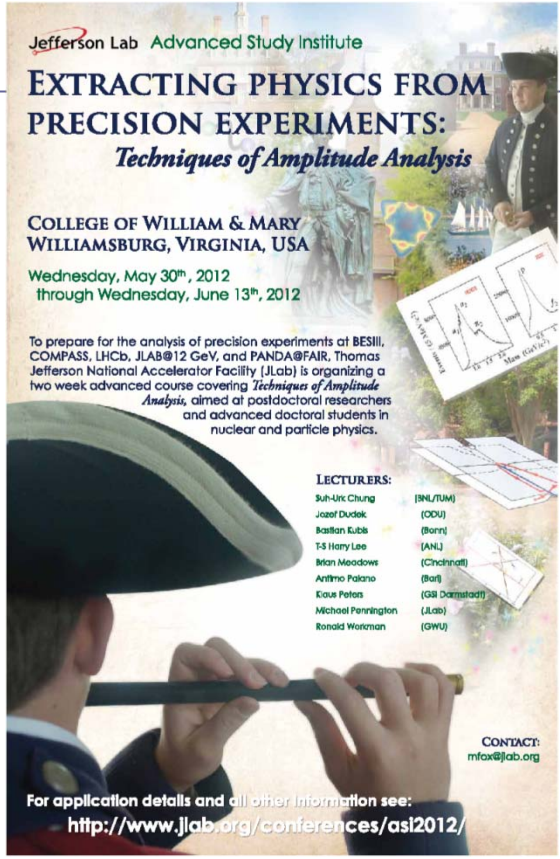
Amplitude Analysis
An Experimentalists View

K. Peters



Part V

K-Matrix

K-Matrix

The case

Derivation

Examples

Properties

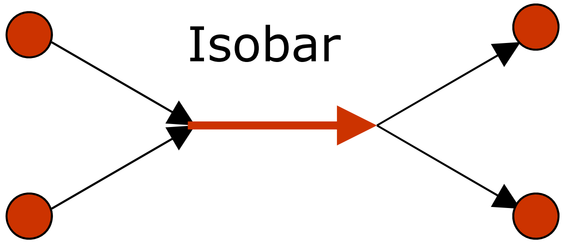
Interpretation

Problems

Isobar Model

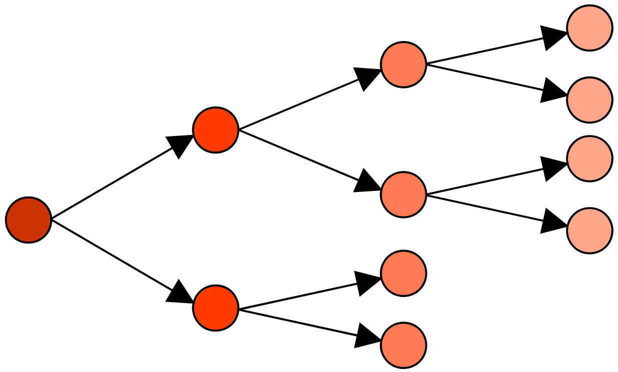
Generalization

- construct any many-body system as a tree of subsequent two-body decays
- the overall process is dominated by two-body processes
- the two-body systems behave identical in each reaction
- different initial states may interfere



We need

- need two-body “spin”-algebra
  - various formalisms
- need two-body scattering formalism
  - final state interaction, e.g. Breit-Wigner



Properties of Dalitz Plots

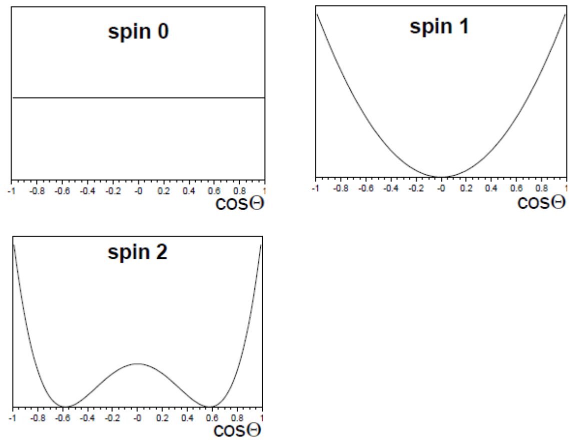
For the process $M \to R m_3$, $R \to m_1 m_2$ the matrix element can be expressed like

$$\mathcal{M}_R(L, m_{12}, m_{23}) = Z(L, \vec{p}, \vec{q}) \cdot B_L^M(p) \cdot B_L^R(q) \cdot T_R(m_{12})$$

Winkelverteilung (Legendre Polyn.)	Formfaktor (Blatt-Weisskopf-F.)	Resonanz-Fkt. (z.B. Breit Wigner)

J → L+l	Z
0 → 0 + 0	1
0 → 1 + 1	$\cos^2\theta$
0 → 2 + 2	$[\cos^2\theta - 1/3]^2$

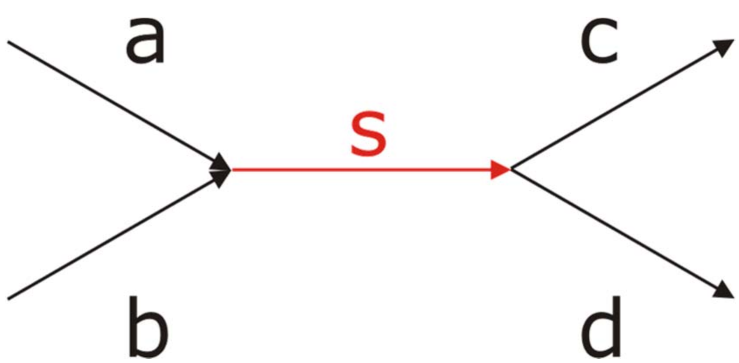
- $Z(L, \vec{p}, \vec{q})$ — decay angular distribution of R →
- $B_L^M(p)$ — Form-(Blatt-Weisskopf)-Factor for $M \to R m_3$, $p = p_3$ in R_{12}
- $B_L^R(q)$ — Form-(Blatt-Weisskopf)-Factor for $R \to m_1 m_2$, $q = p_1$ in R_{12}
- $T_R(m_{12})$ — Dynamical Function (Breit-Wigner, K-Matrix, Flatté)



S-Matrix

Differential cross section

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(8\pi)^2 s}\left(\frac{q_f}{q_i}\right)|\mathcal{M}_{fi}|^2 = |f_{fi}(\Omega)|^2$$



Scattering amplitude

$$f_{fi}(\Omega) = \frac{1}{q_i}\sum_J (2J+1) T^J_{fi}(s) D^{J\,*}_{\lambda\mu}(\phi, \theta, 0)$$

S-Matrix $\quad S = I + 2\imath\, T$

Total scattering cross section

$$\sigma^J_{fi} = \left(\frac{4\pi}{q_i^2}\right)(2J+1)|T^J_{fi}(s)|^2$$

with

$$|i\rangle = |ab, JM\lambda_a\lambda_b\rangle$$
$$|f\rangle = |cd, JM\lambda_c\lambda_d\rangle$$
$$\langle f|i\rangle = \delta_{ij}$$

and

$$S_{fi} = \langle f|S|i\rangle \qquad S\; S^\dagger = S^\dagger S = I$$

Argand Plot

$$z = (a, b) = (a = \Re[z], b = \Im[z]) \Rightarrow (r, \varphi)$$

$$z = a + \iota b = re^{\iota\varphi} = \cos\varphi + \iota\,\mathrm{sn}\,\varphi$$

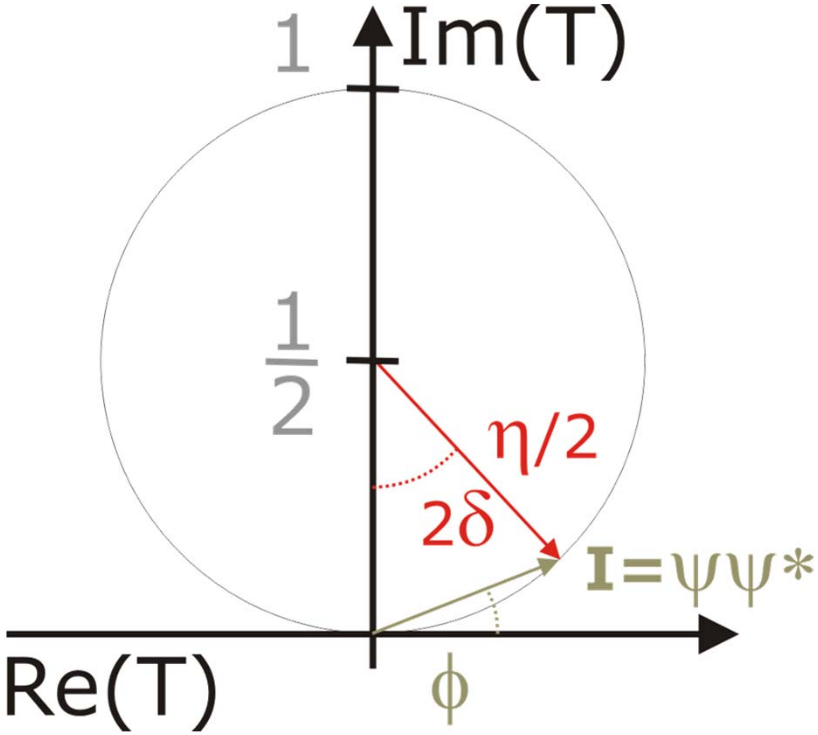
$$(a, b) \Rightarrow (r, \varphi)$$

$$r = \sqrt{a^2 + b^2}$$

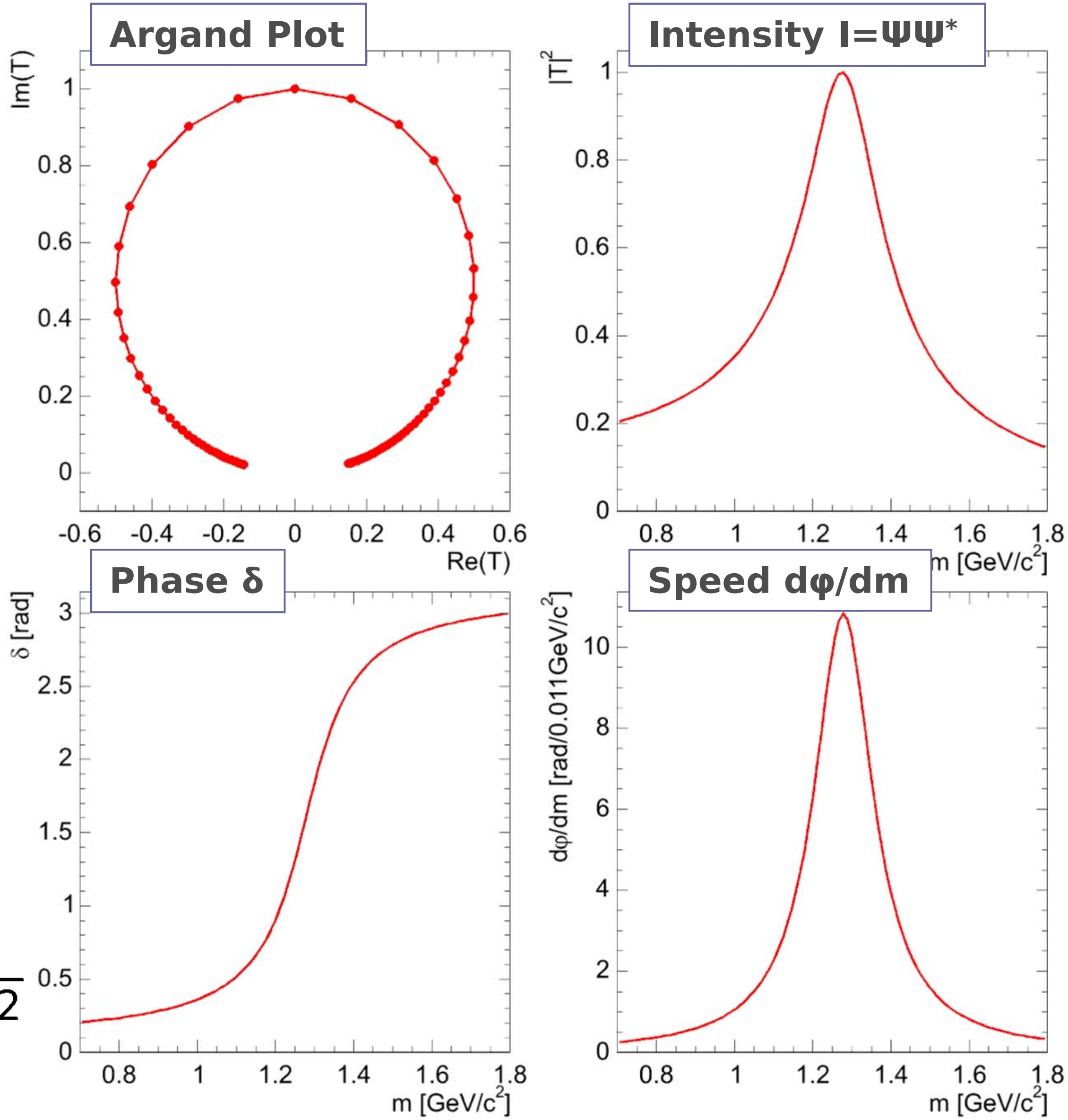
$$\varphi = \tan^{-1}\frac{b}{a}$$

$$\eta = 2\sqrt{a^2 + \left(b - \frac{1}{2}\right)^2}$$

$$\delta = \frac{1}{2}\tan^{-1}\left(\frac{b - \frac{1}{2}}{a}\right) + \frac{\pi}{4}$$



Standard Breit-Wigner



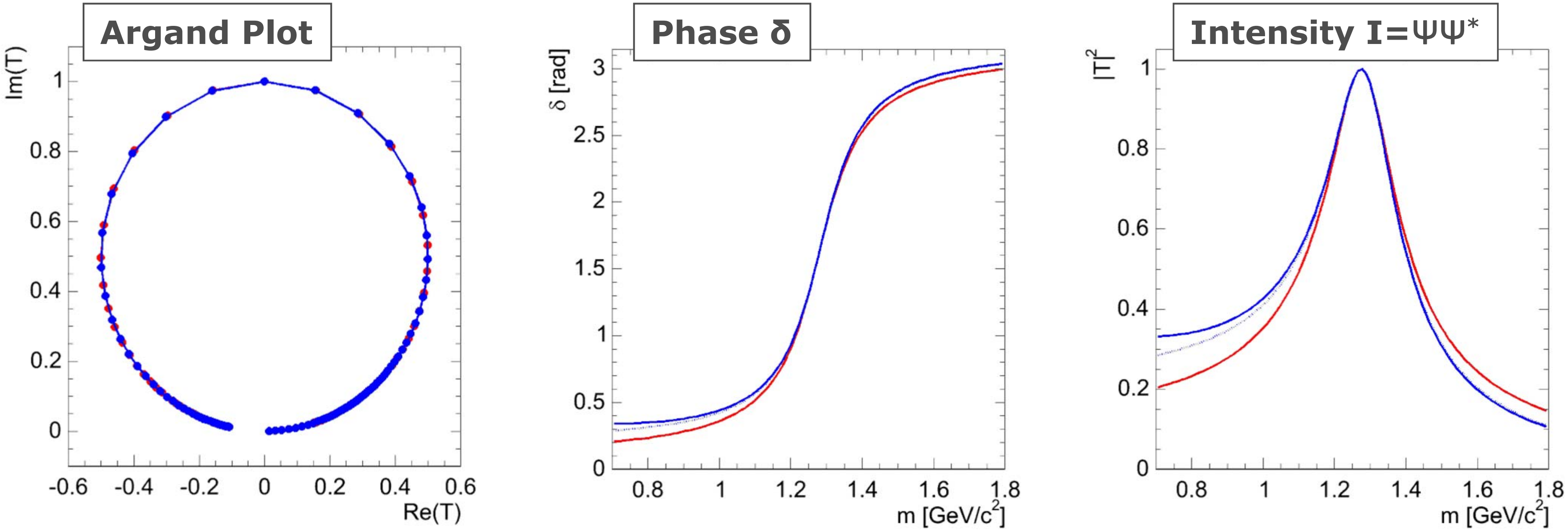
Full circle in the Argand Plot

$$T(m) = \frac{\frac{\Gamma}{2}}{m_0 - m - \imath\frac{\Gamma}{2}}$$

to π

$$I(m) = |T(m)|^2 = \frac{\left(\frac{\Gamma}{2}\right)^2}{(m_0 - m)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

Relativistic Breit-Wigner



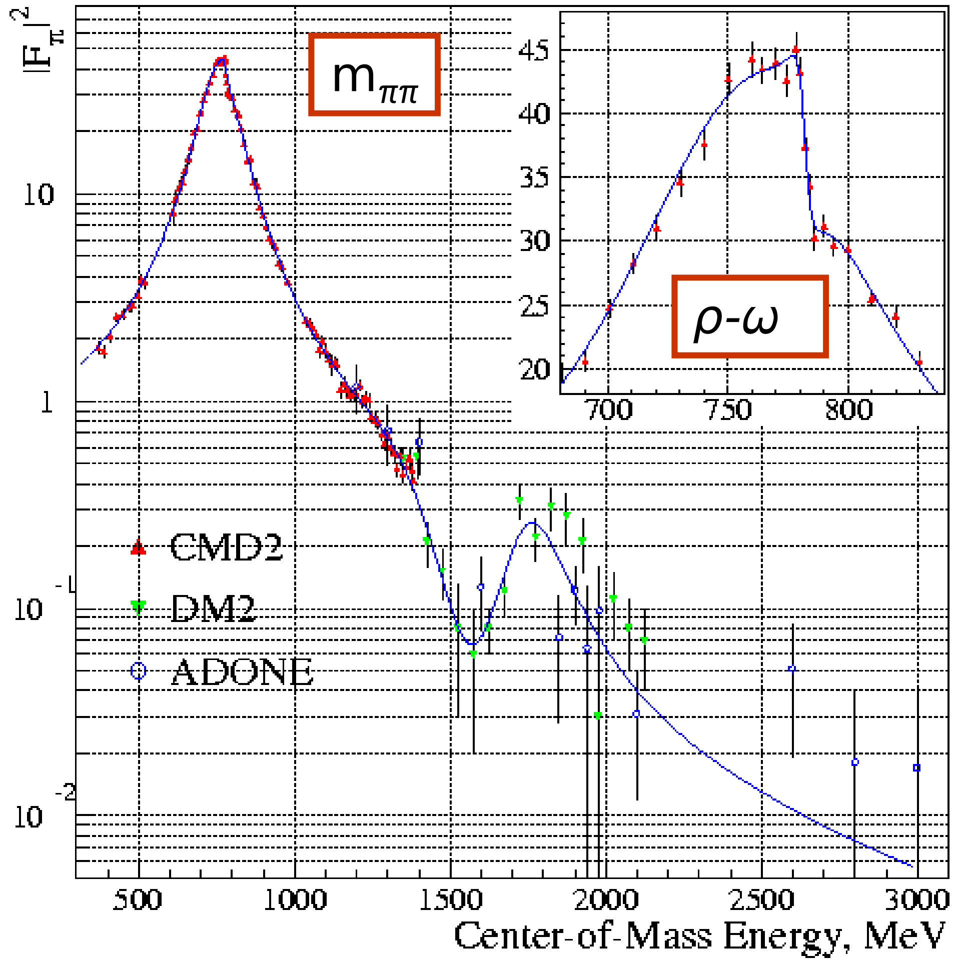
By migrating from Schrödinger‘s equation (non-relativistic) to Klein-Gordon‘s equation (relativistic) the energy term changes different energy-momentum relation $E=p^2/m$ vs. $E^2=m^2c^4+p^2c^2$

The propagators change to s_R-s from m_R-m

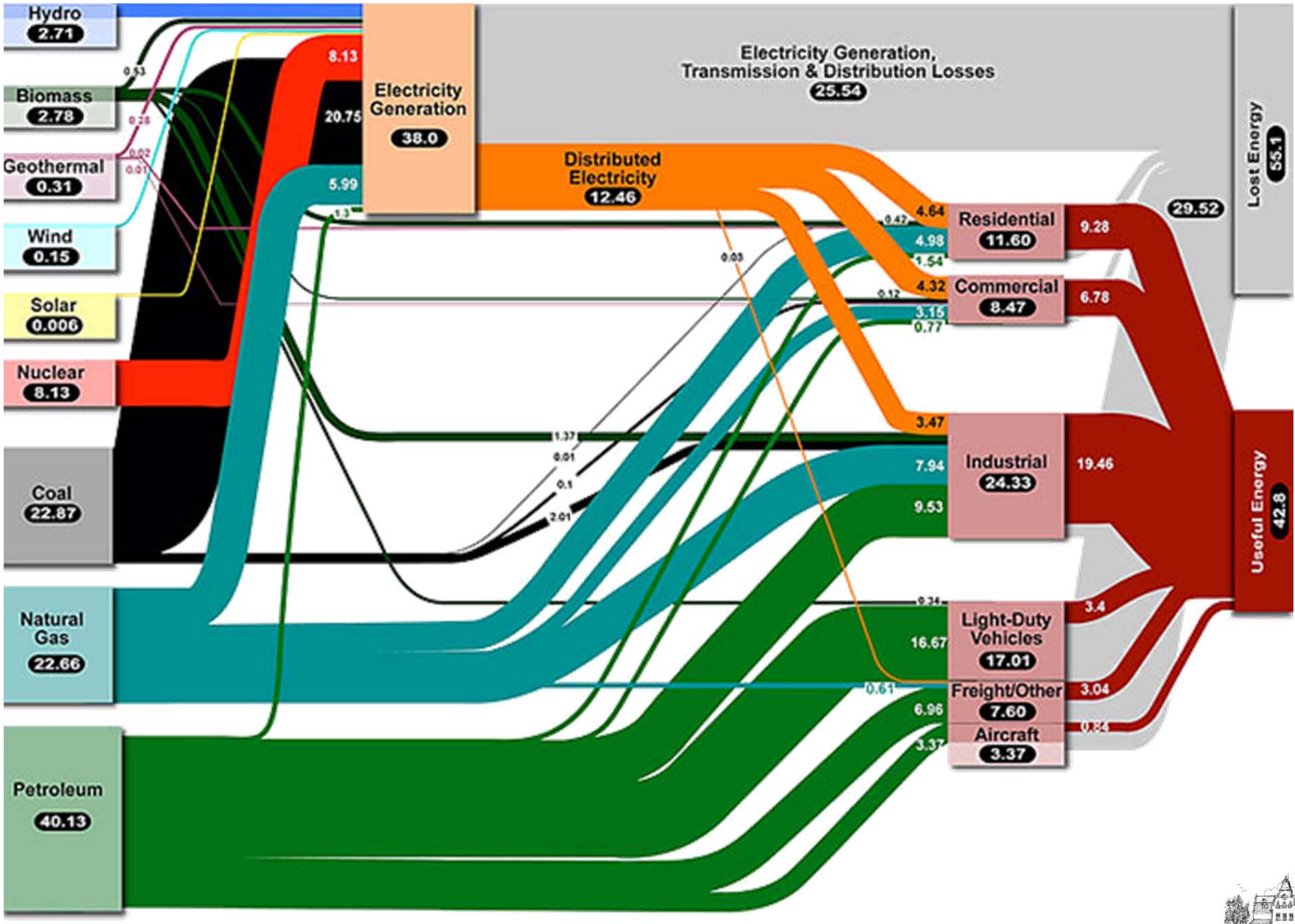
$$T(s) = \frac{\gamma}{s_r - s - \imath\frac{2q\gamma}{\sqrt{s}}} = \frac{\Gamma}{m_r^2 - m^2 - \imath\rho m_0 \Gamma}$$

Breit-Wigner in the Real World

$e^+e^- \rightarrow \pi\pi$



Input = Output



Outline of the Unitarity Approach

The most basic feature of an amplitude is UNITARITY

Everything which comes in has to get out again

no source and no drain of probability

Idea: Model a unitary amplitude

Realization: n-Rank Matrix of analytic functions, T_{ij}

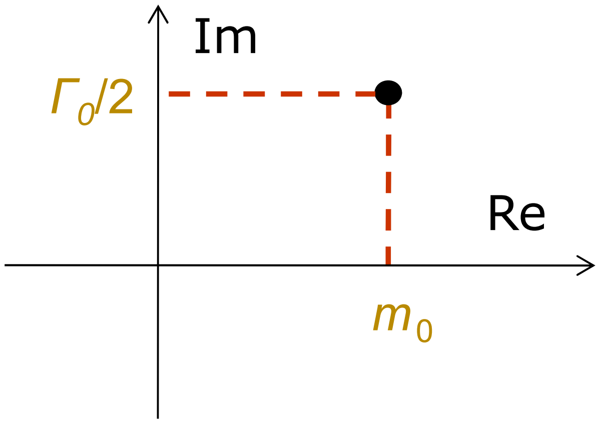
one row (column) for each decay channel

What is a resonance?

A pole in the complex energy plane $T_{ij}(m)$
with m being complex

Parameterizations: e.g. sum of poles

$$\frac{1}{m_0 - i\frac{\Gamma_0}{2}}$$



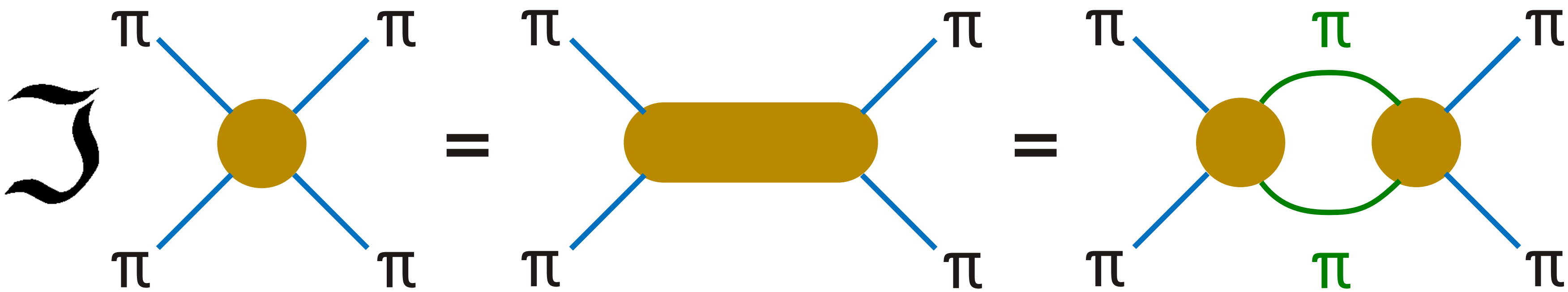
T-Matrix Unitarity Relations

Unitarity is a basic feature since probability has to be conserved

T is unitary if S is unitary

$$\sum_{j=0}^{n} S^*_{kj} S_{ij} = \delta_{ik} = \sum_{j=0}^{n} T^*_{kj} T_{ij}$$

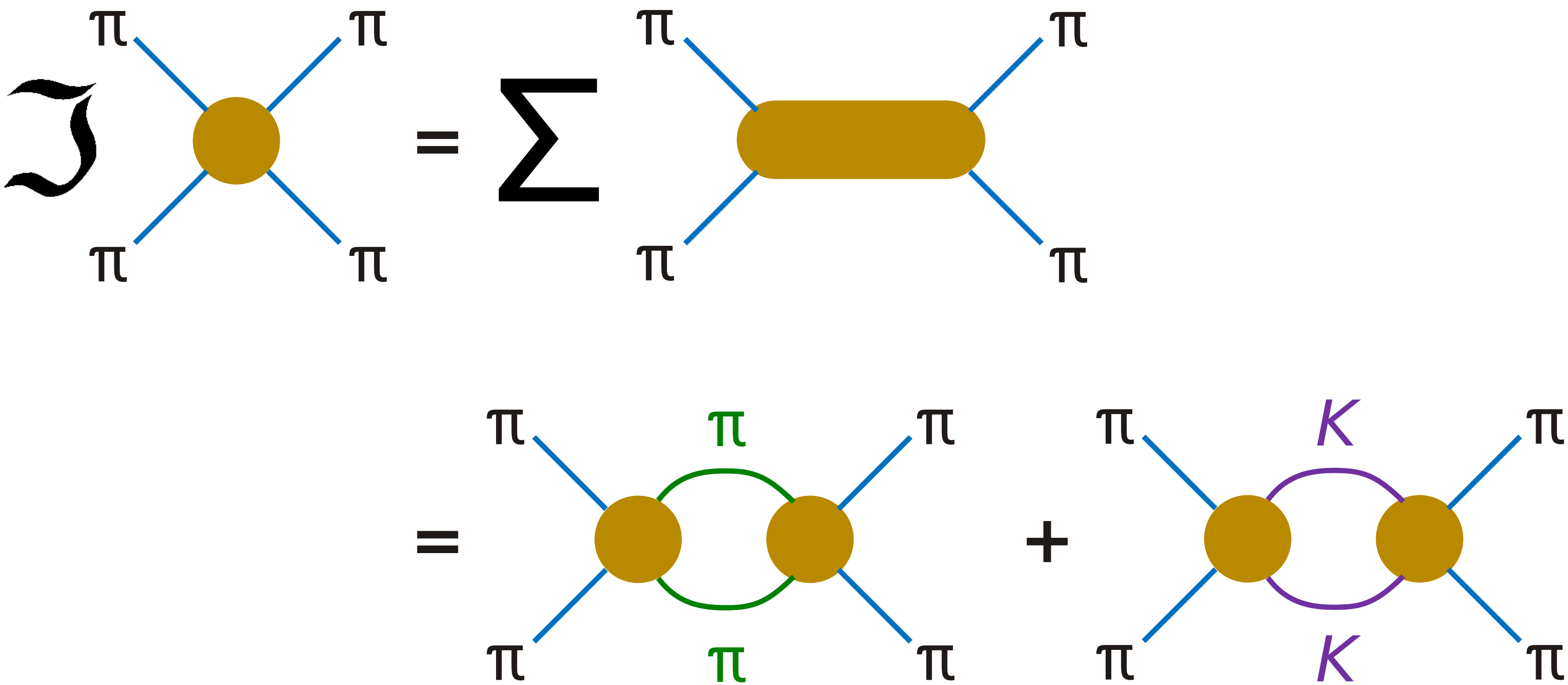
since $S = I + 2\imath\, T$ we get in addition $\Im\left[T_{ij}\right] = \sum_{n} T^*_{nj} T_{ni}$



for a single channel $\Im\left[T_{11}\right] = T^*_{11} T_{11}$

but there a more than one
channel involved....

$$\Im\left[T_{ij}\right] = T_{i1}^{*}T_{1j} + T_{i2}^{*}T_{2j} + \ldots$$



T-Matrix Dispersion Relations

Cauchy Integral on a closed contour

$$T_l(s) = \frac{1}{2\imath\pi}\int_C \frac{T_l(s')ds'}{s'-s}$$

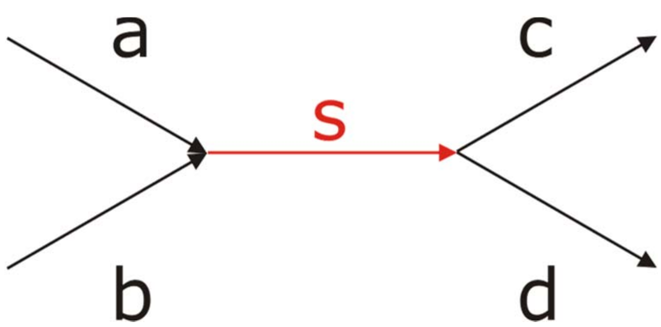
By choosing proper contours and some limits one obtains the dispersion relation for $T_l(s)$

$$T_l(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} \frac{\Im\left[T_l(s')\right]}{s'-s}ds' + \frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty} \frac{\Im\left[T_l(s')\right]}{s'-s}ds'$$

Satisfying this relation with an arbitrary parameterization is extremely difficult and is dropped in many approaches

much more elsewhere….

S-Matrix and Unitarity



$$S_{fi} = \langle f | S | i \rangle$$

$$S = I + 2iT$$

$$SS^{\dagger} = S^{\dagger}S = I$$

$$T - T^{\dagger} = 2iT^{\dagger}T = 2iTT^{\dagger}$$

$$(T^{\dagger})^{-1} - T^{-1} = 2iI$$

$$(T^{-1} + iI)^{\dagger} = T^{-1} + iI$$

$$K^{-1} = T^{-1} + iI$$

$$K = K^{\dagger}$$

$$T = K + iTK = K + iKT$$

$$[K, T] = 0$$

$$T = K(I - iK)^{-1}$$

$$= (I - iK)^{-1}K$$

$$S = (I + iK)(I - iK)^{-1}$$

$$= (I - iK)^{-1}(I + iK)$$

$$\Re(T) = (I + K^2)K = K(I + K^2)^{-1}$$

$$\Im(T) = (I + K^2)K^2 = K^2(I + K^2)^{-1}$$

$$\Im(T) = T^{*}T = TT^{*}$$

$$\Im(T^{-1}) = -I$$

K-Matrix Definition

S (and T) is **n x n** matrix representing
n incoming and **n** outgoing channel

the Caley transformation generates a unitary matrix from a real and symmetric matrix K

$$S = (I + \imath K)(I - \imath K)^{-1} = (I - \imath K)^{-1}(I + \imath K)$$

then T commutes with K $\quad [K, T] = 0$

and is defined like

$$T = K(I - \imath K)^{-1} = (I - \imath K)^{-1} K$$

then T is also unitary by design

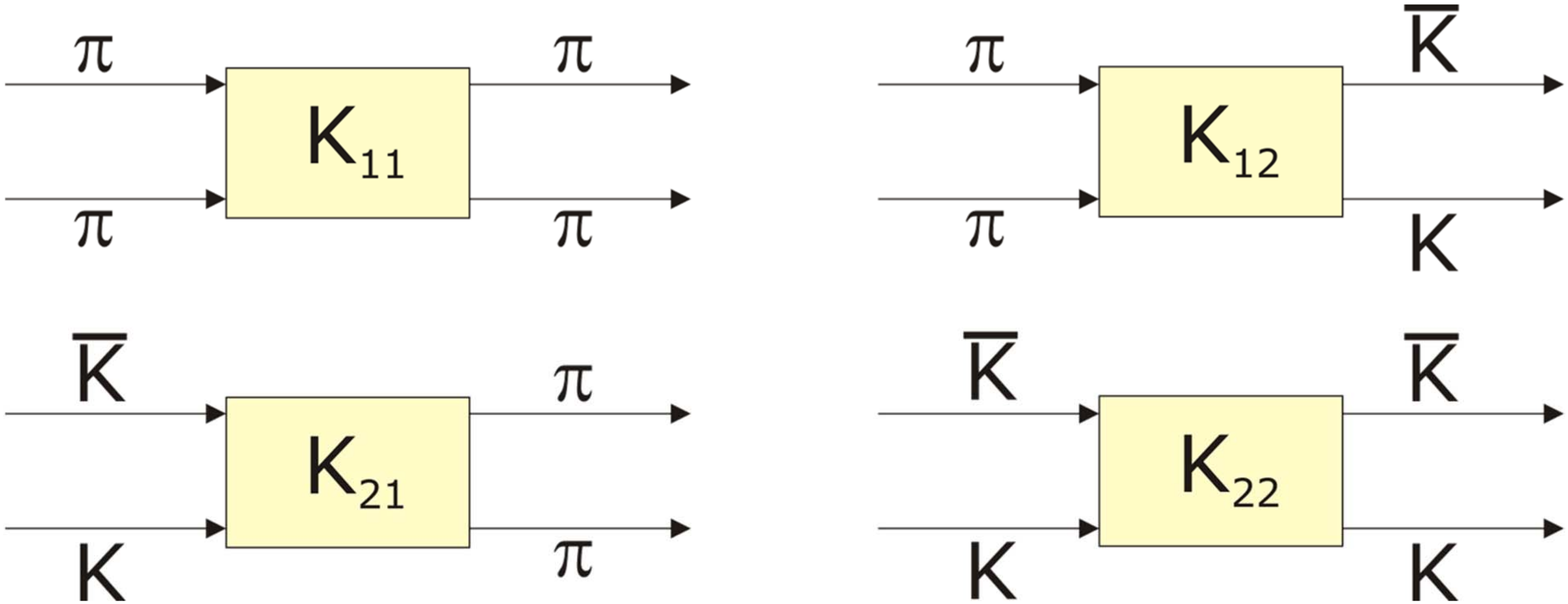
Some more properties

$$\begin{aligned} \Re[T] &= (I + K^2)^{-1} K = K(I + K^2)^{-1} \\ \Im[T] &= (I + K^2)^{-1} K^2 = K^2 (I + K^2)^{-1} \end{aligned}$$

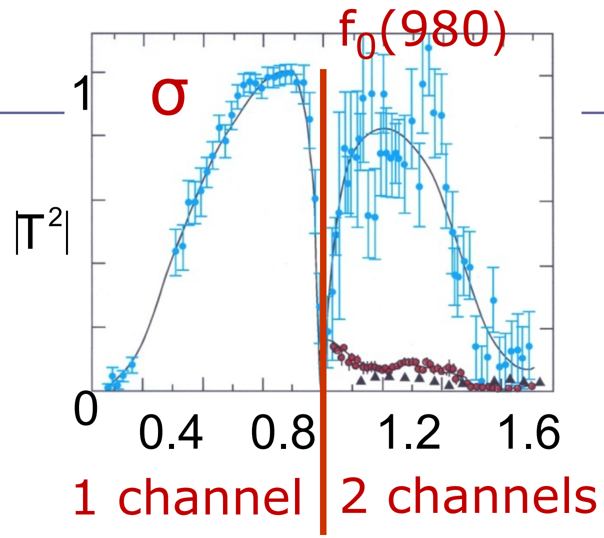
it can be shown, that this leads to $\quad \Im[T] = T^* T = T T^*$

K-Matrix - Interpretation

Each element of the *K*-matrix describes
one particular propagation from initial to final states



Example: ππ-Scattering



1 channel

$$|S| = 1$$

$$S = e^{2\iota\delta}$$

$$K = \tan\delta$$

$$T = e^{\iota\delta}\sin\delta$$

$$\sigma = \left(\frac{4\pi}{q_i^2}\right)\sin^2\delta$$

2 channels

$$S_{ik}S^*_{jk} = \delta_{ij}$$

$$S_{11} = \eta\ e^{2i\delta_1}$$

$$S_{22} = \eta\ e^{2i\delta_2}$$

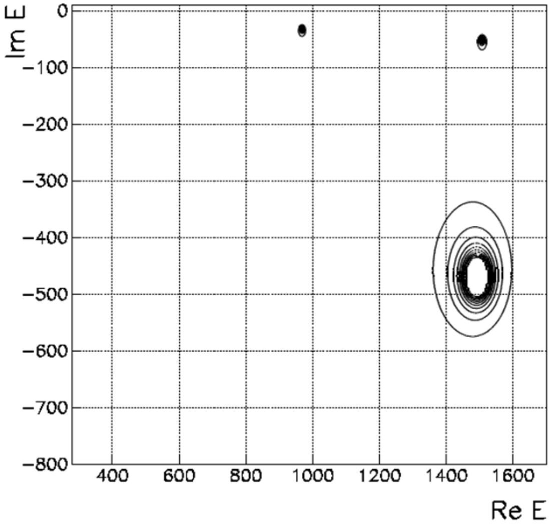
$$S_{12} = \iota\sqrt{1-\eta^2}\ e^{\iota\varphi_{12}}, \qquad \varphi_{12} = \delta_1 + \delta_2$$

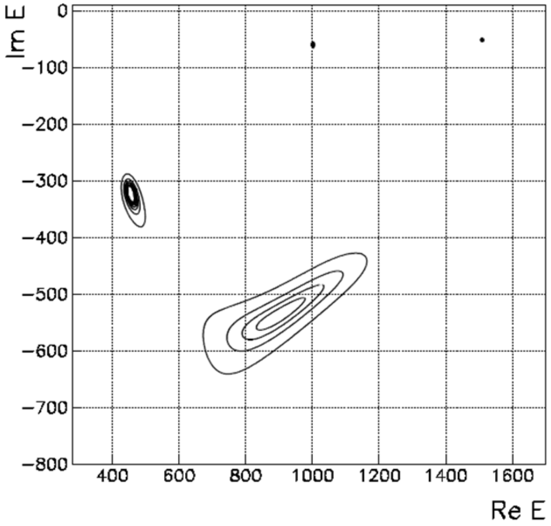
$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

$$T = \frac{1}{1 - D - \iota(K_{11}+K_{22})}\begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix}$$

$$D = K_{11}K_{22} - K_{12}^2$$

Unitarity, cont‘d





Goal: Find a reasonable parameterization

The parameters are used to model the analytic function to follow the data

Only a tool to identify the resonances in the complex energy plane

Does not necessarily help to interpret the data!

Poles and couplings have not always a direct physical meaning

Problem: Freedom and unitarity

Find an approach where unitarity is preserved by construction

And leave a lot of freedom for further extension

Relativistic Treatment

So far we did not care about relativistic kinematics

covariant description $$T = \{\rho\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}}$$

or $$T_{ij} = \{\rho_i\}^{\frac{1}{2}} \ \widehat{T}_{ij} \ \{\rho_j\}^{\frac{1}{2}}$$

and $$S = I + 2\imath \ \{\rho\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}}$$

with $\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$ $\quad \rho_1 = \frac{2q_1}{m}$ and $\rho_2 = \frac{2q_2}{m}$

therefore $\Im\left[\widehat{T}\right] = \widehat{T}^* \rho \widehat{T} = \widehat{T} \rho \widehat{T}^*$ $\quad \Im\left[\widehat{T}^{-1}\right] = -\rho$

and K is changed as well $\quad K = \{\rho\}^{\frac{1}{2}} \ \widehat{K} \ \{\rho\}^{\frac{1}{2}}$ $\quad$ and

$$\widehat{K}^{-1} = \widehat{T}^{-1} + \imath\rho$$

$$\boxed{\widehat{T} = \widehat{K}(I - \imath\rho\widehat{K})^{-1} = (I - \imath\widehat{K}\rho)^{-1}\widehat{K}}$$

Relativistic Treatment (cont’d)

So far we did not care about relativistic kinematics

covariant description $$T = \{\rho\}^{\frac{1}{2}}\ \hat{T}\ \{\rho\}^{\frac{1}{2}}$$

with

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \qquad \rho_1 = \frac{2q_1}{m} \quad \text{and} \quad \rho_2 = \frac{2q_2}{m}$$

in detail

$$\rho_1 = \frac{2q_1}{m} = \sqrt{\left[1 - \left(\frac{m_a + m_b}{m}\right)^2\right]\left[1 - \left(\frac{m_a - m_b}{m}\right)^2\right]}$$

$$\rho_2 = \frac{2q_2}{m} = \sqrt{\left[1 - \left(\frac{m_c + m_d}{m}\right)^2\right]\left[1 - \left(\frac{m_c - m_d}{m}\right)^2\right]}$$

$$\rho_i \to 1 \quad as \quad m^2 \to \infty$$

Relativistic Treatment – 2 channel

S-Matrix

$$
\begin{aligned}
S &= (I + \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})(I - \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1} \\
&= (I - \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1}(I + \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})
\end{aligned}
$$

2 channel T-Matrix

$$
\widehat{T} = \frac{1}{1 - \rho_1\rho_2\widehat{D} - \iota(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \begin{pmatrix} \widehat{K}_{11} - \iota\rho_2\widehat{D} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} - \iota\rho_1\widehat{D} \end{pmatrix}
$$

$$
\widehat{D} = \widehat{K}_{11}\widehat{K}_{22} - \widehat{K}_{12}^2
$$

to be compared with the non-relativistic case

$$
T = \frac{1}{1 - D - \iota(K_{11} + K_{22})} \begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix}
$$

$$
D = K_{11}K_{22} - K_{12}^2
$$

K-Matrix Poles

Now we introduce resonances
as poles (propagators)

$$K_{ij} = \sum_R \frac{g_{Ri}(m) g_{Rj}(m)}{m_R^2 - m^2} + c_{ij}$$

One may add c_{ij} a real polynomial
of m^2 to account for
slowly varying background
(not experimental background!!!)

$$\widehat{K}_{ij} = \sum_R \frac{g_{Ri}(m) g_{Rj}(m)}{(m_R^2 - m^2)\sqrt{\rho_i \rho_j}} + \hat{c}_{ij}$$

$$g_{Ri}^2(m) = m_R \Gamma_{Ri}(m)$$

Width/Lifetime

$$\Gamma_R(m) = \sum_i \Gamma_{Ri}(m)$$

$$\Gamma_{Ri}(m) = \frac{g_{Ri}^2(m)}{m_R} = \gamma_{Ri}^2 \Gamma_R^0 \left[B_{Ri}^l(q, q_R) \right]^2 \rho_i$$

For a single channel and one pole we get

$$T = e^{i\delta} \sin\delta = \left[\frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} \right] [B^l(q, q_0)]^2 \left(\frac{\rho}{\rho_0} \right)$$

- using $g_{\alpha i}^0$ the Lorentz invariant K-Matrix gets a simple form
- It is possible to parametrize non-resonant backgrounds by additional unitless real constants or functions c_{ij}
- Unitarity is still preserved

$$\hat{K}_{ij} = \sum_{\alpha} \frac{\gamma_{\alpha i}\gamma_{\alpha j}\Gamma_{\alpha}^{0} B^{l}_{\alpha i}(q,q_{\alpha}) B^{l}_{\alpha j}(q,q_{\alpha})}{m_{\alpha}^{2}-m^{2}} = \sum_{\alpha} \frac{g^{0}_{\alpha i} g^{0}_{\alpha j} B^{l}_{\alpha i}(q,q_{\alpha}) B^{l}_{\alpha j}(q,q_{\alpha})}{m_{\alpha}^{2}-m^{2}}$$

$$\hat{K}_{ij} \rightarrow \hat{K}_{ij} + c_{ij}$$

- In the trivial case of only one resonance in a single channel the classical Breit-Wigner is retained with

$$K = \frac{m_0 \Gamma(m)}{m_0^2 - m^2} = \tan\delta$$

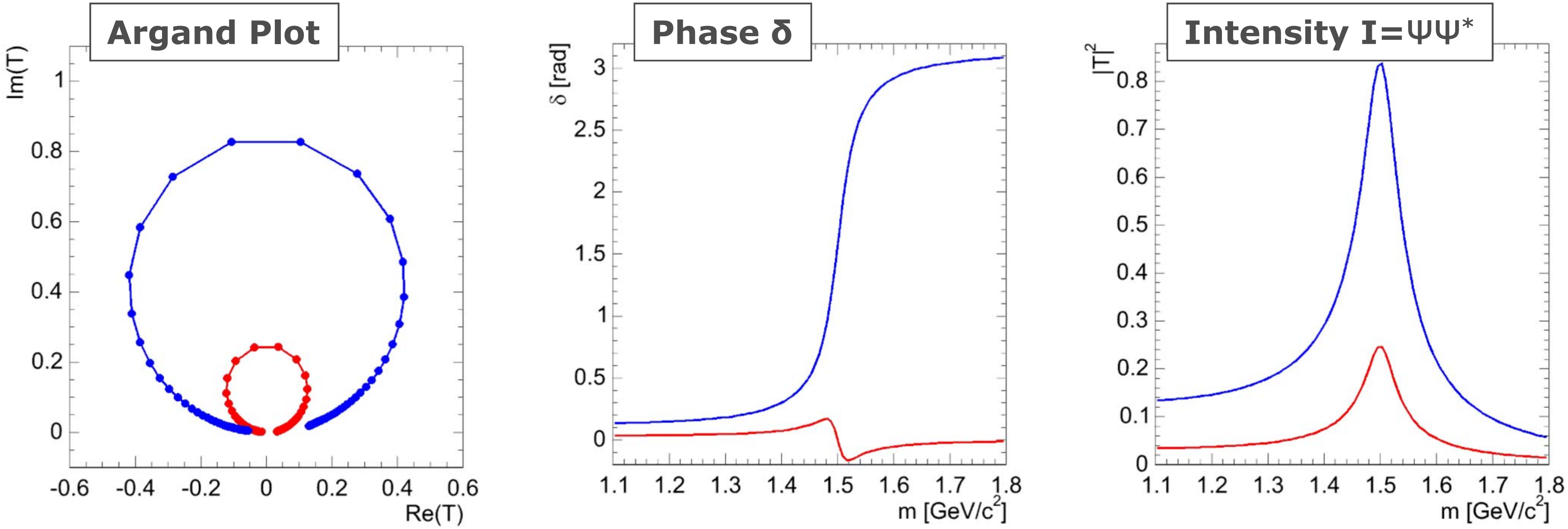
$$\Gamma(m) = \tilde{\Gamma}_0 \left(\frac{\rho}{\rho_0}\right) \left[B^1(q,q_0)\right]^2$$

$$T = e^{i\delta}\sin\delta = \left[\frac{m_0\tilde{\Gamma}_0}{m_0^2 - m^2 - i m_0 \Gamma(m)}\right] \left[B^1(q,q_0)\right]^2 \left(\frac{\rho}{\rho_0}\right)$$

$$T = +i \quad \text{and} \quad \hat{T} = \frac{+i}{\rho}$$

$$\text{at} \quad m = m_0$$

Example: 1x2 K-Matrix



Strange effects in subdominant channels

Scalar resonance at 1500 MeV/c^2, Γ=100 MeV/c^2

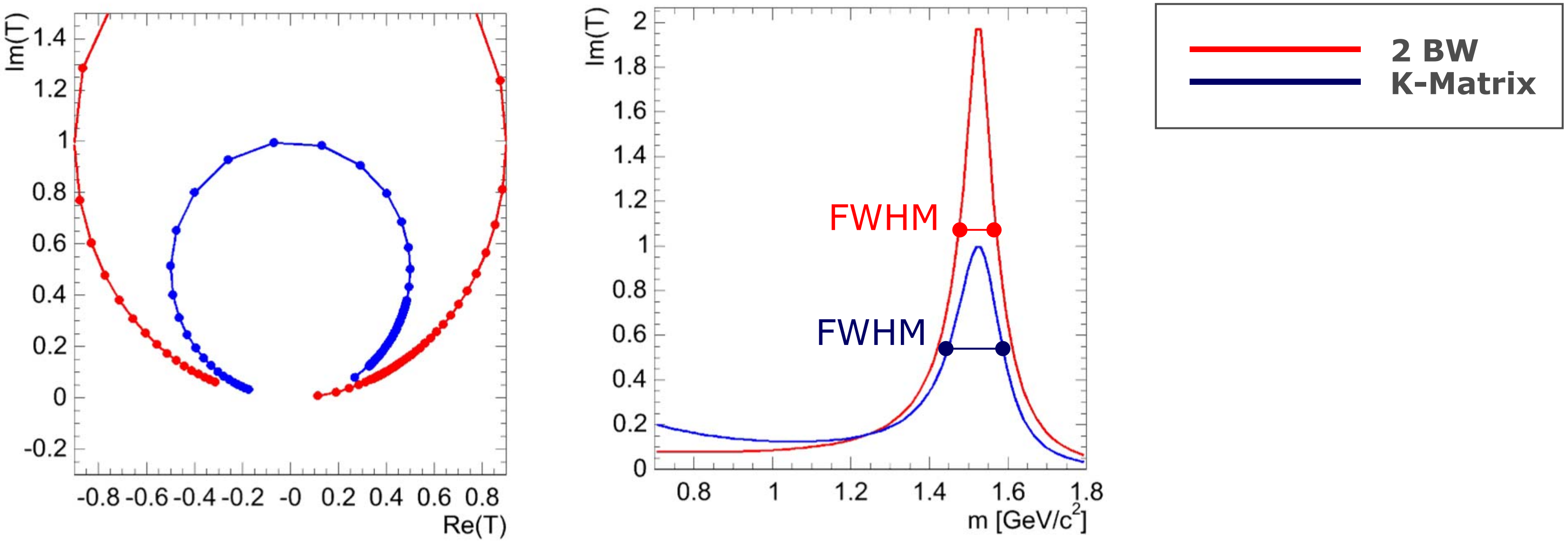
All plots show ππ channel

Blue: $\pi\pi$ dominated resonance ($\Gamma_{\pi\pi}$=80 MeV and $\Gamma_{K\bar{K}}$=20 MeV)

Red: $K\bar{K}$ dominated resonance ($\Gamma_{K\bar{K}}$=80 MeV and $\Gamma_{\pi\pi}$=20 MeV)

Look at the tiny phase motion in the subdominant channel

Example: 2x1 K-Matrix Overlapping Poles



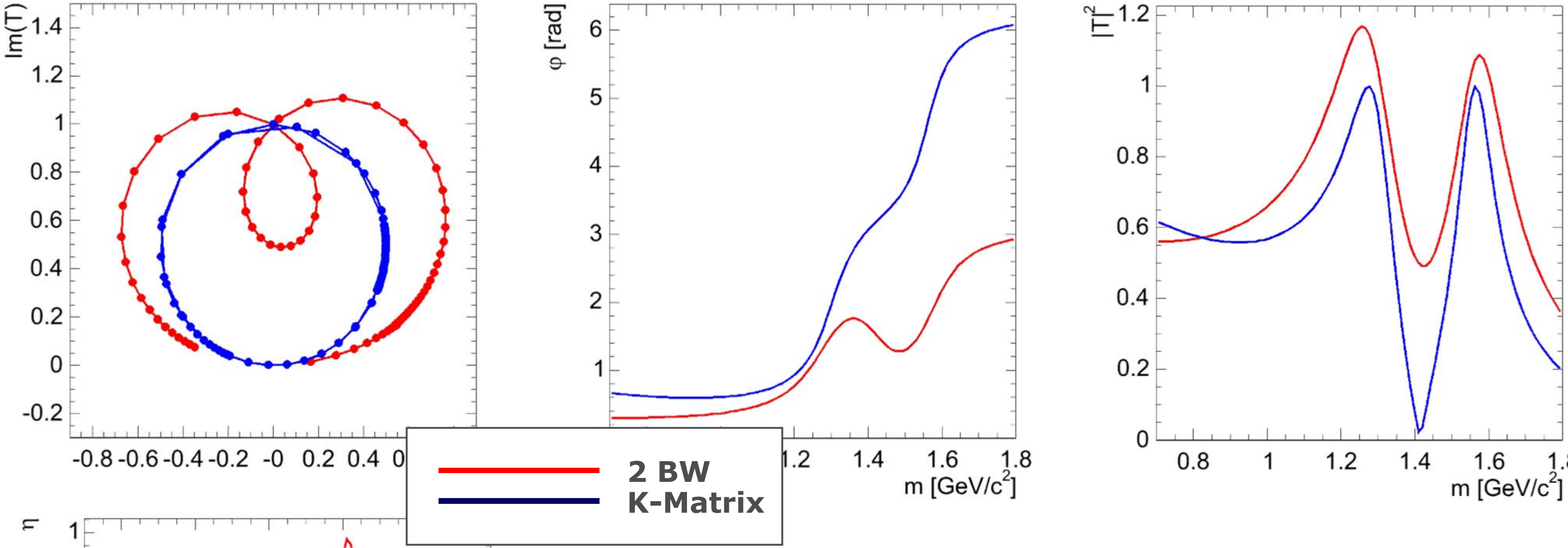
two resonances overlapping with different (100/50 MeV/c^2)

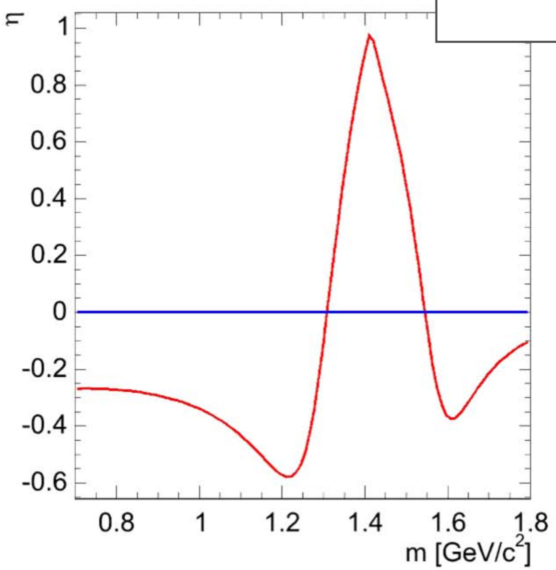
widths are not so dramatic (except the strength)

The width is basically added

$$T = \frac{m_0[\Gamma_a(m) + \Gamma_b(m)]}{m_0^2 - m^2 - i m_0[\Gamma_a(m) + \Gamma_b(m)]}$$

Example: 1x2 K-Matrix Nearby Poles





Two nearby poles (m=1.27 and 1.5 GeV/c^2) show nicely the effect of unitarization

$$K = \frac{m_a \Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2}$$

$$\Gamma_R(m) = \Gamma_R^0 \left(\frac{m_a}{m}\right) \left(\frac{q}{q_R}\right) [B^2(q, q_R)]^2$$

Example: Flatté 1x2 K-Matrix

2 channels for a single resonance at the threshold of one of the channels

with $\gamma_1^2 + \gamma_2^2 = 1$

$$\hat{K}_{11} = \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\hat{K}_{22} = \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\hat{K}_{12} = \hat{K}_{21} = \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2}$$

Leading to the *T*-Matrix

$$\hat{T} = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma_0 (\rho_1 \gamma_1^2 + \rho_2 \gamma_2^2)} \begin{pmatrix} \gamma_1^2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_2^2 \end{pmatrix}$$

and with

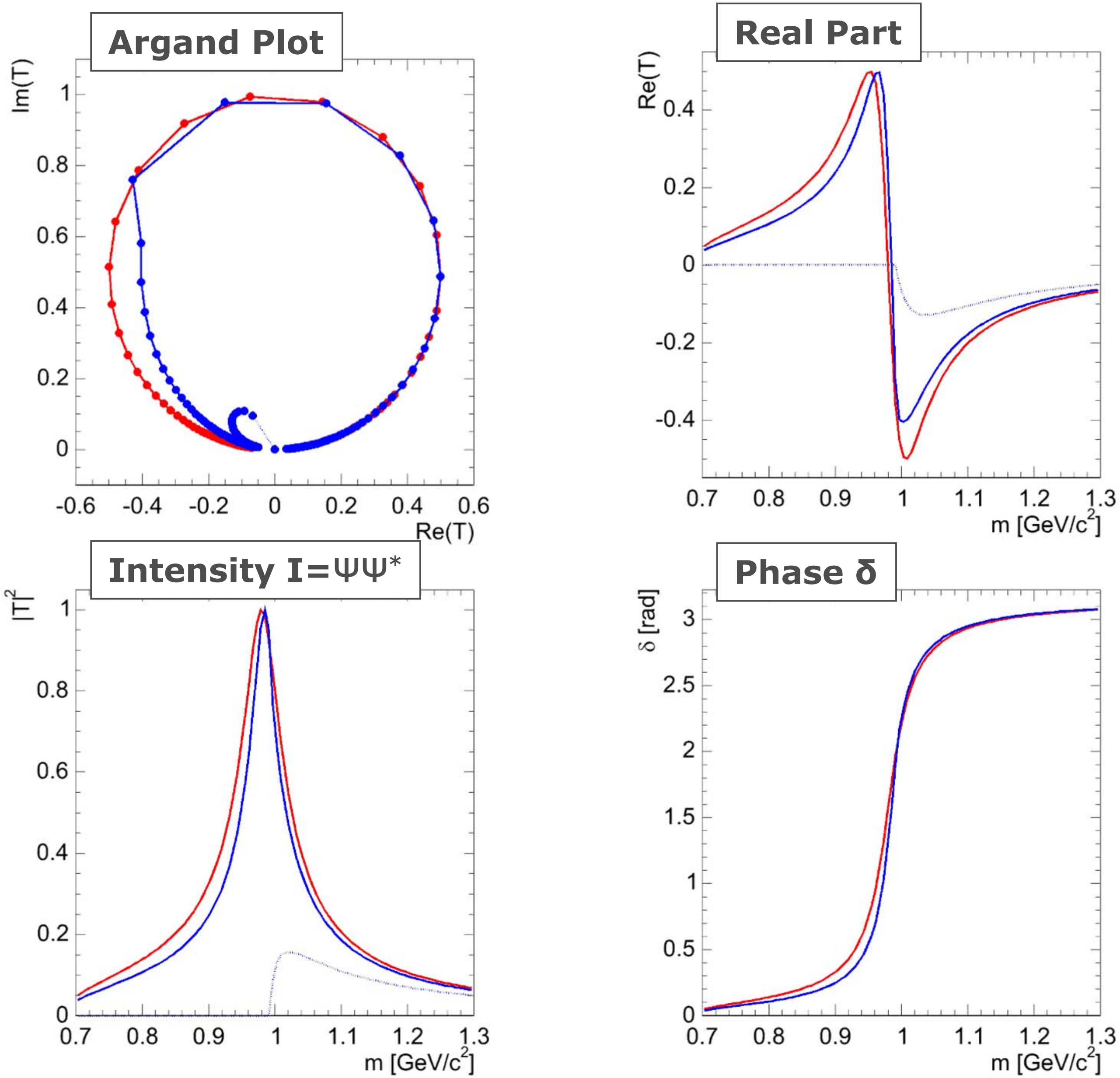
$$g_i = \gamma_i \sqrt{m_0 \Gamma_0}$$

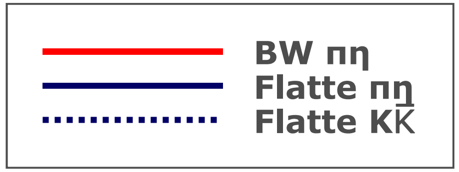
$$g_1^2 + g_2^2 = m_0 \Gamma_0$$

we get

$$\hat{T} = \frac{\begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix}}{m_0^2 - m^2 - \imath (\rho_1 g_1^2 + \rho_2 g_2^2)}$$

Flatté





Example

$a_0(980)$ decaying into $\pi\eta$ and $K\bar{K}$

Flatté Formula, cont‘d

$a_0(980)$ appears as a „regular“ resonance in the $\pi\eta$ system (channel 1)

comparable BW denominator for m near m_R is

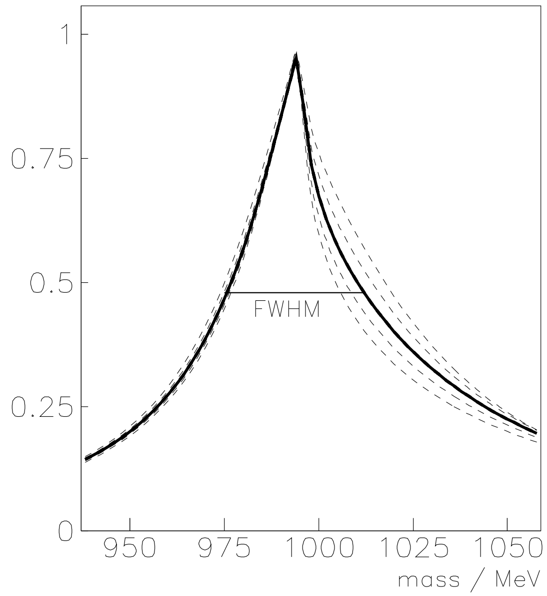
$$m_c^2 - m^2 - m_c \Gamma_c$$

$$m_0^2 = m_c^2 + \left(\frac{\gamma_1}{\gamma_2}\right)^2 \left[\frac{|\rho_2(m_c)|}{\rho_1(m_c)}\right] m_c \Gamma_c$$

$$\Gamma_0 = \frac{m_c \Gamma_c}{m_0 \rho_1(m_c)\gamma_1^2}$$

Simulated mass distributions in the $a_0(980)$ region using the Flatté formula

dashed lines correspond to different ratios of γ_2^2/γ_1^2



Flatté Formula, Pole Structure

Due to the simple form, the pole structure can be explored analytically

4 Riemann sheets (I-IV)

identified with real and imaginary part of q_2

(+,+), (-,+), (+,-) (-,-)

$$\rho_1 \approx 1 \quad q_2 \ll m_K \quad for \quad m \simeq 2m_K$$

$$2m_K q_1 \simeq 2m_K^2 + q_2^2$$

$$for \quad q^2 = re^{i\varphi}$$

$$\Im\left(q_1\right) \simeq \left(\frac{r^2}{2m_K}\right)\sin 2\varphi$$

$$q_a = -\alpha + i\beta$$

$$q_b = +\alpha - i\gamma$$

$$g_1^2 = 4\alpha\left(\gamma + \beta\right)$$

$$g_1^2 = 4m_K\left(\gamma - \beta\right)$$

$$m_0 \simeq 2m_K + \frac{\alpha^2 - \beta\gamma}{m_K}$$

$$\Rightarrow \alpha, \beta, \gamma > 0$$

$$\Rightarrow \gamma > \beta$$

Flatté Formula, Pole Structure, cont'd

Flatté formula entails two poles in sheet II (for q_a) and sheet III (for q_b)

$$m_a \simeq 2m_K + \frac{\alpha^2 - \beta^2}{m_K}$$

$$m_b \simeq 2m_K + \frac{\alpha^2 - \gamma^2}{m_K}$$

$$\Gamma_a \simeq \frac{4\alpha\beta}{m_K}$$

$$\Gamma_b \simeq \frac{4\alpha\gamma}{m_K}$$

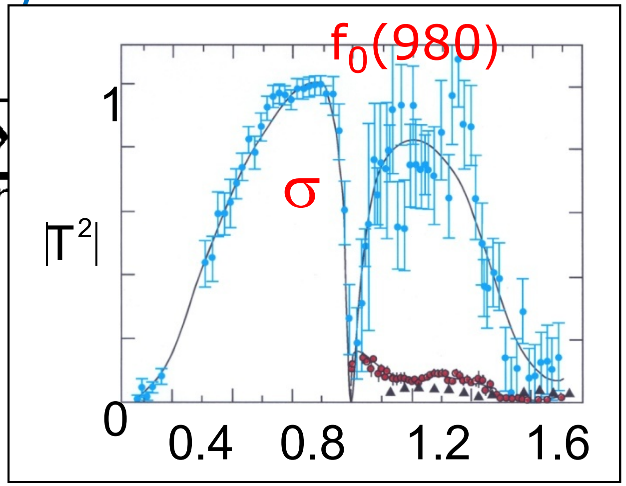
$$m_0 \simeq \frac{m_a + m_b}{2} + \frac{(\gamma - \beta)^2}{2m_K}$$

$$\Gamma_0 \simeq \left(\frac{2m_K}{m_0}\right)\left[\frac{\Gamma_a + \Gamma_b}{2} + 2(\gamma - \beta)\right]$$

Au, Morgan and Pennington (1987)

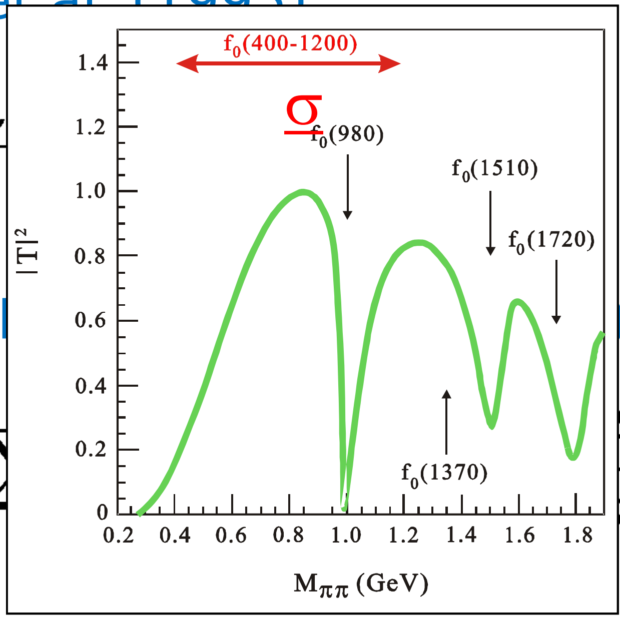
$$K_{ij} = \frac{s-s_0}{4m_K^2}\sum_r \frac{g_{r,i}g_{r,j}}{(s_r-s)(s_r-s_0)} + \sum_r \cdots$$

$$\equiv (s-s_0)\widehat{K}_{ij}$$



Amsler et al. (1995)

$$K_{ij} = \sum_r \frac{g_{r,} \cdots}{\cdots}$$

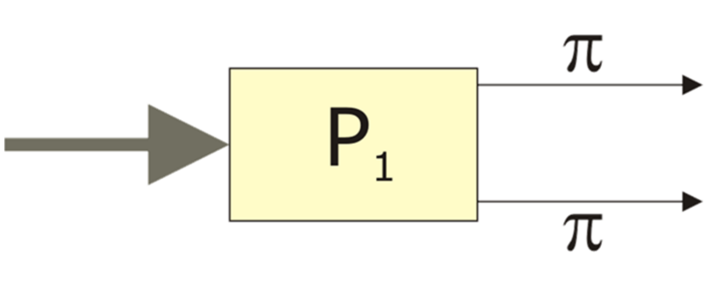


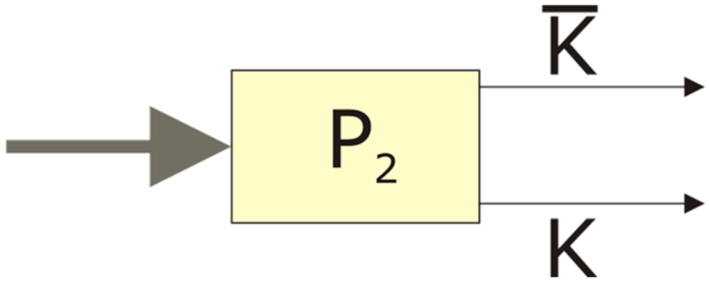
Anisovic…

$$K_{ij}(s) = \left(\sum \cdots \right) \cdots \frac{s-s_A}{s+s_{A0}}$$

P-Vector Definition

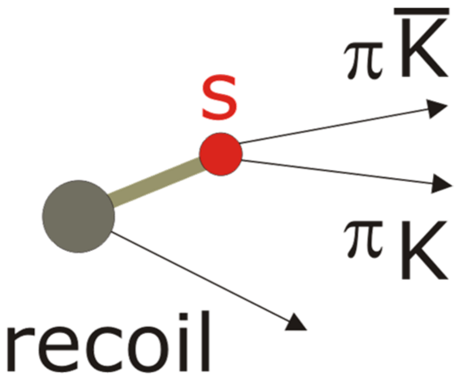
But in many reactions there is no scattering process but a production process, a resonance is produced with a certain strength and then decays

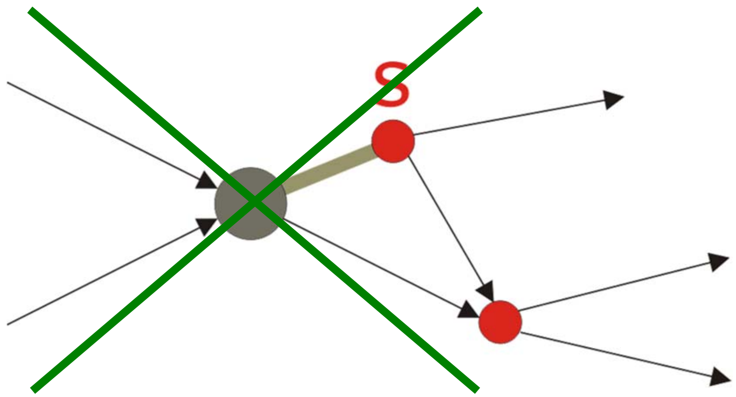




Aitchison (1972) $F = (I - iK)^{-1} P = T K^{-1} P$

$\hat{F} = (I - i\hat{K}\rho)^{-1} \hat{P} = \hat{T} \hat{K}^{-1} \hat{P}$ with $F = \{\rho\}^{\frac{1}{2}} \hat{F}$ and $P = \{\rho\}^{\frac{1}{2}} \hat{P}$





P-Vector Poles

The resonance poles are constructed as in the *K*-Matrix

$$P_i = \sum_R \frac{\beta_R^0 \; g_{Ri}(m)}{m_R^2 - m^2} \qquad \widehat{P}_i = \sum_R \frac{\beta_R^0 \; g_{Ri}(m)}{(m_R^2 - m^2)\sqrt{\rho_i}}$$

and one may add a polynomial ***d_i*** again $P_i \rightarrow P_i + d_i$

For a single channel and a single pole

$$\widehat{F}(m) = \beta \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} B^l(q, q_0)$$

If the *K*-Matrix contains fake poles...

for non *s*-channel processes modeled in an s-channel model

...the corresponding poles in *P* are different

Q-Vector

A different Ansatz with a different picture: channel *n* is produced and undergoes final state interaction

$$Q = K^{-1}P \quad \text{and} \quad \{\rho\}^{\frac{1}{2}} Q = \widehat{Q} \quad \text{and} \quad \widehat{Q} = \widehat{K}^{-1}\widehat{P}$$

$$F = TQ \quad \text{and} \quad \widehat{F} = \widehat{T}\widehat{Q}$$

For channel 1 in 2 channels

$$F_1 = T_{11}Q_1 + T_{12}Q_2$$

The Breit-Wigner example

$$T = e^{i\delta}\sin\delta = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)}\right] \left[B^l(q, q_0)\right]^2 \left(\frac{\rho}{\rho_0}\right)$$

shows, that *Γ(m)* implies *ρ(m)*

$$\Gamma_{Ri}(m) = \frac{g_{Ri}^2(m)}{m_R} = \gamma_{Ri}^2 \Gamma_R^0 \left[B_{Ri}^l(q, q_R)\right]^2 \rho_i$$

Each *ρ(m)* which is a square root,

one obtains two solutions for $p>0$ or $p<0$ respectively

Complex Analysis Revisited (cont'd)

one obtains two solutions for *p>0* or *p<0* respectively

$$p > 0 \qquad\qquad p < 0$$

$$\begin{aligned} \rho_a &= \sqrt{\frac{2|q|}{m}} & \qquad \rho_a &= \iota\sqrt{\frac{2|q|}{m}} \\ \rho_b &= -\sqrt{\frac{2|q|}{m}} & \qquad \rho_b &= -\iota\sqrt{\frac{2|q|}{m}} \end{aligned}$$

But the two values (*w=2q/m*) have some phase in between
and are not identical

$$\sqrt{w} - \sqrt{w^*} = \pm\sqrt{|w|}\left(e^{\iota\frac{\varphi}{2}} + e^{-\iota\frac{\varphi}{2}}\right) = \cosh\frac{\varphi}{2}\bigg|_{\varphi=0} \neq 0$$

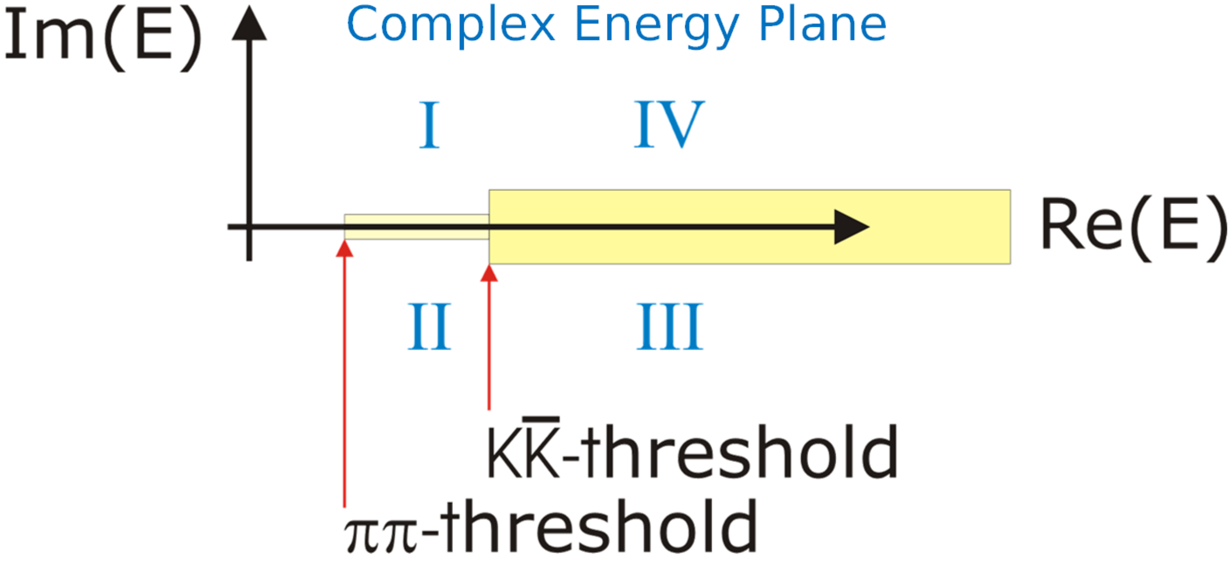
So you define a new complex plane for each solution,
which are 2^n complex planes, called Riemann sheets
they are continuously connected. The borderlines are called CUTS.

Riemann Sheets in a 2 Channel Problem

Usual definition

sheet	$sgn(q_1)$	$sgn(q_2)$
I	+	+
II	−	+
III	−	−
IV	+	+

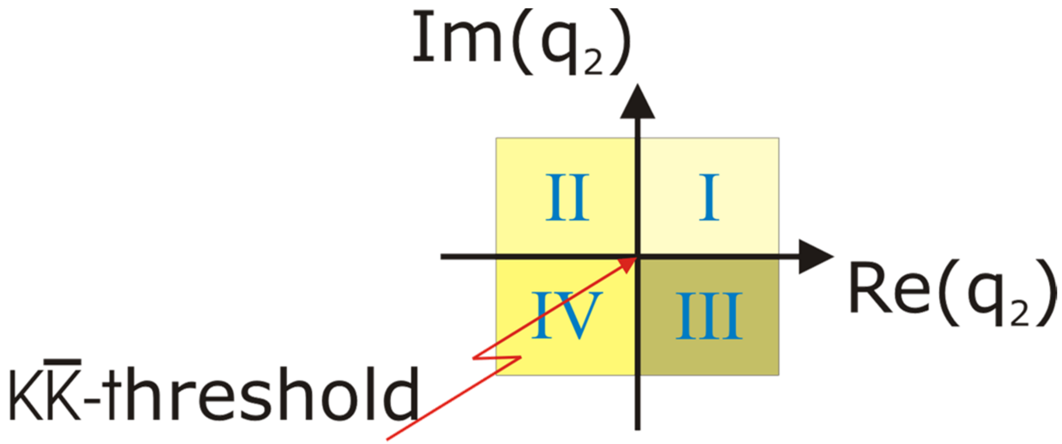
Complex Energy Plane



This implies for the *T*-Matrix

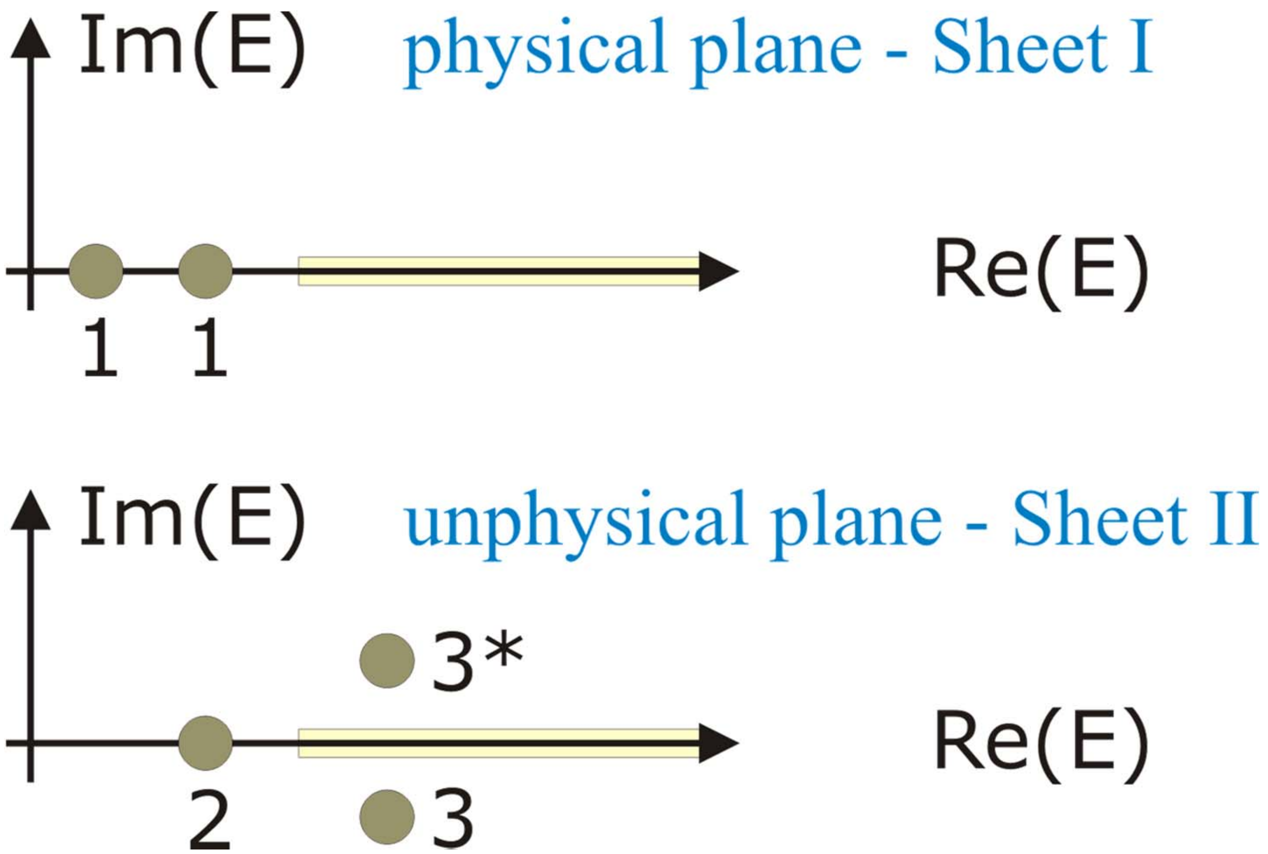
$$\left(\hat{T}^{III}\right)^{-1} = \left(\hat{T}^{II}\right)^{-1} + \imath\rho_2$$

Complex Momentum Plane



Singularities appear naturally where

$$T(E + \iota\frac{\Gamma}{2}) = 0$$



Singularities might be

1 – bound states

2 – anti-bound states

3 – resonances

or

artifacts due to wrong treatment of the model

States on Momentum Sheets

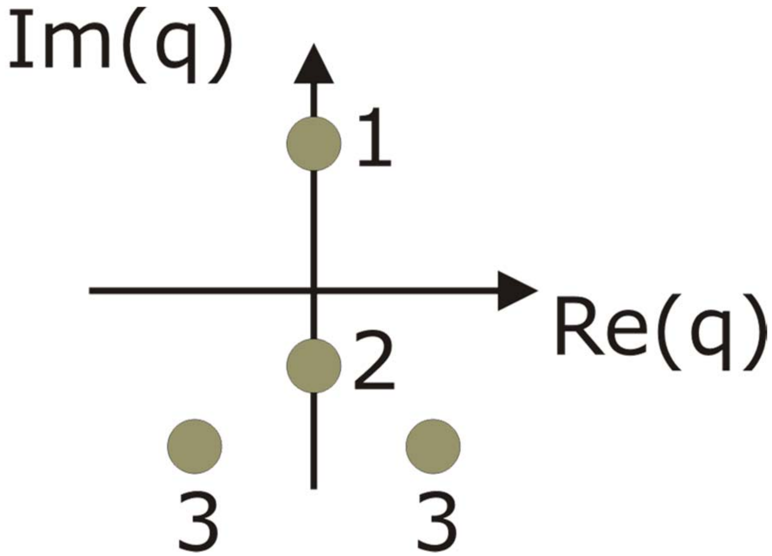
Or in the complex momentum plane

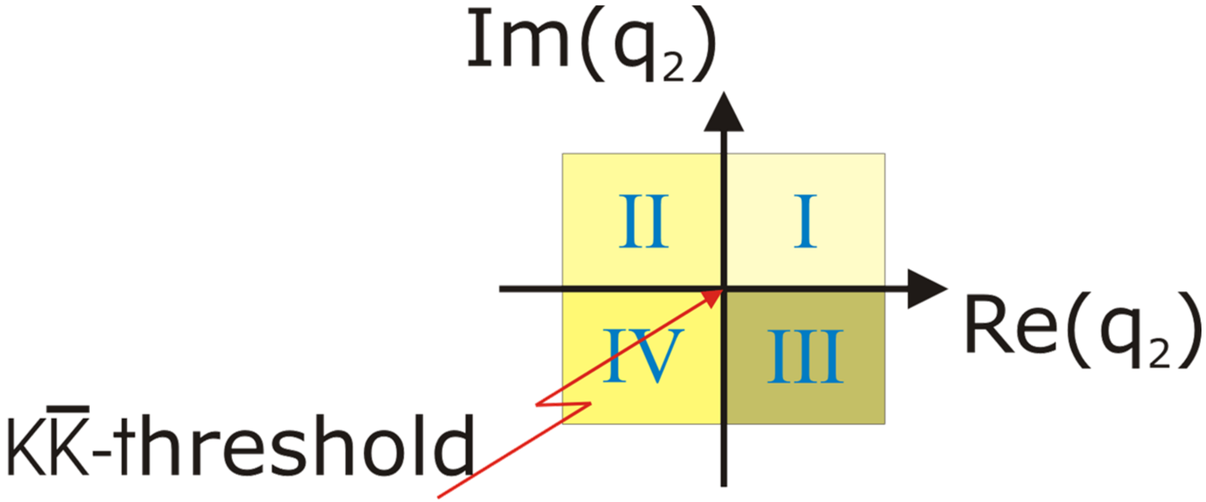
Singularities might be

1 – bound states

2 – anti-bound states

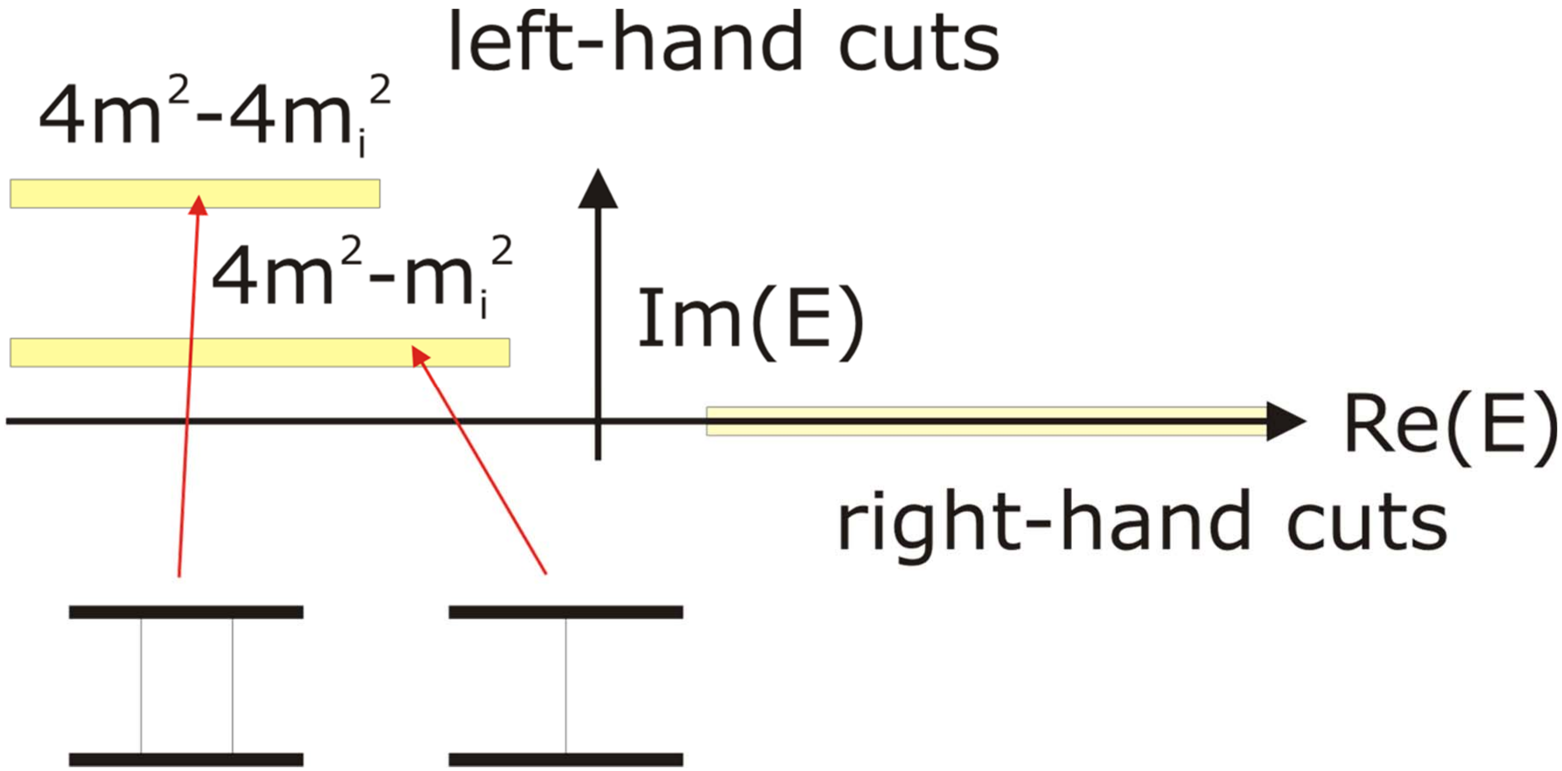
3 – resonances





Left-hand and Right-hand Cuts

The right hand CUTS (RHC) come from the open channels in an n channel problem



But also exchange processes and other effects introduce CUTS on the left-hand side (LHC)

N/D Method

To get the proper behavior for the left-hand cuts

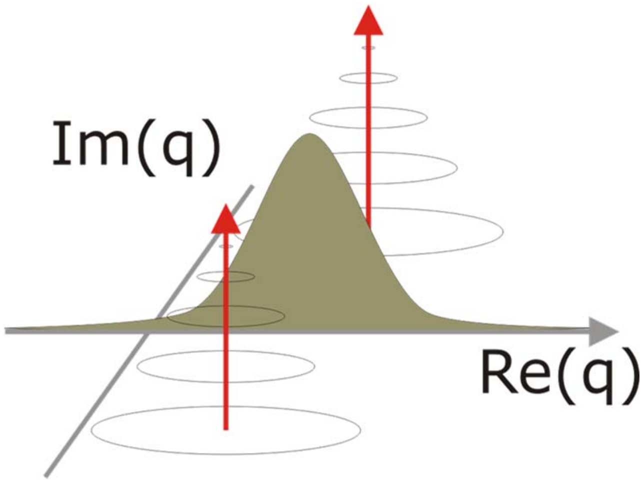
Use $N_l(s)$ and $D_l(s)$ which are correlated by dispersion relations

$$T_l(s) = \frac{N_l(s)}{D_l(s)}$$

An example for this is the work of Bugg and Zhou (1993)

$$\begin{aligned}
K_{ij} &= \left(\frac{s-2m_\pi^2}{s}\right)\left(\frac{\alpha_i\alpha_j}{s_A - s}\frac{\beta_i\beta_j}{s_B - s}\frac{\gamma_i\gamma_j}{s_C - s} + a_{ij} + b_{ij}s\right) \\
N_{\pi\pi}(s) &= N_{11}(s) = (c_1 + c_2 s)K_{11} + i\rho_2(c_3 + c_4 s) \\
&\quad (K_{11}K_{22} - K_{12}K_{21}) \\
N_{\eta\eta}(s) &= N_{22}(s) = c_1 K_{22} + i\rho_2 c_3 (K_{11}K_{22} - K_{12}K_{21})
\end{aligned}$$

Nearest Pole Determines Real Axis



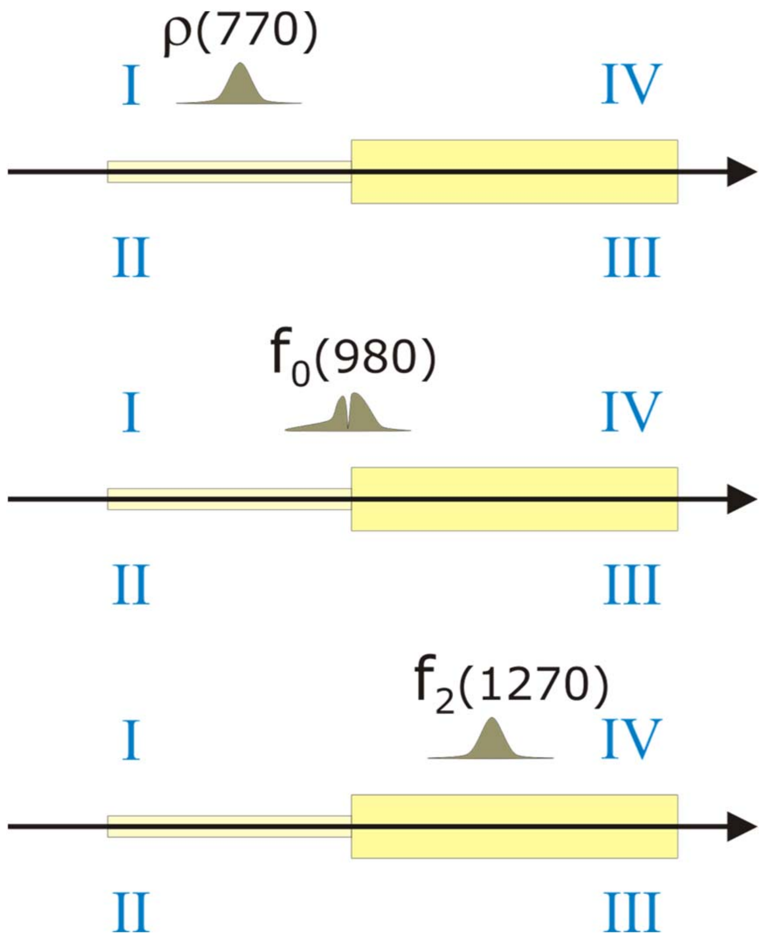
The pole nearest to the real axis

or more clearly to a point with mass *m* on the real axis

determines your physics results

Far away from thresholds this works nicely

At thresholds, the world is more complicated



While $\rho(770)$ in between two thresholds has a beautiful shape the $f_0(980)$ or $a_0(980)$ have not

Pole and Shadows near Threshold (2 Channels)

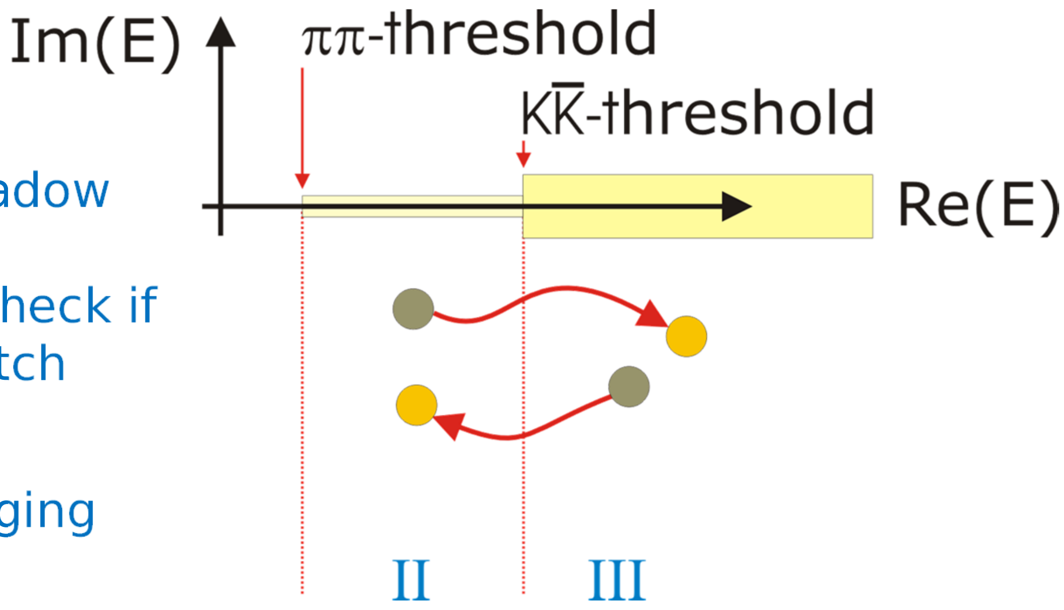
For a real resonance one always obtains poles on sheet II and III

due to symmetries in T_l

$$\hat{T}_l(q) = \hat{T}_l^*(-q^*) \qquad \text{and} \qquad \hat{T}_l(s) = \hat{T}_l^*(s^*)$$

Usually

$$\Gamma_r^{\mathrm{BW}} \approx \frac{1}{2}\left(\Gamma_r^{II} + \Gamma_r^{III}\right)$$



To make sure that pole and shadow match and form an s-channel resonance, it is mandatory to check if the pole on sheets II and III match

This is done by artificially changing ρ_2 smoothly from q_2 to $-q_2$

t-channel Effects (also u-channel)

They may appear resonant and non-resonant

Formally they cannot be used with Isobars

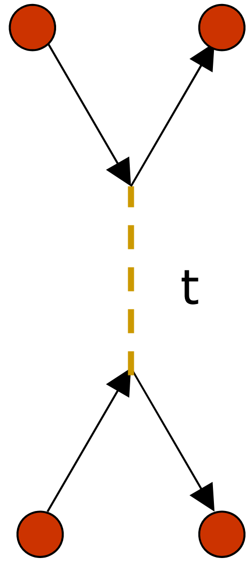
But the interaction is among two particles

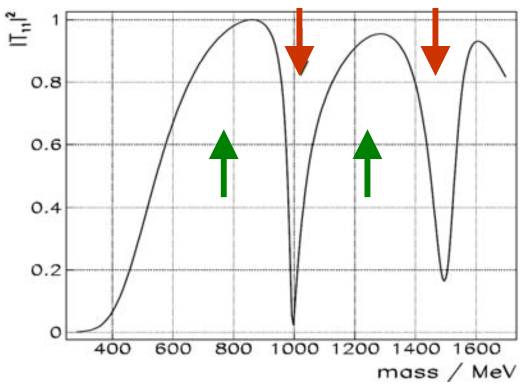
To save the Isobar Ansatz (workaround)

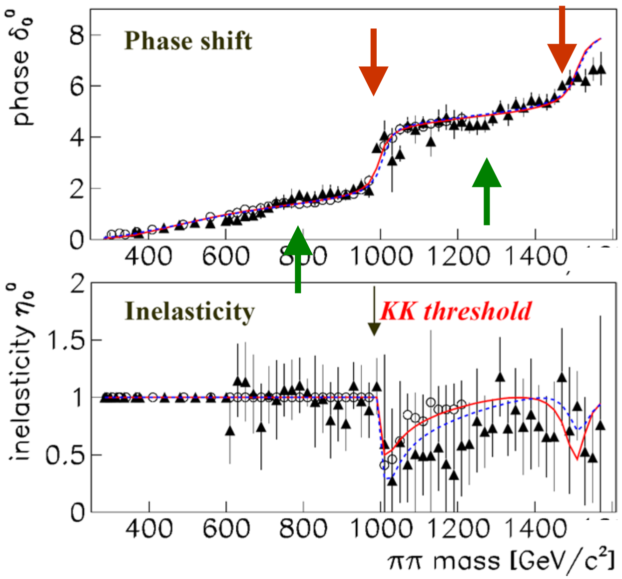
they may appear as unphysical poles in *K*-Matrices

or as polynomial of s in *K*-Matrices

background terms in unitary form

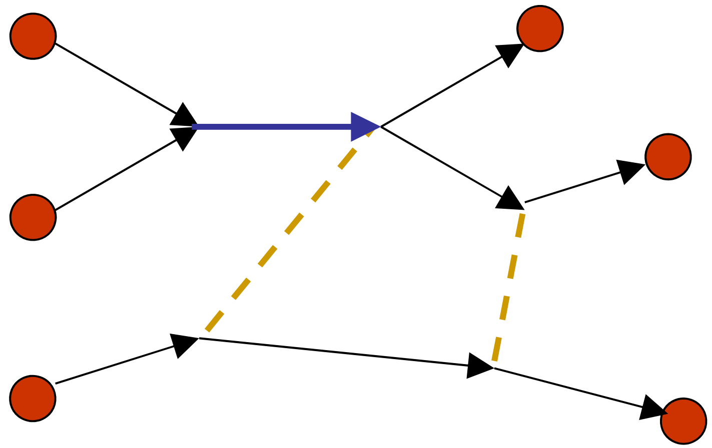


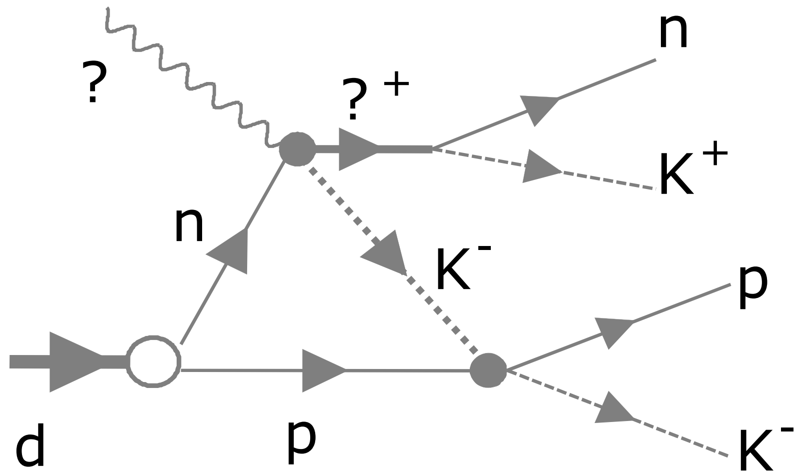




No general solution

Specific models needed





Handling K-Matrices and P-Vectors

Problems of the method are

- performance (complex matrix-inversions!)
- numerical instabilities
  - singularities
- unitarity constraints
  - for *P*-Vectors
- cut structure
  - behavior at left- and right-hand cuts

Handling K-Matrices and P-Vectors

Problems of the method are

- unmeasured channels
  - yield huge problems if numerous or dominant
- systematic errors of the experiment
  - relative efficiency, shift in mass, different resolutions
- damping factors (sizes) for respective objects

Handling K-Matrices and P-Vectors

Problems in terms of interpretation are

- mapping *K*-Matrix to *T*-Matrix poles
  - number might be different
- branching ratios
  - *K*-matrix strength is unequal T-matrix coupling

Handling K-Matrices and P-Vectors

Problems in terms of interpretation are

- validity of *P*-vectors
  - all channels need to have identical production processes
  - FSI has to be dominant
- singularities
  - not all are resonances ⇒ limit of the isobar model

Summary

K-Matrix is a good tool

if one obeys a few rules

ideally one would like to use an unbiased parameterization which fulfills everything

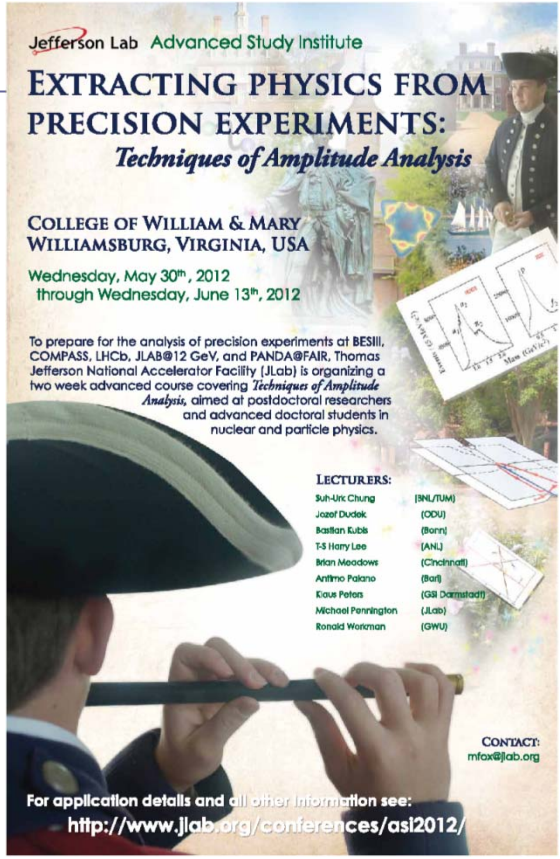
use the best you can for your case and document well, what you have done

THANK YOU

for today

Amplitude Analysis
An Experimentalists View

K. Peters



Part VI

Experiments

Experiments

Background

Numerical Issues

Goodbess-of-Fit

Computers

Phase space

do you expect phase space distortions?

- for example from varying efficiencies
  - example: $\epsilon(p) \neq$ const.

how strong is the event displacement?

- due to resolution
  - example: m^2 has Gaussian smeared
  - may end up in a different bin
- due to wrong particle assignments
  - example: 15 combinations of 6γ may form $3\pi^0$
  - a wrong assignment is still reconstructed but with different coordinates
- has it impact on the model and/or the method?

Finally: Coupled channels

Coupling can occur in initial and final states

same intermediate state, but everything else is different

coupling due to related production mechanisms

is a very important tool, but not the focus of this talk.

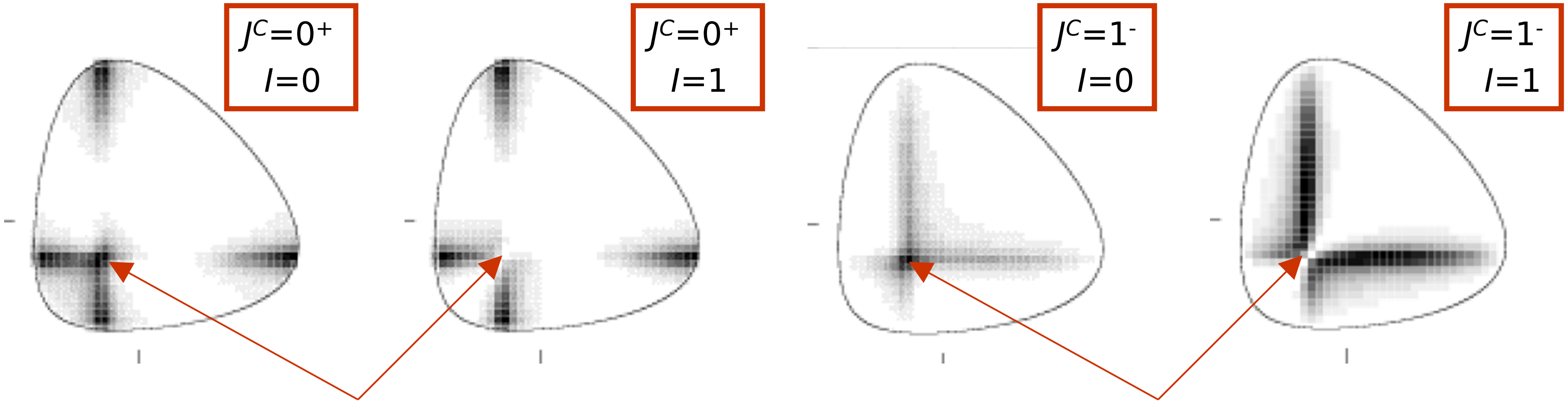
Isospin relations (pure hadronic)

combine different channels of the same gender, like

$\pi^+\pi^-$ and $\pi^0\pi^0$ (as intermediate states)

or combining pp̄, p̄n and nn̄

or X⁰→$KK\pi$, Example K^* in $K^+K_L\pi^-$



Fitting

There are many programs and packages on the market

but there are a few importat aspects which should be mentioned

Having an good (algebraic) description of the Hesse-Matrix is vital for fast and stable convergence

- MINUIT does not allow for them → need for improved version
  - now: only numerical calculation of 2nd derivatives
- FUMILI uses an approximation → good convergence
  - even if the approximation is not always correct

$$f = -\sum \log L\left(x_1, \ldots, x_2\right) \Rightarrow$$

$$\frac{\partial f}{\partial x_i} = \sum \frac{1}{L}\frac{\partial L}{\partial x_i} \Rightarrow$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum\left(\frac{1}{L}\underbrace{\frac{\partial^2 L}{\partial x_i \partial x_j}}_{0} - \frac{1}{L^2}\left(\frac{\partial L}{\partial x_i}\frac{\partial L}{\partial x_j}\right)\right)$$

$$\frac{\partial^2 a^2}{\partial \alpha^2} = 1 \neq 4a^2 = \left(\frac{\partial a^2}{\partial \alpha}\right)^2$$

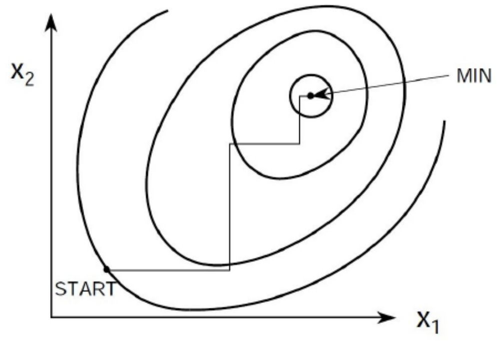
$$\frac{\partial^2 \sin\alpha}{\partial \alpha^2} = -\sin\alpha \neq \cos^2\alpha = \left(\frac{\partial \sin\alpha}{\partial \alpha}\right)^2$$

$$\frac{\partial^2 \sin^2\alpha}{\partial \alpha^2} \neq \left(\frac{\partial \sin^2\alpha}{\partial \alpha}\right)^2$$

Minimization

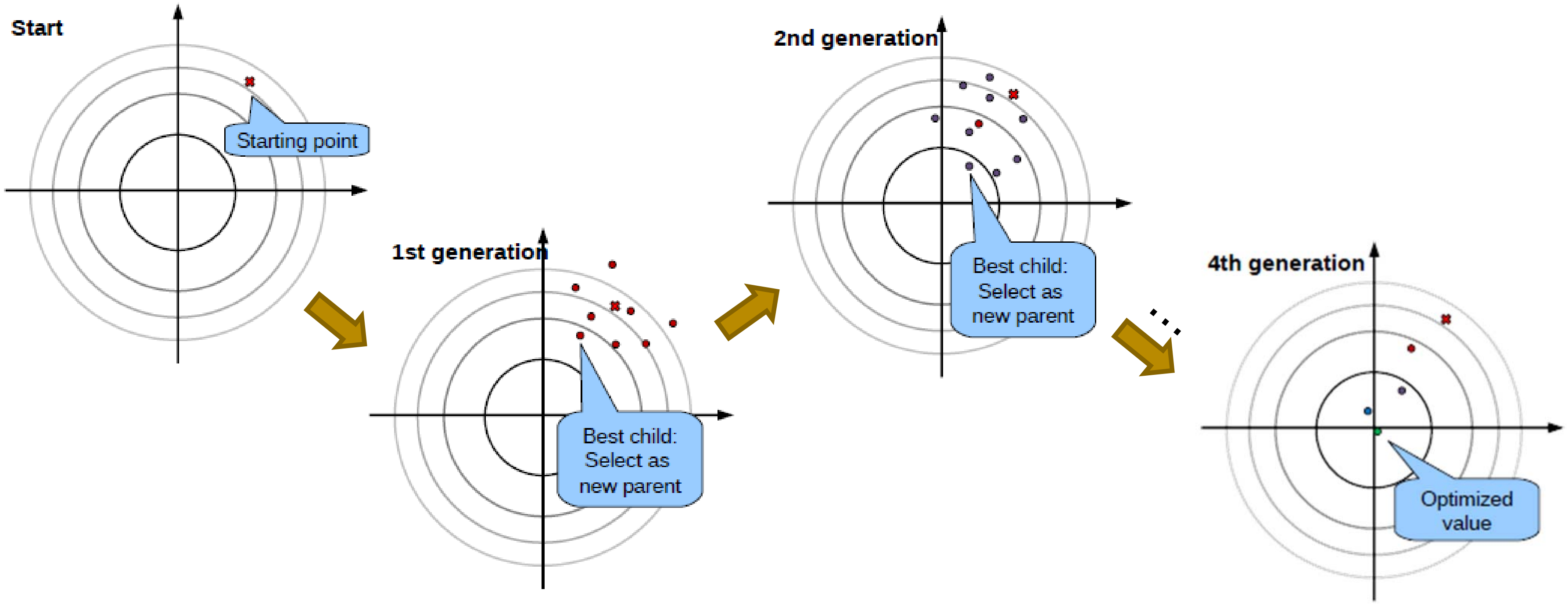
MINUIT2 = classical gradient descent

Sometimes gets stuck in local minima



Alternative: Evolutionary Strategy GenEvA

→ new solutions created from previous ones (offspring)

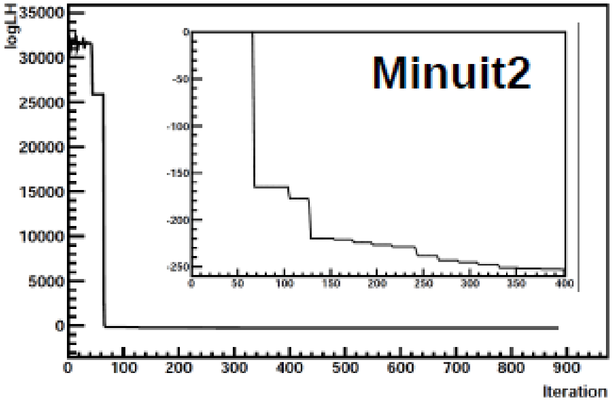


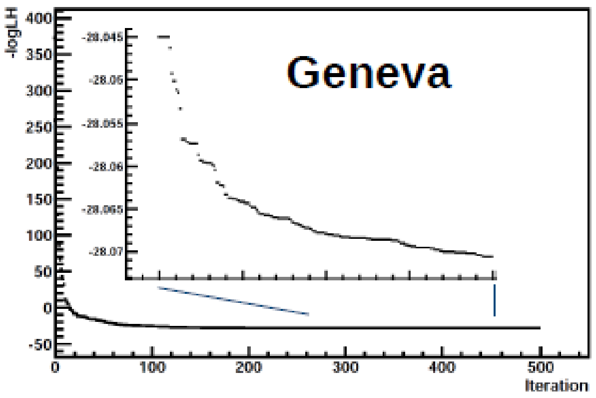
GenEvA Example

Example: Angular distribution + maximum spin of

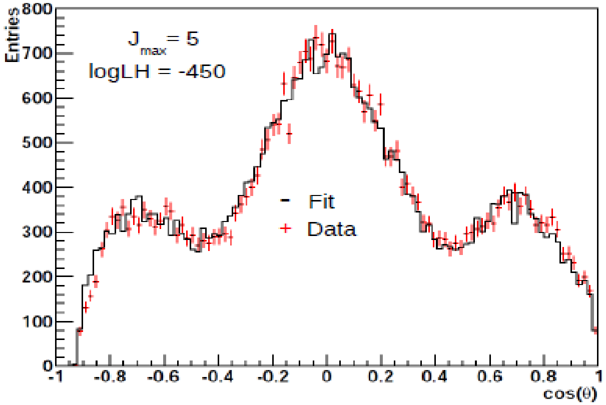
$$\bar{p}p \rightarrow \omega\pi^0,\ \omega \rightarrow \pi^0\gamma$$ @ 1940 MeV/c (LEAR data)

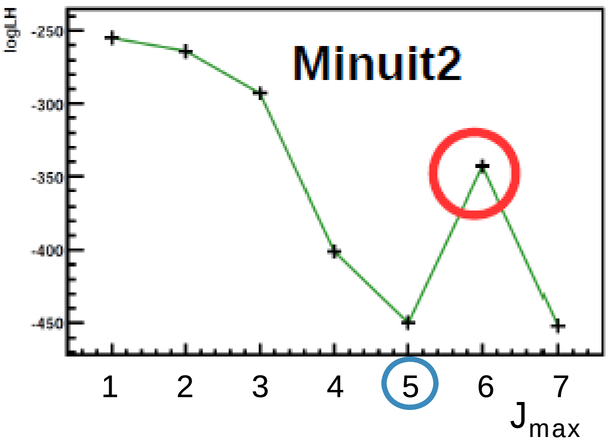
Convergence behaviour of minimizing log(LH)

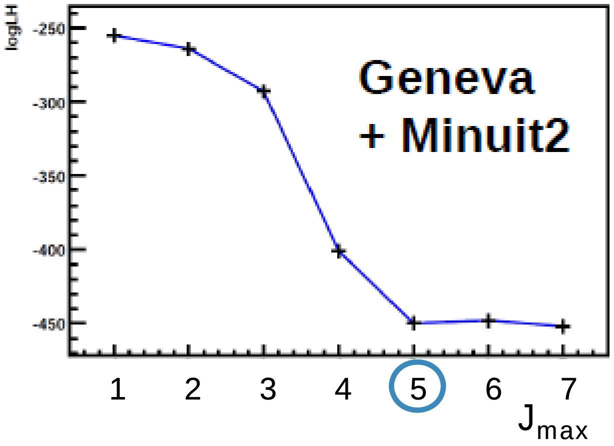




Result: $J_{max} = 5$







Less probability to get stuck in local minima!

Adaptive binning

Finite size effects in a bin of the Dalitz plot

limited line shape sensitivity for narrow resonances

Entry cut-off for bins of a Dalitz plots

χ^2 makes no sense for small #entries

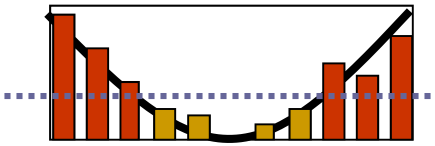
cut-off usually 10 entries

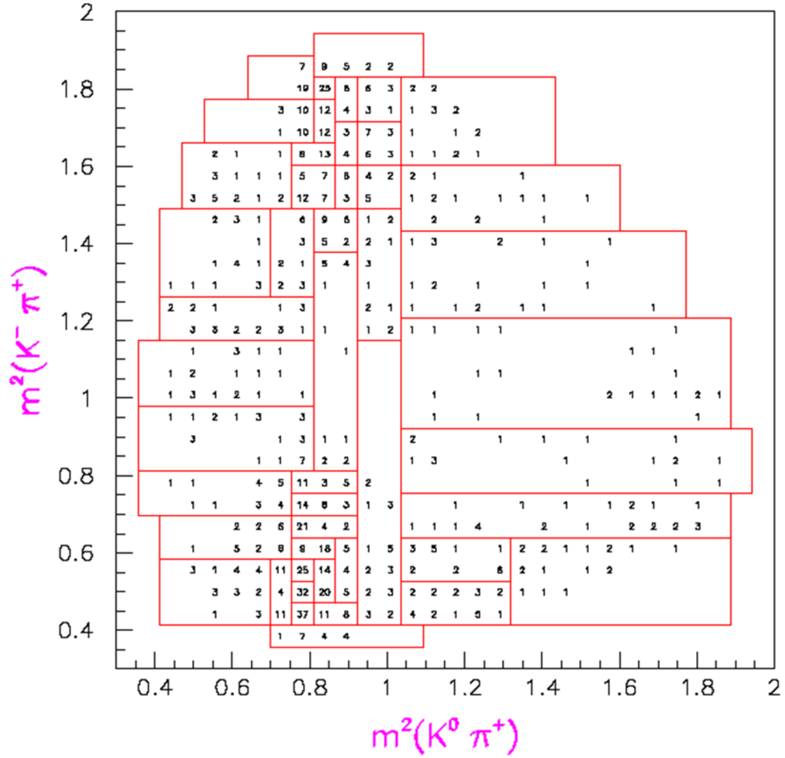
Problems

the cut-off method may deplete important regions of the plot to much

circumvent this by using a bin-by-bin Poisson-test for these areas

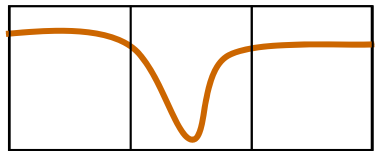
cut-off





alternatively: adaptive Dalitz plots, but one may miss narrow depleted regions, like the $f_0(980)$ dip

systematic choice-of-binning-errors



another Caveat in χ^2 Fits of Dalitzplots

Don‘t forget the non-statistical bin-by-bin errors

statistical error from the MC events

systematic error of a MC efficiency parameterization

statistical error (propagation) from a background subtraction

systematic error from background parameterization

Finite Resolution

Due to resolution or wrong matching:

- True phase space coordinates of MC events are different from the reconstructed coordinates
- In principle amplitudes of MC-events have to be calculated at the generated coordinate, not the reconstructed location
- But they are plotted at the reconstructed location

Applies to:

- Experiments with “bad” resolution (like Asterix)
- For narrow resonances [like Φ or $f_1(1285)$ or $f_0(980)$]
- Wrongly matched tracks

Cures phase-smearing and non-isotropic resolution effects

Is one more resonance significant ?

Base your decision on

- objective bin-by-bin χ^2 and χ^2/N_{dof}
- visual quality
  - is the trend right?
  - is there an imbalance between different regions
- compatibility with expected ΔL structure

Produce Toy MC for Likelihood Evaluation

- many sets with full efficiency and Dalitz plot fit
  - each set of events with various amplitude hypotheses
  - calc ΔL expectation
- ΔL expectation is usually not just ½/dof
  - sometimes adding a wrong (not necessary) resonance can lead to values over 100!
  - compare this with data

Result: a probability for your hypothesis!

ToyMC Significance Test

Your experiment may yield a certain likelihood pattern

- Hypo 1 $-\log L_1=-5123$
- Hypo 2 $-\log L_2=-4987$ ($\Delta L=136$)
- Hypo 3 $-\log L_3=-4877$ ($\Delta L=110$)

Is Hypo 3 really needed? What is the significance

- ToyMC create independent toy data sets which have exactly the same composition as solutions 1,2 and 3
- If 3 is the right solution find out how often $-\log L_3$ is smaller than $-\log L_2$, the percentage gives the confidence level → significance

α	δ
0.3173	1σ
4.55×10^{-2}	2σ
2.7×10^{-3}	3σ
6.3×10^{-5}	4σ
5.7×10^{-7}	5σ
2.0×10^{-9}	6σ

α	δ
0.2	1.28σ
0.1	1.64σ
0.05	1.96σ
0.01	2.58σ
0.001	3.29σ
10^{-4}	3.89σ

table from PDG06 for ±δ

Indications for a bad solution

Plus

one indication can be a large branching fraction of interference terms

Definition of BF of channel j

$$BF_j = \int |A_j|^2 d\Omega \Big/ \int |\sum_i A_i|^2$$

But due to interferences, something is missing

Incoherent $I=|A|^2+|B|^2$

Coherent $I=|A+e^{i\varphi}B|^2 = |A|^2+|B|^2 + 2[\mathrm{Re}(AB^*)\sin\varphi+\mathrm{Im}(AB^*)\cos\varphi]$

If $\sum_j BF_j$ is much different from 100% there might be a problem

- The sum of interference terms must vanish if integrated from $-\infty$ to $+\infty$
- But phase space limits this region
- If the resonances are almost covered by phase space then the argument holds...
- ...and large residual interference intensities signal overfitting

Where to stop

Apart from what was said before

Additional hypothetical trees (resonances, mechanisms) do not improve the description considerably

Don‘t try to parameterize your data with inconsistent techniques

If the model don‘t match, the model might be the problem
reiterate with a better model

Performance Issues

Problem:

Slow convergence

Solution(s):

proper parameterizations

calculate only function branches which depend
on the actually changed parameter

multi-stage fits, increasing number of free parameters

intermediate steps are unimportant, stop early! $\Delta\chi^2$ cut-off

oscillation around the minimum with decreasing distance
due to numerical deriviatives

- may improve with analytical expressions (rarely done)
- more speed by approximating second derivative (FUMILI)
  (wrong for phases! only Re/Im-parameterizations!)

Determining Branching Fractions

QM prevents us from explicitly saying which
slit was more often used than the other one

Dealing with interferences

No correct way to determine the relative couplings
in fits without a coupled channel approach

Even with K-Matrix approach, the
couplings are at K-Matrix-poles and don’t have a priori meaning
→ Residues of the Singularities of the T-Matrix

Other important topics

Amplitude calculation

Symbolic amplitude manipulations (Mathematica, etc.)

On-the-fly amplitude construction (qft++,…etc.)

CPU demand

Minimization strategies and derivatives → GPUs

Coupled channel implementation

Variants, Pros and Cons

Numerical instabilities

Unitarity constraints

Constraining ambiguous solutions with external information

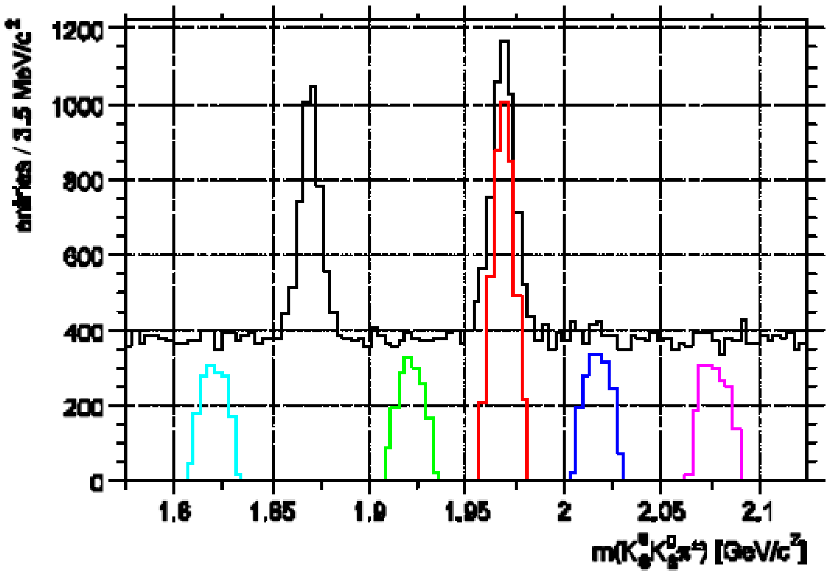
Constraining resonance parameters

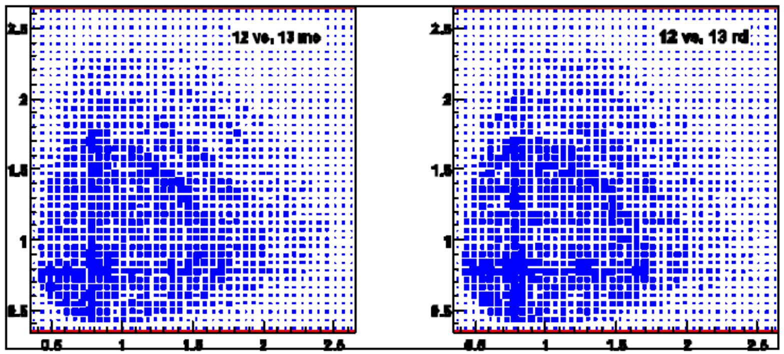
systematic impact if wrong masses are used

Background

Various possibilities (depending on data and process) to account for background

- as part of the data preparation
  - subtraction of background phase space distributions (from MC)
  - subtraction of background phase space distributions (from sidebands)
- background hypotheses as part of the model
  - functional description (parameterized distribution)
  - either form MC or from extra- or interpolated sidebands
  - (or multidimensional extensions like 9-tile etc.)





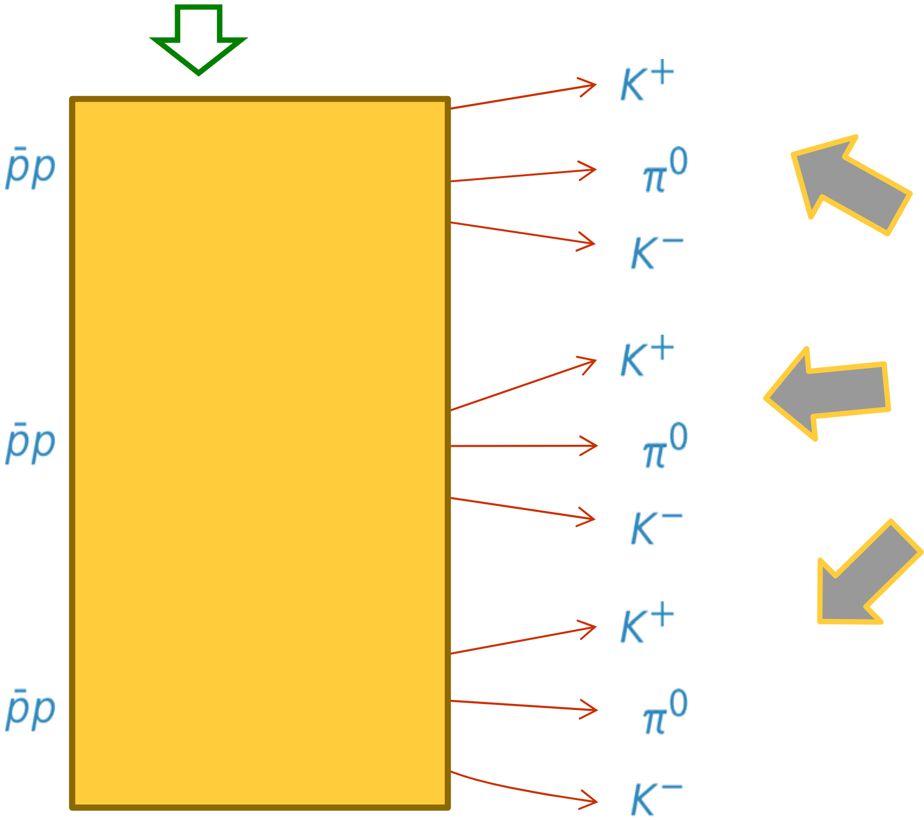
Next Generation PWA Software

see poster!!

The Need for Partial Wave Analysis

Example: Consider the reaction $\bar{p}p \to K^+K^-\pi^0$

What *really* happened...



What you see is always the same ...

...
etc.

PWA = technique to find out what happens in between

Lot of material

Use what you have learned,

but use it

and use it with care

THANK YOU

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