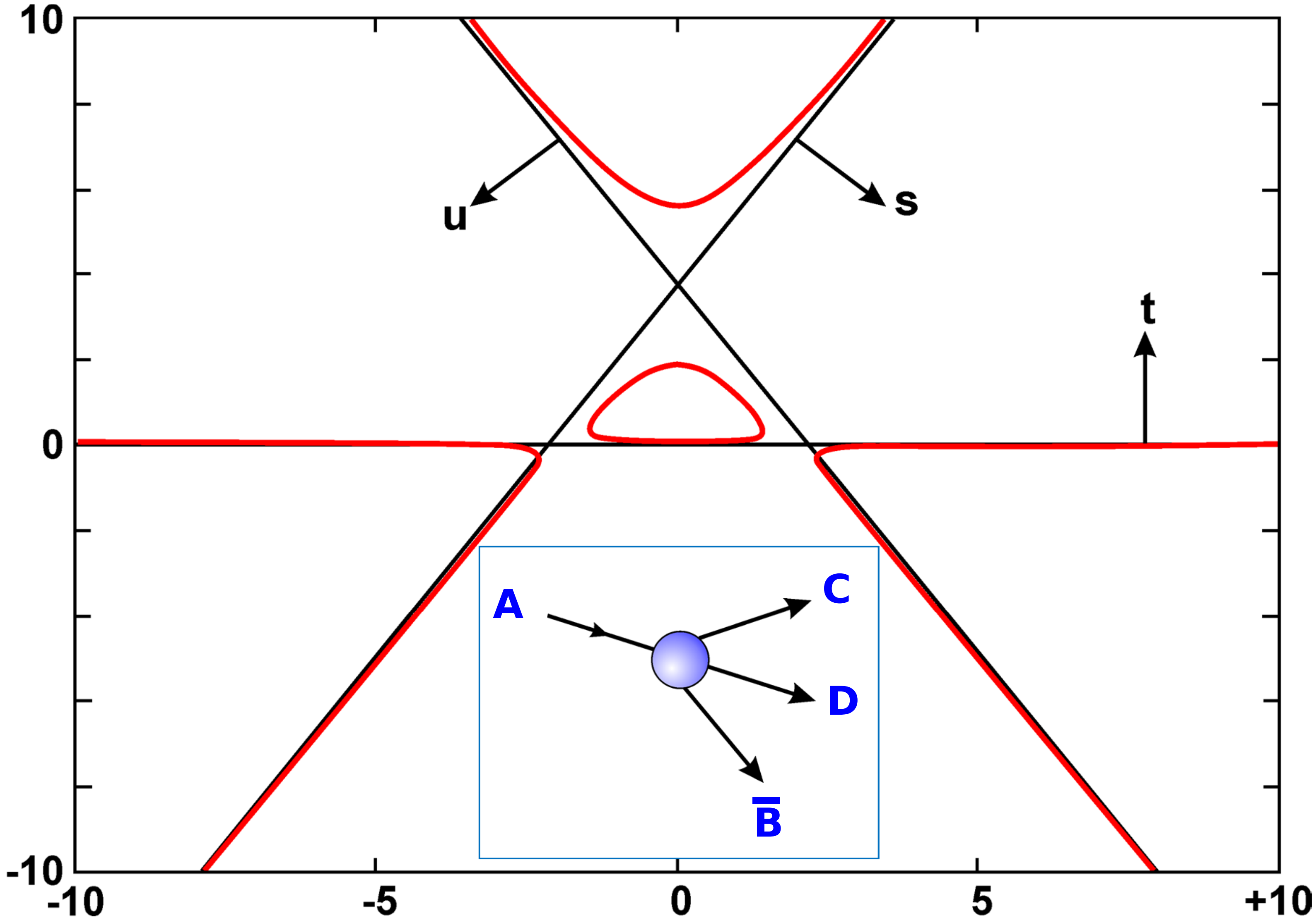
Scattering & decay regions



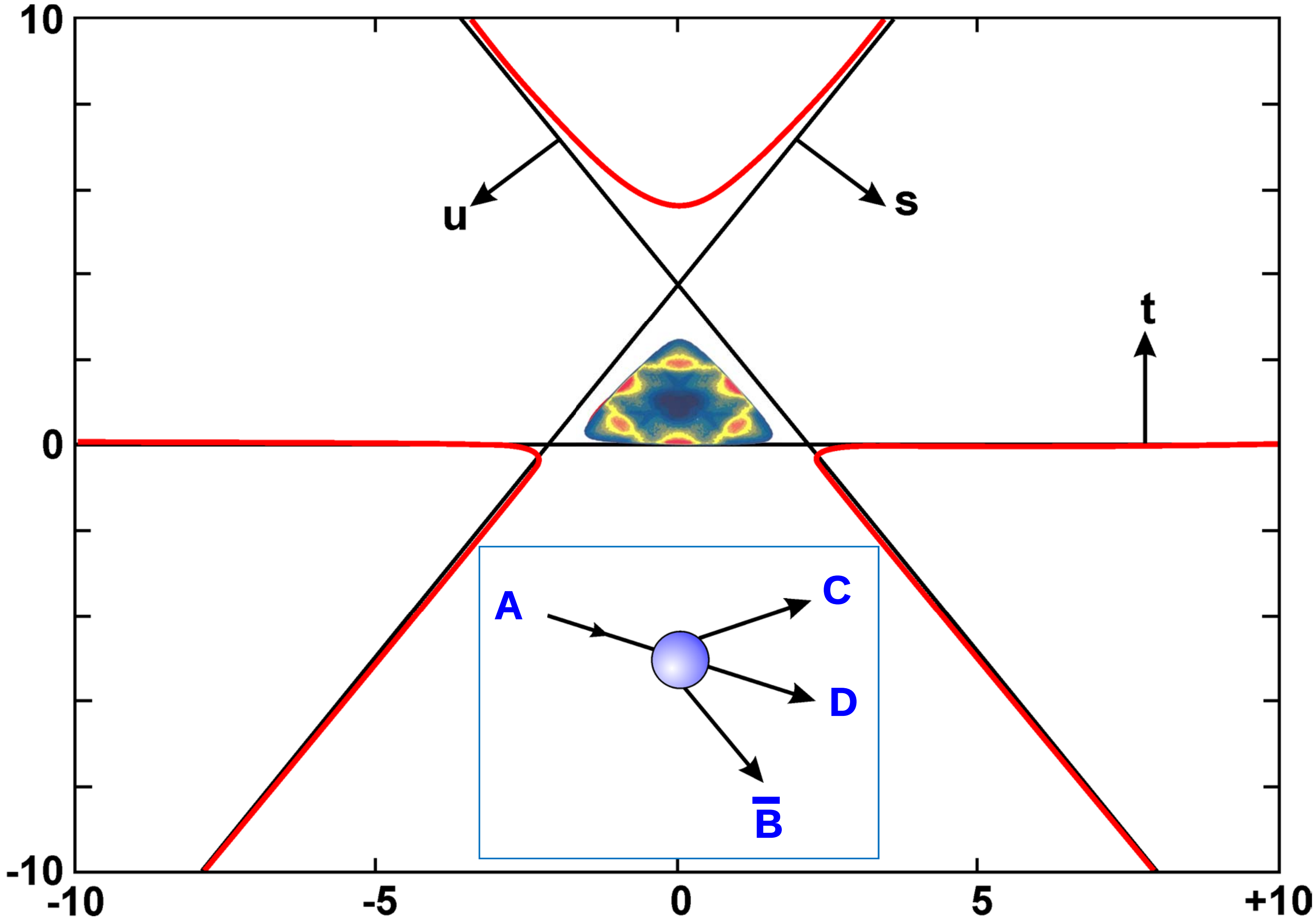
Scattering regions



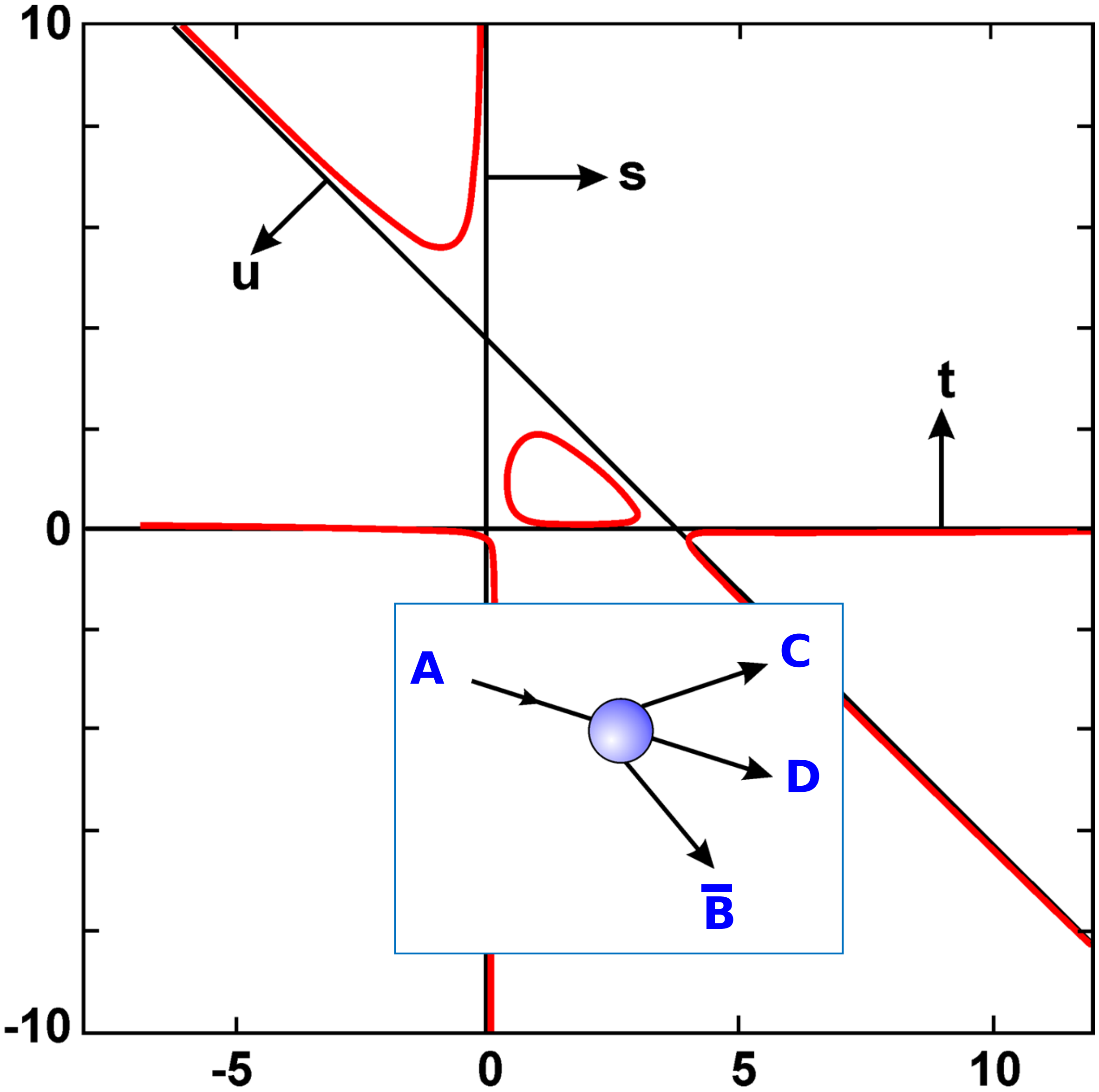
Decay region



Decay region

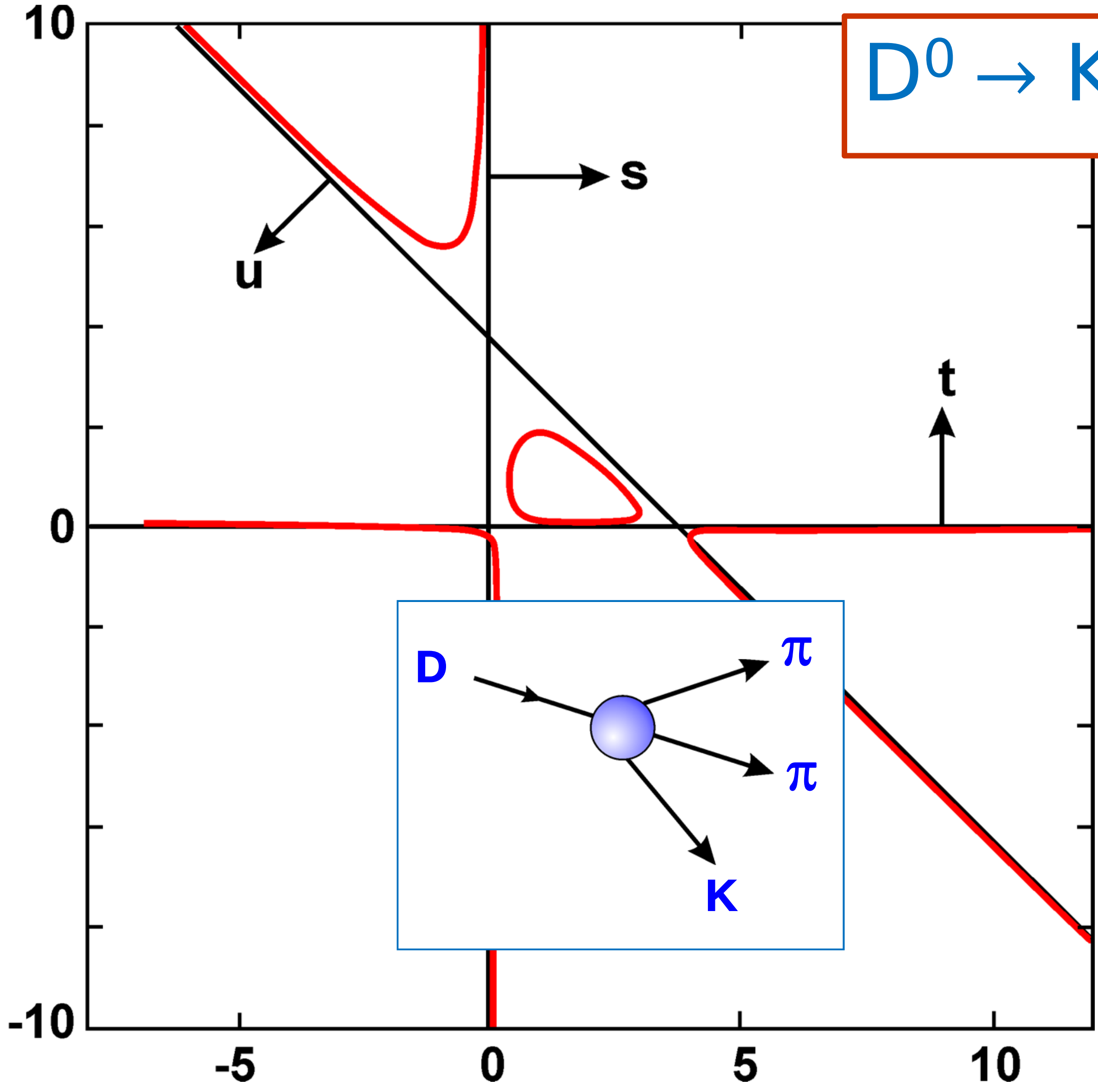


Decay region

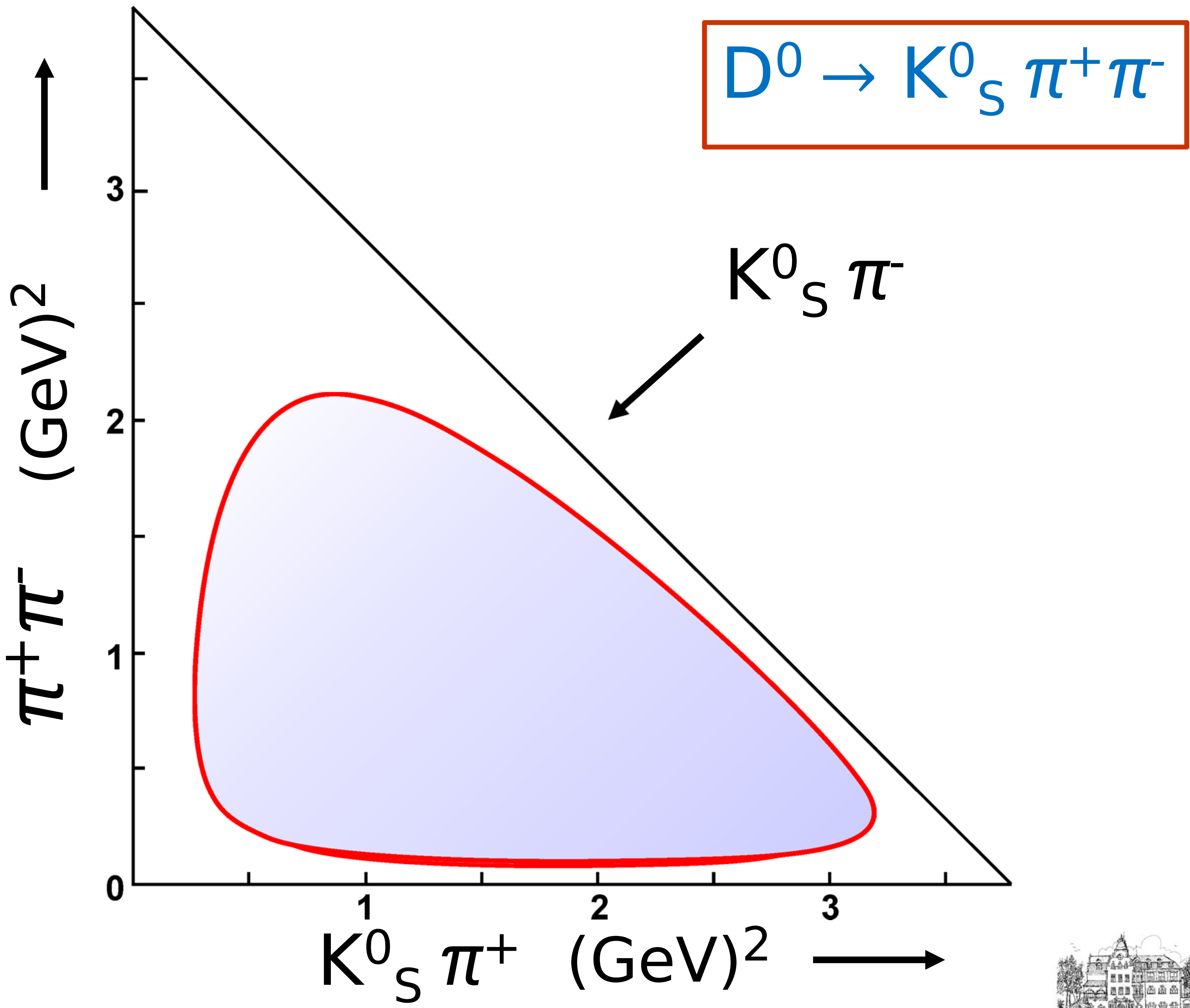


Decay region

$D^0 \to K^0_S \pi^+\pi^-$

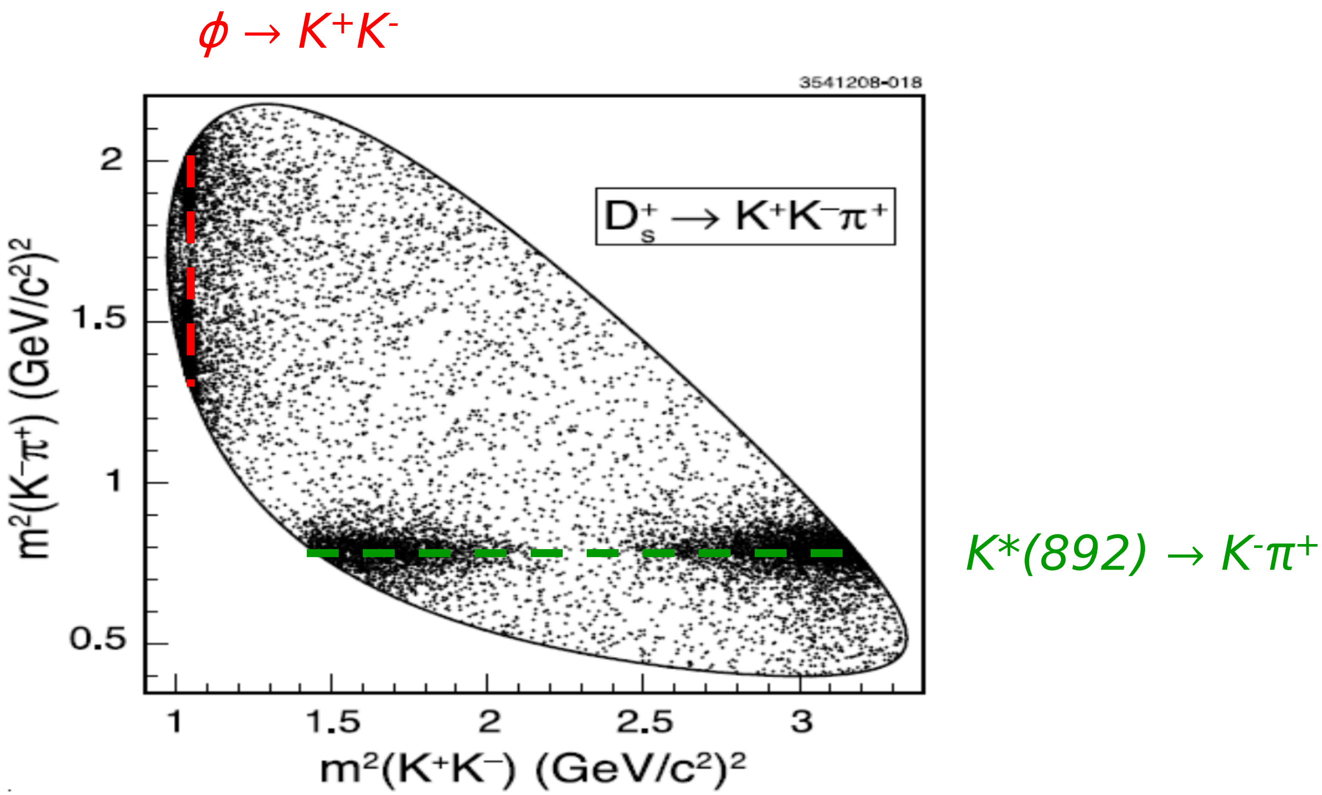


Dalitz plot



visual inspection of the phase space distribution

- are the structures?
- structures from signal or background?
- are there strong interferences, threshold effects, potential resonances?

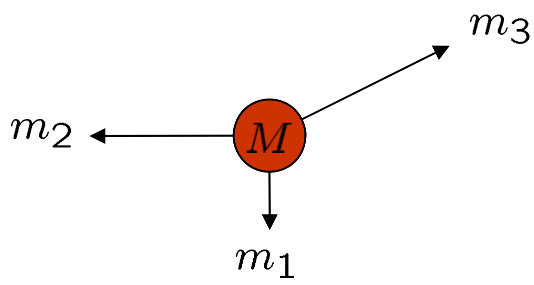


Dalitz Plot – 2D Phase Space

How can resonances be studied in multi-body decays?

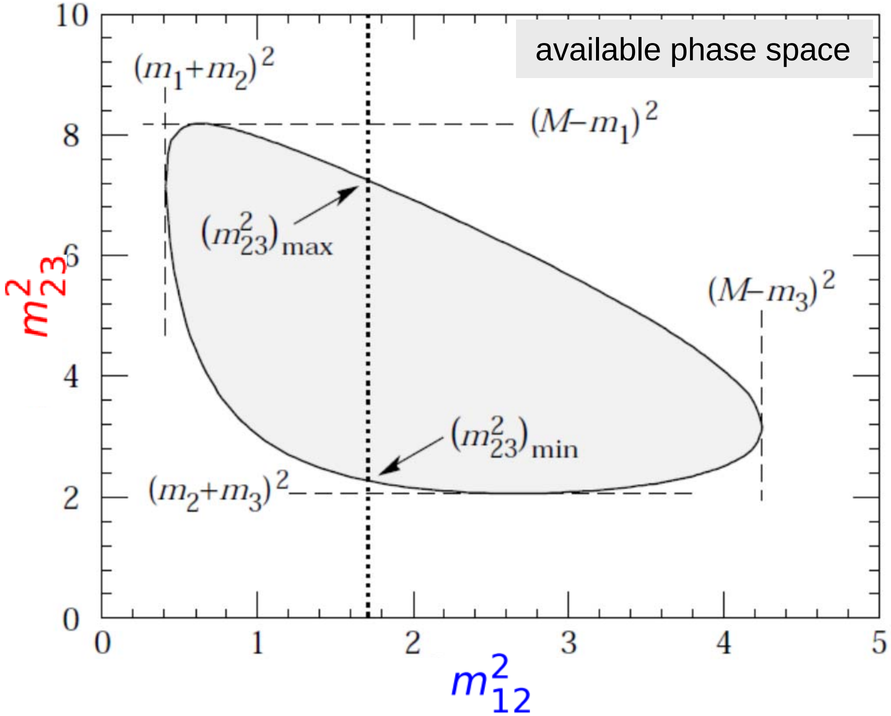
Consider 3 body decay $M \to m_1 m_2 m_3$ *(all spin 0)*

Degrees of freedom



3 Lorentz-vectors	12
3 Masses	-3
Energy conserv.	-1
Momentum conserv.	-3
3 Euler angles	-3
Remaining d.o.f	2

Dalitz-Plot



Complete dynamics described by two variables!

Usual choice

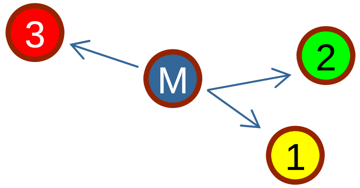
m_{12}^2 vs. m_{23}^2

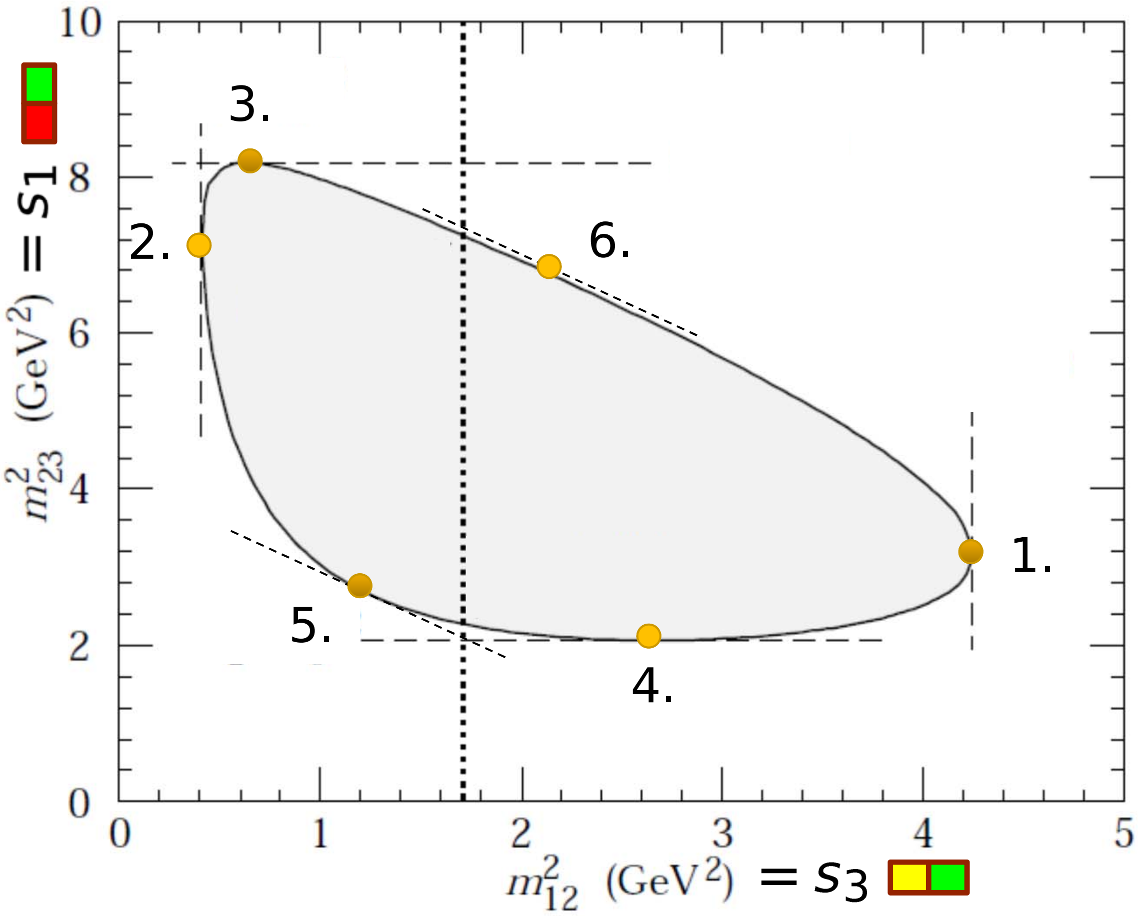
$$m_{ij}^2 = (E_i + E_j)^2 - (\vec{p}_i + \vec{p}_j)^2$$

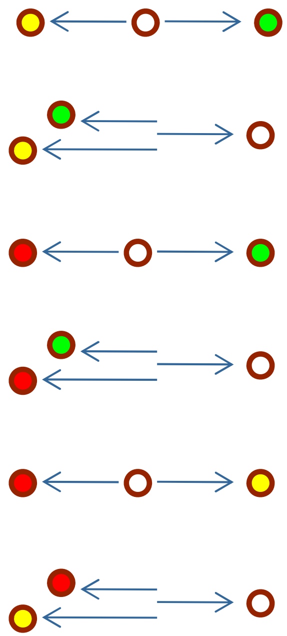
Kinematics in the Dalitz-Plot

$$s = p^2 = M^2$$
$$\sum s_i = M^2 + \sum m_i^2$$

$$s_1 = (p - p_1)^2 = (p_2 + p_3)^2$$
$$s_2 = (p - p_2)^2 = (p_1 + p_3)^2$$
$$s_3 = (p - p_3)^2 = (p_1 + p_2)^2$$







1. $s_{3,max}$
2. $s_{3,min}$
3. $s_{1,max}$
4. $s_{1,min}$
5. $s_{2,max}$
6. $s_{2,min}$

Kinematics in the Dalitz-Plot

Phasespace limits (example: s_1):

Pos 3: *M rest system:*

$$s_1 = (p - p_1)^2 = M^2 - 2EE_1 + m_1^2 = M^2 + m_1^2 - 2ME_1, \qquad E_1 = \sqrt{m_1^2 + p_1^2} \geq m_1$$

s_1 maximum, if E_1 is at minimum

$$s_{1,max} = M^2 + m_1^2 - 2Mm_1 = (M - m_1)^2$$

Pos 4: *m_{23} rest system (= Jackson-Frame R_{23}), p_2=-p_3*

$$s_1 = (p_2 + p_3)^2 = (\hat{E}_2 + \hat{E}_3) \geq (m_2 + m_3)^2$$

$$s_{1,min} = (m_2 + m_3)^2$$

full picture

$$s_1 \in \left[(m_2 + m_3)^2, (M - m_1)^2\right]$$
$$s_2 \in \left[(m_1 + m_3)^2, (M - m_2)^2\right]$$
$$s_3 \in \left[(m_1 + m_2)^2, (M - m_3)^2\right]$$

but:
not the whole
cube is accessible!

Kinematics in the Dalitz-Plot

Example: need $s_{2,\pm}(s_1)$; calculate in Restsystem R_{23} (p_2=-p_3, p_1=p)

$$s_1 = (p - \hat{p}_1)^2 = (\hat{E} - \hat{E}_1)^2 = \left(\sqrt{M^2 + \hat{p}_1^2} - \sqrt{m_1^2 + \hat{p}_1^2}\right)^2$$

$$\Rightarrow \hat{p}_1^2 = \frac{1}{4s_1}\left[s_1 - (M - m_1)^2\right]\left[s_1 - (M + m_1)^2\right]$$

$$= \frac{1}{4s_1}\lambda(s1, M^2, m_1^2)$$

kinematical function

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

$$s_1 = (p_2 + p_3)^2 = (\hat{E}_2 + \hat{E}_3)^2 \qquad \Rightarrow \hat{p}_2^2 = \hat{p}_3^2 = \frac{1}{4s_1}\lambda(s_1, m_2^2, m_3^2)$$

$$s_2 = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2(\hat{E}_1\hat{E}_3 - \hat{p}_1\hat{p}_3\cos\alpha)$$

$$\angle(\hat{\vec{p}}_1, \hat{\vec{p}}_3) \in [-1, 1]$$

$$s_2 = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2(\hat{E}_1\hat{E}_3 \pm \hat{p}_1\hat{p}_3)$$

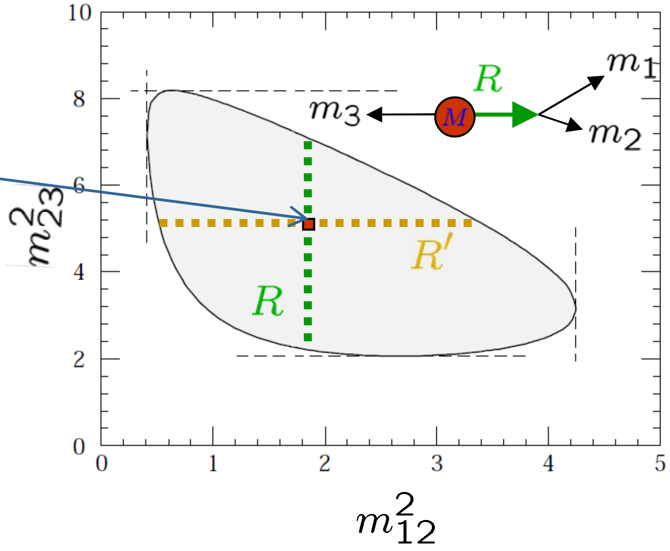
with $\hat{E}_1 = \frac{1}{2s_1}(s - s_1 - m_1^2)$ and $\hat{E}_3 = \frac{1}{2s_1}(s_1 + m_3^2 - m_2^2)$

$$s_{2,\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[(s - s_1 - m_1^2)(s_1 - m_2^2 + m_3^2) \pm \lambda^{\frac{1}{2}}(s_1, s, m_1^2)\lambda^{\frac{1}{2}}(s_1, m_2^2, m_3^2)\right]$$

Density distribution in the Dalitz Plot given by

$$\Gamma(m_{12}^2, m_{23}^2) = \frac{1}{(2\pi)^3 32 M^3} |\mathcal{M}|^2 \, dm_{12}^2 dm_{23}^2$$

for Spin-0 Particles M, m_1, m_2, m_3



Dynamics is contained by the matrix element $\mathcal{M}$

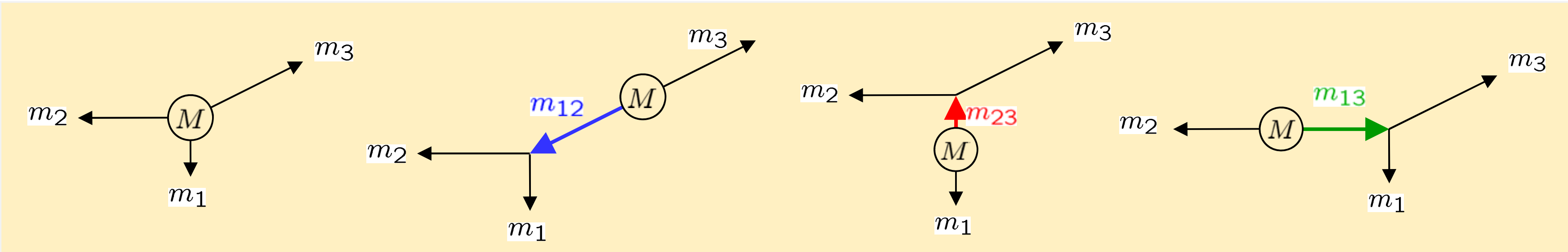
non-resonant processes ⇒ $\mathcal{M}$=const., uniform distribution

resonant processes ⇒ bands (horizontal, vertical, diagonal)

spins ⇒ Density distribution along the bands

Dalitz Plot – 2D Phase space

Possible decay patterns:



Non-Resonant (or background)

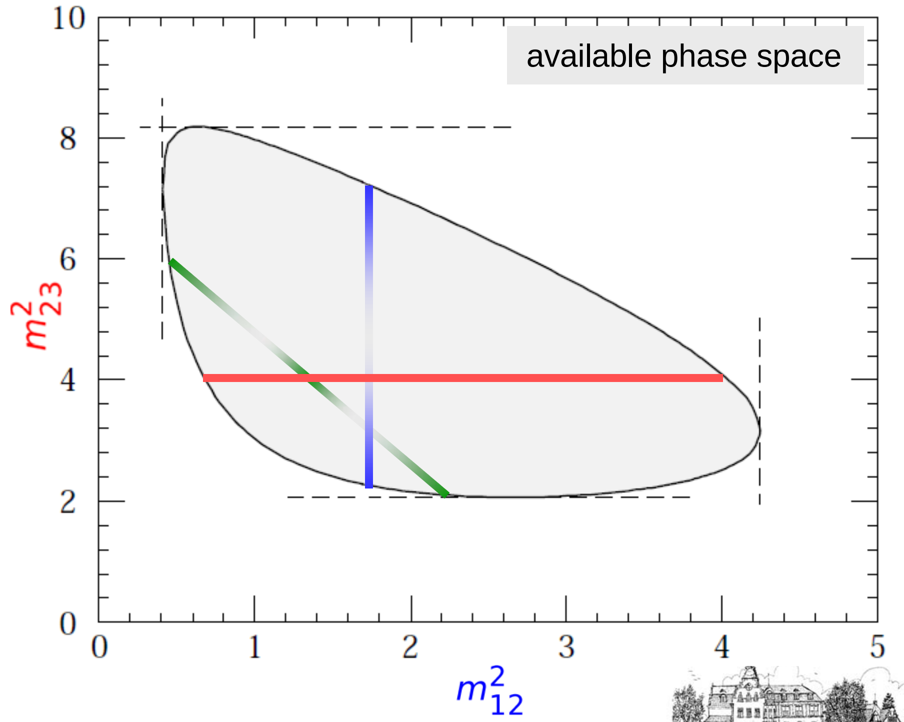
flat (homogeneous) distribution

Resonance *R* in m_{12}, m_{13}, m_{23}

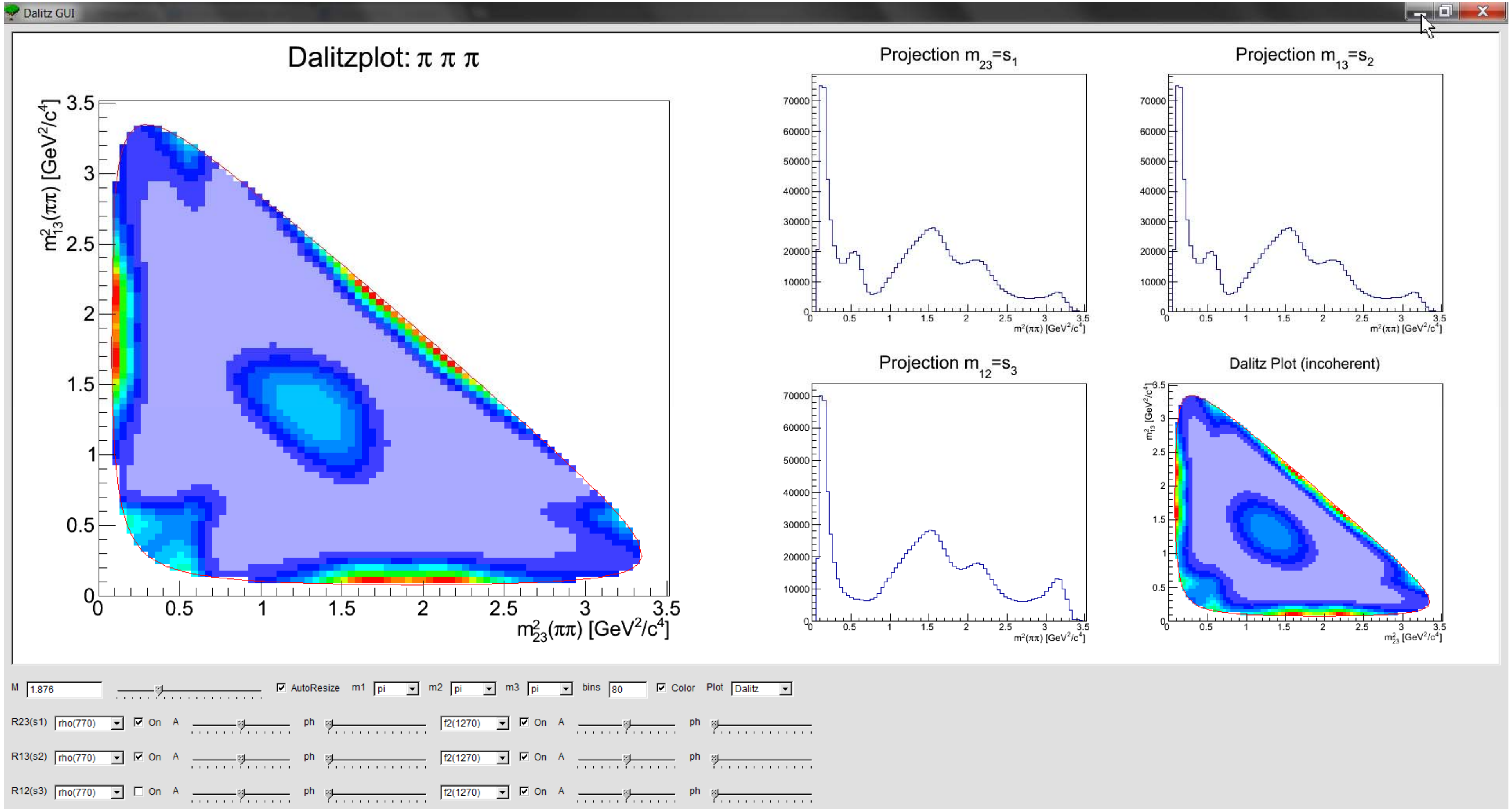
Band Structures

Position: Mass of *R*

Density: Spin of *R*



Dalitz-Plot Tool (Root) Examples....



Properties of Dalitz Plots

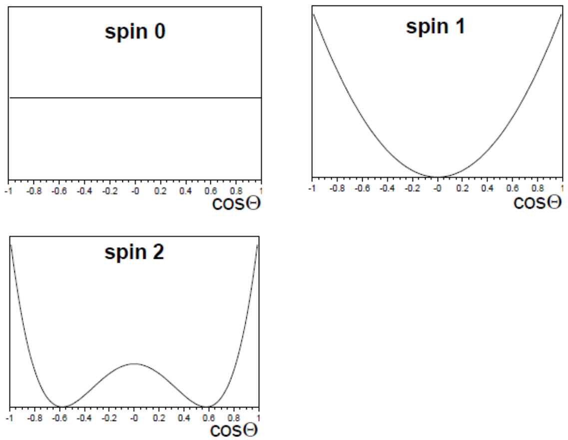
For the process $M \to Rm_3$, $R \to m_1 m_2$ the matrix element can be expressed like

$$\mathcal{M}_R(L, m_{12}, m_{23}) = Z(L, \vec{p}, \vec{q}) \cdot B_L^M(p) \cdot B_L^R(q) \cdot T_R(m_{12})$$

Winkelverteilung (Legendre Polyn.) — Formfaktor (Blatt-Weisskopf-F.) — Resonanz-Fkt. (z.B. Breit Wigner)

$Z(L, \vec{p}, \vec{q})$ decay angular distribution of R →

J → L+l	Z
0 → 0 + 0	1
0 → 1 + 1	$\cos^2\theta$
0 → 2 + 2	$[\cos^2\theta - 1/3]^2$



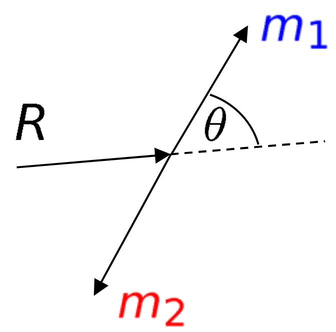
$B_L^M(p)$ Form-(Blatt-Weisskopf)-Factor for $M \to Rm_3$, $p=p_3$ in R_{12}

$B_L^R(q)$ Form-(Blatt-Weisskopf)-Factor for $R \to m_1 m_2$, $q=p_1$ in R_{12}

$T_R(m_{12})$ Dynamical Function (Breit-Wigner, K-Matrix, Flatté)

Angular Distributions

density distribution along the band =
decay angular distribution of *R*

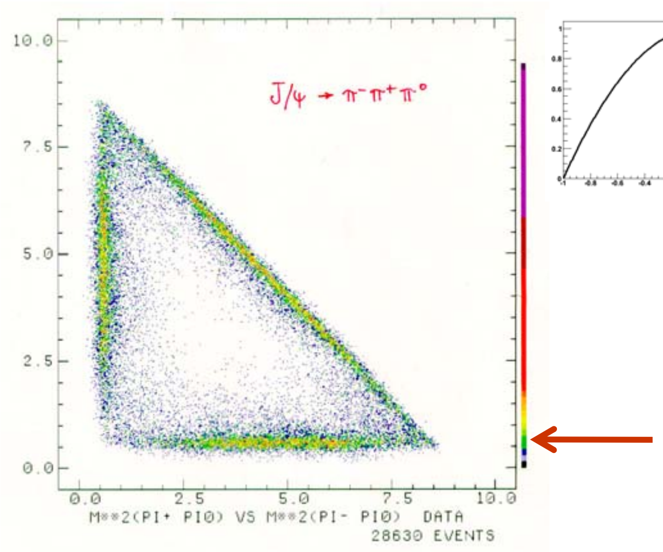


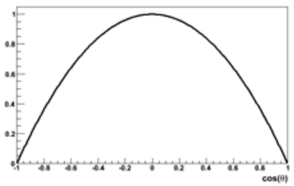
results from Spin of *R*, the spin configuration and polarization of initial and final state(s)

Compare $R=\rho$ and ϕ (both 1^{--}) angular distributions are different !!

$J/\psi \rightarrow \rho \quad \pi$

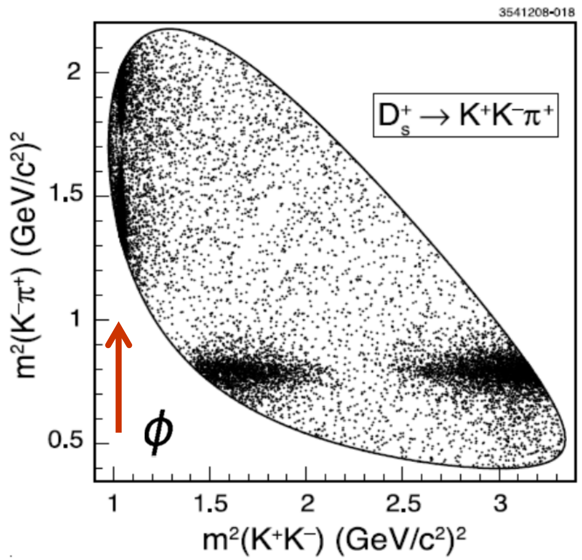
$1^{--} \rightarrow 1^{--} 0^{-+}$

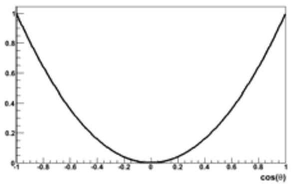




$D_S \rightarrow \phi \quad \pi$

$0^{-+} \rightarrow 1^{--} 0^{-+}$



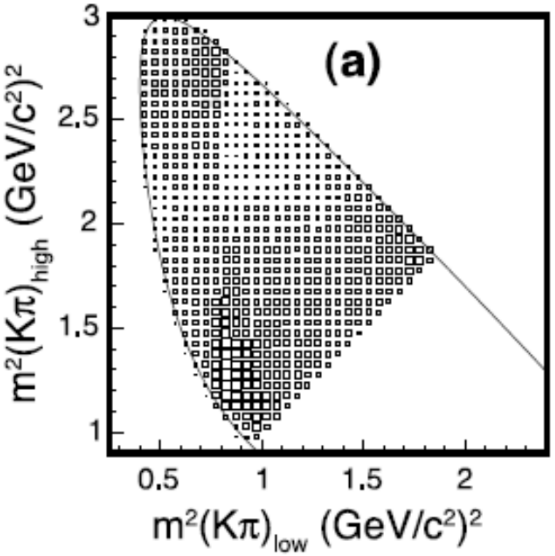


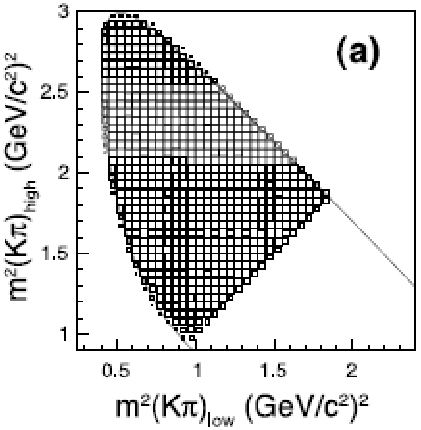
Dalitz-Plot-Analysis

Simultaneous fit of all resonant structures in a Dalitz-Plot

Takes into account interference between resonances!

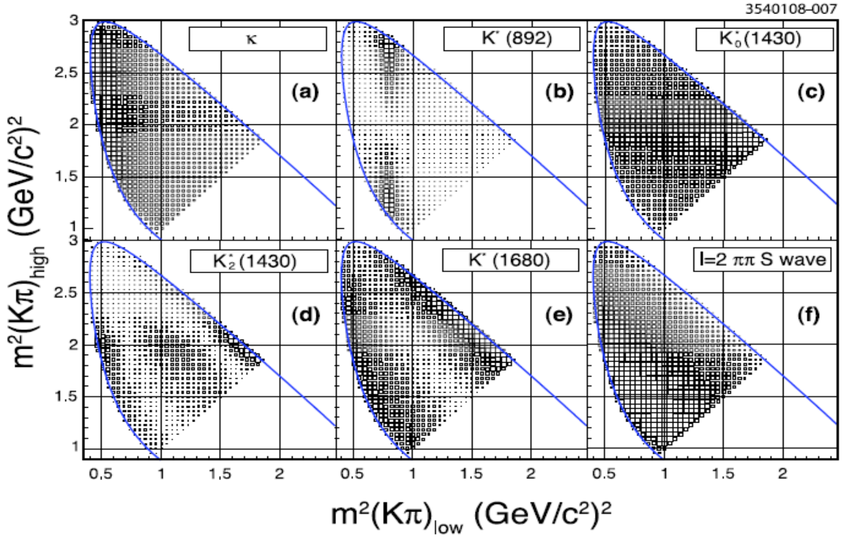
The Dalitz Plot





Reco Efficiency

The Ingredients



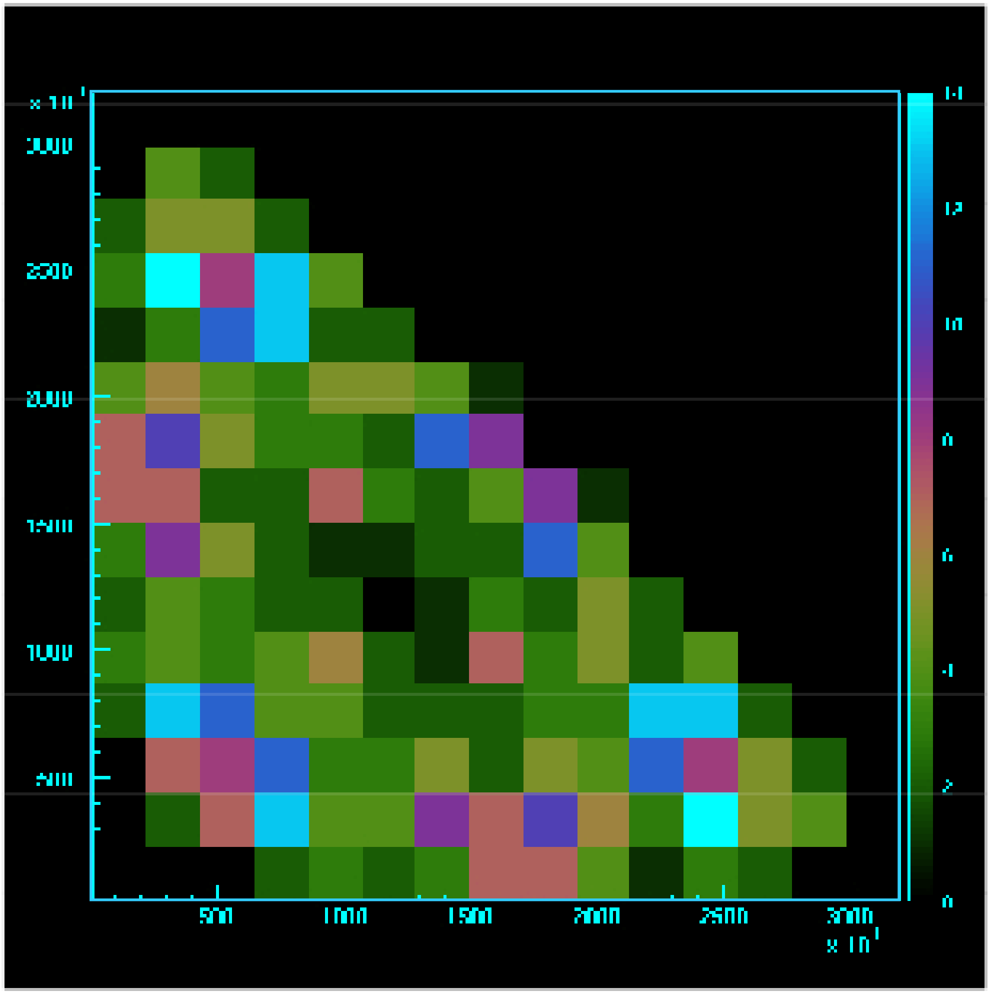
Mode	Parameter	Model I2 (B-W for κ)	Model I2	QMIPWA
$\overline{K}^*(892)\pi^+$	a	1 – fixed	1 – fixed	1 – fixed
	ϕ (°)	0 – fixed	0 – fixed	0 – fixed
	FF (%) 2×	5.15±0.24	5.27±0.08±0.15	4.94±0.23
	m (MeV/c^2)	895.4±0.2	895.7±0.2±0.3	895.7 – fixed
	Γ (MeV/c^2)	44.5±0.7	45.3±0.5±0.6	45.3 – fixed
$\overline{K}^*(1680)\pi^+$	a	4.45±0.23	3.38±0.16±0.78	2.88±0.84
	ϕ (°)	43.3±3.6	68.2±1.6±13	113±14
	FF (%) 2×	0.238±0.024	0.144±0.013±0.12	0.098±0.059
$\overline{K}^*_2(1430)\pi^+$	a	0.866±0.030	0.915±0.025±0.04	0.794±0.073
	ϕ (°)	–17.4±3.5	–17.4±2.3±2.0	14.8±9.0
	FF (%) 2×	0.124±0.011	0.145±0.009±0.03	0.102±0.020
$\overline{K}^*_0(1430)\pi^+$	a	3.97±0.15	3.74±0.02±0.06	3.74 – fixed
	ϕ (°)	45.1±0.9	51.1±0.3±1.6	51.1 – fixed
	FF (%) 2×	7.53±0.65	7.05±0.14±0.55	6.65±0.31
	m (MeV/c^2)	1461.1±1.0	1466.6±0.7±3.4	1466.6 – fixed
	Γ (MeV/c^2)	177.9±3.1	174.2±1.9±3.2	174.2 – fixed

Fit Results

It’s All a Question of Statistics ...

$p\bar{p} \rightarrow 3\pi^0$ with

100 events

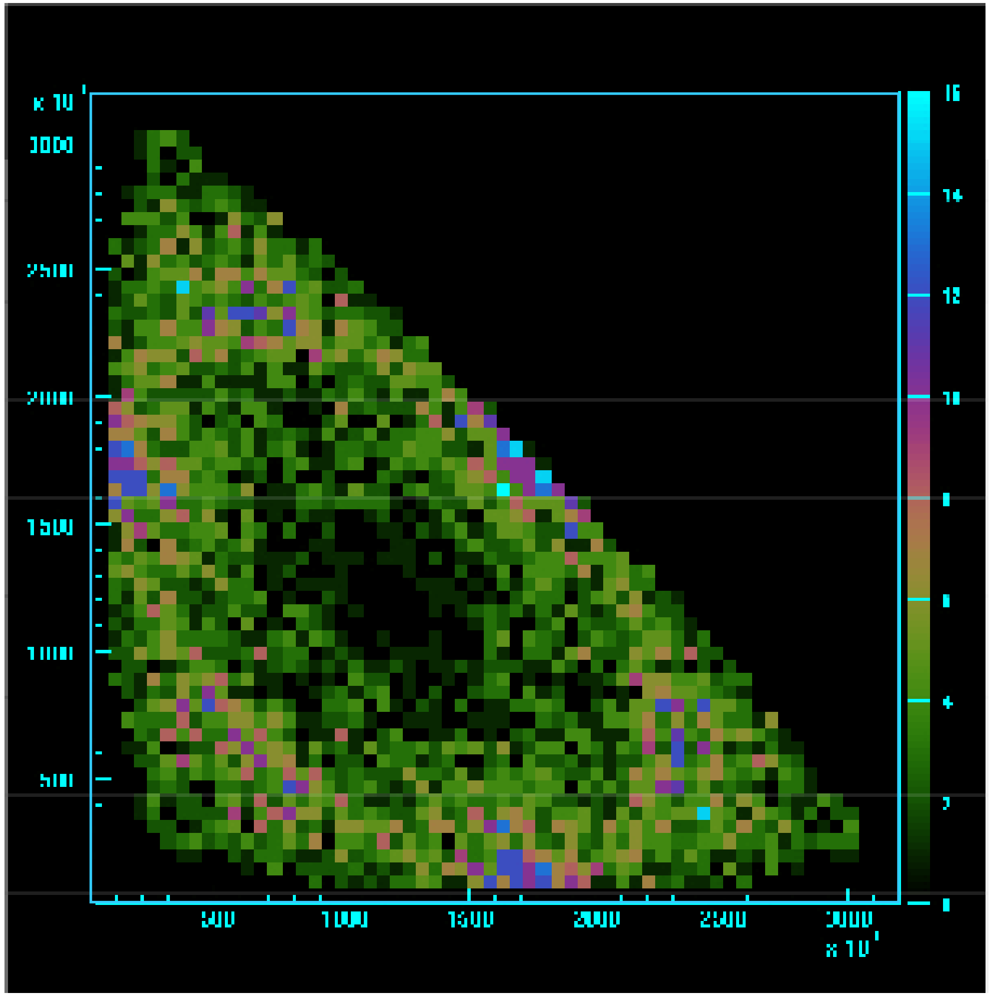


It’s All a Question of Statistics ... ...

$p\bar{p} \rightarrow 3\pi^0$ with

~~100 events~~

1000 events



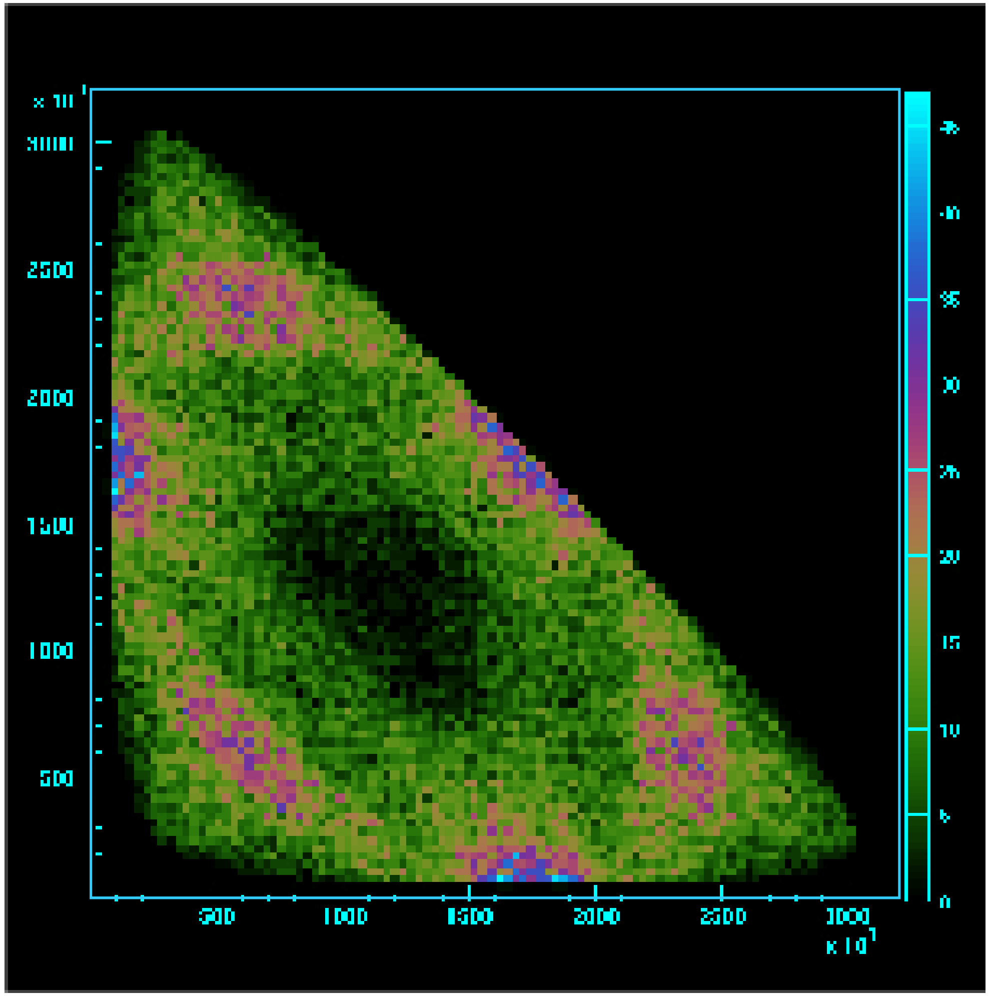
It’s All a Question of Statistics ... ...

$p\bar{p} \rightarrow 3\pi^0$ with

~~100 events~~

~~1000 events~~

10000 events



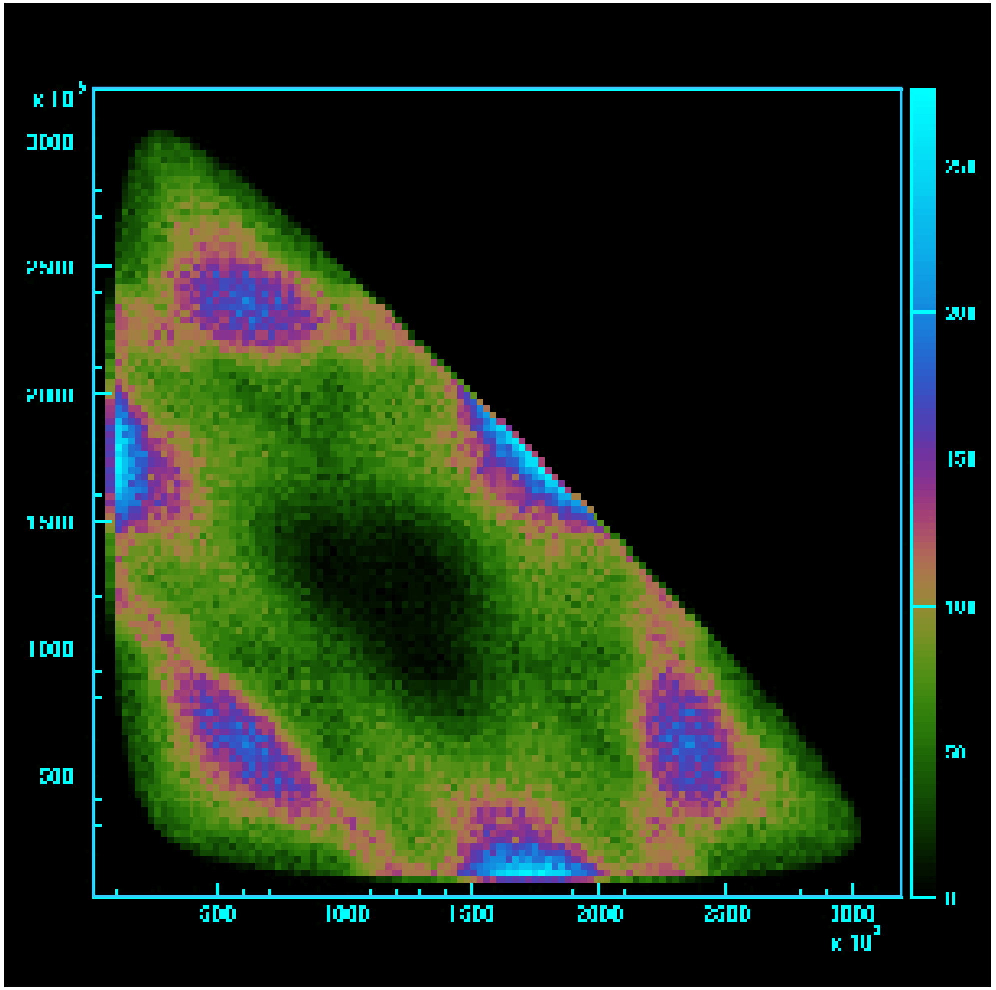
$p\bar{p} \rightarrow 3\pi^0$ with

~~100 events~~

~~1000 events~~

~~10000 events~~

100000 events



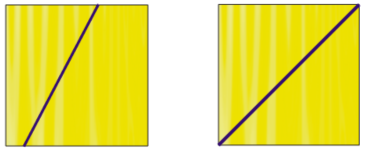
Observables, cont’d

are there symmetries in the phase space?

unique assignment of phase space coordinates
is important to avoid double counting

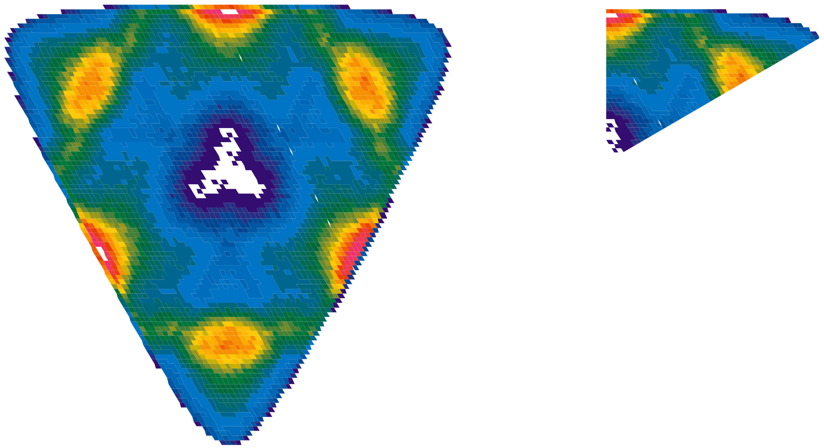
transformation necessary?

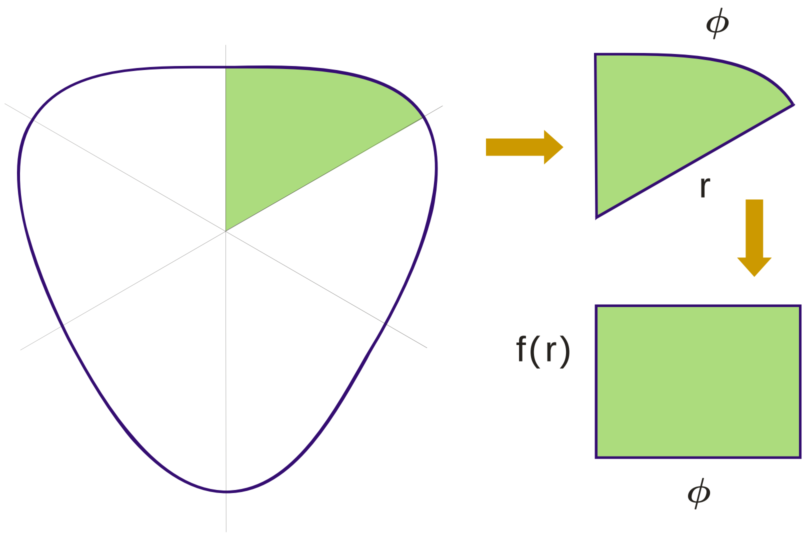
Most Dalitz plots are symmetric:



Problem: sharing of events

Possible solution: transform DP





Spin Part in a nutshell

just a few words…

usually this is not part of your job

there are very many formalisms and packages

finally it‘s just a decomposition of the phase space
which obeys all the necessary symmetries of the reaction

thus (in principle) straigth forward

and usually done by other people ☺

Formalisms – an overview (very limited)

Non-relativistic Tensor formalisms

in non-relativistic (Zemach) or covariant flavor

Fast computation, simple for small *L* and *S*

Spin-projection formalisms

where a quantization axis is chosen and proper rotations are used to define a two-body decay

Efficient formalisms, even large *L* and *S* easy to handle

Relativistic Tensor Formalisms based on Lorentz invariants (Rarita-Schwinger)

where each operator is constructed from Mandelstam variables only

Elegant, but extremely difficult for large *L* and *S*

Spin-Projection Formalisms

Differ in choice of quantization axis

Helicity Formalism

parallel to its own direction of motion

Transversity Formalism

the component normal to the scattering plane is used

Canonical (Orbital) Formalism

the component m in the incident z-direction is diagonal

Zemach Formalism

For particle with spin S traceless tensor of rank S

Similar for orbital angular momentum L

$$l = 0 \qquad A^0 = 1$$

$$l = 1 \qquad A^1(\vec{q}) = \vec{q}$$

$$l = 2 \qquad A^2(\vec{q}) = \frac{3}{2}\left[\vec{q}\cdot\vec{q}^{\,T} - \underbrace{\frac{1}{3}|\vec{q}|^2}_{\text{for tracelessness}}\right]$$

$$\vec{q}\cdot\vec{p}^{\,T} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} q_1p_1 & q_1p_2 & q_1p_3 \\ q_2p_1 & q_2p_2 & q_2p_3 \\ q_3p_1 & q_3p_2 & q_3p_3 \end{pmatrix}$$

with indices

$$l = 0 \qquad A^0 = 1$$

$$l = 1 \qquad A^1_i = q_i$$

$$l = 2 \qquad A^2_{ij} = \frac{3}{2}q_iq_j - \frac{1}{2}|q_i|^2\delta_{ij}$$

The Original Zemach Paper

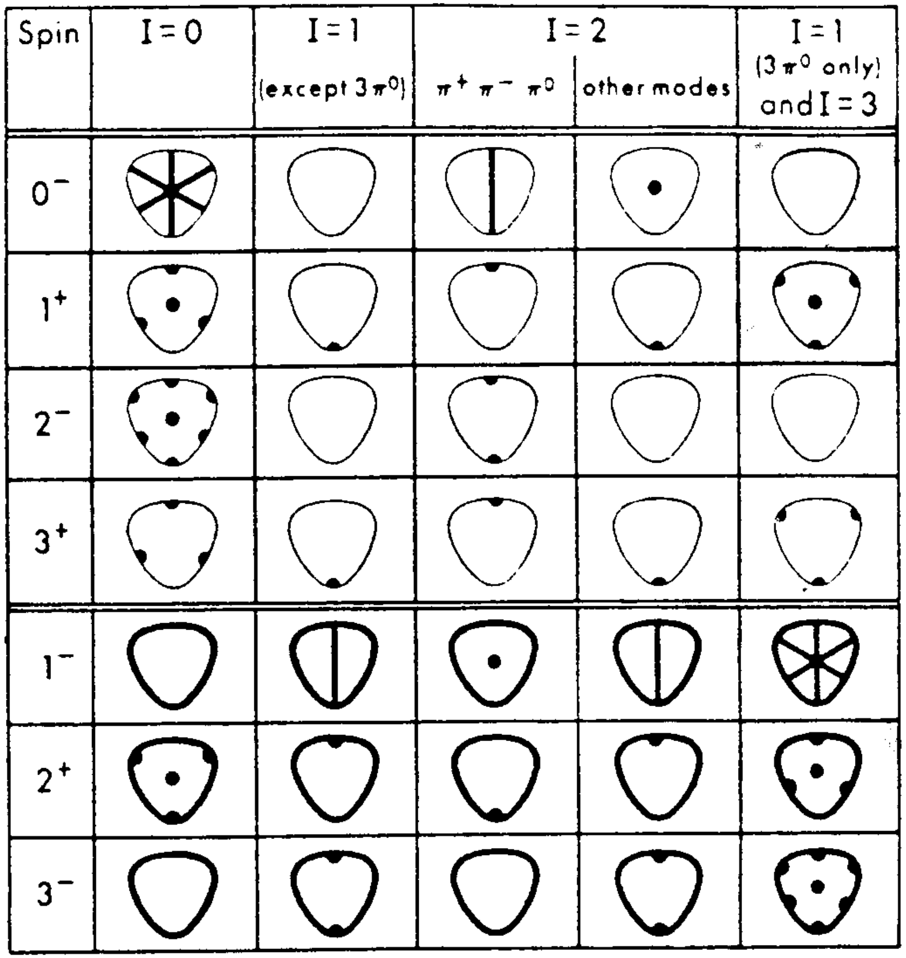


FIG. 2. Regions of the 3π Dalitz plot where the density must vanish because of symmetry requirements are shown in black. The vanishing is of higher order (stronger) where black lines and dots overlap. In each isospin and parity state, the pattern for a spin of $J+$ even integer is identical to the pattern for spin J, provided $J \geqq 2$. (Exception: vanishing at the center is not required for $J \geqq 4$.)

Tensors revisited

The Zemach amplitudes are only valid in the rest frame of the resonance.

- Thus they are not covariant
- Retain covariance by adding the time component and use 4-vectors
- Behavior under spatial rotations dictates that the time component of the decay momentum vanishes in the rest frame
- This condition is called Rarita Schwinger condition

For Spin-1 it reads $Su = S_\mu p^\mu = 0$

- with $p = (p_a + p_b)/m$ the 4-momentum of the resonance

- The vector $S_{\mu\mu}$ is orthogonal to the timelike vector p_μ and is therefore spacelike, thus $S^2 < 0$

Covariant Tensor Formalism

The most simple spin-1 covariant tensor with above properties is

$S_\mu = q_\mu - (qp) p_\mu$

with $q = (p_a - p_b)$

The negative norm is assured by the equation

$$S^2 = q^2 - (qp)^2 = -|q_R|^2$$

where q_R is the break-up three-momentum

the general approach and recipe is a lecture of its own and you should refer to the primary literature for more information

to calculate the amplitudes and intensities you may use qft++

qft++ = Numerical Object Oriented Quantum Field Theory

(by Mike Williams, Carnegie Mellon Univ.)

Calculation of the matrices, tensors, spinors, angular momentum tensors etc. with C++ classes

qft++ Class	Symbol	Concept
`Matrix<T>`	a_{ij}	matrices of any dimension
`Tensor<T>`	x_μ	tensors of any rank
`MetricTensor`	$g_{\mu\nu}$	Minkowski metric
`LeviCivitaTensor`	$\epsilon_{\mu\nu\alpha\beta}$	totally anti-symmetric Levi-Civita tensor
`DiracSpinor`	$u_{\mu_1\ldots\mu_{J-1/2}}(p,m)$	half-integral spin wave functions
`DiracAntiSpinor`	$v(p,m)$	spin-1/2 anti-particle wave functions
`DiracGamma`	γ^μ	Dirac matrices
`DiracGamma5`	γ^5	
`DiracSigma`	$\sigma^{\mu\nu}$	
`PolVector`	$\epsilon_{\mu_1\ldots\mu_J}(p,m)$	integral spin wave functions
`OrbitalTensor`	$L^{(\ell)}_{\mu_1\ldots\mu_\ell}$	orbital angular momentum tensors

Example: $X(2^-) \to \omega K \to \pi^+\pi^-\pi^0 K$

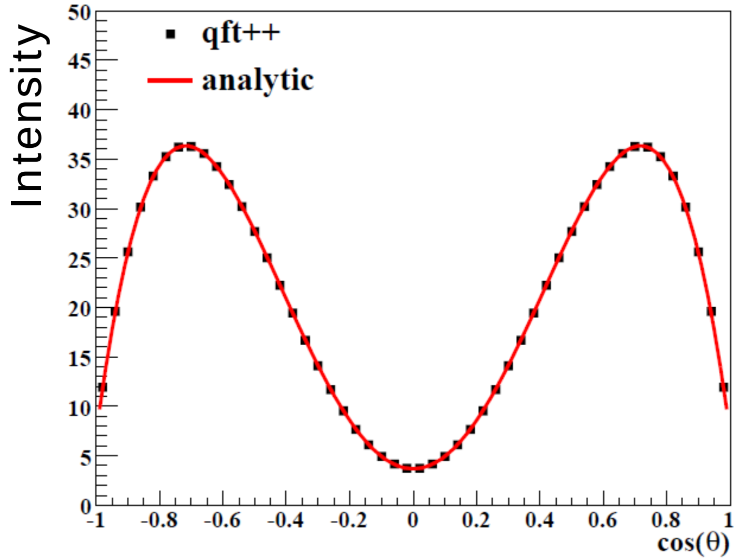
Amplitude and Intensity given by

$$\mathcal{A} \propto \epsilon^*_\mu(p_\omega, m_\omega) L^{(3)\mu\nu\alpha}(p_{\omega K})\epsilon_{\nu\alpha}(P, M) \quad \text{and} \quad \mathcal{I} \propto \sum_{M=\pm 1}\ \sum_{m_\omega=\pm 1,0} |\mathcal{A}|^2$$

qft++: Declaration and Calculation

```
PolVector epso; // omega
PolVector epsx(2); // X
OrbitalTensor orb3(3); // L^3
Tensor<complex<double> > amp;
Vector4<double> p4o,p4k,p4x;

double intensity = 0.;
for(Spin m = -1; m <= 1; m+=2){
  for(Spin mo = -1; mo <= 1; mo++){
    amp = conj(epso(mo))*orb3|epsx(m);
    intensity += norm(amp());
  }
}
```



Angular distribution of $X \to \omega K$

Helicity Decay Amplitudes

Helicity

From two-particle state

$$A = \sum_{\lambda_s,\lambda_t} \langle \vec{p}_s, s\lambda_s | \langle -\vec{p}_s, t\lambda_t | \mathcal{M} | JM \rangle$$

$$\langle \Omega_s, s\lambda_s t\lambda_t | JM\lambda_{s'}\lambda_{t'} \rangle = N_J D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s)$$

$$\begin{aligned} A^{JM}_{\lambda_s\lambda_t} &= \frac{4\pi}{\rho_s} \langle \Omega_s, s\lambda_s t\lambda_t | \mathcal{M} | JM \rangle \\ &= \sum_{\lambda_{s'},\lambda_{t'}} \langle \Omega_s, s\lambda_s t\lambda_t | JM\lambda_{s'} m_{t'} \rangle \frac{4\pi}{\rho_s} \langle JM\lambda_{s'}\lambda_{t'} | \mathcal{M} | JM \rangle \\ &= \sqrt{\frac{4\pi}{\rho_s}(2J+1)} \langle JM\lambda_s\lambda_t | \mathcal{M} | JM \rangle D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s) \end{aligned}$$

$$A^{JM}_{\lambda_s\lambda_t} = N_J f_{\lambda_s\lambda_t} D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s)$$

Helicity amplitude

$$N_J f_{\lambda_s\lambda_t} = \sqrt{\frac{4\pi}{\rho_s}(2J+1)} \langle JM\lambda_s\lambda_t | \mathcal{M} | JM \rangle$$

$f_2 \rightarrow \pi\pi$ (Ansatz)

Initial: $f_{2(1270)}$ $I^G(J^{PC}) = 0^+(2^{++})$

Final: π^0 $I^G(J^{PC}) = 1^-(0^{-+})$

Only even angular momenta, since $\eta_f = \eta_\pi^{\,2}(-1)^l$

Total spin $s = 2s_\pi = 0$

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\varphi,\theta)$$

$$\lambda = \lambda_1 - \lambda_2 = 0$$

$$J = 2$$

$$A^{2M}_{00} = N_2 F^2_{00} D^{2*}_{M0}(\varphi,\theta)$$

$$N_2 F^2_{00} = \sqrt{5}\underbrace{\left(20\quad 00\middle|20\right)}_{1}\underbrace{\left(00\quad 00\middle|00\right)}_{1} a_{20} = \sqrt{5}a_{20}$$

$$A^{2M}_{00} = \sqrt{5}a_{20} D^{2*}_{M0}(\varphi,\theta)$$

$f_2 \to \pi\pi$ (Rates)

$$A_{00}^{1M} = \sqrt{5} a_{20} \begin{bmatrix} d^2_{(-2)0}(\theta) e^{-2i\varphi} \\ d^2_{(-1)0}(\theta) e^{-i\varphi} \\ d^2_{00}(\theta) \\ d^2_{10}(\theta) e^{i\varphi} \\ d^2_{20}(\theta) e^{2i\varphi} \end{bmatrix}$$

Amplitude has to be symmetrized because of the final state particles

$$I(\theta) = \sum_{M,M'} A_{00}^{1M} \rho_{MM'} A_{00}^{1M'*}$$

$$\rho = \frac{1}{2J+1} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$d^2_{(\pm 2)0}(\theta) = \frac{\sqrt{6}}{4} \sin^2\theta$$

$$d^2_{(\pm 1)0}(\theta) = -\sqrt{\frac{3}{2}} \sin\theta \cos\theta$$

$$d^2_{00}(\theta) = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right)$$

$$I(\theta) = |a_{20}|^2 \underbrace{\left(\frac{15}{4}\sin^4\theta + 15\sin^2\theta\cos^2\theta + 5\left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right)^2 \right)}_{15\left(\frac{1}{4}\sin^4\theta + \sin^2\theta\cos^2\theta + \frac{3}{4}\cos^4\theta - \frac{1}{2}\cos^2\theta + \frac{1}{12}\right)}$$

$$= |a_{20}|^2 = const$$

THANK YOU

for today