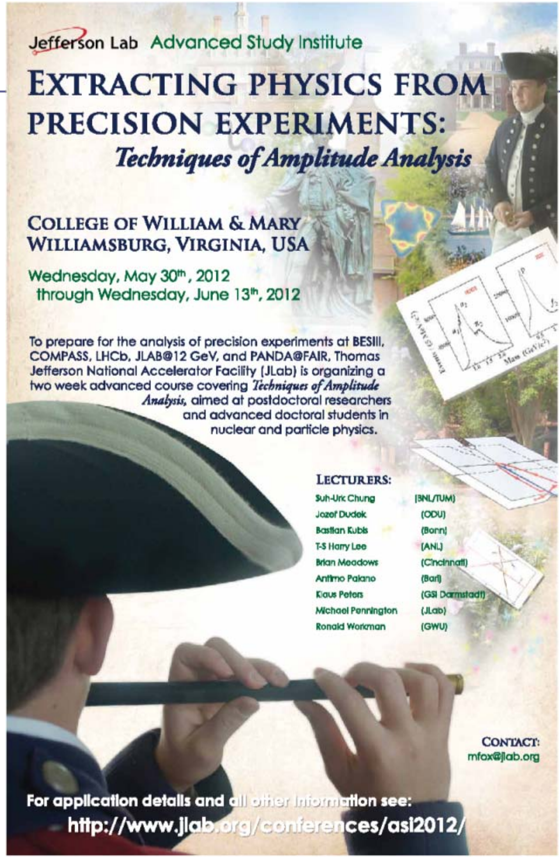
Amplitude Analysis
An Experimentalists View

Lectures at the "Extracting Physics from Precision Experiments Techniques of Amplitude Analysis"



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Williamsburg, June 2012

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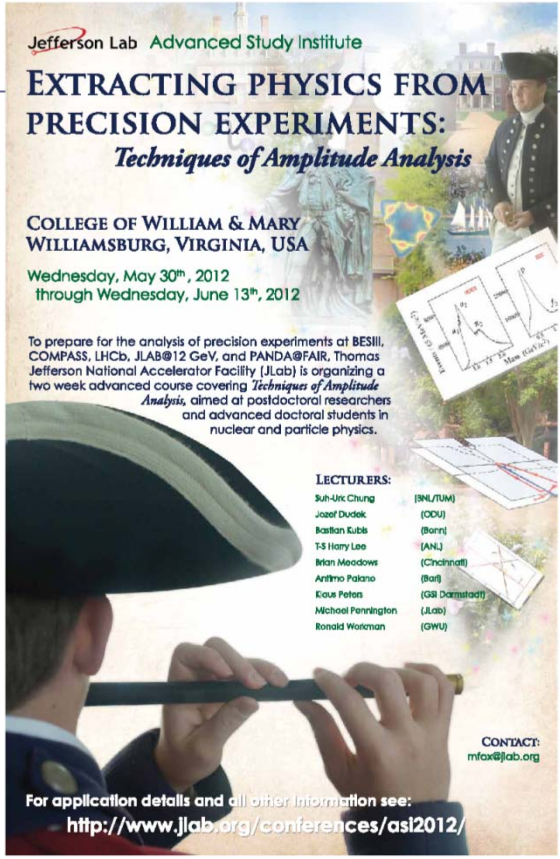
An Experimentalists View

K. Peters



Part III

Spin

Spin-Parity

Basics

Formalisms

Examples

Formalisms – an overview (very limited)

Non-relativistic Tensor formalisms

in non-relativistic (Zemach) or covariant flavor

Fast computation, simple for small *L* and *S*

Spin-projection formalisms

where a quantization axis is chosen and proper rotations are used to define a two-body decay

Efficient formalisms, even large *L* and *S* easy to handle

Relativistic Tensor Formalisms based on Lorentz invariants (Rarita-Schwinger)

where each operator is constructed from Mandelstam variables only

Elegant, but extremely difficult for large *L* and *S*

Zemach Formalism

For particle with spin S

traceless tensor of rank S

Similar for orbital angular momentum L

$$l = 0 \qquad A^0 = 1$$

$$l = 1 \qquad A^1(\vec{q}) = \vec{q}$$

$$l = 2 \qquad A^2(\vec{q}) = \frac{3}{2}\left[\vec{q}\cdot\vec{q}^T - \underbrace{\frac{1}{3}|\vec{q}|^2}_{\text{for tracelessness}}\right]$$

$$\vec{q}\cdot\vec{p}^T = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} q_1p_1 & q_1p_2 & q_1p_3 \\ q_2p_1 & q_2p_2 & q_2p_3 \\ q_3p_1 & q_3p_2 & q_3p_3 \end{pmatrix}$$

with indices

$$l = 0 \qquad A^0 = 1$$

$$l = 1 \qquad A^1_i = q_i$$

$$l = 2 \qquad A^2_{ij} = \frac{3}{2}q_iq_j - \frac{1}{2}|q_i|^2\delta_{ij}$$

Example: Zemach – pp̄ $(0^{-+}) \rightarrow f_2\pi^0$

Construct total spin 0 amplitude

$$
\begin{aligned}
A^0 &= A^2_{f_2\pi^0,ij} A^2_{\pi^+\pi^-,kl} \underbrace{\delta_{ik}\delta_{jl}}_{\text{unpolarized}} \\
&= \sum_{i,j,k,l} A^2_{f_2\pi^0,ij} A^2_{\pi^+\pi^-,kl} \delta_{ik}\delta_{jl} \\
&= \sum_{i,j} A^2_{f_2\pi^0,ij} A^2_{\pi^+\pi^-,ij}
\end{aligned}
$$

$\tau\pi$

$$
A^2_{f_2\pi^0,ij} = \frac{3}{2} p_i p_j - \frac{1}{2} |p_i|^2 \delta_{ij} \qquad A^2_{\pi^+\pi^-,kl} = \frac{3}{2} q_k q_l - \frac{1}{2} |q_l|^2 \delta_{kl}
$$

Example: Zemach – p$\bar{p}$ (0^{-+}) → $f_2\pi^0$

$$A^0 = \left(\frac{3}{2}p_i p_j - \frac{1}{2}|p_i|^2\delta_{ij}\right)\left(\frac{3}{2}q_i q_j - \frac{1}{2}|q_i|^2\delta_{ij}\right)$$

$$= \frac{9}{4}(\vec{q}\cdot\vec{p})^2 - \frac{3}{4}\vec{q}^2\vec{p}^2 - \frac{3}{4}\vec{q}^2\vec{p}^2 + 3\frac{1}{4}|\vec{q}|^2|\vec{p}|^2 = \frac{9}{4}(\vec{q}\cdot\vec{p})^2 - \frac{3}{4}\vec{q}^2\vec{p}^2$$

$$I = \frac{9}{4}\left[(\vec{q}\cdot\vec{p})^2 - \frac{1}{3}\vec{q}^2\vec{p}^2\right]^2 = \frac{9}{4}\left[(qp\cos\vartheta)^2 - \frac{1}{3}q^2p^2\right]^2$$

$$= \frac{9}{4}(\cos^2\vartheta - \frac{1}{3})^2 = P_2^0(\vartheta)^2$$

Angular distribution (Intensity)

The Original Zemach Paper

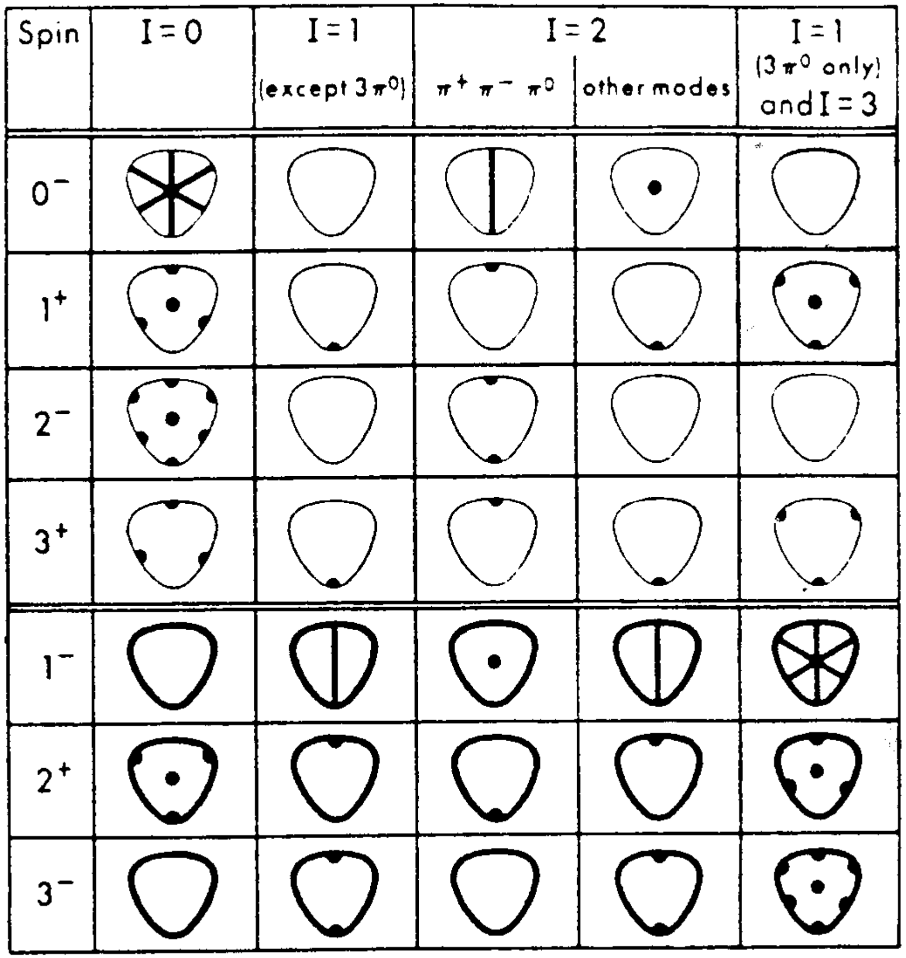


FIG. 2. Regions of the 3π Dalitz plot where the density must vanish because of symmetry requirements are shown in black. The vanishing is of higher order (stronger) where black lines and dots overlap. In each isospin and parity state, the pattern for a spin of $J+$ even integer is identical to the pattern for spin J, provided $J \geqq 2$. (Exception: vanishing at the center is not required for $J \geqq 4$.)

Spin-Projection Formalisms

Differ in choice of quantization axis

Helicity Formalism

parallel to its own direction of motion

Transversity Formalism

the component normal to the scattering plane is used

Canonical (Orbital) Formalism

the component m in the incident z-direction is diagonal

Differ in choice of quantization axis

Helicity Formalism

parallel to its own direction of motion

$$\Psi_\lambda = |\tilde{p}, \lambda\rangle = \widehat{R}(\phi, \theta, -\phi)\widehat{B}(0, 0, p)|m\rangle \equiv \widehat{H}(\tilde{p})|\lambda\rangle$$

Transversity Formalism

the component normal to the scattering plane is used

$$\Psi_\tau = |\tilde{p}, \tau\rangle = \sum_\lambda |\tilde{p}\lambda\rangle \Delta^J_{\lambda\tau} = \widehat{\Delta}\widehat{H}(\tilde{p})\widehat{\Delta}^{-1}|\tau\rangle = \widehat{T}|\tau\rangle$$

Canonical (Orbital) Formalism

the component m in the incident z-direction is diagonal

$$\Psi_m = |\tilde{p}m\rangle = \sum_\lambda |\tilde{p}, \lambda\rangle D^{J\,*}_{\lambda\tau}\widehat{R}(\phi, \theta, -\phi) = \widehat{R}^{-1}(\phi, \theta, -\phi)\widehat{H}(\tilde{p})|m\rangle = \widehat{O}|m\rangle$$

How To Construct a Formalism

Key steps are

Definition of single particle states of given momentum and spin component (momentum-states),

Definition of two-particle momentum-states in the *s*-channel center-of-mass system and of amplitudes between them,

Transformation to states and amplitudes of given total angular momentum (*J*-states),

Symmetry restrictions on the amplitudes,

Derive Formulae for observable quantities.

Generalized Single Particle State

In general all single particle states
are derived from a lorentz transformation
and the rotation of the basic state

$$L|\tilde{p}\xi\rangle = X(L\tilde{p})R(L,\tilde{p})|0\xi\rangle = \sum_{\xi'} |L\tilde{p},\xi'\rangle D^{J}_{\xi'\xi}(r)$$

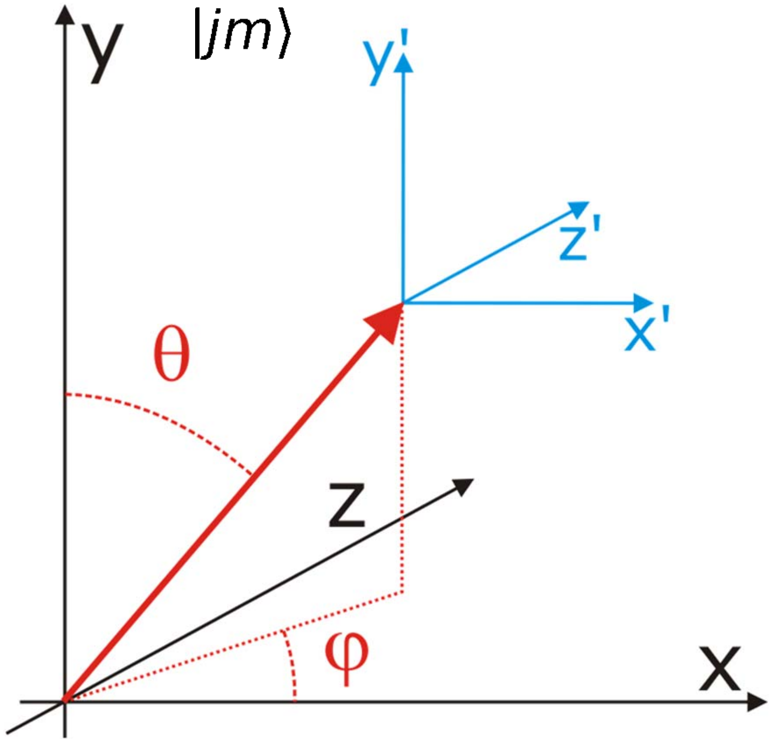
with the Wigner rotation

$$R = X^{-1}(L\tilde{p})LX(\tilde{p})$$

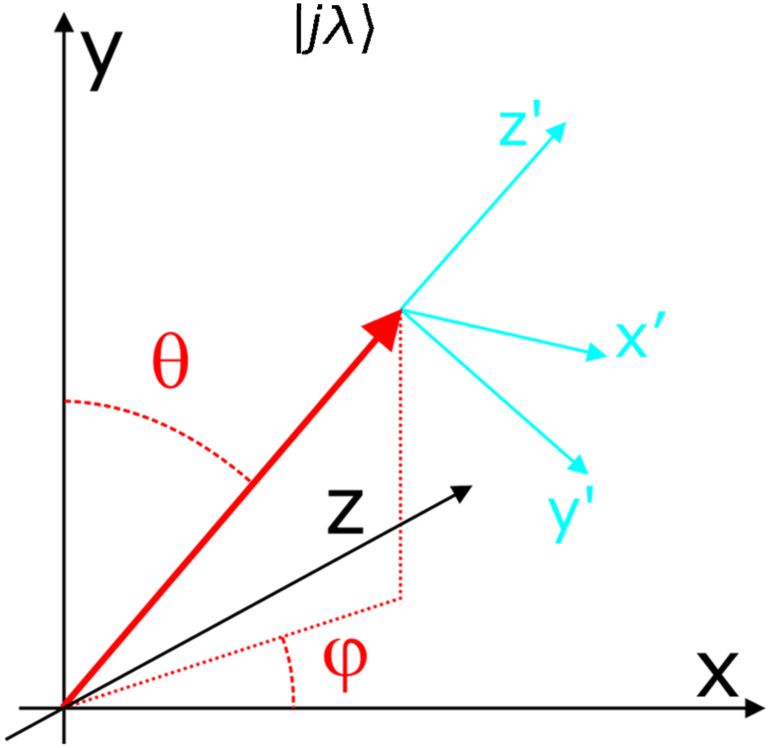
Properties

	Helicity	Transversity	Canonical
property	possibility/simplicity		
partial wave expansion	simple	complicated	complicated
parity conservation	no	yes	yes
crossing relation	no	good	bad
specification of kinematical constraints	no	yes	yes

Rotation of States



Canonical System



Helicity System

Rotations

Single particle states

$$\langle j'm' | jm \rangle = \delta_{jj'}\delta_{mm'}$$

$$1 = \sum_{j,m} | jm \rangle \langle jm |$$

Rotation R

Unitary operator U

$$U\left[R_2 R_1\right] = U\left[R_2\right]U\left[R_1\right]$$

$$U\left[R(\alpha,\beta,\gamma)\right] = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$$

D function represents the rotation in the angular momentum space

$$U\left[R(\alpha,\beta,\gamma)\right] | jm \rangle = \sum_{m'} | jm' \rangle D^{*J}_{m',m}(\alpha,\beta,\gamma)$$

$$D^{*J}_{m',m}(\alpha,\beta,\gamma) = \langle jm' | U\left[R(\alpha,\beta,\gamma)\right] | jm \rangle$$

$$= e^{-im'\alpha} d^{j}_{m'm}(\beta) e^{-im'\gamma}$$

Valid in an inertial system

Relativistic state

$$| p, jm \rangle = U\left[R(\Omega) L_z(\beta) R^{-1}(\Omega)\right] | jm \rangle$$

$$| p, j\lambda \rangle = D^{j}_{m\lambda}(\Omega) | p, jm \rangle$$

Single Particle State

Canonical

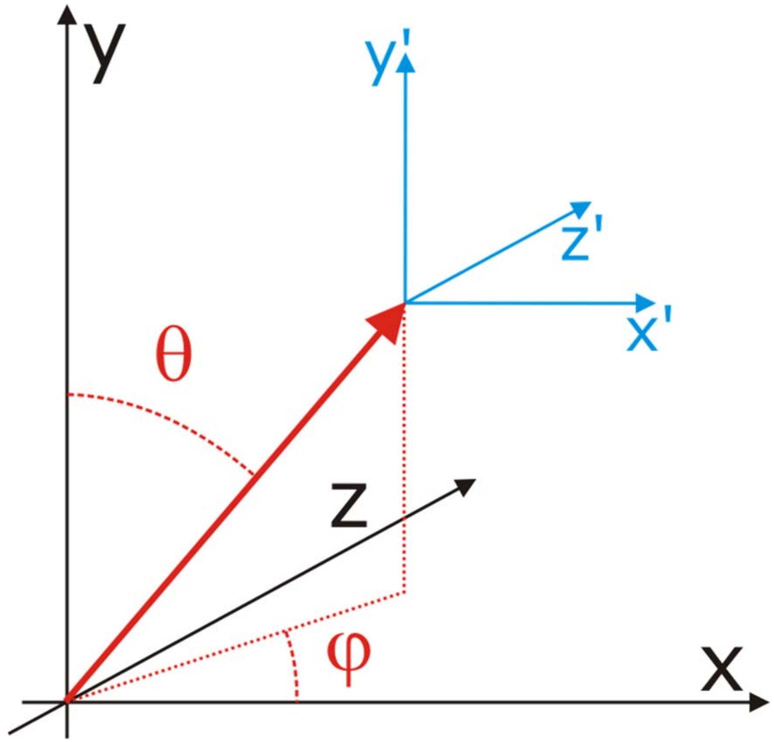
$$|\tilde{p}, jm\rangle \stackrel{def}{=} L\tilde{p}|jm\rangle$$
$$= \hat{R}_0 L_z p \hat{R}_0^{-1}|jm\rangle$$

1) momentum vector is rotated via z-direction.
2) absolute value of the momentum is Lorentz boosted along z
3) z-axis is rotated to the momentum direction

$$\hat{R}_0 = \hat{R}_0(\varphi, \vartheta, 0)$$

$$\vec{e}_{\vec{p}} = \hat{R}_0 \vec{e}_z$$

$$\hat{R}|\vec{p}, m\rangle = \sum_{m'} D^j_{m'm}|\hat{R}p, m\rangle$$



Two-Particle State

Canonical

constructed from two single-particle states (back to back)

$$\kappa = \frac{1}{4\pi}\sqrt{\frac{p_s}{m_J}} = \frac{1}{4\pi}\sqrt{\rho_s}$$

$$|\Omega_s^0, sm_s tm_t\rangle \overset{def}{=} \kappa \left[L \underbrace{\tilde{p}_s}_{(E_s,\tilde{p}_s)} |sm_s\rangle L \underbrace{\tilde{p}_t}_{(E_t,-\tilde{p}_s)} |tm_t\rangle \right]$$

$$|\Omega, Sm_S\rangle = \sum_{m_s,m_t} (sm_s\ tm_t|Sm_S)|\Omega, sm_s tm_t\rangle$$ Couple ***s*** and ***t*** to ***S***

$$|Lm_L Sm_S\rangle = \int d\Omega \quad Y^L_{m_L}(\Omega)|\Omega, Sm_S\rangle$$ Couple ***L*** and ***S*** to ***J***

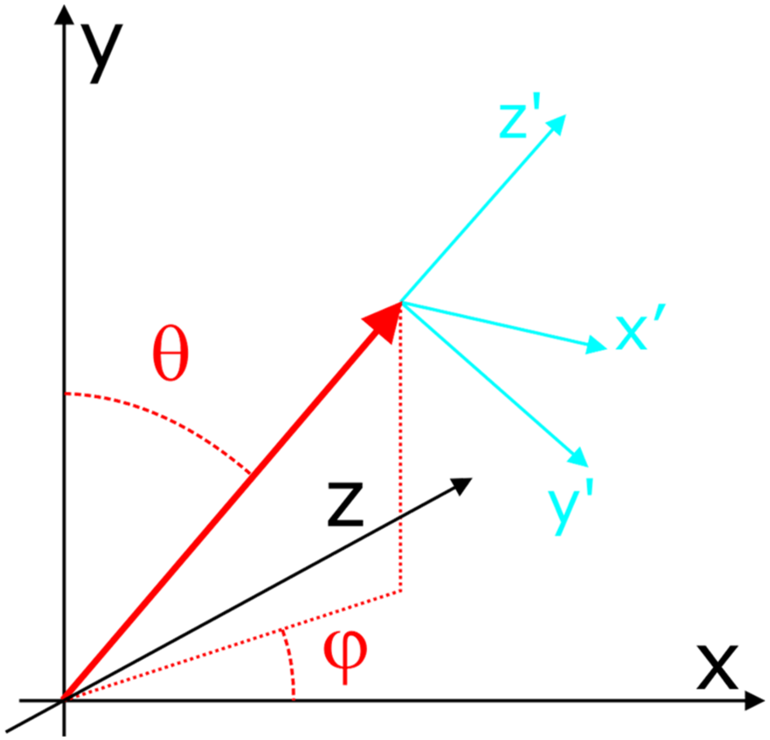
$$|JMLS\rangle = \sum_{m_L,m_S} (Lm_L\ Sm_S|JM)|Lm_L Sm_S\rangle$$

$$= \sum_{m_L,m_S,m_s,m_t} (Lm_L\ Sm_S|JM)(sm_s\ tm_t|Sm_S) \int d\Omega Y^L_{m_L}(\Omega)|\Omega_s^0, sm_s tm_t\rangle$$

al Harmonics

Single Particle State

Helicity



1) *z*-axis is rotated to the momentum direction
2) Lorentz Boost
   Therefore the new *z*-axis, *z*', is parallel to the momentum

$$\hat{R}|\vec{p},\lambda\rangle = |\hat{R}\vec{p},\lambda\rangle$$

$$|\vec{p},\lambda\rangle = \sum_m D^j_{m\lambda}(R_0)|\vec{p},m\rangle$$

Two-Particle State

Helicity

similar procedure

$$|\Omega_s, s\lambda_s t\lambda_t\rangle \overset{def}{=} \kappa \widehat{R}_0 \left[L_z p_s |s\lambda_s\rangle L_z p_t |t\lambda_t\rangle \right]$$

$$= \widehat{R}_0(\Omega_s)|\Omega = (0,0), s\lambda_s t\lambda_t\rangle$$

no recoupling needed

$$|JM\lambda_s\lambda_t\rangle = N_J \int d\Omega \quad D^{J\,*}_{M,\lambda_s-\lambda_t} |\Omega, s\lambda_s t\lambda_t\rangle$$

normalization

$$N_J = \sqrt{\frac{2J+1}{4\pi}}$$

Completeness and Normalization

Canonical

completeness

$$1 = \sum_{J,M,L,S} |JMLS\rangle\langle JMLS|$$

Helicity

completeness

$$1 = \sum_{J,M,\lambda_s,\lambda_t} |JM\lambda_s\lambda_t\rangle\langle JM\lambda_s\lambda_t|$$

normalization

$$\begin{aligned}
\langle \Omega_{s'}, s'm_{s'}t'm_{t'}|\Omega_s, sm_stm_t\rangle &= \delta(\Omega_{s'}-\Omega_s)\delta_{ss'}\delta_{tt'}\delta_{m_sm_{s'}}\delta_{m_tm_{t'}} \\
\langle J'M'L'S'|JMLS\rangle &= \delta_{JJ'}\delta_{MM'}\delta_{LL'}\delta_{SS'}
\end{aligned}$$

normalization

$$\begin{aligned}
\langle \Omega_{s'}, s'\lambda_{s'}t'\lambda_{t'}|\Omega_s, s\lambda_st\lambda_t\rangle &= \delta(\Omega_{s'}-\Omega_s)\delta_{ss'}\delta_{tt'}\delta_{\lambda_s\lambda_{s'}}\delta_{\lambda_t\lambda_{t'}} \\
\langle J'M'\lambda_{s'}\lambda_{t'}|JM\lambda_s\lambda_t\rangle &= \delta_{JJ'}\delta_{MM'}\delta_{\lambda_s\lambda_{s'}}\delta_{\lambda_t\lambda_{t'}}
\end{aligned}$$

Canonical Decay Amplitudes

Canonical

From two-particle state

$$A = \sum_{m_s,m_t} \langle \vec{p}_s, sm_s | \langle -\vec{p}_s, tm_t | \mathcal{M} | JM \rangle$$

$$\langle \Omega_s, sm_s tm_t | JMLS \rangle = \sum_{m_L,m_S} (Lm_L \; Sm_S | JM)(sm_s \; tm_t | Sm_S) Y^L_{m_L}(\Omega_s)$$

$$A^{JM}_{m_s m_t} = \frac{4\pi}{\sqrt{\rho_s}} \langle \Omega_s, sm_s tm_t | \mathcal{M} | JM \rangle$$

$$= \sum_{L,S} \langle \Omega_s, sm_s tm_t | JMLS \rangle \frac{4\pi}{\sqrt{\rho_s}} \langle JMLS | \mathcal{M} | JM \rangle$$

$$\stackrel{def}{=} \sum_{L,S} \sqrt{4\pi} \alpha^J_{LS} \langle \Omega_s, sm_s tm_t | JMLS \rangle$$

$$A^{JM}_{m_s m_t} \stackrel{def}{=} \sum_{L,S,m_L,m_S} \sqrt{4\pi} \alpha^J_{LS} (Lm_L \; Sm_S | JM)(sm_s \; tm_t | Sm_S) Y^L_{m_L}(\Omega_s)$$

LS-Coefficients

$$\alpha^J_{LS} \stackrel{def}{=} \sqrt{\frac{4\pi}{\rho_s}} \langle JMLS | \mathcal{M} | JM \rangle$$

Helicity

From two-particle state

$$A = \sum_{\lambda_s,\lambda_t} \langle \vec{p}_s, s\lambda_s | \langle -\vec{p}_s, t\lambda_t | \mathcal{M} | JM \rangle$$

$$\langle \Omega_s, s\lambda_s t\lambda_t | JM\lambda_{s'}\lambda_{t'} \rangle = N_J D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s)$$

$$\begin{aligned} A^{JM}_{\lambda_s\lambda_t} &= \frac{4\pi}{\rho_s} \langle \Omega_s, s\lambda_s t\lambda_t | \mathcal{M} | JM \rangle \\ &= \sum_{\lambda_{s'},\lambda_{t'}} \langle \Omega_s, s\lambda_s t\lambda_t | JM\lambda_{s'} m_{t'} \rangle \frac{4\pi}{\rho_s} \langle JM\lambda_{s'}\lambda_{t'} | \mathcal{M} | JM \rangle \\ &= \sqrt{\frac{4\pi}{\rho_s}(2J+1)} \langle JM\lambda_s\lambda_t | \mathcal{M} | JM \rangle D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s) \end{aligned}$$

$$A^{JM}_{\lambda_s\lambda_t} = N_J f_{\lambda_s\lambda_t} D^{J\,*}_{M,\lambda_{s'}-\lambda_{t'}}(\Omega_s)$$

Helicity amplitude

$$N_J f_{\lambda_s\lambda_t} = \sqrt{\frac{4\pi}{\rho_s}(2J+1)} \langle JM\lambda_s\lambda_t | \mathcal{M} | JM \rangle$$

Spin Density and Observed # of Events

To finally calculate the intensity
i.e. the number of events
observed

Spin density of the initial state

$$\rho_{MM'} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Sum over all unobserved states

$$I(\vartheta)_{\lambda\lambda'} = \sum_{M,M',\lambda_s\lambda_{s'},\lambda_t\lambda_{t'}} A^{JM}_{\lambda_s\lambda_t}(\varphi,\vartheta)\rho_{MM'}A^{JM'*}_{\lambda_{s'},\lambda_{t'}}(\varphi,\vartheta)$$

taking into account

$$\begin{aligned} \lambda &= \lambda_s - \lambda_t \\ \lambda' &= \lambda_{s'} - \lambda_{t'} \end{aligned}$$

Relations Canonical ⇔ Helicity $f_{\lambda_s\lambda_t} \Leftrightarrow a^J_{LS}$

Recoupling coefficients

Start with

$$\langle JMLS|JM\lambda_s\lambda_t\rangle = \sqrt{\frac{2L+1}{2J+1}}(L0\ S(\lambda_s-\lambda_t)|J(\lambda_s-\lambda_t))$$
$$(s\lambda_s\ t(-\lambda_t)|S(\lambda_s-\lambda_t))$$

Canonical to Helicity

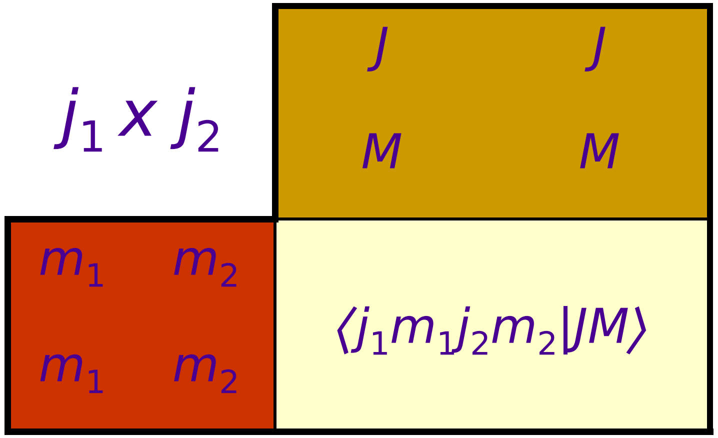
$$N_J f^J_{\lambda_s\lambda_t} = \sum_{L,S}\sqrt{2L+1}(L0\ S(\lambda_s-\lambda_t)|J(\lambda_s-\lambda_t))$$
$$(s\lambda_s\ t(-\lambda_t)|S(\lambda_s-\lambda_t))a^J_{LS}$$

Helicity to Canonical

$$a^J_{LS} = N_J\sum_{\lambda_s,\lambda_t}\frac{\sqrt{2L+1}}{2J+1}(L0\ S(\lambda_s-\lambda_t)|J(\lambda_s-\lambda_t))$$
$$(s\lambda_s\ t(-\lambda_t)|S(\lambda_s-\lambda_t))f^J_{\lambda_s\lambda_t}$$

Clebsch-Gordan Tables

Clebsch-Gordan Coefficients are usually tabled in a graphical form (like in the PDG)



Two cases

- coupling two initial particles with $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ to final system $\langle JM|$

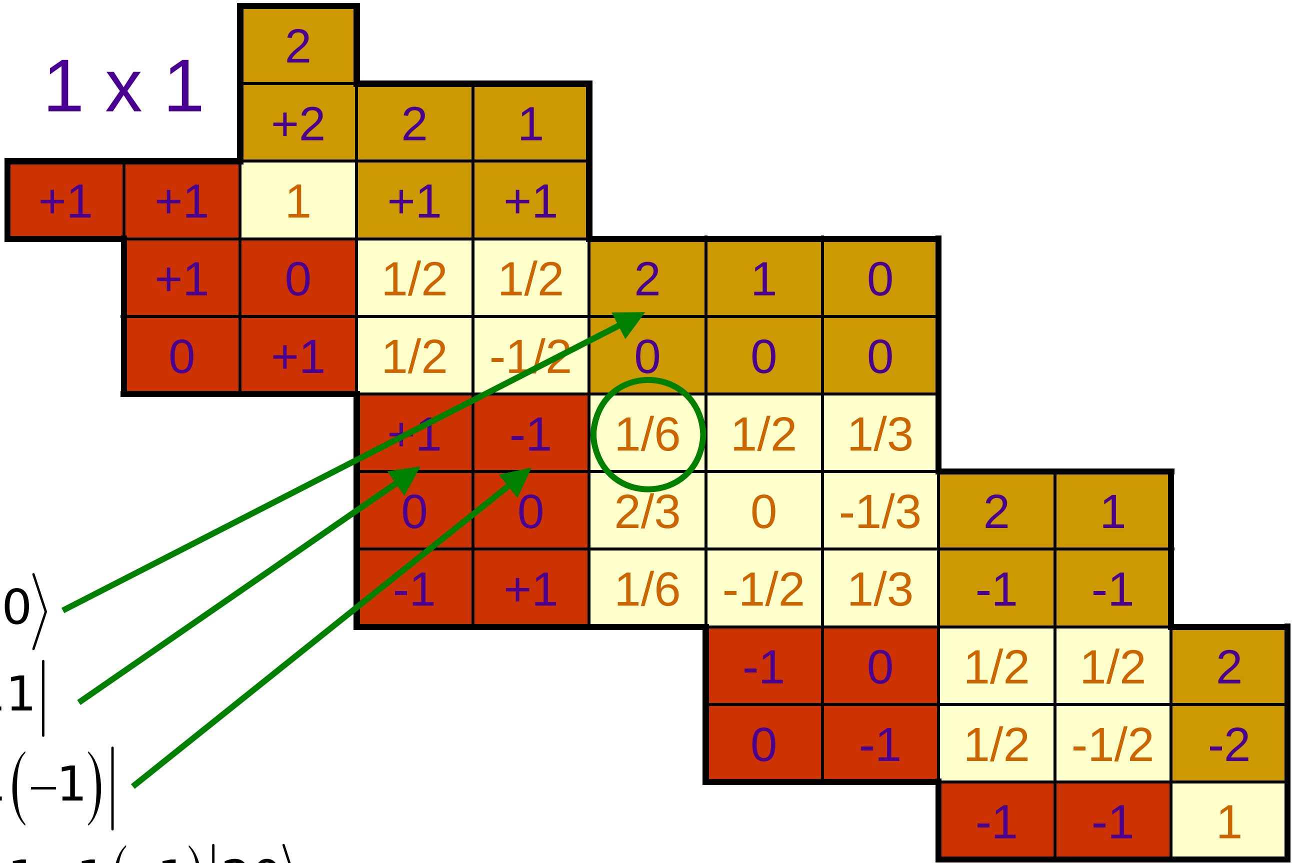
- decay of an initial system $|JM\rangle$ to $\langle j_1 m_1|$ and $\langle j_2 m_2|$

j_1 and j_2 do not explicitly appear in the tables

all values implicitly contain a square root

Minus signs are meant to be used in front of the square root

Using Clebsch-Gordan Tables, Case 1



$$\left|JM\right\rangle = \left|20\right\rangle$$

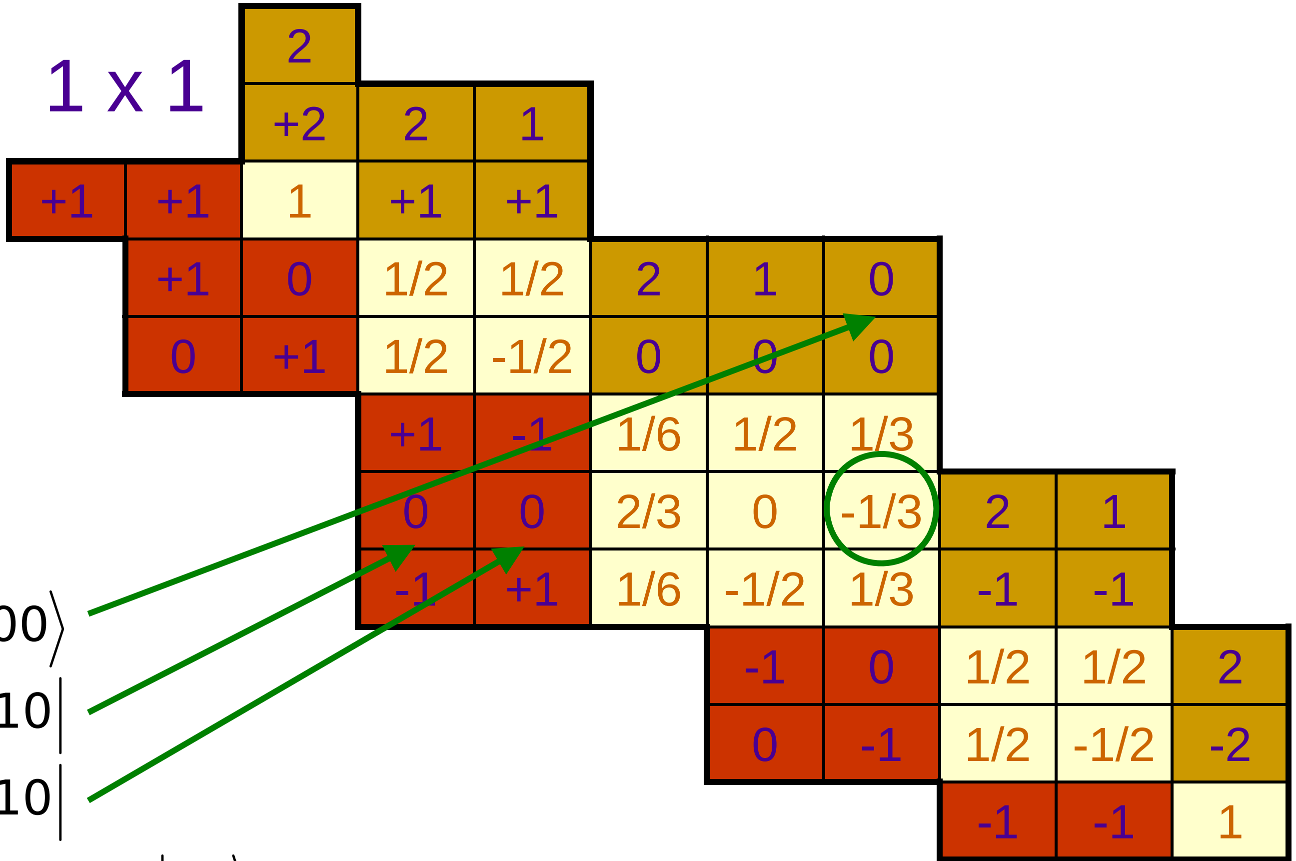
$$\left\langle j_1 m_1\right| = \left\langle 11\right|$$

$$\left\langle j_2 m_2\right| = \left\langle 1(-1)\right|$$

$$\left\langle j_1 m_1 \quad j_2 m_2 \middle| JM\right\rangle = \left\langle 11 \quad 1(-1) \middle| 20\right\rangle$$

$$= \sqrt{1/6}$$

Using Clebsch-Gordan Tables, Case 2



$$\left| JM \right\rangle = \left| 00 \right\rangle$$

$$\left\langle j_1 m_1 \right| = \left\langle 10 \right|$$

$$\left\langle j_2 m_2 \right| = \left\langle 10 \right|$$

$$\left\langle j_1 m_1 \quad j_2 m_2 \middle| JM \right\rangle = \left\langle 10 \quad 10 \middle| 00 \right\rangle$$

$$= -\sqrt{1/3}$$

Parity Transformation and Conservation

Parity transformation

single particle

$$P|\vec{p}, jm\rangle = \eta|\varphi+\pi, \pi-\vartheta, p, jm\rangle$$

$$P|\vec{p}, j\lambda\rangle = \eta e^{-\iota\pi j}|\varphi+\pi, \pi-\vartheta, |\vec{p}|, j-\lambda\rangle$$

two particles

$$P|JMls\rangle = \eta_1\eta_2(-1)^l|JMls\rangle$$

$$|JM\lambda_1\lambda_2\rangle = \sum_{l,s}\sqrt{\frac{2l+1}{2J+1}}(l0\ s\lambda|JM)(s_1\lambda_1\ s_2(-\lambda_2)|s\lambda)|JMls\rangle$$

$$P|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J+s_1+s_2}|JMls\rangle$$

helicity amplitude relations (for P conservation)

$$F^J_{\lambda_1\lambda_2} = \eta\eta_1\eta_2(-1)^{J+s_1+s_2}F^J_{(-\lambda_1)(-\lambda_2)}$$

$$F^J_{\lambda_1\lambda_2} \overset{1\equiv 2}{\equiv} \eta(-1)^J F_{\lambda_2\lambda_1}$$

$f_2 \to \pi\pi$ (Ansatz)

Initial: $f_{2(1270)}$ $I^G(J^{PC}) = 0^+(2^{++})$

Final: π^0 $I^G(J^{PC}) = 1^-(0^{-+})$

Only even angular momenta, since $\eta_f = \eta_\pi^{\,2}(-1)^l$

Total spin $s = 2s_\pi = 0$

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\varphi,\theta)$$

$$\lambda = \lambda_1 - \lambda_2 = 0$$

$$J = 2$$

$$A^{2M}_{00} = N_2 F^2_{00} D^{2*}_{M0}(\varphi,\theta)$$

$$N_2 F^2_{00} = \sqrt{5}\underbrace{\left(20\quad 00\middle|20\right)}_{1}\underbrace{\left(00\quad 00\middle|00\right)}_{1} a_{20} = \sqrt{5}a_{20}$$

$$A^{2M}_{00} = \sqrt{5}a_{20} D^{2*}_{M0}(\varphi,\theta)$$

$f_2 \to \pi\pi$ (Rates)

$$A_{00}^{1M} = \sqrt{5} a_{20} \begin{bmatrix} d^2_{(-2)0}(\theta) e^{-2i\varphi} \\ d^2_{(-1)0}(\theta) e^{-i\varphi} \\ d^2_{00}(\theta) \\ d^2_{10}(\theta) e^{i\varphi} \\ d^2_{20}(\theta) e^{2i\varphi} \end{bmatrix}$$

Amplitude has to be symmetrized because of the final state particles

$$I(\theta) = \sum_{M,M'} A_{00}^{1M} \rho_{MM'} A_{00}^{1M'*}$$

$$\rho = \frac{1}{2J+1} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$d^2_{(\pm 2)0}(\theta) = \frac{\sqrt{6}}{4} \sin^2\theta$$

$$d^2_{(\pm 1)0}(\theta) = -\sqrt{\frac{3}{2}} \sin\theta \cos\theta$$

$$d^2_{00}(\theta) = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$I(\theta) = \left|a_{20}\right|^2 \underbrace{\left(\frac{15}{4}\sin^4\theta + 15\sin^2\theta\cos^2\theta + 5\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)^2\right)}_{15\left(\frac{1}{4}\sin^4\theta + \sin^2\theta\cos^2\theta + \frac{3}{4}\cos^4\theta - \frac{1}{2}\cos^2\theta + \frac{1}{12}\right)}$$

$$= \left|a_{20}\right|^2 = const$$

$\omega \to \pi^0\gamma$ (Ansatz)

Initial:	ω	$I^G(J^{PC}) = 0^-(1^{--})$
Final:	π^0	$I^G(J^{PC}) = 1^-(0^{-+})$
	γ	$I^G(J^{PC}) = 0(1^{--})$

Only odd angular momenta, since $\eta_\omega = \eta_\pi \eta_\gamma (-1)^l$

Only photon contributes to total spin $s = s_\pi + s_\gamma$

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\varphi,\theta)$$

$$\lambda = \lambda_1 - \lambda_2 = \lambda_\gamma = \lambda_1$$

$$J = 1$$

$$A^{1M}_{\lambda 0} = N_1 F^1_{\lambda 0} D^{1*}_{M\lambda}(\varphi,\theta)$$

$$N_1 F^1_{\lambda 0} = \sqrt{3}\underbrace{\left(10 \quad 1\lambda \middle| J\lambda\right)}_{-\frac{\lambda}{\sqrt{2}}}\underbrace{\left(1\lambda \quad 00 \middle| 1\lambda\right)}_{1} a_{11} = -\lambda\sqrt{\frac{3}{2}}a_{11}$$

$$A^{1M}_{\lambda 0} = -\lambda\sqrt{\frac{3}{2}}a_{11} D^{1*}_{M\lambda}(\varphi,\theta)$$

$\lambda_\gamma=\pm1$ do not interfere, $\lambda_\gamma=0$ does not exist for real photons

Rate depends on density matrix

Choose uniform density matrix as an example

$$A^{1M}_{\lambda 0} = -\sqrt{\frac{3}{2}}\begin{bmatrix} -d^1_{(-1)(-1)}(\theta)e^{-i\varphi} & 0 & d^1_{(-1)1}(\theta)e^{-i\varphi} \\ -d^1_{0(-1)}(\theta) & 0 & d^1_{01}(\theta) \\ -d^1_{1(-1)}(\theta)e^{i\varphi} & 0 & d^1_{11}(\theta)e^{i\varphi} \end{bmatrix}$$

$$I(\theta) = \sum_{M,M',\lambda,\lambda'} A^{1M}_{\lambda 0}\rho_{MM'}A^{1M'*}_{\lambda' 0}\delta_{\lambda\lambda'}$$

$$\rho = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$d^1_{1(\pm1)}(\theta) = \frac{1\pm\cos\theta}{2} = d^1_{(\mp1)1}(\theta)$$

$$d^1_{10}(\theta) = -d^1_{01}(\theta) = d^1_{0(-1)}(\theta) = \frac{\sin\theta}{\sqrt{2}}$$

$$I = \frac{1}{2}\left[2\left(\frac{1-\cos\theta}{2}\right)^2 + 2\left(\frac{1+\cos\theta}{2}\right)^2 + \not{2}\,\frac{\sin^2\theta}{\not{2}}\right]$$

$$= \frac{1}{2}\left[1+\cos^2\theta+\sin^2\theta\right] = 1 = \boxed{const}$$

$f_{0,2} \to \gamma\gamma$ (Ansatz)

Initial: $f_{0,2}$ $I^G(J^{PC}) = 0^+(0,2^{++})$

Final: γ $I^G(J^{PC}) = 0(1^{--})$

Only even angular momenta, since $\eta_f = \eta_\gamma^2(-1)^l$

Total spin $s = 2s_\gamma = 2$, $l=0,2$ (f_0), $l=0,2,4$ (f_2)

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\varphi,\theta)$$

$$\lambda = \lambda_1 - \lambda_2$$

$$J = 0$$

$$A^{00}_{\lambda_1\lambda_2} = N_1 F^0_{\lambda_1\lambda_2} D^{0*}_{0\lambda}(\varphi,\theta)$$

$$\begin{aligned} N_0 F^0_{\lambda_1\lambda_2} &= \sum_{ls} \left(l0 \quad s\lambda \middle| J\lambda\right)\left(s_1\lambda_1 \quad s_2(-\lambda_2)\middle| s\lambda\right) a_{ls} \\ &= \sqrt{1} a_{00} \left(00 \quad 00 \middle| 00\right)\left(1\lambda_1 \quad 1(-\lambda_2)\middle| 0\lambda\right) \\ &\quad + \sqrt{5} a_{22} \left(20 \quad 20 \middle| 00\right)\left(1\lambda_1 \quad 1(-\lambda_2)\middle| 2\lambda\right) \\ &= \sqrt{\frac{1}{3}} a_{00} + \sqrt{\frac{1}{6}} a_{22} \end{aligned}$$

$f_{0,2} \to \gamma\gamma$ (cont'd)

$$
\begin{aligned}
A^{00}_{\lambda_1\lambda_1} &= N_1 F^{0}_{\lambda_1\lambda_1} D^{0*}_{00}\left(\varphi,\theta\right) \\
&= \left[\sqrt{\frac{1}{3}}a_{00} + \sqrt{\frac{1}{6}}a_{22}\right] \underbrace{D^{0*}_{00}\left(\varphi,\theta\right)}_{const} \\
&= const
\end{aligned}
$$

Ratio between a_{00} and a_{22} is not measurable

Problem even worse for J=2

$$
\begin{aligned}
J &= 2 \\
A^{2M}_{\lambda_1\lambda_2} &= N_2 F^{2}_{\lambda_1\lambda_2} D^{2*}_{M\lambda}\left(\varphi,\theta\right) \\
N_2 F^{2}_{\lambda_1\lambda_2} &= \sum_{ls} \left(l0 \quad s\lambda \middle| J\lambda\right)\left(s_1\lambda_1 \quad s_2\left(-\lambda_2\right) \middle| s\lambda\right) a_{ls} \\
&= \sqrt{5}a_{20}\left(20 \quad 00 \middle| 2\lambda\right)\left(1\lambda_1 \quad 1\left(-\lambda_2\right) \middle| 00\right) \\
&\quad + \sqrt{5}a_{22}\left(20 \quad 2\lambda \middle| 2\lambda\right)\left(1\lambda_1 \quad 1\left(-\lambda_2\right) \middle| 2\lambda\right) \\
&\quad + \sqrt{9}a_{42}\left(40 \quad 2\lambda \middle| 2\lambda\right)\left(1\lambda_1 \quad 1\left(-\lambda_2\right) \middle| 2\lambda\right)
\end{aligned}
$$

$f_{0,2} \to \gamma\gamma$ (cont'd)

Usual assumption $J=\lambda=2$

$$N_2 F^2_{1(-1)} = \sum_{ls} \left(l0 \quad s2 \middle| 2\right)\left(s_1 1 \quad s_2 1 \middle| s2\right) a_{ls}$$

$$= +\sqrt{5} a_{22} \underbrace{\left(20 \quad 22 \middle| 22\right)}_{\sqrt{\frac{2}{7}}} \underbrace{\left(11 \quad 11 \middle| 22\right)}_{1}$$

$$+\sqrt{9} a_{42} \underbrace{\left(40 \quad 22 \middle| 22\right)}_{t.b.d.} \underbrace{\left(11 \quad 11 \middle| 22\right)}_{1}$$

Symmetrization

$$A^{2M} = N_2 \left(F^2_{1(-1)} + F^2_{(-1)1}\right) D^{2*}_{M2}(\varphi,\theta)$$

$$= N_2' D^{2*}_{M2}(\varphi,\theta)$$

Comparison

$$A^{00} = N_0'$$

p̄p (2^{++}) → $\pi\pi$

Proton antiproton in flight into two pseudo scalars

Initial: p̄p $J,M=0,\pm 1$

Final: π $I^G(J^{PC}) = 1^-(0^{-+})$

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\varphi,\theta)$$

$$\lambda = \lambda_1 - \lambda_2 = 0$$

$$J = l$$

$$A^{JM}_{00} = N_J F^J_{00} D^{J*}_{M0}(\varphi,\theta)$$

$$N_J F^J_{00} = \sum_l \sqrt{2J+1}\,\underbrace{\left(l0 \quad 00 \middle| J0\right)}_{\delta_{lJ}}\underbrace{\left(00 \quad 00 \middle| 00\right)}_{1} a_{l0} = \sqrt{2J+1}\,a_{J0}$$

$$A^{JM}_{00} = \sqrt{2J+1}\,a_{J0} D^{J*}_{M0}(\varphi,\theta) = \sqrt{2J+1}\,a_{J0} d^{J*}_{M0}(\theta) e^{-iM\varphi}$$

Problem: *d*-functions are not orthogonal, if φ is not observed
ambiguities remain in the amplitude – polarization is needed

$\bar{p}p \rightarrow \pi^0\omega$

Two step process

First step $p\bar{p} \rightarrow \pi^0\omega$ - Second step $\omega \rightarrow \pi^0\gamma$

Combine the amplitudes

$$
\begin{aligned}
A^{JM}_{\lambda_\omega\lambda_\gamma}\left(\Omega_\omega,\Omega_\gamma\right) &= A^{1\lambda_\omega}_{\lambda_\gamma 0}\left(\Omega_\gamma\right)A^{JM}_{\lambda_\omega 0}\left(\Omega_\omega\right) \\
&= N_{\omega,1}F^{1}_{\lambda_\gamma 0}D^{J*}_{\lambda_\omega\lambda_\gamma}\left(\Omega_\gamma\right)N_{p\bar{p},J}F^{J}_{\lambda_\omega 0}D^{J*}_{M\lambda_\omega}\left(\Omega_\omega\right) \\
&= -\lambda\sqrt{\frac{3}{2}}a_{\omega,11}D^{1*}_{\lambda_\omega\lambda_\gamma}\left(\Omega_\gamma\right)\sum_{l}\sqrt{2l+1}\left(l0 \quad 1\lambda_\omega \middle| J\lambda_\omega\right)a_{p\bar{p},l1}D^{J*}_{M\lambda_\omega}\left(\Omega_\omega\right) \\
&= -\lambda\sqrt{\frac{3}{2}}a_{\omega,11}D^{1*}_{\lambda_\omega\lambda_\gamma}\left(\Omega_\gamma\right)D^{J*}_{M\lambda_\omega}\left(\Omega_\omega\right)\sum_{l}\sqrt{2l+1}\left(l0 \quad 1\lambda_\omega \middle| J\lambda_\omega\right)a_{p\bar{p},l1}
\end{aligned}
$$

helicity constant $a_{\omega,11}$ factorizes and is unimportant for angular distributions

Initial:	$\bar{p}p$	$I^G(J^{PC}) = 1^-(0^{-+})$
Final:	$f_{2(1270)}$	$I^G(J^{PC}) = 0^+(2^{++})$
	π^0	$I^G(J^{PC}) = 1^-(0^{-+})$

is only possible from $L=2$

Ansatz

$$A^{JM}_{\lambda_1\lambda_2} = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\Omega)$$

$$\lambda = \lambda_1 - \lambda_2 = 0$$

$$J_{p\bar{p}} = 0$$

$$J_{f_2} = 2$$

$$A^{00}_{00}(\Omega_{f_2})\,A^{20}_{00}(\Omega_\pi) = N_{p\bar{p},0}F^0_{00}D^{0*}_{00}(\Omega_{f_2})\,N_{f_2,2}F^2_{00}D^{2*}_{00}(\Omega_\pi)$$

$$N_{p\bar{p},0}F^0_{00} = \sqrt{1}\,\underbrace{(20\ \ 20|00)}_{\sqrt{\frac{1}{5}}}\,\underbrace{(20\ \ 00|20)}_{1}\,a_{p\bar{p},22}$$

$$N_{f_2,2}F^2_{00} = \sqrt{5}\,\underbrace{(20\ \ 00|20)}_{1}\,\underbrace{(00\ \ 00|00)}_{1}\,a_{f_2,20}$$

$$A^{00}_{00}(\Omega_{f_2})\,A^{20}_{00}(\Omega_\pi) = \sqrt{5}a_{p\bar{p},22}a_{f_2,20}D^{2*}_{00}(\Omega_\pi) = \sqrt{5}a_{p\bar{p},22}a_{f_2,20}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$I(\cos\theta) = 5\left|a_{p\bar{p},22}a_{f_2,20}\right|^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)^2$$

Flat angular distributions

General rules for spin 0

initial state has spin 0

$0 \rightarrow \text{any}$

both final state particles have spin 0

$J \rightarrow 0+0$

Special rules for isotropic density matrix and unobserved azimuth angle

one final state particle has spin 0 and the second carries the same spin as the initial state

$J \rightarrow J+0$

Moments Analysis

Consider reaction

$$\pi^- + \mathrm{p} \to M^0 + \mathrm{n} \qquad \text{and} \qquad M^0 \to a + b$$

Total differential cross section

$$I(t, M, \vartheta, \varphi) = \frac{\partial^4 \sigma}{\partial t \partial M \partial \cos\vartheta \partial \varphi} = \frac{1}{2} \sum_{\lambda_\mathrm{p}, \lambda_\mathrm{n}} |H_{\lambda_\mathrm{p}\lambda_\mathrm{n}}(t, M, \vartheta, \varphi)|^2$$

expand H

$$\sqrt{4\pi} H_{\lambda_\mathrm{p}\lambda_\mathrm{n}}(t, M, \vartheta, \varphi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sqrt{2j+1} H^j_{\lambda_\mathrm{p}\lambda_\mathrm{n},m} d^j_{m0}(\vartheta) e^{\imath m \varphi}$$

leading to

$$\begin{aligned} 4\pi I(\vartheta, \varphi) &= \frac{1}{2} \sum_{\lambda_\mathrm{p}, \lambda_\mathrm{n}} \sum_{j_2, m_2} \sum_{j_1, m_1} \sqrt{2j_1+1} \sqrt{2j_2+1} e^{\imath(m_1 - m_2)\varphi} \\ &\quad \times H^{j_1 *}_{\lambda_\mathrm{p}, \lambda_\mathrm{n}, m_1} H^{j_2}_{\lambda_\mathrm{p}, \lambda_\mathrm{n}, m_2} d^{j_1}_{m_1 0}(\vartheta) d^{j_2}_{m_2 0}(\vartheta) \end{aligned}$$

Moments Analysis cont‘d

Define now a density tensor

$$\rho^{j_1 j_2}_{m_1 m_2} = \frac{1}{2N} \sum_{\lambda_\mathrm{p},\lambda_\mathrm{n}} H^{j_1\,*}_{\lambda_\mathrm{p},\lambda_\mathrm{n},m_1} H^{j_2}_{\lambda_\mathrm{p},\lambda_\mathrm{n},m_2}$$

the d-function products

can be expanded in spherical harmonics

and the density matrix gets absorbed in a spherical moment

$$I(t, M, \vartheta, \varphi) = N \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle Y_l^m \rangle Y_l^m(\vartheta, \varphi)$$

$$I_{prod}(t, M, \vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} t_l^m Y_l^m(\vartheta, \varphi)$$

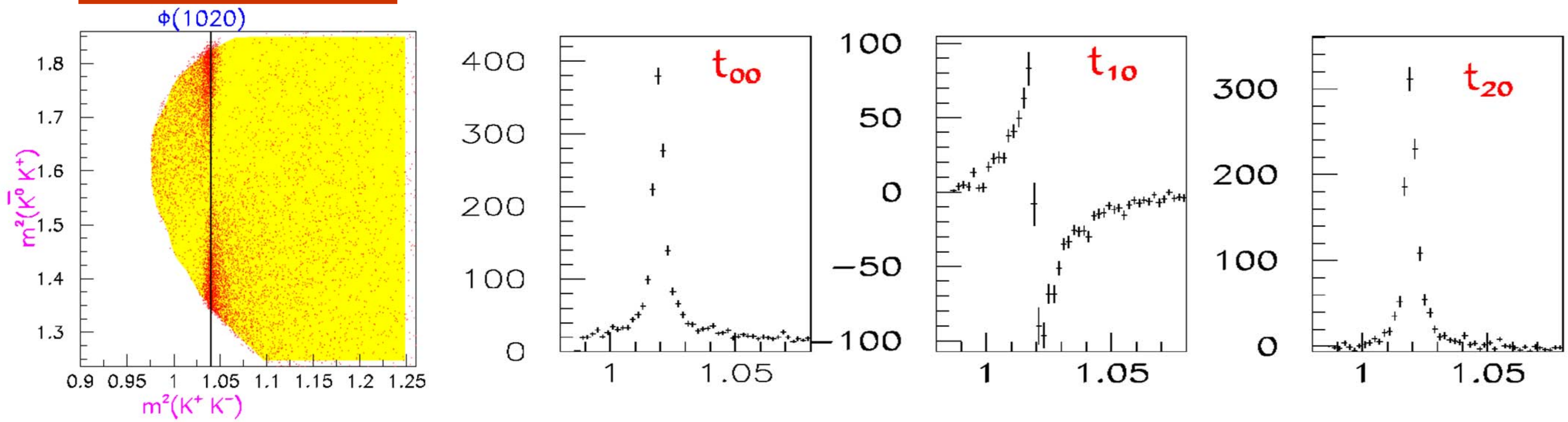
Example: Where to start in Dalitz plot anlysis

Sometimes a moment-analysis can help to find important contributions

best suited if no crossing bands occur

$$t\left(LM\right) = \left\langle D^{L}_{M0}\left(\varphi,\theta,0\right)\right\rangle$$
$$= \int I\left(\Omega\right) D^{L}_{M0}\left(\varphi,\theta,0\right) d\Omega$$

$D^0 \rightarrow K_S K^+ K^-$



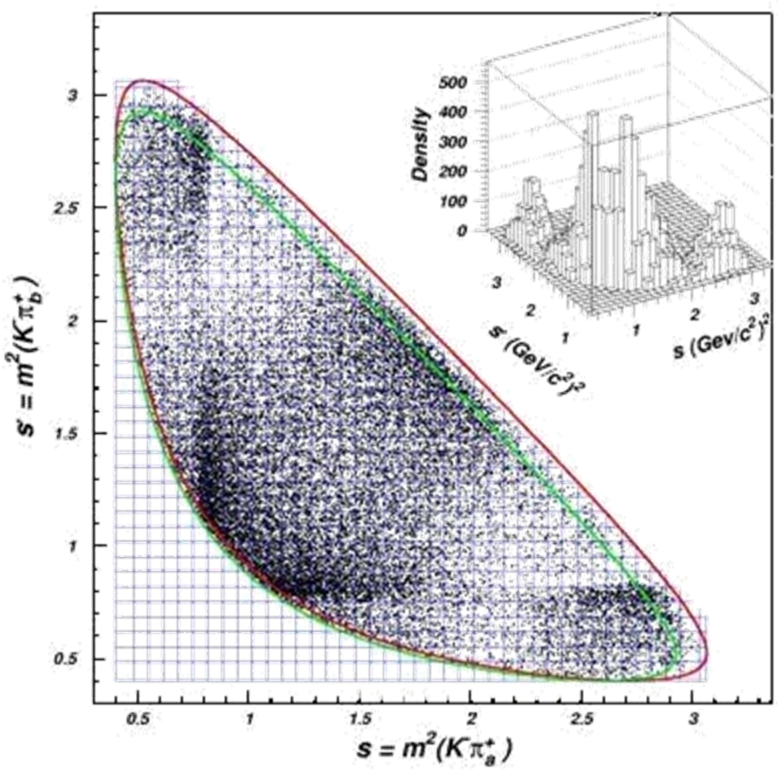
E791: $D^+ \rightarrow K^- \pi^+ \pi^+$ Dalitz

Striking K^*(892) bands

asymmetry implies strong S-wave interference (in $K\pi$)

Dalitz plot analysis as an Interferometer

Model-independent
analysis by using
interference to fix the S-wave



Recipe

Create slices in $m^2(K^-\pi^+)$

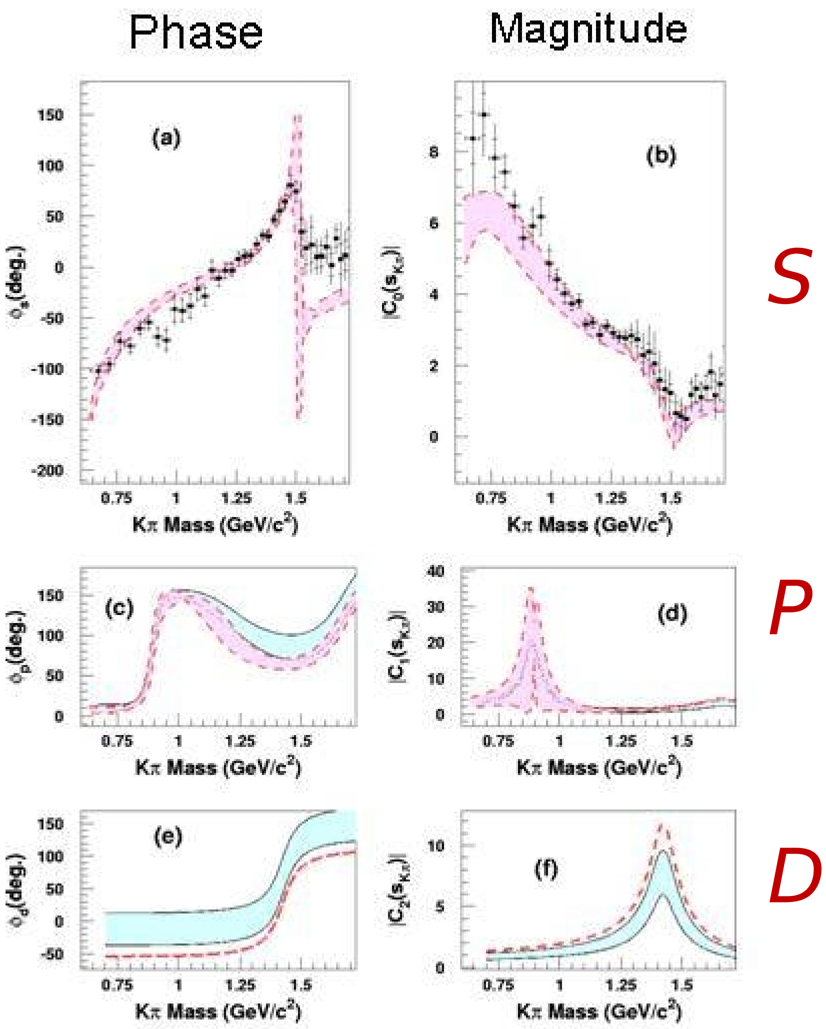
S-wave is than a binned function with parameters c_k and γ_k

$$S = c_k e^{i\gamma_k}$$

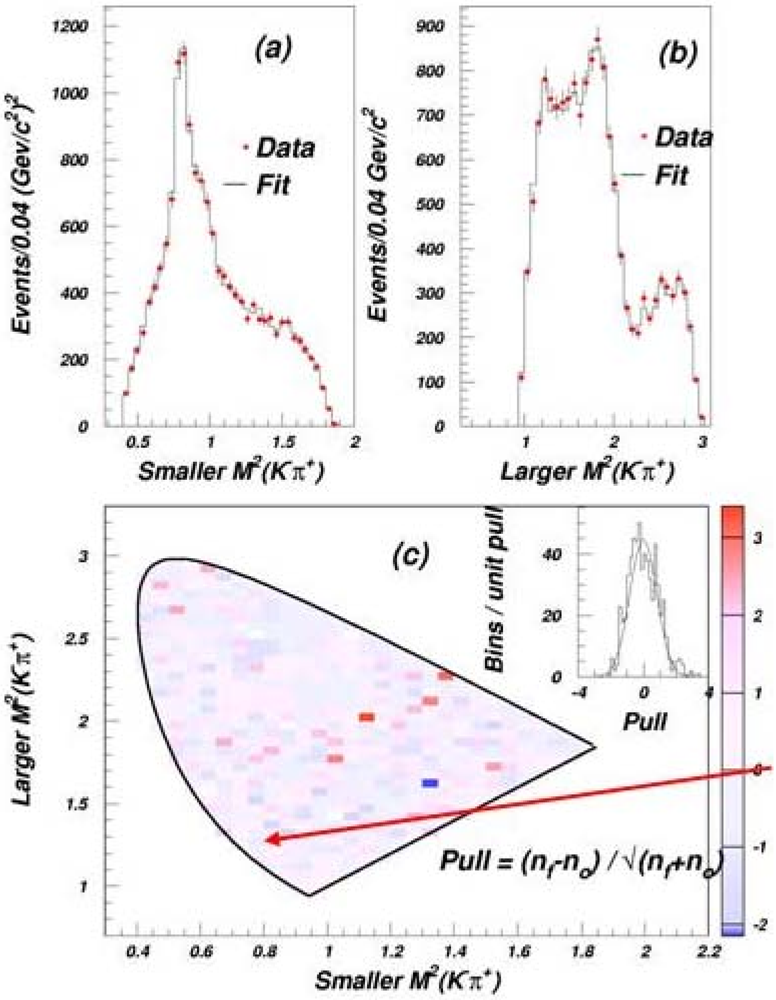
Model well known
P- and *D*-wave (K^*, K_1 and K^*_2)

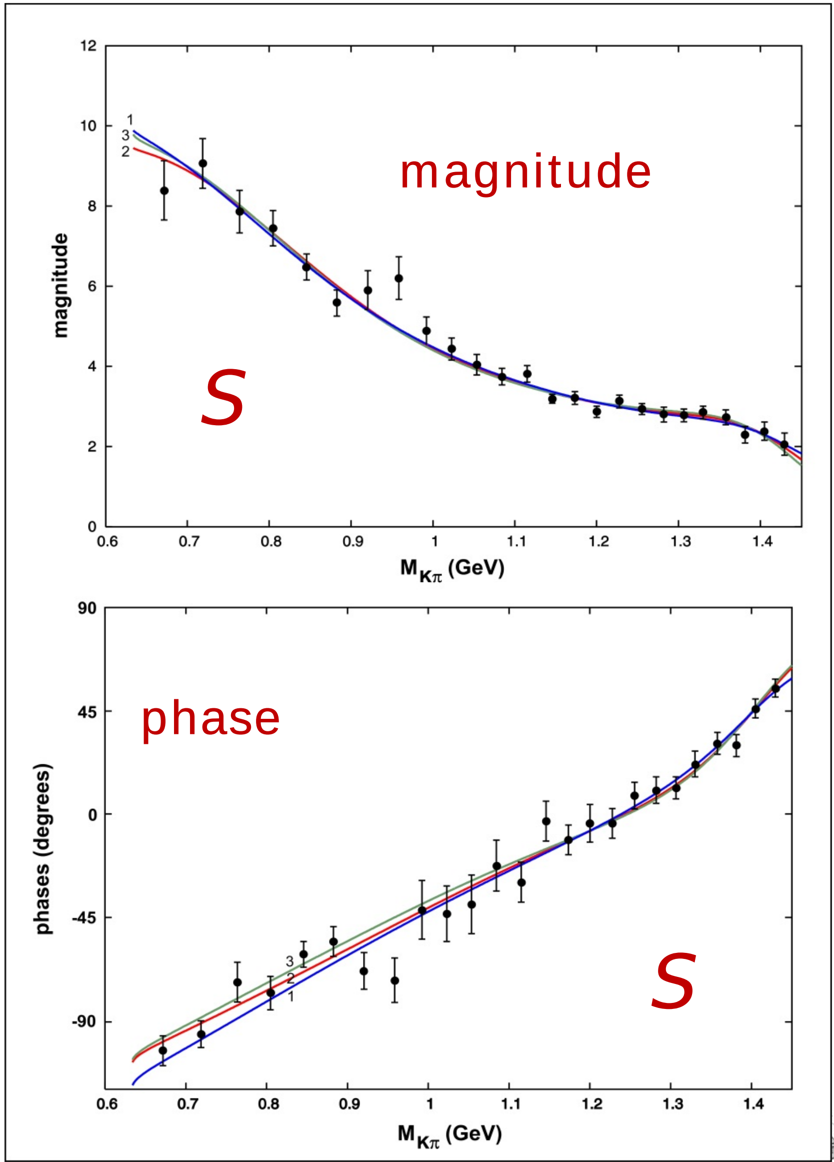
add form factors and
put this into the fit

main uncertainty from K^* and K_1



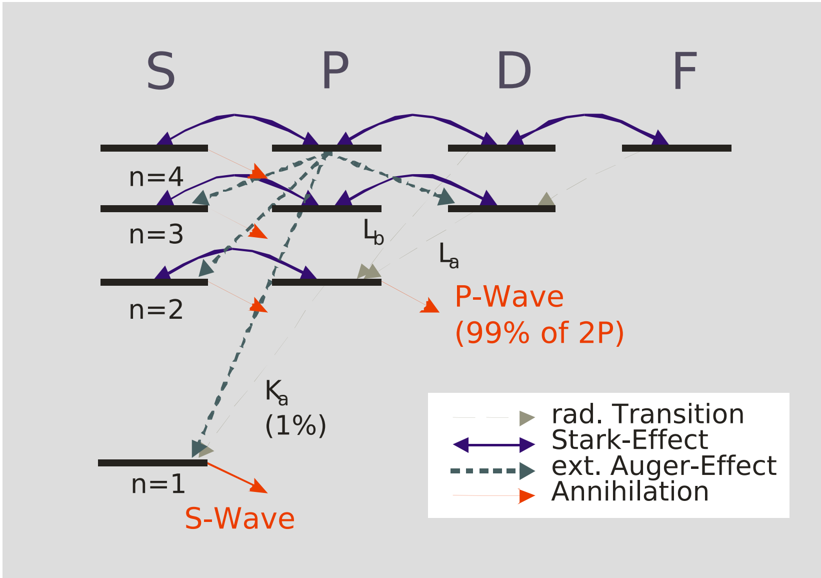
Comparison with Data





Atomic initial system

- formation at high *n, l* (*n*~30)
- slow radiative transitions
- de-excitation through collisions (Auger effect)
- Stark mixing of *l*-levels (Day, Snow, Sucher, 1960)



Advantages

- J^{PC} varies with target density
- isospin varies with n (d) or p target
- incoherent initial states
- unambiguous PWA possible

Disadvantages

- phase space very limited
- small kaon yield

p̄p Initial States @ Rest

Quantumnumbers

$G=(-1)^{I+L+S}$

$P=(-1)^{L+1}$

$C=(-1)^{L+S}$ $CP=(-1)^{2L+S+1}$

I=0

$$|i\rangle = 1/\sqrt{2}\left(|p\bar{p}\rangle + |n\bar{n}\rangle\right)$$

I=1

$$|i\rangle = 1/\sqrt{2}\left(|p\bar{p}\rangle - |n\bar{n}\rangle\right)$$

	J^{PC}		I^G	L	S
$^1S_0$	0^{-+}	pseudo scalar	$1^-;0^+$	0	0
$^3S_1$	1^{--}	vector	$1^+;0^-$	0	1
$^1P_1$	1^{+-}	axial vector	$1^+;0^-$	1	0
$^3P_0$	0^{++}	scalar	$1^-;0^+$	1	1
$^3P_1$	1^{++}	axial vector	$1^-;0^+$	1	1
$^3P_2$	2^{++}	tensor	$1^-;0^+$	1	1

Proton-Antiproton Annihilation in Flight

Annihilation in flight

- scattering process:
  no well defined initial state
- maximum angular momentum
  rises with energy

Advantages

- larger phase space
- formation experiments

Disadvantages

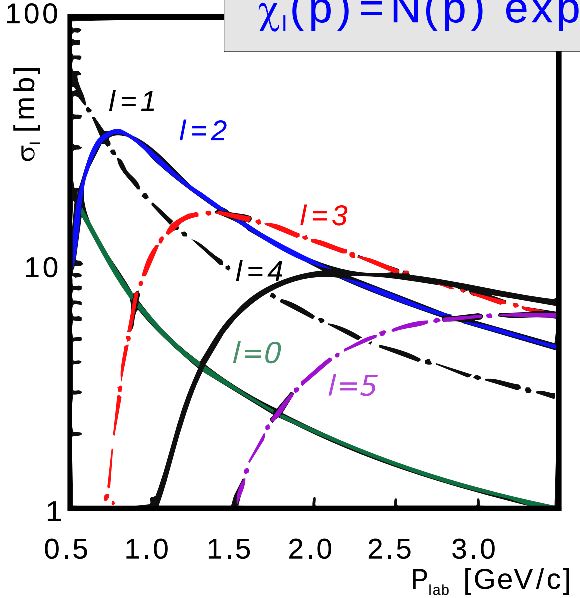
- many waves interfere
  with each other
- many waves due
  to large phase space

$\sigma_{ann} = \Sigma_l \sigma_l$

$\sigma_l(p) = (2l+1)\ \pi\ [1-\exp(-\chi_l(p))]\ /\ p^2$

$\chi_l(p) = N(p)\ \exp(-3l(l+1)/4p^2R^2)$

with $R^2 = \langle r^2 \rangle$ (Baryon)



ang. mom. l
$l \sim p_{cms}/0.2$ GeV/c

p̄p helicity amplitude

$$H^J_{\nu_1\nu_2} = \sum_{L,S} \frac{\sqrt{2L+1}}{\sqrt{2J+1}} \left(L0S\nu \middle| J\nu\right) \left(s_1\nu_1 s_2 -\nu_2 \middle| S\nu\right) \left(JMLS \middle| \mathrm{M} \middle| JM\right)$$

CP transform

only H_{++} and H_{+-} exist

$CP=(-1)^{2L+S+1}$

$$H^J_{\nu_1\nu_2} = \eta_J \left(-1\right)^J H^J_{-\nu_2(-\nu_1)}$$

S and CP directly correlated

CP conserved in strong int.

singlet and triplet decoupled

C-Invariance

$H_{++}=0$ if $L+S+J$ odd

CP-Invariance

$H_{+-}=0$ if $S=0$ and/or $J=0$

C transform

L and P directly correlated

C conserved in strong int.

(if total charge is $q=0$)

odd and even L decouples

4 incoherent sets of coherent amplitudes

Scattering Amplitudes in p̄p in Flight (II)

Singlett even L	J^{PC}	L	S	H_{++}	H_{+-}
$^1S_0$	0^{-+}	0	0	Yes	No
$^1D_2$	2^{-+}	2	0	Yes	No
$^1G_4$	4^{-+}	4	0	Yes	No

Triplett even L	J^{PC}	L	S	H_{++}	H_{+-}
$^3S_1$	1^{--}	0	1	Yes	Yes
$^3D_1$	1^{--}	2	1	Yes	Yes
$^3D_2$	2^{--}	2	1	Yes	Yes
$^3D_3$	3^{--}	2	1	Yes	Yes

Singlett odd L	J^{PC}	L	S	H_{++}	H_{+-}
$^1P_1$	1^{+-}	1	0	Yes	No
$^1F_3$	3^{+-}	3	0	Yes	No
$^1G_5$	5^{+-}	5	0	Yes	No

Triplett odd L	J^{PC}	L	S	H_{++}	H_{+-}
$^3P_0$	0^{++}	1	1	Yes	No
$^3P_1$	1^{++}	1	1	No	Yes
$^3P_2$	2^{++}	1	1	Yes	Yes
$^3F_2$	2^{++}	3	1	Yes	No
$^3F_3$	3^{++}	3	1	No	Yes
$^3F_4$	4^{++}	3	1	Yes	Yes

Tensors revisited

The Zemach amplitudes are only valid in the rest frame of the resonance.

- Thus they are not covariant
- Retain covariance by adding the time component and use 4-vectors
- Behavior under spatial rotations dictates that the time component of the decay momentum vanishes in the rest frame
- This condition is called Rarita Schwinger condition

For Spin-1 it reads $Su = S_\mu p^\mu = 0$

- with $p = (p_a+p_b)/m$ the 4-momentum of the resonance

- The vector $S_{\mu\mu}$ is orthogonal to the timelike vector p_μ and is therefore spacelike, thus $S^2 < 0$

Covariant Tensor Formalism

The most simple spin-1 covariant tensor with above properties is

$S_\mu = q_\mu - (qp)p_\mu$

with $q = (p_a - p_b)$

The negative norm is assured by the equation

$$S^2 = q^2 - (qp)^2 = -|q_R|^2$$

where q_R is the break-up three-momentum

the general approach and recipe is a lecture of its own and you should refer to the primary literature for more information

to calculate the amplitudes and intensities you may use qft++

qft++ = Numerical Object Oriented Quantum Field Theory

(by Mike Williams, Carnegie Mellon Univ.)

Calculation of the matrices, tensors, spinors, angular momentum tensors etc. with C++ classes

qft++ Class	Symbol	Concept
Matrix<T>	a_{ij}	matrices of any dimension
Tensor<T>	x_{μ}	tensors of any rank
MetricTensor	$g_{\mu\nu}$	Minkowski metric
LeviCivitaTensor	$\epsilon_{\mu\nu\alpha\beta}$	totally anti-symmetric Levi-Civita tensor
DiracSpinor	$u_{\mu_1 \ldots \mu_{J-1/2}}(p, m)$	half-integral spin wave functions
DiracAntiSpinor	$v(p, m)$	spin-1/2 anti-particle wave functions
DiracGamma	γ^{μ}	
DiracGamma5	γ^5	Dirac matrices
DiracSigma	$\sigma^{\mu\nu}$	
PolVector	$\epsilon_{\mu_1 \ldots \mu_J}(p, m)$	integral spin wave functions
OrbitalTensor	$L^{(\ell)}_{\mu_1 \ldots \mu_\ell}$	orbital angular momentum tensors

qft++ Package

Example: $X(2^-) \rightarrow \omega K \rightarrow \pi^+\pi^-\pi^0 K$

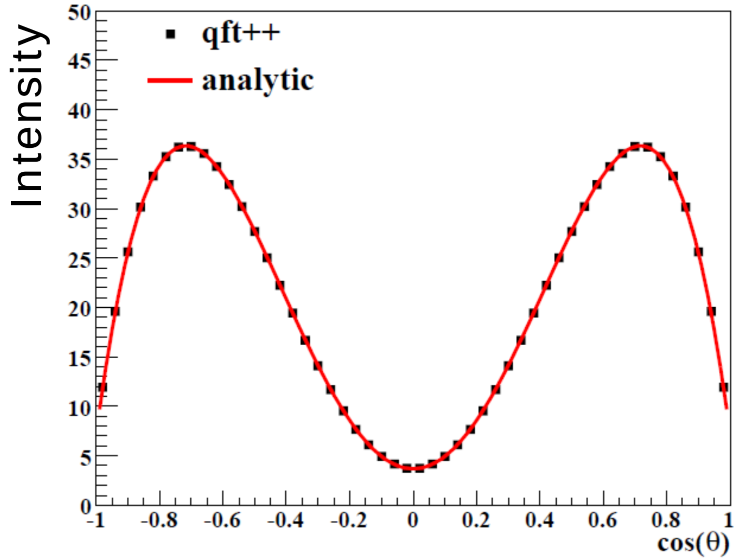
Amplitude and Intensity given by

$$\mathcal{A} \propto \epsilon^*_\mu(p_\omega, m_\omega) L^{(3)\mu\nu\alpha}(p_{\omega K})\epsilon_{\nu\alpha}(P, M) \quad \text{and} \quad \mathcal{I} \propto \sum_{M=\pm 1} \sum_{m_\omega = \pm 1, 0} |\mathcal{A}|^2$$

qft++: Declaration and Calculation

```
PolVector epso; // omega
PolVector epsx(2); // X
OrbitalTensor orb3(3); // L^3
Tensor<complex<double> > amp;
Vector4<double> p4o,p4k,p4x;

double intensity = 0.;
for(Spin m = -1; m <= 1; m+=2){
   for(Spin mo = -1; mo <= 1; mo++){
      amp = conj(epso(mo))*orb3|epsx(m);
      intensity += norm(amp());
   }
}
```



Angular distribution of $X \rightarrow \omega K$

Differences to the Zemach formalism

$$I(0 \to 1+1) \;\propto\; (1+z^2)\cos^2\theta$$

$$I(1 \to 1+0) \;\propto\; 1+z^2\cos^2\theta$$

$$I(1 \to 1+1) \;\propto\; 1-\cos^2\theta$$

$$I(2 \to 2+0) \;\propto\; 1+z^2\left(\frac{1}{3}+\cos^2\theta\right)+z^4\left(\cos^2\theta-\frac{1}{3}\right)^2$$

it is possible to show, that

$$z^2 \;=\; \gamma^2-1$$

for $\gamma = E/m$ for the resonant system formed by (a+b)

Comparison γ=1 and γ=∞

γ=1 (non-relativistic case)

$$I(0 \rightarrow 1+1) \;\propto\; \cos^2\theta$$
$$I(1 \rightarrow 1+0) \;\propto\; 1$$
$$I(1 \rightarrow 1+1) \;\propto\; 1-\cos^2\theta$$
$$I(2 \rightarrow 2+0) \;\propto\; 1$$

γ=∞ (ultra-relativistic)

$$I(0 \rightarrow 1+1) \;\propto\; \cos^2\theta$$
$$I(1 \rightarrow 1+0) \;\propto\; \cos^2\theta$$
$$I(1 \rightarrow 1+1) \;\propto\; 1-\cos^2\theta$$
$$I(2 \rightarrow 2+0) \;\propto\; (\cos^2\theta-\frac{1}{3})^2$$

the angular distributions can be radically different

it depends on the available phase space of a resonance,
if this effect is actually measurable

Covariant extension of the helicity formalism

in non-covariant description we obtained to relationsship

$$N_J f^J_{\lambda_s \lambda_t} = \sum_{L,S} \sqrt{2L+1}\,(L0\;S(\lambda_s-\lambda_t)|J(\lambda_s-\lambda_t))\,(s\lambda_s\;t(-\lambda_t)|S(\lambda_s-\lambda_t))\,\alpha^J_{LS}$$

where a_{LS} is a constant for each J

in covariant description a_{LS} depend on λ_s,λ_L !!!

the formula for a_{LS} reads then

$$a^J_{LS} = g_{LS} N_J \sum_{\lambda_s,\lambda_t} \sqrt{\frac{2L+1}{2J+1}} (L0\ S(\lambda_s-\lambda_t)|J(\lambda_s-\lambda_t))$$

$$(s\lambda_s\ t(-\lambda_t)|S(\lambda_s-\lambda_t))(\frac{W}{W^0})^n B_L(q,q_0) f_{\lambda_s}(\gamma_1)^S f_{\lambda_t}(\gamma_2)^T$$

with

- n=1 if S+$\lambda_{s+L+}\lambda_L$=odd and n=0 otherwise
- $W = \sqrt{s}$ of the two-body system and $W_0 = W(m_0)$
- q = two-body breakup momentum and $q_0 = q(m_0)$
- B_L = Form-factor
- $f_\lambda(\gamma)$ = f-function for given daughter particle with Lorentz-factors γ

Definition

$$f^j_n(\gamma) = a^j(n) \sum_{n_0} b^j(n,n_0)(2\gamma)^{n_0}$$

with

$$a^j(n) = \frac{(j+m)!(j-m)}{(2j)!}$$

$$b^j(n,n_0) = \frac{j!}{n_+!n_0!n_-!}$$

$$2n_\pm = J \pm n - n_0$$

THANK YOU

for today

Amplitude Analysis
An Experimentalists View

Lectures at the "Extracting Physics from Precision Experiments Techniques of Amplitude Analysis"

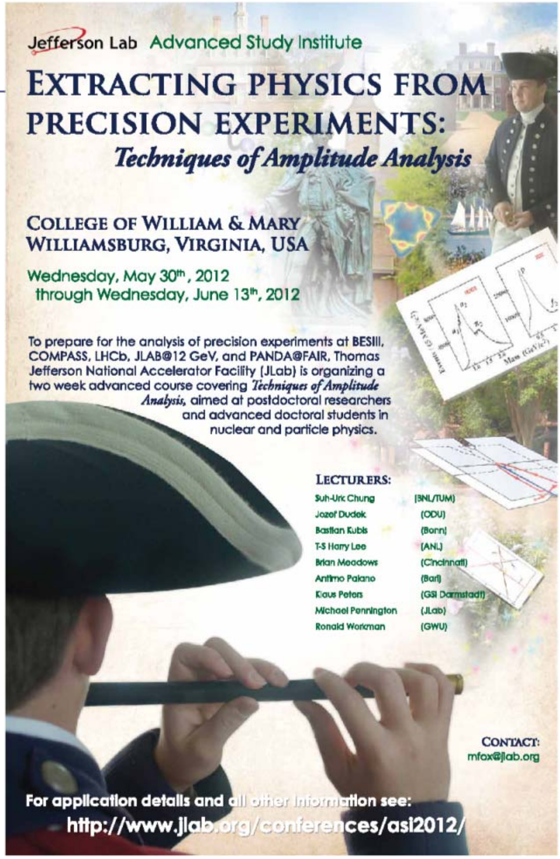
Klaus Peters

GSI Darmstadt and GU Frankfurt

Williamsburg, June 2012

Amplitude Analysis
An Experimentalists View

K. Peters



Part IV

Dynamics

Dynamics

Scattering

T-Matrix

Breit-Wigner

Blatt-Weisskopf

Properties of Dalitz Plots

For the process $M \to R m_3$, $R \to m_1 m_2$ the matrix element can be expressed like

$$\mathcal{M}_R(L, m_{12}, m_{23}) = Z(L, \vec{p}, \vec{q}) \cdot B_L^M(p) \cdot B_L^R(q) \cdot T_R(m_{12})$$

Winkelverteilung (Legendre Polyn.)	Formfaktor (Blatt-Weisskopf-F.)	Resonanz-Fkt. (z.B. Breit Wigner)

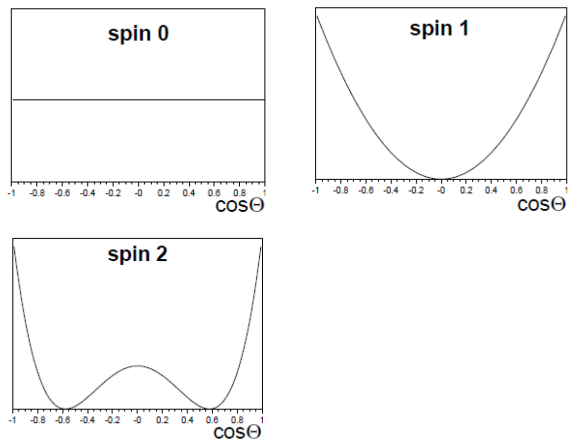
$Z(L, \vec{p}, \vec{q})$ decay angular distribution of R →

$B_L^M(p)$ Form-(Blatt-Weisskopf)-Factor for $M \to R m_3$, $p = p_3$ in R_{12}

$B_L^R(q)$ Form-(Blatt-Weisskopf)-Factor for $R \to m_1 m_2$, $q = p_1$ in R_{12}

$T_R(m_{12})$ Dynamical Function (Breit-Wigner, K-Matrix, Flatté)

J → L+l	Z
0 → 0 + 0	1
0 → 1 + 1	$\cos^2\theta$
0 → 2 + 2	$[\cos^2\theta - 1/3]^2$



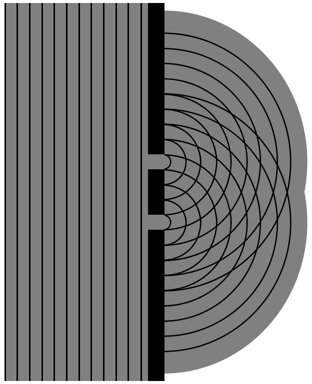
Interference problem

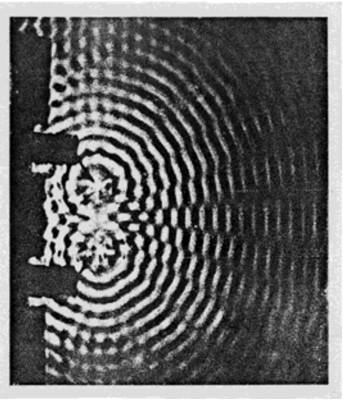
PWA

The phase space diagram in hadron physics shows a pattern due to interference and spin effects

This is the unbiased measurement

What has to be determined ?





Analogy Optics ⇔ PWA

lamps ⇔ # level

slits ⇔ # resonances

positions of slits ⇔ masses

sizes of slits ⇔ widths

Optics

$$I(x) = \left|A_1(x) + A_2(x)e^{\iota\varphi}\right|^2$$

Dalitz plot

$$I(m) = \left|A_1(m) + A_2(m)e^{\iota\varphi}\right|^2$$

but only if spins are properly assigned

bias due to hypothetical spin-parity assumption

Introducing Partial Waves, cont'd

Schrödinger's Equation

$$-\frac{\hbar}{2\mu}\nabla^2\Psi(\vec{r}) + V(\vec{r})\Psi(\vec{r}) = E\Psi(\vec{r})$$

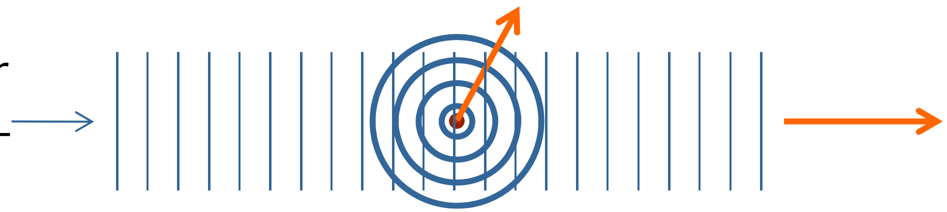
$$V(\vec{r}) = 0$$

$$\vec{k} = \frac{\vec{p}}{\hbar} = \mu\frac{\vec{v}}{\hbar} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Scattering of particles on
a spherical potential

incoming planar wave
outgoing spherical wave

$$\Psi_i = e^{ikz} \ ; \ \Psi_f = f(\theta)\frac{e^{ikr}}{r}$$



Compose planar wave in terms of partial waves with given L

spherical Besselfct. Legendre-Polyn.

$$e^{ikz} = e^{ikr\cos\theta} = \sum_l U_l(r) P_l(\cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\cos\theta)$$

with $j_l(kr) \xrightarrow{kr\to\infty} \dfrac{\sin(kr - \frac{l\pi}{2})}{kr} = \dfrac{1}{2ikr}\left[e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}\right]$

$$e^{ikz} = \sum_l \frac{(2l+1)\, i^l}{2ikr}\left[e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}\right] P_l(\cos\theta)$$

Introducing Partial Waves, cont'd

wave without scattering

outgoing incoming

$$\psi \xrightarrow{kr\to\infty} \sum_l \frac{(2l+1)\,i^l}{2ikr}\left[e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}\right] P_l(\cos\theta)$$

wave with scattering (only outgoing part is modified)

$$\psi' \xrightarrow{kr\to\infty} \sum_l \frac{(2l+1)\,i^l}{2ikr}\left[\eta_l e^{2i\delta_l}\cdot e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}\right] P_l(\cos\theta)$$

Inelasticity + Phaseshift

for the scattered wave ψ_S one gets

$$e^{ikr}\left(e^{-\frac{i\pi}{2}}\right)^l = e^{ikr}(-i)^l$$

$$\psi_S = \psi' - \psi = f(\theta)\frac{e^{ikr}}{r} = \sum_l \frac{(2l+1)\,i^l}{2ikr}(\eta_l e^{2i\delta_l} - 1)e^{i(kr-\frac{l\pi}{2})}$$

$$\psi_S = \left[\frac{1}{k}\sum_l (2l+1)\frac{\eta_l e^{2i\delta_l}-1}{2i}P_l(\cos\theta)\right]\cdot\frac{e^{ikr}}{r} \qquad T_l = \frac{\eta_l e^{2i\delta_l}-1}{2i}$$

Argand Plot

$$z = (a, b) = (a = \Re[z], b = \Im[z]) \Rightarrow (r, \varphi)$$

$$z = a + \iota b = re^{\iota\varphi} = \cos\varphi + \iota \, \mathrm{sn}\, \varphi$$

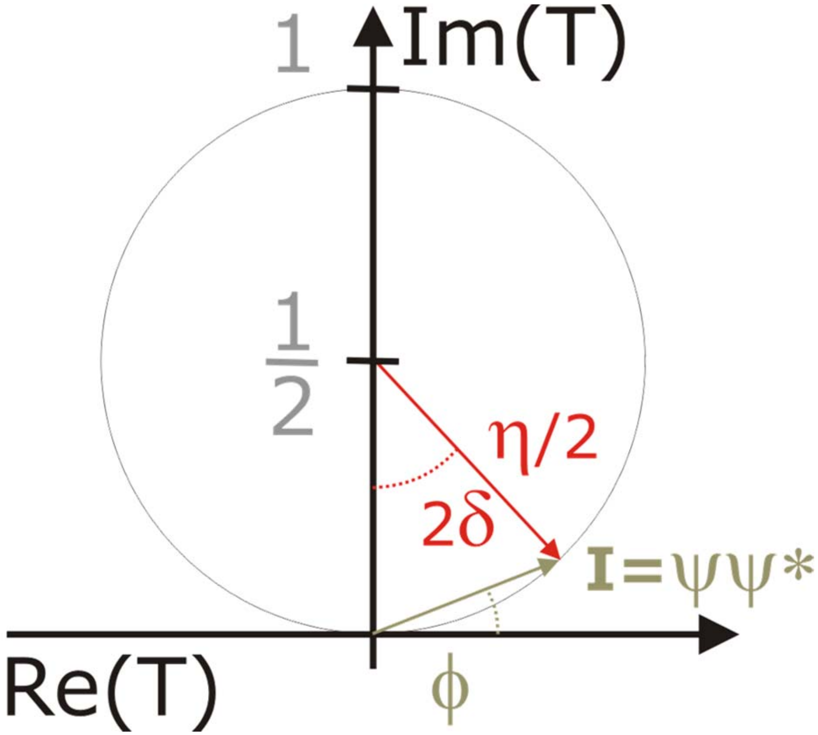
$$(a, b) \Rightarrow (r, \varphi)$$

$$r = \sqrt{a^2 + b^2}$$

$$\varphi = \tan^{-1}\frac{b}{a}$$

$$\eta = 2\sqrt{a^2 + \left(b - \frac{1}{2}\right)^2}$$

$$\delta = \frac{1}{2}\tan^{-1}\left(\frac{b - \frac{1}{2}}{a}\right) + \frac{\pi}{4}$$



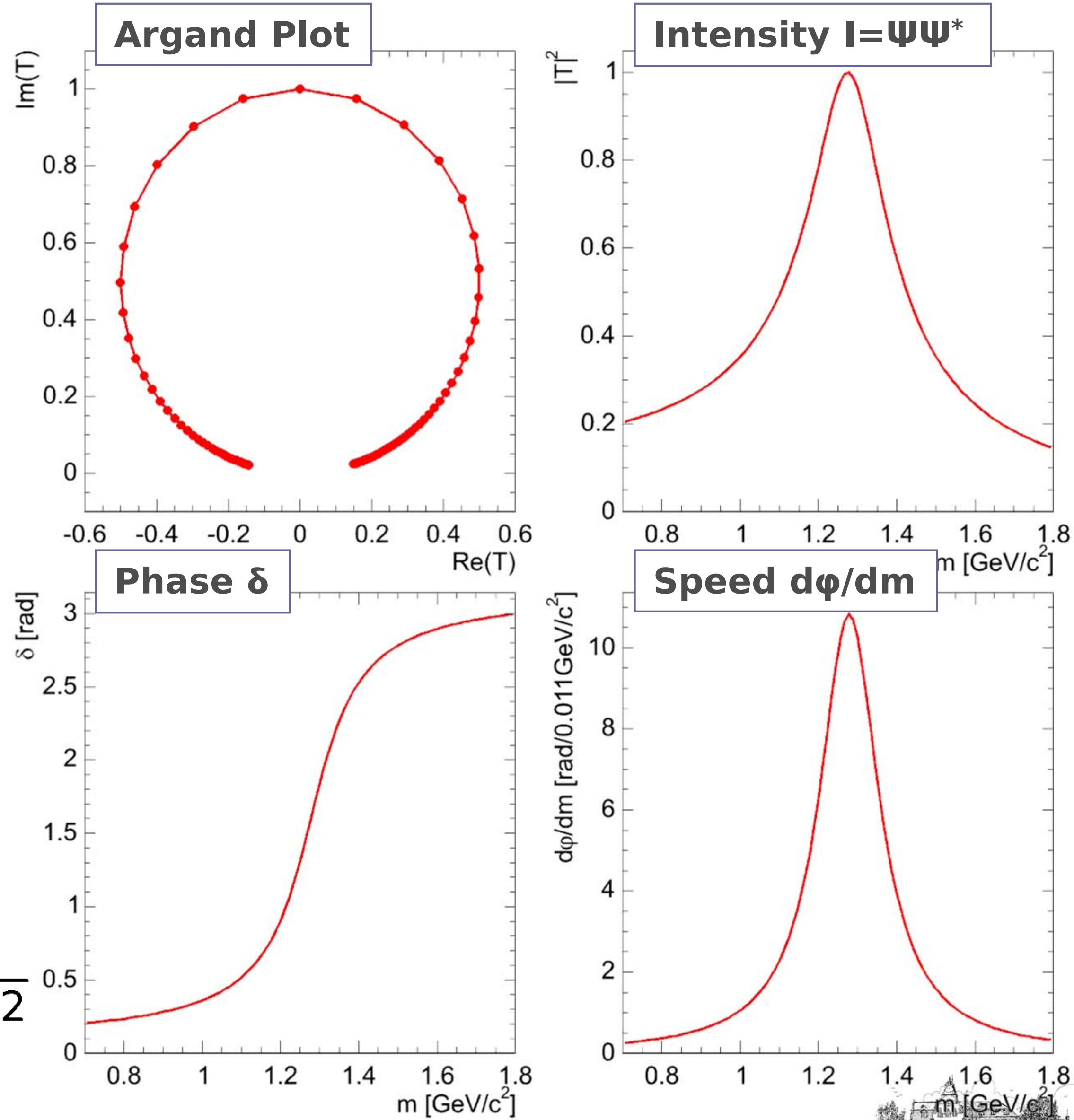
Standard Breit-Wigner

Full circle in the Argand Plot

$$T(m) = \frac{\frac{\Gamma}{2}}{m_0 - m - \imath\frac{\Gamma}{2}}$$ to π

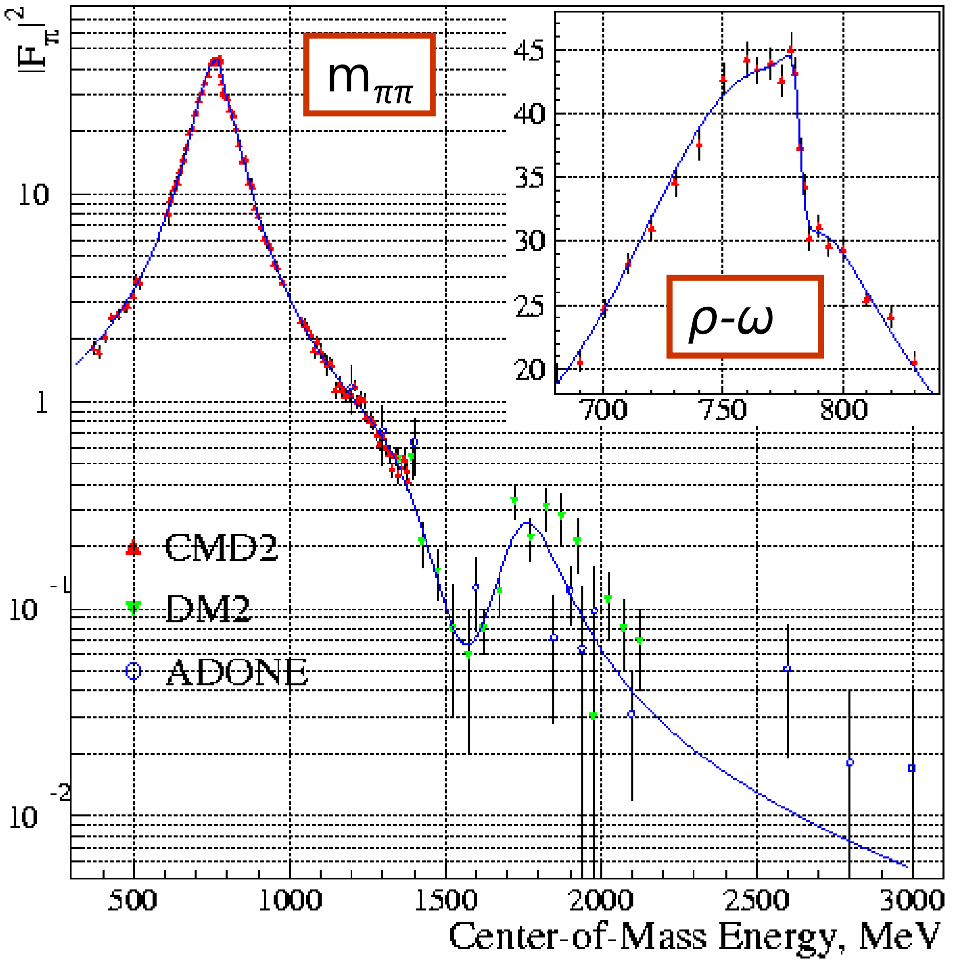
$$I(m) = |T(m)|^2$$

$$= \frac{\left(\frac{\Gamma}{2}\right)^2}{(m_0 - m)^2 + \left(\frac{\Gamma}{2}\right)^2}$$



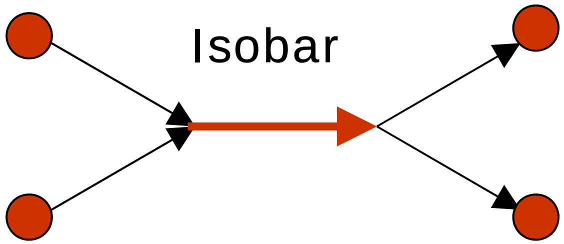
Breit-Wigner in the Real World

$e^+e^- \to \pi\pi$



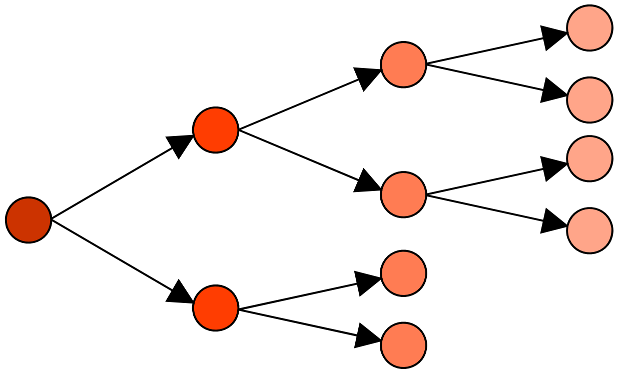
Generalization

- construct any many-body system as a tree of subsequent two-body decays
- the overall process is dominated by two-body processes
- the two-body systems behave identical in each reaction
- different initial states may interfere

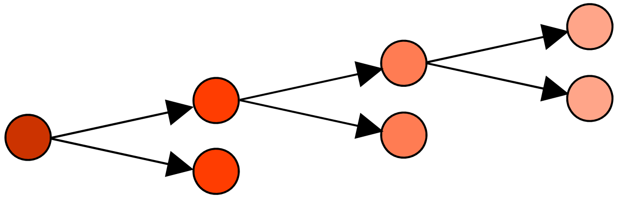


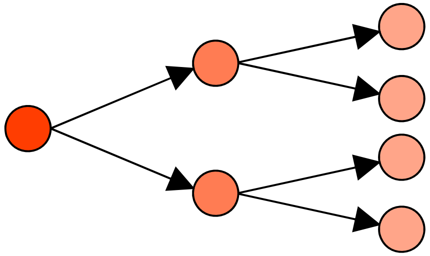
We need

- need two-body "spin"-algebra
  - various formalisms
- need two-body scattering formalism
  - final state interaction, e.g. Breit-Wigner



The Full Amplitude $f(I, I_3, s, \Omega) = I_l(I, I_3) T_l(s) R_l(\Omega)$

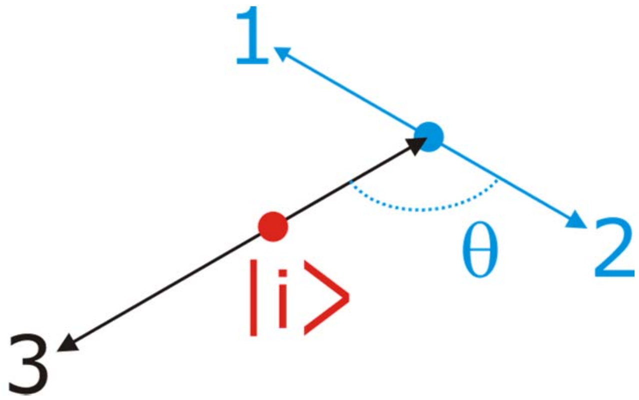




For each node an amplitude $f(I,I_3,s,\Omega)$ is obtained.

The full amplitude is the sum of all nodes.

Summed over all unobservables



Dynamical Functions are Complicated

Search for resonance enhancements
is a major tool in meson spectroscopy

The Breit-Wigner Formula was derived
for a single resonance
appearing in a single channel

But: Nature is more complicated

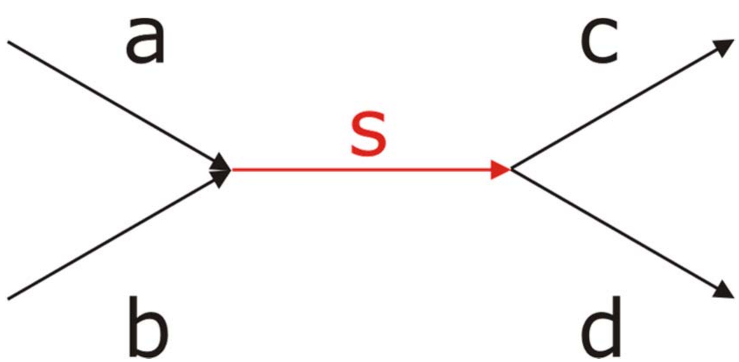
- Resonances decay into several channels
- Several resonances appear within the same channel
- Thresholds distort line shapes due to available phase space

A more general approach is needed
for a detailed understanding (see last lecture!)

S-Matrix

Differential cross section

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(8\pi)^2 s}\left(\frac{q_f}{q_i}\right)|\mathcal{M}_{fi}|^2 = |f_{fi}(\Omega)|^2$$



Scattering amplitude

$$f_{fi}(\Omega) = \frac{1}{q_i}\sum_J (2J+1) T^J_{fi}(s) D^{J\,*}_{\lambda\mu}(\phi,\theta,0)$$

S-Matrix $\quad S = I + 2\iota\, T$

Total scattering cross section

$$\sigma^J_{fi} = \left(\frac{4\pi}{q_i^2}\right)(2J+1)|T^J_{fi}(s)|^2$$

with

$$\begin{aligned} |i\rangle &= |ab, JM\lambda_a\lambda_b\rangle \\ |f\rangle &= |cd, JM\lambda_c\lambda_d\rangle \\ \langle f|i\rangle &= \delta_{ij} \end{aligned}$$

and

$$S_{fi} = \langle f|S|i\rangle \qquad S\ S^\dagger = S^\dagger S = I$$

Harmonic Oscillator (classics revisited)

Free oscillator

$$\ddot{x} + \omega_0^2 x = 0$$

Damped oscillator

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

Solution

$$x(t) = Ae^{-\lambda t}\cos(\omega t + \alpha) \qquad \text{with} \qquad \omega = \sqrt{\omega_0^2 + \lambda^2}$$

External periodic force

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f}{m}\cos\omega_R t = \frac{f}{m}\Re\left[e^{\imath\omega_R t}\right]$$

Oscillation strength

and phase shift

Lorentz function

$$I(\omega_R) = \frac{f^2}{4m}\frac{\lambda}{(\omega_R - \omega_0)^2 + \lambda^2} \qquad \tan\delta = \frac{2\lambda\omega_R}{\omega_0^2 - \omega_R^2}$$

Breit-Wigner Function

Wave function for an unstable particle

$$\Psi(t) \;=\; \Psi_0 e^{-\iota\omega_R t} e^{-\frac{t}{2\tau}} = \Psi_0 e^{-\iota\omega_R t} e^{-\frac{\Gamma}{2}t}$$

Fourier transformation for E dependence

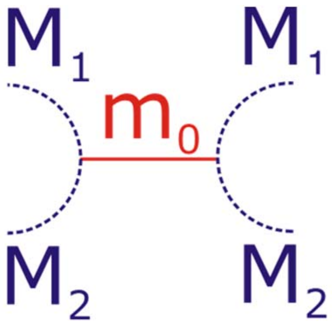
$$\begin{aligned}
\Psi(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(t) e^{\iota\omega t} dt = \frac{\Psi_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota\left(\omega-\omega_R+\iota\frac{\Gamma}{2}\right)t} dt \\
&= \frac{\Psi_0}{\omega-\omega_R-\iota\frac{\Gamma}{2}} \left[\frac{1}{\sqrt{2\pi}} e^{\iota\left(\omega-\omega_R+\iota\frac{\Gamma}{2}\right)t}\right]_{-\infty}^{\infty} \\
&= \frac{\kappa}{(E_R-E)-\iota\frac{\Gamma}{2}}
\end{aligned}$$

Finally our first Breit-Wigner

$$\Psi(E) = \frac{\frac{\Gamma}{2}}{(E_R-E)-\iota\frac{\Gamma}{2}}$$

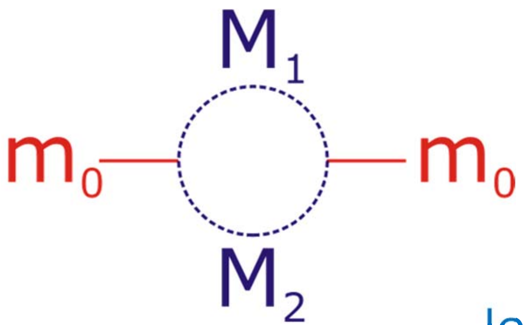
Dressed Resonances – T meets Field Theory

Suppose we have a resonance with mass m_0



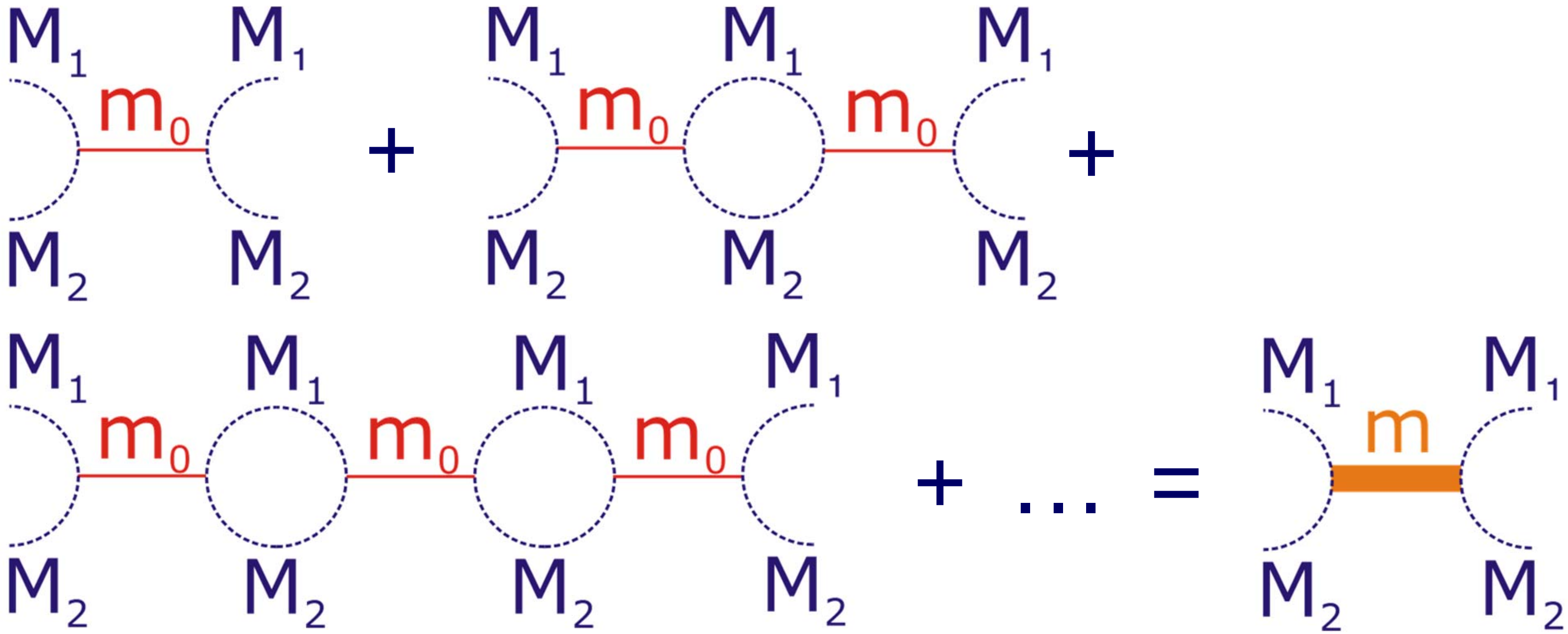
We can describe this with a propagator $$T = V_{12}\frac{1}{E_0 - E}V_{12}$$

But we may have a self-energy term



leading to $$T = V_{12}\frac{1}{E_0 - E}b\frac{1}{E_0 - E}V_{12} = \frac{V_{12}bV_{12}}{(E_0 - E)^2}$$

T-Matrix Perturbation



We can have an infinite number of loops inside our propagator

$$T = V_{12}\frac{1}{E_0 - E}V_{12} + \frac{V_{12}bV_{12}}{(E_0 - E)^2} + \frac{V_{12}b^2V_{12}}{(E_0 - E)^3} + \ldots$$

every loop involves a coupling ***b***,

so if ***b*** is small, this converges like a geometric series

T-Matrix Perturbation – Retaining BW

So we get

$$T = \frac{V_{12}V_{12}}{E_0 - E}\left(1 + \frac{b}{E_0 - E} + \frac{b^2}{(E_0 - E)^2} + \ldots\right)$$

$$= \frac{V_{12}V_{12}}{E_0 - E}\left(\frac{1}{1 - \frac{b}{E_0 - E}}\right)$$

$$= \frac{V_{12}V_{12}}{E_0 - E - b}$$

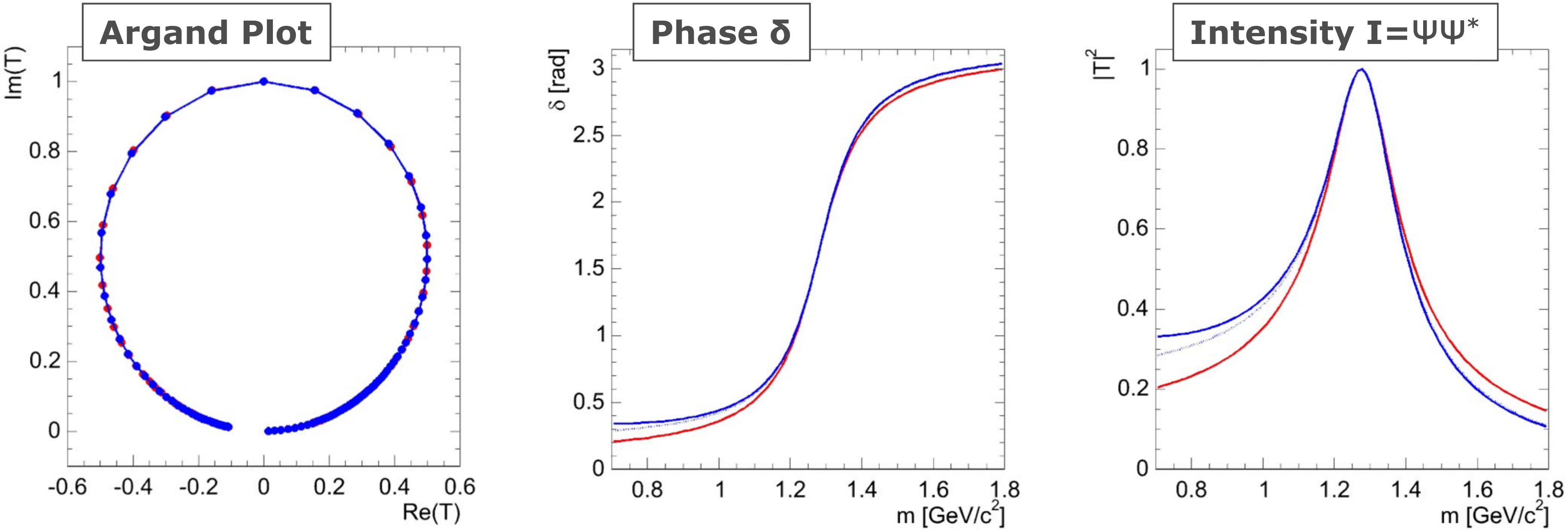
and the full amplitude with a “dressed propagator” leads to

$$T = \frac{V_{12}V_{12}}{E_0 - \Re[b] - E - \imath\Im[b]}$$

$$= \frac{V_{12}V_{12}}{E_R - E - \imath\Im[b]}$$

which is again a Breit-Wigner like function,
but the bare energy ***E_0*** has now changed into ***E_0-$\Re${b}***

Relativistic Breit-Wigner



By migrating from Schrödinger's equation (non-relativistic) to Klein-Gordon's equation (relativistic) the energy term changes different energy-momentum relation $E=p^2/m$ vs. $E^2=m^2c^4+p^2c^2$

The propagators change to s_R-s from m_R-m

$$T(s) = \frac{\gamma}{s_r - s - \imath\frac{2q\gamma}{\sqrt{s}}} = \frac{\Gamma}{m_r^2 - m^2 - \imath\rho m_0 \Gamma}$$

Barrier Factors - Introduction

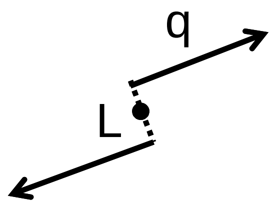
At low energies, near thresholds $\quad \Gamma_r \propto q^{2l+1} = \rho q^{2l}$

- but is not valid far away from thresholds -- otherwise the width would explode and the integral of the Breit-Wigner diverges
- It reflects the non-zero size of the object

Need more realistic centrifugal barriers

- known as Blatt-Weisskopf damping factors

We start with the semi-classical impact parameter



$$b = [L(L+1)]^{\frac{1}{2}}/q$$

and use the approximation for the stationary solution of the radial differential equation

$$\frac{\partial^2}{\partial \rho^2} U_l^n \rho \simeq \left(\frac{b_n^2}{r^2} - 1\right) U_l^n \rho \quad U_l^n \rho \overset{r>R}{\simeq} iC_n \rho h_l^{(1)}(\rho) \sim C_n e^{\iota\left(\rho - \frac{1}{2} L\pi\right)}$$

with

$$\left[H_l^n(R/b)\right]^{-1} \equiv \rho^2 |h_l^{(1)}(\rho)|^2$$ we obtain

$$\Gamma_n(q_n) \quad = \quad \Gamma_n^0 \frac{\frac{q_n}{m} H_l^n(R/b_n)}{\frac{q_n^0}{m} H_l^n(R/b_n^0)}$$

The energy dependence is usually parameterized in terms of spherical Hankel-Functions

$$j_l(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} J_{1+\frac{1}{2}}(x)$$

$$n_l(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} N_{1+\frac{1}{2}}(x)$$

$$h_l^{(1,2)}(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} \left[J_{1+\frac{1}{2}}(x) \pm N_{1+\frac{1}{2}}(x)\right]$$

$$h_0^{(1)}(x) = \frac{e^{\iota x}}{\iota x}$$

$$h_1^{(1)}(x) = \frac{-e^{\iota x}\left(1+\frac{\iota}{x}\right)}{x}$$

$$h_2^{(1)}(x) = \iota\frac{e^{\iota x}\left(1+\frac{3\iota}{x}-\frac{3}{x^2}\right)}{x}$$

we define $F_l(q)$ with the following features

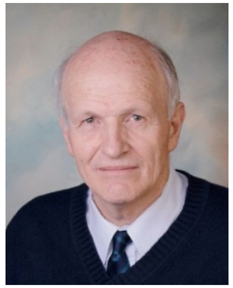
$$F_l(q) \overset{x=\frac{q}{q_{scale}}}{=} \sqrt{\frac{|h_l^{(1)}(x)|^2}{|h_l^{(1)}(x=1)|^2}}$$

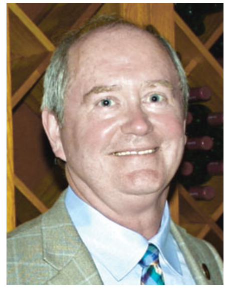
$$F_l(q) \overset{q\to q_{scale}}{=} 1$$

$$F_l(q) \overset{q\to 0}{=} q^l$$

Main problem is the choice of the scale parameter $q_R = q_{scale}$

$$
\begin{aligned}
F_0(x) &= 1 \\
F_1(x) &= \sqrt{\frac{x}{x+1}} \\
F_2(x) &= \sqrt{\frac{13x^2}{(x-3)^2+9x}} \\
F_3(x) &= \sqrt{\frac{277x^3}{x(x-15)^2+9(2x-5)^2}} \\
B_l(q, q_R) &= \frac{F_l(q)}{F_l(q_R)}
\end{aligned}
$$





by Hippel and Quigg (1972)

Usage

$$
T_l(s) = \frac{B_l^2(q)\Gamma}{m_r^2 - m^2 - \imath\rho B_l^2(q) m_0 \Gamma}
$$

Resonant daughters

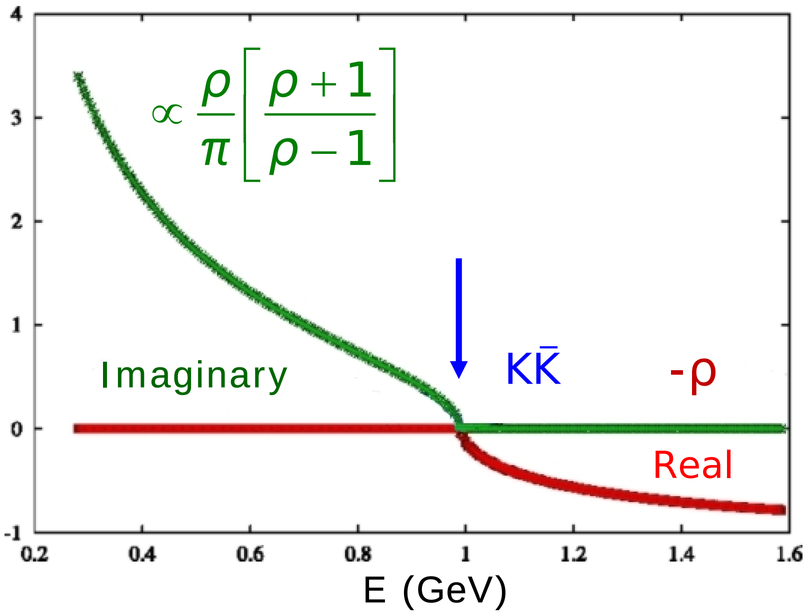
$$\rho_i \to 1 \quad as \quad m^2 \to \infty; \quad \rho_i = \frac{2q_i}{m} = \sqrt{\left[1-\left(\frac{m_a+m_b}{m}\right)^2\right]\left[1-\left(\frac{m_a-m_b}{m}\right)^2\right]}$$

Scales and Formulae

formula was derived from a cylindrical potential

the scale (197.3 MeV/*c*) may be different for different processes

valid in the vicinity of the pole



Breakup-momentum

may become complex (sub-threshold)

need **$\langle F^l(q)\rangle = \int F^l(q)dBW$**

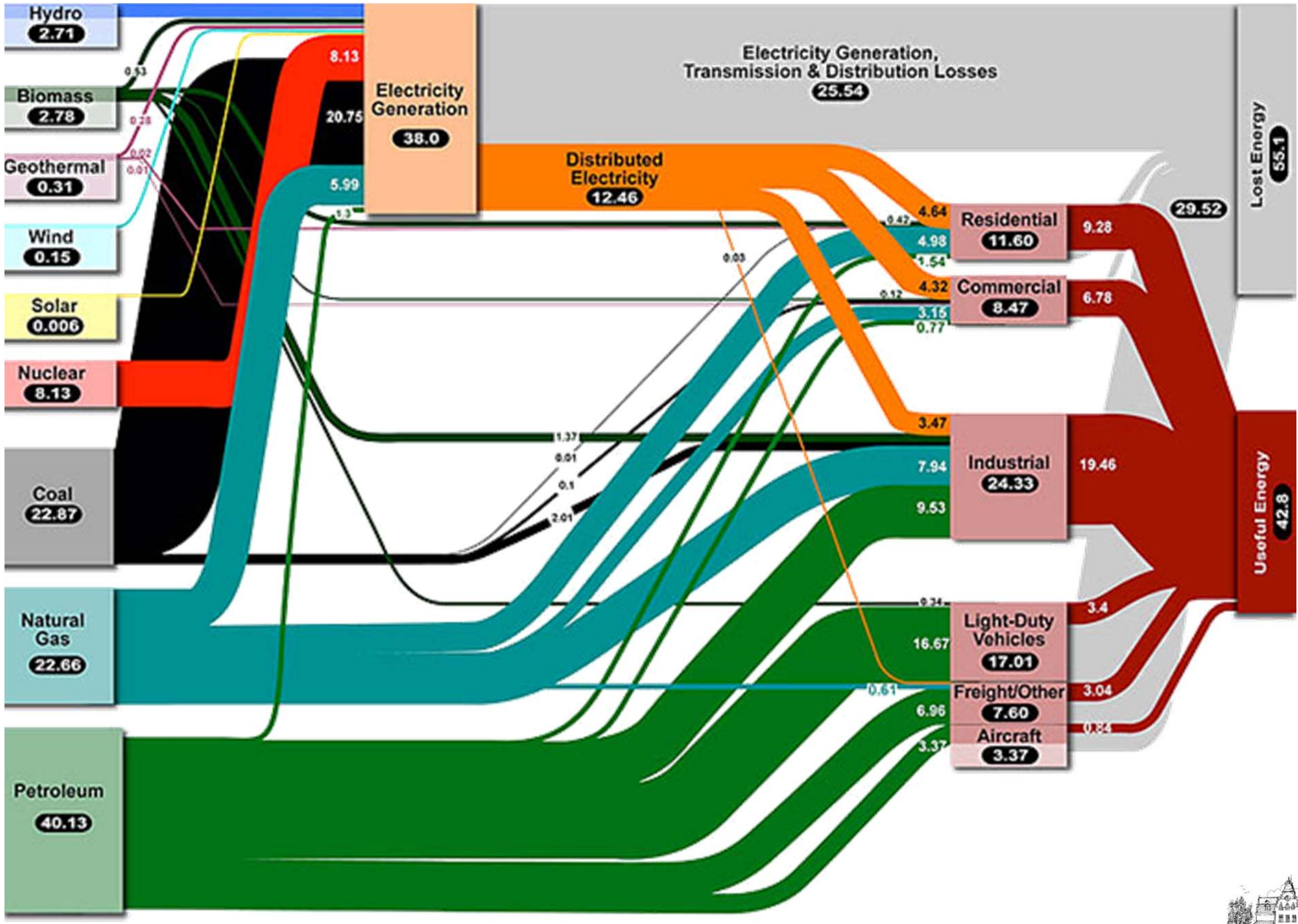
since $F^l(q) \approx q^l$

complex even above threshold

meaning of mass and width are mixed up

needs analytic continuation

Input = Output



Outline of the Unitarity Approach

The most basic feature of an amplitude is UNITARITY

Everything which comes in has to get out again

no source and no drain of probability

Idea: Model a unitary amplitude

Realization: n-Rank Matrix of analytic functions, T_{ij}

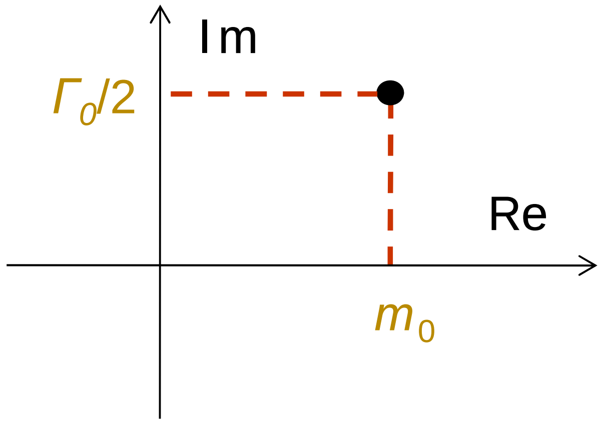
one row (column) for each decay channel

What is a resonance?

A pole in the complex energy plane $T_{ij}(m)$
with m being complex

Parameterizations: e.g. sum of poles

$$\frac{1}{m_0 - i\frac{\Gamma_0}{2}}$$



THANK YOU

for today